Definite integrals

1. $\int_0^{\pi/4} \frac{sinx+cosx}{9+16sin2x}\, dx$

$sol: let$ $\boldsymbol{sinx - cosx = t}$

$diff.w.r.t\,'x'$

$\boldsymbol{(cosx + sinx)dx = dt}$

$L.L: x = 0 \Rightarrow t = sino - coso = 0 - 1 = -1$

$U.L: x = \frac{\pi}{4} \Rightarrow t = \sin\left(\frac{\pi}{4}\right) - \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = 0$

$sinx - cosx = t$ S.O.B

$\Rightarrow (sinx - cosx)^2 = t^2$

$\Rightarrow sin^2x + cos^2x - 2sinxcosx = t^2$

$\Rightarrow 1 - sin2x = t^2$

$\Rightarrow$ $\boldsymbol{1 - t^2 = sin2x}$

$I = \int_0^{\pi/4} \frac{sinx+cosx}{9+16sin2x}\, dx$

$= \int_{-1}^{0} \frac{1}{9+16(1-t^2)} dt$

$= \int_{-1}^{0} \frac{1}{9+16-16t^2} dt$

$= \int_{-1}^{0} \frac{1}{25-16t^2} dt$

$= \int_{-1}^{0} \frac{1}{(5)^2-(4t)^2} dt$

$\therefore \int \frac{1}{a^2-x^2} dx = \frac{1}{2a} \log\left|\frac{a+x}{a-x}\right| + c$

$= \frac{\left[\frac{1}{2[5]} \log\left|\frac{5+4t}{4-5t}\right|\right]}{4}\Bigg|_{-1}^{0}$

$= \frac{1}{40}\left[\log\left|\frac{5+0}{5-0}\right| - \log\left|\frac{5-4}{5+4}\right|\right]$

$= \frac{1}{40}\left[\log 1 - \log\frac{1}{9}\right]$ $\quad$ $\boldsymbol{\log 1 = 0}$

$= \frac{1}{40}[0 - \log 3^{-2}] = \frac{1}{40}[2\log 3]$

$= \frac{1}{20}\log 3$

2. $\int_0^1 \frac{\log(1+x)}{(1+x^2)} dx$

$sol: let$ $\boldsymbol{x = tan\theta}$

$diff.w.r.t\,'x'$

$\boldsymbol{dx = sec^2\theta d\theta}$

$L.L: x = 0 \Rightarrow \theta = 0$

$U.L: x = 1 \Rightarrow \theta = \frac{\pi}{4}$

$\int_0^1 \log\frac{(1+x)}{(1+x^2)}\, dx$

$= \int_0^{\pi/4} \frac{\log(1+tan\theta)}{(1+tan^2\theta)} sec^2\theta d\theta$

$I = \int_0^{\pi/4} \log(1 + tan\theta) d\theta \ldots\ldots\ldots (1)$

$\therefore \int_0^a f(x)dx = \int_0^a f(a-x)dx$

$I = \int_0^{\pi/4} \log\left(1 + tan\left[\frac{\pi}{4} - \theta\right]\right) d\theta$

$I = \int_0^{\pi/4} \log\left(1 + \frac{1-tan\theta}{1+tan\theta}\right) d\theta$

$I = \int_0^{\pi/4} \log\left(\frac{1+tan\theta+1-tan\theta}{1+tan\theta}\right) d\theta$

$I = \int_0^{\pi/4} \log\left(\frac{2}{1+tan\theta}\right) d\theta$

$\boldsymbol{\log(a/b) = \log a - \log b}$

$I = \int_0^{\pi/4} \log 2 d\theta - \int_0^{\pi/4} \log(1 + \tan\theta)\, d\theta$

$I + I = \int_0^{\pi/4} \log 2 d\theta$

$2I = \log 2 \int_0^{\pi/4} (1) d\theta$

$2I = \log 2 [\theta]_0^{\pi/4}$

$2I = \log 2 \left[\frac{\pi}{4} - 0\right]$

$I = \frac{\pi}{8} \log 2$

3. $\int_0^{\pi} \frac{x.sinx}{1+sinx}\, dx$

Sol:

$I = \int_0^{\pi} \frac{x.sinx}{1+sinx}\, dx$

$\therefore \int_0^a f(x)dx = \int_0^a f(a-x)dx$

$I = \int_0^{\pi} \frac{(\pi-x).\sin(\pi-x)}{1+\sin(\pi-x)} dx$

$I = \int_0^{\pi} \frac{(\pi-x).sinx}{1+sinx} dx$

$$I = \int_0^{\pi} \frac{\pi sinx}{1+sinx} dx - \int_0^{\pi} \frac{x.sinx}{1+sinx}\, dx$$

