1. **Solve $(x^3 - 3xy^2)dx + (3x^2y - y^3)dy = 0$.**

Sol: $(x^3 - 3xy^2)dx + (3x^2y - y^3)dy = 0.$

$\Rightarrow \frac{dy}{dx} = -\frac{(x^3-3xy^2)}{(3x^2y-y^3)}$..... (1)

this is homogenous D.E

$let\ y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

Eq'n (1) $\Rightarrow v + x\frac{dv}{dx} = -\frac{x^3-3x(vx)^2}{3x^2(vx)-(vx)^3}$

$\Rightarrow v + x\frac{dv}{dx} = -\frac{x^3(1-3v^2)}{x^3(3v-v^3)}$

$\Rightarrow v + x\frac{dv}{dx} = -\left(\frac{1-3v^2}{3v-v^3}\right)$

$\Rightarrow x\frac{dv}{dx} = \frac{-1+3v^2}{3v-v^3} - \frac{v}{1}$

$\Rightarrow x\frac{dv}{dx} = \frac{-1+3v^2-3v^2+v^4}{3v-v^3}$

$\Rightarrow x\frac{dv}{dx} = \left(\frac{v^4-1}{3v-v^3}\right)$

$\Rightarrow \int \frac{3v-v^3}{v^4-1}dv = \int \frac{1}{x}dx$

By partial fraction

$\Rightarrow \int \frac{3v-v^3}{(v-1)(v+1)(v^2+1)}dv = \int \frac{1}{x}dx$

$\left[\frac{1}{2(v+1)} + \frac{1}{2(v-1)} - \frac{2v}{(v^2+1)}\right]dv = \int \frac{1}{x}dx$

$\Rightarrow \frac{1}{2}\log|v+1| + \frac{1}{2}\log|v-1| - \log|v^2+1| = \log x + logc$

$\Rightarrow \log\left|\frac{\sqrt{v+1}\sqrt{v-1}}{v^2+1}\right| = \log(cx)$

$\Rightarrow \frac{\sqrt{v+1}\sqrt{v-1}}{v^2+1} = cx \quad \Rightarrow \frac{\sqrt{v^2-1}}{v^2+1} = cx$

$\Rightarrow\Rightarrow \frac{v^2-1}{(v^2+1)^2} = (cx)^2$

$\Rightarrow \frac{(y^2-x^2)}{x^2} = c^2x^2\frac{(y^2+x^2)^2}{x^4}$

$\Rightarrow (y^2 - x^2) = (y^2 + x^2)^2$

Which is required general solution.

2. **Given the solution of $x\sin^2\left(\frac{y}{x}\right)dx = ydx - xdy$ Which passes through the point $(1, \frac{\pi}{4})$**

Sol:

$x\sin^2\left(\frac{y}{x}\right)dx = ydx - xdy$

$\Rightarrow xdy = ydx - x\sin^2(\frac{y}{x})dx$

$\Rightarrow xdy = \left[y - x\sin^2\left(\frac{y}{x}\right)\right]dx$

$\Rightarrow dy = \left[\frac{y}{x} - \frac{x\sin^2\left(\frac{y}{x}\right)}{x}\right]dx$

$\Rightarrow \frac{dy}{dx} = \left[\frac{y}{x} - \sin^2\left(\frac{y}{x}\right)\right]$ .....

$let\ y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$\Rightarrow v + x\frac{dv}{dx} = v - \sin^2(v)$

$\Rightarrow x\frac{dv}{dx} = -\sin^2(v)$

$\Rightarrow \int \frac{1}{\sin^2(v)}dv = -\int \frac{1}{x}dx$

$\Rightarrow \int cosec^2 v\, dv = -\int \frac{1}{x}dx$

$\Rightarrow -cotv = -logx + c$

$\Rightarrow -\cot\left(\frac{y}{x}\right) = -logx + c$

this is passing Through the point $(1, \frac{\pi}{4})$

$\Rightarrow -\cot\left(\frac{\pi}{4}\right) = -log1 + c$

$\Rightarrow -1 = 0 + c \Rightarrow c = -1$

$\therefore -\cot\left(\frac{y}{x}\right) = -logx - 1$

3. *solve the differential equation* $\frac{dy}{dx}=\frac{x-y+3}{2x-2y+5}$.

