1. **Solve** \((x^3 - 3xy^2)dx + (3x^2y - y^3)dy = 0.\)

   **Sol:** \((x^3 - 3xy^2)dx + (3x^2y - y^3)dy = 0.\)

   \[\frac{dy}{dx} = -\frac{x^3 - 3xy^2}{3x^2y - y^3} \ldots \ldots \ldots \text{(1)}\]

   This is homogenous D.E.

   Let \(y = vx\) ⇒ \[\frac{dy}{dx} = v + x \frac{dv}{dx}\]

   **Eq’n (1) ⇒** \[v + x \frac{dv}{dx} = -\frac{x^3(1 - 3v^2)}{3x^2(vx) - (vx)^3}\]

   ⇒ \[v + x \frac{dv}{dx} = -(\frac{1 - 3v^2}{3v^3 - v})\]

   ⇒ \[\frac{dv}{dx} = -\frac{1 + 3v^2}{3v^3 - v^2} - \frac{1}{x}\]

   ⇒ \[\frac{dv}{dx} = -\frac{1 + 3v^2 - 3v^2 + v^4}{3v^3 - v^2}\]

   ⇒ \[\frac{dv}{dx} = \left(\frac{v^4 - 1}{3v - 1}\right)\]

   ⇒ \[\int \frac{3v - 1}{v^4 - 1} dv = \int \frac{1}{x} dx\]

   \[\int \left(\frac{3v - 1}{(v-1)(v+1)(v^2+1)}\right) dv = \int \frac{1}{x} dx\]

   \[\left[\frac{1}{2(v+1)} + \frac{1}{2(v-1)} - \frac{2v}{(v^2+1)}\right] dv = \int \frac{1}{x} dx\]

   ⇒ \[\frac{1}{2} \log |v + 1| + \frac{1}{2} \log |v - 1| - \log |v^2 + 1| = \log x + \log c\]

   \[\log \left|\frac{\sqrt{v+1}\sqrt{v-1}}{v^2+1}\right| = \log (cx)\]

   \[\frac{\sqrt{v+1}\sqrt{v-1}}{v^2+1} = cx \Rightarrow \frac{\sqrt{v^2-1}}{v^2+1} = cx\]

   ⇒ \[\frac{v^2 - 1}{v^2 + 1} = (cx)^2\]

   \[\frac{(y^2 - x^2)}{x^2} = c^2 x^2 \left(\frac{(y^2 + x^2)^2}{x^2}\right)\]

   \[\Rightarrow (y^2 - x^2) = (y^2 + x^2)^2\]

   Which is required general solution.

2. **Given the solution of** \(x \sin^2 \left(\frac{y}{x}\right) dx = y dx - x dy\) Which passes through the point \((1, \frac{\pi}{4})\)

   **Sol:**

   \[x \sin^2 \left(\frac{y}{x}\right) dx = y dx - x dy\]

   ⇒ \[\int x \sin^2 \left(\frac{y}{x}\right) dx = \int (y - x \sin \left(\frac{y}{x}\right)) dy\]

   ⇒ \[\int \frac{y}{x} dx = \left[\frac{y}{x} - \sin \left(\frac{y}{x}\right)\right] dx\]

   Let \(y = vx\) ⇒ \[\frac{dy}{dx} = v + x \frac{dv}{dx}\]

   ⇒ \[v + x \frac{dv}{dx} = y - \sin \left(\frac{y}{x}\right)\]

   ⇒ \[\frac{dv}{dx} = -\sin \left(\frac{y}{x}\right)\]

   ⇒ \[\int \frac{1}{\sin \left(\frac{y}{x}\right)} dv = - \int \frac{1}{x} dx\]

   ⇒ \[\int \csc^2 v dv = - \int \frac{1}{x} dx\]

   ⇒ \[\cot v = -\log x + c\]

   ⇒ \[\cot \left(\frac{\pi}{4}\right) = -\log x + c\]

   ∴ \(-\cot \left(\frac{\pi}{4}\right) = -\log 1 + c\)

   \(-1 = 0 + c \Rightarrow c = -1\)

   \[-\cot \left(\frac{\pi}{4}\right) = -\log x - 1\]