$I + I = \pi \int_0^{\pi} \frac{sinx}{1+sinx} dx$

$2I = \pi \int_0^{\pi} \frac{sinx(1-sinx)}{(1+sinx)(1-sinx)} dx$

$2I = \pi \int_0^{\pi} \frac{sinx-\sin^2 x}{(1-\sin^2 x)} dx$

$2I = \pi \int_0^{\pi} \frac{sinx-\sin^2 x}{(\cos^2 x)} dx$

$2I = \pi \int_0^{\pi} \left[\frac{sinx}{cosx}.\frac{1}{cosx} - \frac{\sin^2 x}{\cos^2 x}\right] dx$

$2I = \pi \int_0^{\pi} [tanx.secx - \tan^2 x]dx$

$2I = \pi \int_0^{\pi} tanx.secx.dx$ - $\pi \int_0^{\pi} \tan^2 x\, dx$

$\int tanx.secx.dx = secx + c$

$2I = \pi[secx]_0^{\pi}$ - $\pi \int_0^{\pi} (sex^2 x - 1)dx$

$\int sex^2 x\, dx = tanx + c$

$2I = \pi[sec\pi - sec0]$ - $\pi[tanx]_0^{\pi}$-$\pi[x]_0^{\pi}$

$2I = \pi[-1-1]$ - $\pi[0-0] + \pi[\pi - 0]$

$2I = -2\pi + \pi^2$

$I = \frac{\pi^2}{2} - \pi$

4. $\int_0^{\pi} \frac{x}{1+sinx}\, dx$

Sol:

$I = \int_0^{\pi} \frac{x}{1+sinx}\, dx$

$\therefore \int_0^a f(x)dx = \int_0^a f(a-x)dx$

$I = \int_0^{\pi} \frac{(\pi-x)}{1+\sin(\pi-x)} dx$

$I = \int_0^{\pi} \frac{(\pi-x)}{1+sinx} dx$

$$I = \int_0^{\pi} \frac{\pi}{1+sinx} dx - \int_0^{\pi} \frac{x}{1+sinx}\, dx$$

$I + I = \pi \int_0^{\pi} \frac{1}{1+sinx} dx$

$2I = \pi \int_0^{\pi} \frac{(1-sinx)}{(1+sinx)(1-sinx)} dx$

$2I = \pi \int_0^{\pi} \frac{1-sinx}{(1-\sin^2 x)} dx$

$2I = \pi \int_0^{\pi} \frac{1-sinx}{(\cos^2 x)} dx$

$2I = \pi \int_0^{\pi} \left[\frac{1}{\cos^2 x} - \frac{sinx}{cosx}.\frac{1}{cosx}\right] dx$

$2I = \pi \int_0^{\pi} [sex^2 x - tanx.secx]dx$

$\int tanx.secx.dx = secx + c$

$\int sex^2 x\, dx = tanx + c$

$2I = \pi[tanx]_0^{\pi}$ - $\pi[secx]_0^{\pi}$

$2I = \pi[tan\pi - tan0]$- $\pi[sec\pi - sec0]$

$2I = \pi[0-0]$ - $\pi[-1-1]$

$2I = 2\pi$

$I = \pi$

5. $\int_0^{\pi} \frac{x.sinx}{1+cos^2x}\,dx$

Sol:

$I = \int_0^{\pi} \frac{x.sinx}{1+cos^2x}\,dx$

$\therefore \int_0^a f(x)dx = \int_0^a f(a-x)dx$

$I = \int_0^{\pi} \frac{(\pi-x).\sin(\pi-x)}{1+cos^2(\pi-x)}dx$

$I = \int_0^{\pi} \frac{(\pi-x).sinx}{1+cos^2x}dx$

$I = \int_0^{\pi} \frac{\pi sinx}{1+cos^2x}dx - \int_0^{\pi} \frac{x.sinx}{1+cos^2x}\,dx$