Sol: $\frac{dy}{dx}=\frac{x-y+3}{2x-2y+5}$

$\left[\frac{a}{a'}=\frac{b}{b'}\right]$

$this\ is\ non-homogeneous\ D.E\ of\ case(2)$

$\frac{dy}{dx}=\frac{x-y+3}{2(x-y)+5}$ .....

$let\ (x-y)=v \Rightarrow 1-\frac{dy}{dx}=\frac{dv}{dx}$

$\Rightarrow \frac{dy}{dx}=1-\frac{dv}{dx}$

Now eq''n (1) becomes

$\Rightarrow 1-\frac{dv}{dx}=\frac{v+3}{2v+5}$

$\Rightarrow 1-\frac{v+3}{2v+5}=\frac{dv}{dx}$

$\Rightarrow \frac{2v+5-v-3}{2v+5}=\frac{dv}{dx}$

$\Rightarrow \frac{v+2}{2v+5}=\frac{dv}{dx}$

$\Rightarrow \int 1dx=\int \frac{2v+5}{v+2}dv$

$\Rightarrow \int 1dx=\int \frac{2v+4+1}{v+2}dv$

$\Rightarrow \int 1dx=\int \frac{2(v+2)+1}{v+2}dv$

$\Rightarrow \int 1dx=\int \left(\frac{2(v+2)}{v+2}+\frac{1}{v+2}\right)dv$

$\Rightarrow \int 1dx=\int \left(2+\frac{1}{v+2}\right)dv$

$\Rightarrow x=2v+\log(v+2)+c$

$\Rightarrow x=2(x-y)+\log(x-y+2)+c$

$\therefore x-2y+\log(x-y+2)=c$

4. *solve the diffrential equation* $(2x+y+1)dx+(4x+2y-1)dy=0$.

Sol: $\frac{dy}{dx}=\frac{-(2x+y+1)}{4x+2y-1}$

$$\left[\frac{a}{a'}=\frac{b}{b'}\right]$$

$this\ is\ non-homogeneous\ D.E\ of\ case(2)$

$\frac{dy}{dx}=\frac{-(2x+y+1)}{4x+2y-1}$ .....(1)

$let\ (2x+y)=v \Rightarrow 2+\frac{dy}{dx}=\frac{dv}{dx}$

$\Rightarrow \frac{dy}{dx}=\frac{dv}{dx}-2$

Now eq''n (1) becomes

$\Rightarrow \frac{dv}{dx}-2=\frac{-(v+1)}{2v-1}$

$\Rightarrow \frac{dv}{dx}=\frac{-v-1}{2v-1}+2$

$\Rightarrow \frac{dv}{dx}=\frac{-v-1+4v-2}{2v-1}$

$\Rightarrow \frac{dv}{dx}=\frac{3v-3}{2v-1}$

$\Rightarrow \int \frac{2v-1}{v-1}dv=\int 3dx$

$\Rightarrow \int \frac{2(v-1)+1}{v-1}dv=\int 3dx$

$\Rightarrow \int \left(\frac{2(v-1)}{v-1}+\frac{1}{v-1}\right)dv=\int 3dx$

$\Rightarrow \int 3dx=\int \left(2+\frac{1}{v-1}\right)dv$

$\Rightarrow 3x=2v+\log(v-1)+c$

$\Rightarrow 3x=2(2x+y)+\log(2x+y-1)+c$

$\Rightarrow 3x=4x+2y+\log(2x+y-1)+c$

$\therefore x+2y+\log(2x+y-1)=c$

5. $\boldsymbol{solve\ \frac{dy}{dx}=\frac{2x+y+3}{2y+x+1}}$.