   This is passing through the point \((1, \frac{\pi}{4})\)
3. solve the differential equation
\[ \frac{dy}{dx} = \frac{x-y+3}{2x-2y+5} \]
Sol: \[ \frac{dy}{dx} = \frac{x-y+3}{2x-2y+5} \]
\[ [\frac{a}{a'} = \frac{b}{b'}] \]
this is non – homogeneous D.E of case(2)
\[ \frac{dy}{dx} = \frac{x-y+3}{2(x-y)+5} \]
let \((x-y) = v \Rightarrow 1 - \frac{dy}{dx} = \frac{dv}{dx}
\[ \Rightarrow \frac{dy}{dx} = 1 - \frac{dv}{dx} \]
Now eq’n (1) becomes
\[ \Rightarrow 1 - \frac{dv}{dx} = \frac{v+3}{2v+5} \]
\[ \Rightarrow 1 - \frac{v+3}{2v+5} = \frac{dv}{dx} \]
\[ \Rightarrow 2v+5-v-3 = \frac{dv}{dx} \]
\[ \Rightarrow v+2 = \frac{dv}{dx} \]
\[ \Rightarrow \int 1 \, dx = \int \frac{2v+5}{v+2} \, dv \]
\[ \Rightarrow \int 1 \, dx = \int \frac{2v+4+1}{v+2} \, dv \]
\[ \Rightarrow \int 1 \, dx = \int \frac{2(v+2)+1}{v+2} \, dv \]
\[ \Rightarrow \int 1 \, dx = \int \left( \frac{2(v+2)}{v+2} + \frac{1}{v+2} \right) \, dv \]
\[ \Rightarrow \int 1 \, dx = \int \left( 2 + \frac{1}{v+2} \right) \, dv \]
\[ \Rightarrow x = 2v + \log(v + 2) + c \]
\[ \Rightarrow x = 2(x - y) + \log(x - y + 2) + c \]
\[ \therefore x - 2y + \log(x - y + 2) = c \]

4. solve the differential equation
\[ (2x + y + 1)dx + (4x + 2y - 1)dy = 0. \]
Sol: \[ \frac{dy}{dx} = \frac{-(2x+y+1)}{4x+2y-1} \]
\[ [\frac{a}{a'} = \frac{b}{b'}] \]
this is non – homogeneous D.E of case(2)
\[ \frac{dy}{dx} = \frac{-(2x+y+1)}{4x+2y-1} \]
let \((2x + y) = v \Rightarrow 2 + \frac{dy}{dx} = \frac{dv}{dx}
\[ \Rightarrow \frac{dy}{dx} = 2 - \frac{v+1}{2v-1} \]
Now eq’n (1) becomes
\[ \Rightarrow \frac{dv}{dx} = 2 - \frac{v+1}{2v-1} \]
\[ \Rightarrow \frac{dv}{dx} = \frac{3v-3}{2v-1} \]
\[ \Rightarrow \int \frac{2v-1}{v-1} \, dv = \int 3 \, dx \]
\[ \Rightarrow \int 2(v-1) \, dv = \int 3 \, dx \]
\[ \Rightarrow \int \left( \frac{2(v-1)}{v-1} + \frac{1}{v-1} \right) \, dv = \int 3 \, dx \]
\[ \Rightarrow \int 3 \, dx = \int \left( 2 + \frac{1}{v-1} \right) \, dv \]
\[ \Rightarrow 3x = 2v + \log(v - 1) + c \]
\[ \Rightarrow 3x = 2(2x + y) + \log(2x + y - 1) + c \]
\[ \Rightarrow 3x = 4x + 2y + \log(2x + y - 1) + c \]
\[ \Rightarrow x + 2y + \log(2x + y - 1) = c \]
5. Solve \( \frac{dy}{dx} = \frac{2x+y+3}{2y+x+1} \).

Sol: Given eq”n \( \frac{dy}{dx} = \frac{2x+y+3}{2y+x+1} [a/a' \neq b/b'] 

this is non – homogeneous D.E of case(3)

put \( x = X + h \) and \( y = Y + k \)

\( \frac{dy}{dx} = \frac{2x+y+3}{2y+x+1} \)

\( \Rightarrow \frac{dy}{dx} = \frac{2(X+h)+(Y+k)+3}{2(Y+k)+(X+h)+1} \)

\( \Rightarrow \frac{dy}{dx} = \frac{2X+Y+(2h+k+3)}{X+2Y+(h+2k+1)} \) \( \cdots \) (*)

now choose \( h \) and \( k \) such that

\( 2h + k + 3 = 0 \) ... (1)
and

\( h + 2k + 1 = 0 \) ... (2)

solving (1)& (2)

\[ \begin{array}{cccc}
2 & 1 & 3 & 2 \\
1 & 2 & 1 & 1 \\
\end{array} \]

\( (h, k) = \left[ \begin{array}{cc}
1 & -6 \\
3 & -2 \\
4 & -1 \\
\end{array} \right] = \left[ \begin{array}{cc}
\frac{5}{3} & 1 \\
\frac{1}{3} & 3 \\
\end{array} \right] \)