$I + I = \pi\int_0^{\pi} \frac{sinx}{1+cos^2x}dx$

$2I = \pi\int_0^{\pi} \frac{sinx}{1+cos^2x}dx$

let $cosx = t \Rightarrow -sixdx = dt$

or $\boldsymbol{sinxdx = -dt}$

$L.L: x = 0 \Rightarrow t = 1$; $U.L: x = \pi \Rightarrow t = -1$

$\Rightarrow 2I = \pi\int_1^{-1} \frac{-1}{1+t^2}\,dt$

$\Rightarrow 2I = \pi\int_{-1}^{1} \frac{1}{1+t^2}\,dt$

$2I = \pi[\tan^{-1} t]$

$2I = \pi[\tan^{-1}(1) - \tan^{-1}(-1)]$

$2I = \pi\left[\frac{\pi}{4} + \frac{\pi}{4}\right]$

$2I = \pi\left[2.\frac{\pi}{4}\right] = \pi\left[.\frac{\pi}{4}\right]$

$I = \frac{\pi^2}{4}$

6. $\int_0^{\pi} \frac{x.sin^3x}{1+cos^2x}\,dx$

Sol:

$I = \int_0^{\pi} \frac{x.sin^3x}{1+cos^2x}\,dx$

$\therefore \int_0^a f(x)dx = \int_0^a f(a-x)dx$

$I = \int_0^{\pi} \frac{(\pi-x).sin^3(\pi-x)}{1+cos^2(\pi-x)}dx$

$I = \int_0^{\pi} \frac{(\pi-x).sin^3x}{1+cos^2x}dx$

$I = \int_0^{\pi} \frac{\pi sin^3x}{1+cos^2x}dx - \int_0^{\pi} \frac{x.sin^3x}{1+cos^2x}\,dx$

$I + I = \pi\int_0^{\pi} \frac{sin^3x}{1+cos^2x}dx = \pi\int_0^{\pi} \frac{sin^3x}{1+cos^2x}dx$

let $cosx = t \Rightarrow -sixdx = dt$

or $\boldsymbol{sinxdx = -dt}$

$sin^2x = 1 - cos^2x = 1 - t^2$

$\boldsymbol{sin^3}x = -(1-t^2)dt$

$L.L: x = 0 \Rightarrow t = 1$; $U.L: x = \pi \Rightarrow t = -1$

$\Rightarrow 2I = \pi\int_1^{-1} \frac{-(1-t^2)}{1+t^2}\,dt$

$\Rightarrow 2I = \pi\int_1^{-1} \frac{t^2-1}{1+t^2}\,dt = \pi\int_1^{-1} \frac{t^2+1-2}{1+t^2}\,dt$

$\Rightarrow 2I = \pi\int_1^{-1} \left[1 - \frac{2}{1+t^2}\right]\,dt$

$2I = \pi[t - 2\tan^{-1} t]_1^{-1}$

$2I = \pi[-1-1] - 2\pi[\tan^{-1}(-1) - \tan^{-1}(1)]$

$2I = -2\pi + 2\pi\left[\frac{\pi}{4} + \frac{\pi}{4}\right]$

$2I = -2\pi + 2\pi\left[\frac{\pi}{2}\right]$

$I = -\pi + \frac{\pi^2}{2} = \frac{\pi}{2}(\pi - 2)$

7. $\int_0^{\pi/2} \frac{x}{cosx+sinx}\,dx$

sol:

$$I = \int_0^{\pi/2} \frac{x}{cosx+sinx}\,dx$$

$$\therefore \int_0^a f(x)dx = \int_0^a f(a-x)dx$$

$$I = \int_0^{\pi/2} \frac{(\frac{\pi}{2}-x)}{\cos(\frac{\pi}{2}-x)+\sin(\frac{\pi}{2}-x)}dx$$

$$I = \int_0^{\pi/2} \frac{(\frac{\pi}{2}-x)}{sinx+cosx}dx$$

$$= \int_0^{\pi/2} \frac{\frac{\pi}{2}}{cosx+sinx}dx - \int_0^{\pi/2} \frac{x}{cosx+sinx}\,dx$$

$$I + I = \frac{\pi}{2}\int_0^{\pi/2} \frac{1}{cosx+sinx}dx$$

$$2I = \frac{\pi}{2}\int_0^{\pi/2} \frac{1}{cosx+sinx}dx$$

Let t= $tan\left(\frac{x}{2}\right)$;

$dx = \frac{2dt}{1+t^2}$;