Sol: $Given\ eq''n\frac{dy}{dx}=\frac{2x+y+3}{2y+x+1}[a/a'\neq b/b']$

$this\ is\ non-homogeneous\ D.E\ of\ case(3)$

$\boldsymbol{put\ x=X+h\ and\ y=Y+k}$

$\frac{dy}{dx}=\frac{2x+y+3}{2y+x+1}$

$\Rightarrow \frac{dY}{dX}=\frac{2(X+h)+(Y+k)+3}{2(Y+k)+(X+h)+1}$

$\Rightarrow \frac{dY}{dX}=\frac{2X+Y+(2h+k+3)}{X+2Y+(h+2k+1)}\ldots(*)$

$now\ choose\ h\ and\ k\ such\ that$

$2h+k+3=0\ \ldots(1)$

and

$h+2k+1=0\ldots(2)$

$solving\ (1)\&\ (2)$

2 1 3 2

1 2 1 1

$(h,k)=\left[\frac{1-6}{4-1},\frac{3-2}{4-1}\right]=\left[-\frac{5}{3},\frac{1}{3}\right]$

$Hence\ (*)becomes$

$\Rightarrow \frac{dY}{dX}=\frac{2X+Y}{X+2Y}$ is a homogeneous equation.

$put\ y=VX\Rightarrow \frac{dY}{dX}=V+\frac{XdV}{dX}$

$\Rightarrow V+\frac{XdV}{dX}=\frac{2X+VX}{X+2VX}$

$\Rightarrow V+\frac{XdV}{dX}=\frac{X(2+V)}{X(1+2V)}$

$\Rightarrow V+\frac{XdV}{dX}=\frac{(2+V)}{(1+2V)}$

$\Rightarrow \frac{XdV}{dX}=\frac{(2+V)}{(1+2V)}-\frac{V}{1}$

$\Rightarrow \frac{XdV}{dX}=\frac{2+V-V-2V^2}{(1+2V)}$

$\Rightarrow \frac{XdV}{dX}=\frac{2-2V^2}{(1+2V)}$

$\Rightarrow \frac{XdV}{dX}=\frac{2(1-V^2)}{(1+2V)}$

$\Rightarrow\left[-\frac{1}{2(1+V)}+\frac{3}{2(1-V)}\right]dV=2\frac{dX}{X}$

$\Rightarrow -\frac{1}{2}\int\frac{1}{(1+V)}dV+\frac{3}{2}\int\frac{1}{(1-V)}dV=2\int\frac{1}{X}dX$

$\Rightarrow -\frac{1}{2}\log|1+V|-\frac{3}{2}\log|1-V|=2\log X+\log C$

$\Rightarrow \log|1+V|+3\log|1-V|=-4\log X+\log C$

$\Rightarrow \log(1+V)(1-V)^3=\log(C/X^4)$

$\Rightarrow(1+V)(1-V)^3=\frac{C}{X^4}\ \{V=\frac{Y}{X}\ \}$

$\Rightarrow\left(1+\frac{Y}{X}\right)\left(1-\frac{Y}{X}\right)^3=\frac{C}{X^4}$

$\Rightarrow\frac{(X+Y)(X-Y)^3}{X^4}=\frac{C}{X^4}$

$\Rightarrow(X+Y)(X-Y)^3=C$

$(h,k)=\left[-\frac{5}{3},\frac{1}{3}\right]\ X=x+\frac{5}{3},Y=y-\frac{1}{3}$

$\Rightarrow\left(x+\frac{5}{3}+y-\frac{1}{3}\right)\left(x+\frac{5}{3}-y+\frac{1}{3}\right)^3=C$

$\Rightarrow\left(x+y+\frac{4}{3}\right)(x-y+2)^3=C$

$(3x+3y+4)(x+-y+2)^3=3C$

This is the required solution.

6. $\boldsymbol{solve\ \frac{dy}{dx}=\frac{3y-7x+7}{3x-7y-3}}$.