Hence (*) becomes

\( \Rightarrow \frac{dy}{dx} = \frac{2x+y}{x+2y} \) is a homogeneous equation.

put \( y = VX \) \( \Rightarrow \frac{dy}{dx} = V + \frac{x}{dV}{dx} \)

\( \Rightarrow V + \frac{x}{dV}{dx} = \frac{2x+y}{x+2y} \)

\( \Rightarrow V + \frac{x}{dV}{dx} = \frac{X(2+V)}{X(1+2V)} \)

\( \Rightarrow V + \frac{x}{dV}{dx} = \frac{2+V}{(1+2V)} \)

\( \Rightarrow X \frac{dV}{dX} = \frac{2+V}{(1+2V)} \cdots \frac{V}{1} \)

\( \Rightarrow X \frac{dV}{dX} = \frac{2+V}{(1+2V)} \cdots \frac{2+V}{(1+2V)} \)

\( \Rightarrow X \frac{dV}{dX} = \frac{2-2V^2}{(1+2V)} \)

\( \Rightarrow X \frac{dV}{dX} = \frac{2(1-V^2)}{1+2V} \)

\( \Rightarrow \left[ -\frac{1}{2} + \frac{3}{2(1+V)} \right] dv = 2 \frac{dx}{x} \)

\( \Rightarrow -\frac{1}{2} \int \frac{1}{(1+V)} dv + \frac{3}{2} \int \frac{1}{1-V} dv = 2 \int \frac{1}{x} dx \)

\( \Rightarrow -\frac{1}{2} \log|1 + V| - \frac{3}{2} \log|1 - V| = 2 \log X + \log C \)

\( \Rightarrow \log|1 + V| + 3 \log|1 - V| = -4 \log X + \log C \)

\( \Rightarrow (1 + V)(1 - V)^3 = \log(C/X^4) \)

\( \Rightarrow (1 + V)(1 - V)^3 = \frac{c}{x} \) \( \{ V = \frac{y}{x} \} \)

\( \Rightarrow \left( 1 + \frac{y}{x} \right) \left( 1 - \frac{y}{x} \right)^3 = \frac{c}{x^4} \)

\( \Rightarrow (X + Y)(X - Y)^3 = C \)

\( (h, k) = \left[ \begin{array}{cc}
-\frac{5}{3} & 1 \\
\frac{1}{3} & 3 \\
\end{array} \right] \)

\( \Rightarrow (x + y = \frac{5}{3}) (x + y + \frac{1}{3})^3 = C \)

\( \Rightarrow (x + y + \frac{4}{3}) (x - y + 2)^3 = C \)

(3x + 3y + 4)(x + y + 2) = 3C

This is the required solution.
6. solve \( \frac{dy}{dx} = \frac{3y-7x+7}{3x-7y-3} \)

Sol: Given \( \frac{dy}{dx} = \frac{3y-7x+7}{3x-7y-3} \left[ \frac{a}{a'} \neq b/b' \right] \)

this is non-homogeneous D.E of case (3)

put \( x = X + h \) and \( y = Y + k \)

\[
\frac{dy}{dx} = \frac{3y-7x+7}{3x-7y-3} \\
\Rightarrow \frac{dy}{dx} = \frac{3(Y+k)-7(X+h)+7}{3(X+h)-7(Y+k)-3} \\
\Rightarrow \frac{dy}{dx} = \frac{-7X+3Y+(-7h+3k+7)}{3X-7Y+(3h-7k-3)} \quad \ldots (*) \]

now choose \( h \) and \( k \) such that

\(-7h + 3k + 7 = 0 \ldots (1) \)

and

\(3h - 7k - 3 = 0 \ldots (2)\)

solving (1) & (2)

\[
\begin{align*}
-7 & \quad 3 \quad 7 \quad -7 \\
3 & \quad -7 \quad -3 \quad 3
\end{align*}
\]

\((h, k) = \left[\begin{array}{c} -9+49 \\
49-9 \end{array} \right] = [1, 0] \]