$\cos x = \frac{1-t^2}{1+t^2}$

$sinx = \frac{2t}{1+t^2}$

$$2I = \frac{\pi}{2}\int_0^1 \frac{1}{\left(\frac{1-t^2}{1+t^2}\right)+\left(\frac{2t}{1+t^2}\right)}\left(\frac{2dt}{1+t^2}\right)$$

$$2I = \frac{\pi}{2}\int_0^1 \frac{1}{\left[\frac{1-t^2+2t}{1+t^2}\right]}\left(\frac{2dt}{1+t^2}\right)$$

$$2I = \frac{\pi}{2}2\int_0^1 \frac{1}{-(t^2-2t-1)}dt$$

$$I = \frac{\pi}{2}\int_0^1 \frac{1}{-[t^2-2t+(1)^2-(1)^2-1]}dt$$

$$I = \frac{\pi}{2}\int_0^1 \frac{1}{-[(t-1)^2-(\sqrt{2})^2]}dt$$

$$I = \frac{\pi}{2}\int_0^1 \frac{1}{[(\sqrt{2})^2-(t-1)^2]}dt$$

$$\therefore \int \frac{1}{a^2-x^2}dx = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + c$$

$$2I = \frac{\pi}{2}\frac{1}{2(\sqrt{2})}\left[\log\left|\frac{\sqrt{2}+t-1}{\sqrt{2}-t+1}\right|_0^1\right]$$

$$I = \frac{\pi}{2}\frac{1}{2\sqrt{2}}\left[\log\left|\frac{\sqrt{2}+1-1}{\sqrt{2}-1+1}\right| - \log\left|\frac{\sqrt{2}-1}{\sqrt{2}+1}\right|\right]$$

$$I = \frac{\pi}{2}\frac{1}{2\sqrt{2}}\left[\log|1| + \log\left|\frac{\sqrt{2}+1}{\sqrt{2}-1}\right|\right]$$

$$I = \frac{\pi}{2}\frac{1}{2\sqrt{2}}\log\left|\frac{\sqrt{2}+1}{\sqrt{2}-1}\right|$$

$$I = \frac{\pi}{2}\frac{1}{2\sqrt{2}}\log\left|\frac{(\sqrt{2}+1)(\sqrt{2}+1)}{(\sqrt{2}-1)(\sqrt{2}+1)}\right|$$

$$I = \frac{\pi}{2}\frac{1}{2\sqrt{2}}\log\frac{(\sqrt{2}+1)^2}{\sqrt{2}^2-1^2}$$

$$I = \frac{2}{2\sqrt{2}}\log\frac{(\sqrt{2}+1)}{2-1}$$

$$I = \frac{\pi}{2\sqrt{2}}\log(\sqrt{2}+1)$$

8. $\int_0^{\pi/2} \frac{\sin^2 x}{\cos x+\sin x}\, dx$

sol:

$I = \int_0^{\pi/2} \frac{\sin^2 x}{\cos x+\sin x}\, dx \ldots (1)$

$\therefore \int_0^a f(x)dx = \int_0^a f(a-x)dx$

$I = \int_0^{\pi/2} \frac{\sin^2(\frac{\pi}{2}-x)}{\cos(\frac{\pi}{2}-x)+\sin(\frac{\pi}{2}-x)} dx$

$I = \int_0^{\pi/2} \frac{\cos^2 x}{\sin x+\cos x} dx \ldots (2)$

Adding (1) & (2)

$= \int_0^{\pi/2} \frac{\sin^2 x}{\cos x + \sin x} dx + \int_0^{\pi/2} \frac{\cos^2 x}{\cos x + \sin x}\, dx$

$I + I = \int_0^{\pi/2} \frac{\sin^2 x+\cos^2 x}{\cos x+\sin x} dx$

$2I = \int_0^{\pi/2} \frac{1}{\cos x+\sin x} dx$

Let t= $\tan\left(\frac{x}{2}\right)$;

$dx = \frac{2dt}{1+t^2}$;

$\cos x = \frac{1-t^2}{1+t^2}$

$\sin x = \frac{2t}{1+t^2}$

$2I = \int_0^1 \frac{1}{\left(\frac{1-t^2}{1+t^2}\right)+\left(\frac{2t}{1+t^2}\right)} \left(\frac{2dt}{1+t^2}\right)$

$2I = \int_0^1 \frac{1}{\left[\frac{1-t^2+2t}{1+t^2}\right]} \left(\frac{2dt}{1+t^2}\right)$