Sol: $Given\ eq''n\frac{dy}{dx}=\frac{3y-7x+7}{3x-7y-3}[a/a'\neq b/b']$

$this\ is\ non-homogeneous\ D.E\ of\ case(3)$

$\boldsymbol{put\ x=X+h\ and\ y=Y+k}$

$\frac{dy}{dx}=\frac{3y-7x+7}{3x-7y-3}$

$\Rightarrow \frac{dY}{dX}=\frac{3(Y+k)-7(X+h)+7}{3(X+h)-7(Y+k)-3}$

$\Rightarrow \frac{dY}{dX}=\frac{-7X+3Y+(-7h+3k+7)}{3X-7Y+(3h-7k-3)}\ldots(*)$

$now\ choose\ h\ and\ k\ such\ that$

$-7h+3k+7=0\ \ldots(1)$

and

$3h-7k-3=0\ldots(2)$

$solving\ (1)\&\ (2)$

-7 3 7 -7

3 -7 -3 3

$(h,k)=\left[\frac{-9+49}{49-9},\frac{21-21}{49-9}\right]=[1,0]$

$Hence\ (*)becomes$

$\Rightarrow \frac{dY}{dX}=\frac{-7X+3Y}{3X-7Y}$ is a homogeneous equation.

$put\ y=VX\Rightarrow \frac{dY}{dX}=V+\frac{XdV}{dX}$

$\Rightarrow V+\frac{XdV}{dX}=\frac{-7X+3VX}{3X-7VX}$

$\Rightarrow V+\frac{XdV}{dX}=\frac{X(-7+3V)}{X(3-7V)}$

$\Rightarrow V+\frac{XdV}{dX}=\frac{(-7+3V)}{(3-7V)}$

$\Rightarrow \frac{XdV}{dX}=\frac{(-7+3V)}{(3-7V)}-\frac{V}{1}$

$\Rightarrow \frac{XdV}{dX}=\frac{-7+3V-3V+7V^2}{(3-7V)}$

$\Rightarrow \frac{XdV}{dX}=\frac{-7+7V^2}{3-7V}$

$\Rightarrow \frac{XdV}{dX}=\frac{-7(V^2-1)}{3-7V}$

$\Rightarrow \int\frac{3-7V}{V^2-1}dV=-7\frac{dX}{X}$

$\Rightarrow \left[-\frac{1}{2(1+V)}+\frac{3}{2(1-V)}\right]dV=-7\frac{dX}{X}$

$\Rightarrow -\frac{1}{2}\int\frac{1}{(1+V)}dV+\frac{3}{2}\int\frac{1}{(1-V)}dV=2\int\frac{1}{X}dX$

$\Rightarrow -\frac{1}{2}\log|1+V|-\frac{3}{2}\log|1-V|=2\log X+\log C$

$\Rightarrow \log|1+V|+3\log|1-V|=-4\log X+\log C$

$\Rightarrow \log(1+V)(1-V)^3=\log(C/X^4)$

$\Rightarrow (1+V)(1-V)^3=\frac{C}{X^4}\ \{V=\frac{Y}{X}\ \}$

$\Rightarrow \left(1+\frac{Y}{X}\right)\left(1-\frac{Y}{X}\right)^3=\frac{C}{X^4}$

$\Rightarrow \frac{(X+Y)(X-Y)^3}{X^4}=\frac{C}{X^4}$

$\Rightarrow (X+Y)(X-Y)^3=C$

$(h,k)=\left[-\frac{5}{3},\frac{1}{3}\right]\ X=x+\frac{5}{3},Y=y-\frac{1}{3}$

$\Rightarrow \left(x+\frac{5}{3}+y-\frac{1}{3}\right)\left(x+\frac{5}{3}-y+\frac{1}{3}\right)^3=C$

$(3x+3y+4)(x+-y+2)^3=3C$

This is the required solution.