Hence (*) becomes

\[
\Rightarrow \frac{dy}{dx} = \frac{-7X+3Y}{3X-7Y} \text{ is a homogeneous equation.}
\]

put \( VX = \frac{dy}{dx} = V + \frac{x \frac{dv}{dx}}{dx} \)

\[
\Rightarrow V + \frac{x \frac{dv}{dx}}{dx} = \frac{-7X+3YX}{3X-7YX} \\
\Rightarrow V + \frac{x \frac{dv}{dx}}{dx} = \frac{X(-7+3V)}{X(3-7V)} \\
\Rightarrow V + \frac{x \frac{dv}{dx}}{dx} = \frac{(-7+3V)}{(3-7V)} \\
\Rightarrow \frac{x \frac{dv}{dx}}{dx} = \frac{-7+3V}{(3-7V)} - V \\
\Rightarrow \frac{x \frac{dv}{dx}}{dx} = \frac{-7+3V+3V+7V^2}{(3-7V)}
\]

Diff. equations
7. solve the differential equation
\[ \frac{dy}{dx} = \frac{x^2 + y^2 + 3}{2x + 3y + 4} \]

Sol: Given eq”n \[ \frac{dy}{dx} = \frac{x^2 + y^2 + 3}{2x + 3y + 4} [a/a' ≠ b/b'] \]

This is non – homogeneous D.E of case (3)

put \( x = X + h \) and \( y = Y + k \)

\[ \frac{dy}{dx} = \frac{x^2 + y^2 + 3}{2x + 3y + 4} \]
\[ \Rightarrow \frac{dY}{dX} = \frac{X^2 + Y^2 + (h + 2k + 3)}{2X + 3Y + (2h + 3k + 4)} \quad (\ast) \]

now choose \( h \) and \( k \) such that

\[ h + 2k + 3 = 0 \quad \ldots (1) \]
and
\[ 2h + 3k + 4 = 0 \quad \ldots (2) \]

solving (1) & (2)

\[ 1 \quad 2 \quad 3 \quad 1 \]
\[ 2 \quad 3 \quad 4 \quad 2 \]

\( (h, k) = \left[ \frac{8 - 9}{3 - 4}, \frac{6 - 4}{3 - 4} \right] = \left[ 1, -2 \right] \)

Hence (\( \ast \)) becomes

\[ \Rightarrow \frac{dY}{dX} = \frac{X^2 + Y^2}{2X + 3Y} \] is a homogeneous equation.

put \( y = VX \) \( \Rightarrow \frac{dY}{dX} = V + \frac{xdV}{dx} \)

\[ \Rightarrow V + \frac{xdV}{dx} = \frac{x^2 + 2Vx}{2x + 3Vx} \]
\[ \Rightarrow V + \frac{xdV}{dx} = \frac{x(1 + 2V)}{x(2 + 3V)} \]
\[ \Rightarrow V + \frac{xdV}{dx} = \frac{(1 + 2V)}{(2 + 3V)} \]
\[ \Rightarrow x\frac{dV}{dx} = \frac{(1 + 2V)}{(2 + 3V)} - \frac{V}{1} \]
\[ \Rightarrow x\frac{dV}{dx} = \frac{1 + 2V - 2V - 3V^2}{2 + 3V} \]
\[ \Rightarrow x\frac{dV}{dx} = \frac{1 - 3V^2}{2 + 3V} \]

\[ \Rightarrow \int \frac{2 + 3V}{1 - 3V^2} dV = \frac{dx}{x} \]
\[ \Rightarrow \left[ \frac{2}{1 - 3V^2} + \frac{3V}{1 - 3V^2} \right] dV = \frac{dx}{x} \]
\[ \Rightarrow \frac{2}{3} \int \frac{1}{V} dV - \frac{1}{2} \int \frac{6V}{1 - 3V^2} dV = \int \frac{1}{x} dx \]
\[ \Rightarrow \frac{2}{3} \cdot \frac{1}{2V} \log \left| \frac{1 + V}{1 - V} \right| - \frac{1}{2} \log |1 - 3V^2| = \log CX \]
\[ \Rightarrow \frac{1}{(\sqrt{3})} \log \left| \frac{1 + \sqrt{3}V}{1 - \sqrt{3}V} \right| - \frac{1}{2} \log |1 - 3V^2| = \log CX \]
\[ \Rightarrow \frac{1}{(\sqrt{3})} \log \left| \frac{1 + \sqrt{3}V}{1 - \sqrt{3}V} \right| - \frac{1}{2} \log \left| \frac{x^2 - 3y^2}{x^2} \right| = \log CX \]

\( X = x + 1 \) and \( Y = y + 2 \)

\[ \Rightarrow \frac{1}{(\sqrt{3})} \log \left| \frac{x + 1 + \sqrt{3}(y - 2)}{x + 1 - \sqrt{3}(y - 2)} \right| - \frac{1}{2} \log \left| \frac{(x + 1)^2 - 3(y - 2)^2}{(x + 1)^2} \right| = \log CX \]