$2I = 2\int_0^1 \frac{1}{-(t^2-2t-1)} dt$

$I = \int_0^1 \frac{1}{-[t^2-2t+(1)^2-(1)^2-1]} dt$

$I = \int_0^1 \frac{1}{-[(t-1)^2-(\sqrt{2})^2]} dt$

$I = \int_0^1 \frac{1}{[(\sqrt{2})^2-(t-1)^2]} dt$

$\therefore \int \frac{1}{a^2-x^2} dx = \frac{1}{2a} \log\left|\frac{a+x}{a-x}\right| + c$

$2I = \frac{1}{2(\sqrt{2})}\left[\log\left|\frac{\sqrt{2}+t-1}{\sqrt{2}-t+1}\right|_0^1\right]$

$I = \frac{1}{2\sqrt{2}}\left[\log\left|\frac{\sqrt{2}+1-1}{\sqrt{2}-1+1}\right| - \log\left|\frac{\sqrt{2}-1}{\sqrt{2}+1}\right|\right]$

$I = \frac{1}{2\sqrt{2}}\left[\log|1| + \log\left|\frac{\sqrt{2}+1}{\sqrt{2}-1}\right|\right]$

$I = \frac{1}{2\sqrt{2}}\log\left|\frac{\sqrt{2}+1}{\sqrt{2}-1}\right|$

$I = \frac{1}{2\sqrt{2}}\log\left|\frac{(\sqrt{2}+1)(\sqrt{2}+1)}{(\sqrt{2}-1)(\sqrt{2}+1)}\right|$

$I = \frac{1}{2\sqrt{2}}\log\frac{(\sqrt{2}+1)^2}{\sqrt{2}^2-1^2}$

$I = \frac{2}{2\sqrt{2}}\log\frac{(\sqrt{2}+1)}{2-1}$

$I = \frac{1}{\sqrt{2}}\log(\sqrt{2}+1)$

9. $\int_3^7 \sqrt{\frac{7-x}{x-3}}\, dx.$

Sol:

$let\ x = 3cos^2\theta + 7sin^2\theta$

$dx = 8sin\theta cos\theta . d\theta$

$L.L: x = 3 \Rightarrow \theta = 0$

$U.L: x = 7 \Rightarrow \theta = \frac{\pi}{2}$

$7 - x = 7 - 3cos^2\theta - 7sin^2\theta$

$= 7(1 - sin^2\theta) - 3cos^2\theta$

$= 7(cos^2\theta) - 3cos^2\theta$

$= 4cos^2\theta$

$x - 3 = 3cos^2\theta + 7sin^2\theta - 3$

$= 7sin^2\theta - 3(1 - cos^2\theta)$

$= 7sin^2\theta - 3(sin^2\theta)$

$= 4sin^2\theta$

$\int_3^7 \sqrt{\frac{7-x}{x-3}}\, dx$

$= \int_0^{\pi/2} \sqrt{\frac{4cos^2\theta}{4sin^2\theta}}\, 8sin\theta cos\theta d\theta$

$= 8\int_0^{\pi/2} cos^2\theta \ \ d\theta$

$$\int_0^{\pi/2} cos^n x \ \ dx = \frac{n-1}{n}.\frac{n-3}{n-2}.\frac{n-5}{n-4}\cdots\frac{\pi}{2}$$

$= 8.\left(\frac{1}{2}\right)\frac{\pi}{2} = 2\pi$

10. $\int_4^9 \frac{1}{\sqrt{(9-x)(x-4)}}\, dx.$

Sol:

$let\ x = 4cos^2\theta + 9sin^2\theta$

$dx = 10sin\theta cos\theta . d\theta$

$L.L: x = 4 \Rightarrow \theta = 0$

$U.L: x = 9 \Rightarrow \theta = \frac{\pi}{2}$

$9 - x = 9 - 4cos^2\theta - 9sin^2\theta$

$= 9(1 - sin^2\theta) - 3cos^2\theta$

$= 9(cos^2\theta) - 4cos^2\theta$

$= 5cos^2\theta$

$x - 4 = 4cos^2\theta + 9sin^2\theta - 4$

$= 9sin^2\theta - 4(1 - cos^2\theta)$

$= 9sin^2\theta - 4(sin^2\theta)$

$= 5sin^2\theta$

$\int_4^9 \frac{1}{\sqrt{(9-x)(x-4)}}\, dx.$

$= \int_0^{\pi/2} \frac{1}{\sqrt{5cos^2\theta 5sin^2\theta}}\, 10sin\theta cos\theta d\theta$