7. *solve the diffrential equation*

$\frac{dy}{dx} = \frac{x+2y+3}{2x+3y+4}$

Sol: $Given\ eq''n \frac{dy}{dx} = \frac{x+2y+3}{2x+3y+4} [a/a' \neq b/b']$

$this\ is\ non-homogeneous\ D.E\ of\ case(3)$

$\boldsymbol{put\ x = X + h\ and\ y = Y + k}$

$\frac{dy}{dx} = \frac{x+2y+3}{2x+3y+4}$

$\Rightarrow \frac{dY}{dX} = \frac{(X+h)+2(Y+k)+3}{2(X+h)+(Y+k)+4}$

$\Rightarrow \frac{dY}{dX} = \frac{X+2Y+(h+2k+3)}{2X+3Y+(2h+3k+4)} \ldots (*)$

$now\ choose\ h\ and\ k\ such\ that$

$h + 2k + 3 = 0 \ \ldots (1)$
and
$2h + 3k + 4 = 0 \ldots (2)$

$solving\ (1)\&\ (2)$

1 2 3 1

2 3 4 2

$(h, k) = \left[\frac{8-9}{3-4}, \frac{6-4}{3-4}\right] = [1, -2]$

$Hence\ (*) becomes$

$\Rightarrow \frac{dY}{dX} = \frac{X+2Y}{2X+3Y}$ is a homogeneous equation.

$put\ y = VX \Rightarrow \frac{dY}{dX} = V + \frac{XdV}{dX}$

$\Rightarrow V + \frac{XdV}{dX} = \frac{X+2VX}{2X+3VX}$

$\Rightarrow V + \frac{XdV}{dX} = \frac{X(1+2V)}{X(2+3V)}$

$\Rightarrow V + \frac{XdV}{dX} = \frac{(1+2V)}{(2+3V)}$

$\Rightarrow \frac{XdV}{dX} = \frac{(1+2V)}{(2+3V)} - \frac{V}{1}$

$\Rightarrow \frac{XdV}{dX} = \frac{1+2V-2V-3V^2}{(2+3V)}$

$\Rightarrow \frac{XdV}{dX} = \frac{1-3V^2}{2+3V}$

$\Rightarrow \int \frac{2+3V}{1-3V^2} dV = \frac{dX}{X}$

$\Rightarrow \left[\frac{2}{1-3V^2} + \frac{3V}{1-3V^2}\right] dV = \frac{dX}{X}$

$\Rightarrow \frac{2}{3} \int \frac{1}{\left[\frac{1}{\sqrt{3}}\right]^2 - V^2} dV - \frac{1}{2} \int \frac{6V}{1-3V^2} dV = \int \frac{1}{X} dX$

$\Rightarrow \frac{2}{3} . \frac{1}{2(\frac{1}{\sqrt{3}})} \log \left|\frac{\frac{1}{\sqrt{3}}+V}{\frac{1}{\sqrt{3}}-V}\right| - \frac{1}{2} \log|1 - 3V^2| = \log CX$

$\Rightarrow \frac{1}{(\sqrt{3})} \log \left|\frac{1+\sqrt{3}V}{1-\sqrt{3}V}\right| - \frac{1}{2} \log|1 - 3V^2| = \log CX$

$\Rightarrow \frac{1}{(\sqrt{3})} \log \left|\frac{1+\sqrt{3}\left(\frac{Y}{X}\right)}{1-\sqrt{3}\left(\frac{Y}{X}\right)}\right| - \frac{1}{2} \log \left|1 - 3\left(\frac{Y}{X}\right)^2\right| = \log CX$

$\Rightarrow \frac{1}{(\sqrt{3})} \log \left|\frac{X+\sqrt{3}Y}{X-\sqrt{3}Y}\right| - \frac{1}{2} \log \left|\frac{X^2-3Y^2}{X^2}\right| = \log CX$

$\boldsymbol{X = x + 1\ and Y = y + 2}$

$\Rightarrow \frac{1}{(\sqrt{3})} \log \left|\frac{x+1+\sqrt{3}(y-2)}{x+1-\sqrt{3}(y-2)}\right| - \frac{1}{2} \log \left|\frac{(x+1)^2-3(y-2)^2}{(x+1)^2}\right| = \log CX$