$== \int_0^{\pi/2} \frac{1}{5sin\theta cos\theta}\, 10sin\theta cos\theta d\theta$

$$I = 2\int_0^{\pi/2} 1 \ \ dx$$

$= 2[x]_0^{\pi/2} = \frac{2\pi}{2} = \pi$

11. Evaluate $\int_a^b \sqrt{(x-a)(x-b)}dx$

Sol: $\int_a^b \sqrt{-(x-a)(x-b)}dx$

$$I = \int_a^b \sqrt{-[x^2-(a+b)x+ab]}dx$$

$$= \int_a^b \sqrt{-\left[x^2-(a+b)x+\frac{[a+b]^2}{4}-\frac{[a+b]^2}{4}+ab\right]}dx$$

$$= \int_a^b \sqrt{-\left[\left[x-\frac{a+b}{2}\right]^2-\frac{[b-a]^2}{4}\right]}dx$$

$$= \int_a^b \sqrt{\left[\left[\frac{b-a}{2}\right]^2-\left[x-\frac{a+b}{2}\right]^2\right]}dx$$

$$\therefore \int \sqrt{a^2-x^2}dx = \frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)+c$$

$$I = \left[\frac{\left[x-\frac{a+b}{2}\right]}{2}\sqrt{(a-x)(b-x)}+\frac{[b-a]^2}{4.2}\sin^{-1}\frac{\left(x-\frac{a+b}{2}\right)}{\frac{b-a}{2}}\right]_a^b$$

$$= \left[\frac{\left[b-\frac{a+b}{2}\right]}{2}\sqrt{(a-b)(b-b)}+\frac{[b-a]^2}{4.2}\sin^{-1}\frac{\left(b-\frac{a+b}{2}\right)}{\frac{b-a}{2}}\right]$$
$$-\left[\frac{\left[a-\frac{a+b}{2}\right]}{2}\sqrt{(a-a)(b-a)}+\frac{[b-a]^2}{4.2}\sin^{-1}\frac{\left(a-\frac{a+b}{2}\right)}{\frac{b-a}{2}}\right]$$

$$= \left[\frac{[b-a]^2}{8}\sin^{-1}\frac{\frac{(b-a)}{2}}{\frac{b-a}{2}}\right]-\left[\frac{[b-a]^2}{8}\sin^{-1}\frac{\frac{(a-b)}{2}}{\frac{b-a}{2}}\right]$$

$$= \left[\frac{[b-a]^2}{8}\sin^{-1}1\right]-\left[\frac{[b-a]^2}{8}\sin^{-1}(-1)\right]$$

$$= \left[\frac{[b-a]^2}{8}\left(\frac{\pi}{2}\right)\right]+\left[\frac{[b-a]^2}{8}\left(\frac{\pi}{2}\right)\right] = \frac{\pi[b-a]^2}{8}$$

12. $\int_0^\pi x.\sin^7 x\cos^6 x\ dx$

Sol:

$$I = \int_0^\pi x.\sin^7 x\cos^6 x\ dx$$

$$\therefore \int_0^a f(x)dx = \int_0^a f(a-x)dx$$

$$I = \int_0^\pi (\pi-x)\sin^7(\pi-x)\cos^6(\pi-x)\ dx$$

$$I = \int_0^\pi (\pi-x)\sin^7 x\cos^6 x\ dx$$

$$I = \int_0^\pi \pi\sin^7 x\cos^6 x\ dx - \int_0^\pi x\sin^7 x\cos^6 x\ dx$$

$$I = \int_0^\pi \pi\sin^7 x\cos^6 x\ dx - I$$

$$I + I = \pi\int_0^\pi \sin^7 x\cos^6 x\ dx$$

$$\therefore \int_0^{2a} f(x)dx = 2\int_0^a f(x)dx$$

$$2I = 2\pi\int_0^{\pi/2} \sin^7 x\cos^6 x\ dx$$

$$I = \pi\int_0^{\pi/2} \sin^7 x\cos^6 x\ dx$$

$$\int_0^{\pi/2} \cos^n x\sin^m x\ dx = \frac{(n-1)(n-3)(n-5)\ldots(m-1)(m-3)\ldots}{(m+n)(m+n-2)(m+n-4)\ldots}$$

$$I = \pi\frac{6.4.2.5.3.1}{13.11.9.7.5.3.1}$$

$$I = \pi\frac{6.4.2}{13.11.9.7}$$

$$I = \frac{16\pi}{3003}$$

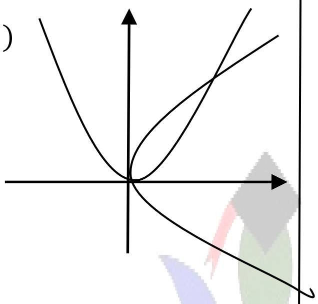
13. Find the area enclosed by the curves $y^2 = 4ax$ and $x^2 = 4by$.

Sol:

Given eq''n

$y^2 = 4ax \Rightarrow y = \sqrt{4ax}$ ... (1)

$x^2 = 4by \Rightarrow y = \frac{x^2}{4b}$ ..... (2)



solving (1) and (2)

$\sqrt{4ax} = \frac{x^2}{4b}$ S.O.B

$\Rightarrow 4ax = \frac{x^4}{16b^2}$

$\Rightarrow 64ab^2x = x^4 \Rightarrow 64ab^2x - x^4 = 0$

$\Rightarrow x(64ab^2 - x^3) = 0$

$x = 0$ or $x^3 = 64ab^2$

$\Rightarrow x = 4a^{1/3}b^{2/3} = u$

Required Area $= \int_0^u [(1) - (2)]\, dx$

$= \int_0^u \left[\sqrt{4ax} - \frac{x^2}{4b}\right] dx$

$= \int_0^u \left[2\sqrt{a}x^{1/2} - \frac{x^2}{4b}\right] dx$

$= \left[\frac{2\sqrt{a}x^{3/2}}{3/2} - \frac{x^3}{4b3}\right]_0^u$

$= \frac{4}{3}\sqrt{a}[u^{3/2} - 0^2] - \frac{1}{12}[u^3 - 0^3]$

$= \frac{4}{3}\sqrt{a}\left[\left[4a^{\frac{1}{3}}b^{\frac{2}{3}}\right]^{\frac{3}{2}} - 0^2\right] - \frac{1}{12}\left[\left[4a^{1/3}b^{2/3}\right]^3 - 0^3\right]$

$= \frac{4}{3}[8ab] - \frac{1}{12b}[64ab^2]$

$= \frac{32ab}{3} - \frac{16ab}{3} = \frac{16ab}{3}$ sq.units

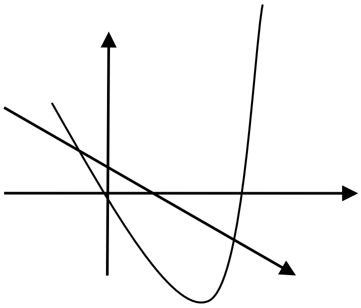
14. Find the area enclosed by the curves $y = 4 - 2x$ and $y = x^2 - 5x$.

Sol:

Given eq''n

$y = 4 - 2x$ ..... (1)

$y = x^2 - 5x$ ... (1)



solving (1) and (2)

$x^2 - 5x = 4 - 2x$

$\Rightarrow x^2 - 3x - 4 = 0$

$\Rightarrow x^2 + 1x - 4x - 4 = 0$

$\Rightarrow x(x + 1) - 4(x+1) = 0$

$x = 4$ or $x = -1$

Required Area $= \int_{-1}^{4} [(1) - (2)]\, dx$

$= \int_{-1}^{4} [4 - 2x - x^2 + 5x]\, dx$

$= \int_{-1}^{4} [4 + 3x - x^2]\, dx$

$= \left[4x + \frac{3x^2}{2} - \frac{x^3}{3}\right]_{-1}^{4}$

$= 4[4 + 1] + \frac{3}{2}[4^2 - (-1)^2] - \frac{1}{3}[4^3 - (-1)^3]$

$= 20 + \frac{3}{2}(16 - 1) - \frac{1}{3}(64 + 1)$

$= 20 + \frac{45}{2} - \frac{65}{3}$

$= \frac{120+135-135}{6}$

$= \frac{125}{6}$ sq.units

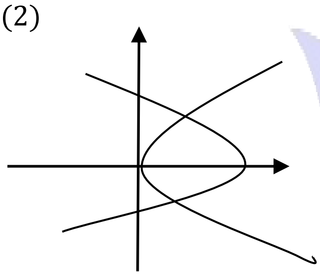
15. Find the area **enclosed by the curves**
$y^2 = 4x$ and $y^2 = 4(4-x)$.
Sol:
Given eq''n
$y^2 = 4x \Rightarrow y = \sqrt{4x}$ ... (1)

$\boldsymbol{y^2 = 4(4-x)} \Rightarrow \boldsymbol{y} = \sqrt{4(4-x)}$ ..... (2)

solving (1)*and* (2)

$4x = \boldsymbol{4(4-x)}$

$\Rightarrow x = 4 - x$



$\Rightarrow 2x = 4$
$x = 2$
Sub x=2 in (1)
$y^2 = 4x = 4(2) = 8$
$y = \sqrt{8} = \pm 2\sqrt{2}$
Two parabolas are symmetric about X-axis
Required Area $=2\left[\int_0^2 (1)dx + \int_2^4 (2)dx\right]$

$=2\left[\int_0^2 \sqrt{4x}dx + \int_2^4 \sqrt{4(4-x)}dx\right]$

$=2\left[2\int_0^2 x^{1/2}dx + 2\int_2^4 (4-x)^{1/2}dx\right]$

$=4\left[\frac{x^{3/2}}{3/2}\right]_0^2 + 4\left[\frac{(4-x)^{3/2}}{-3/2}\right]_2^4$

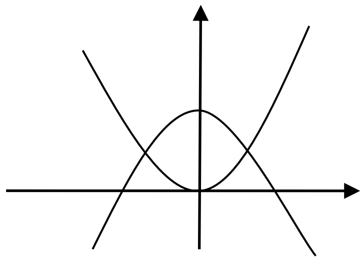
$=\frac{8}{3}\left[2^{3/2} - 0^{3/2}\right]$-$\frac{8}{3}\left[(4-4)^{3/2} - (4-2)^{3/2}\right]$

$=\frac{8}{3}\left[2\sqrt{2}\right] + \frac{8}{3}\left[2\sqrt{2}\right]$

$=\frac{16\sqrt{2}}{2} + \frac{16\sqrt{2}}{2}$
$= \frac{32\sqrt{2}}{2}$ *sq.units*

16. **Find the area enclosed by the curves**
$y = 2 - x^2$ and $y = x^2$.
Sol:
Given eq''n

$\boldsymbol{y = 2 - x^2}$ ..... (1)
$y = x^2$ ... (1)
solving (1)*and* (2)
$2 - x^2 = x^2$
$2 = 2x^2$
$x^2 = 1$
$x = \pm 1$
$\therefore x = 1$ *or* $x = -1$



Required Area $=\int_{-1}^{1}[(1)—(2)]\,dx$
$=\int_{-1}^{1}[2 - x^2 - x^2]\,dx$
$= \int_{-1}^{1}[2 - 2x^2]\,dx$
$= \left[2x - 2\frac{x^3}{3}\right]_{-1}^{1}$

$= 2[1+1] - \frac{2}{3}[(1)^3 - (-1)^3]$
$= 4 - \frac{2}{3}(1+1)$

$= 4 - \frac{4}{3} = \frac{12-4}{3}$
$= \frac{8}{3}$ *sq.unit*

17. Show that the area of the region bounded by $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is πab.also deduce the area of the circle $x^2+y^2=a^2$.

Sol: Given eq"n of the ellipse

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \dots \dots .. (1)$

⇨ $\frac{y^2}{b^2}=1-\frac{x^2}{a^2}$

⇨ $\frac{y^2}{b^2}=\frac{1}{a^2}[a^2-x^2]$

⇨ $y^2=\frac{b^2}{a^2}[a^2-x^2]$

⇨ $y=\pm\frac{b}{a}\sqrt{a^2-x^2}$

Ellipse is symmetric about both the axes.

Required area =4 area of shaded region

Area=$\int_0^a y dx$

=4 $\int_0^a \frac{b}{a}\sqrt{a^2-x^2}\, dx$

=4$\frac{b}{a}\int_0^a \sqrt{a^2-x^2}\, dx$

$\therefore \int\sqrt{a^2-x^2}dx=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)+c$

$=\frac{4b}{a}\left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$

$=\frac{4b}{a}\left[\frac{a}{2}\sqrt{a^2-a^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{a}{a}\right)-0-0\right]$

$=\frac{4b}{a}\frac{a^2}{2}\sin^{-1}(1)$

$=\frac{2ab\pi}{2}$

$=\pi ab\ sq\ units$

If a=b the ellipse becomes a circle

∴Area of the circle $x^2+y^2=a^2$.

is $\pi a.a=\pi a^2\ sq\ .units$