ROTATION OF AXES (CHANGE OF DIRECTION)

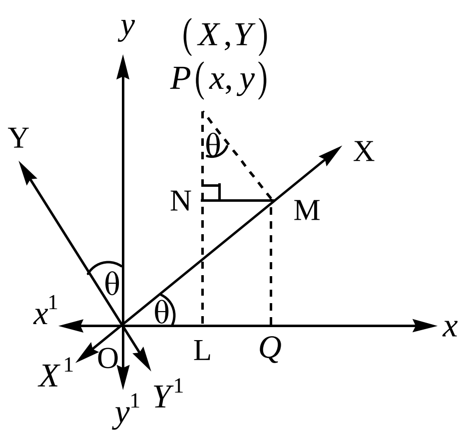
1. Definition: If the axes are rotated through an angle in the same plane by keeping the origin constant, then the transformation is called Rotation of axes.

2. Theorem: To find the co-ordinates of a point (x, y) are transformed to (X, Y) when the axes are rotated through an angle 'θ' about the origin in the same plane.

Proof: Let $x^1Ox,\ yOY^1$ are the original axes

Let $P(x, y)$ be the co-ordinates of the point in the above axes.

After rotating the axes through an angle 'θ', then the co-ordinates of P be (X, Y) w.r.t the new axes X^1OX and YOY^1 as in figure.



Since θ is the angle of rotation, then $\underline{|xOX} = \underline{|yOY} = \theta$ as in the figure.

Since L, M is projections of P on Ox and OX respectively. We can see that $\underline{|LPM} = \underline{|xOX} = \theta$

Let N be the projection to PL from M

Now $x = OL = OQ - LQ = OQ - NM$

$= OM\cos\theta - PM\sin\theta$

$= X\cos\theta - Y\sin\theta$

$y = PL = PN + NL = PN + MQ$

$PM\cos\theta + OM\sin\theta$

$= Y\cos\theta + X\sin\theta$

$\therefore\ x = X\cos\theta - Y\sin\theta$ and

$y = Y\cos\theta + X\sin\theta$ ------- (1)

Solving the above equations to get X and Y, then $X = x\cos\theta + y\sin\theta$ and

$Y = -x\sin\theta + y\cos\theta$ ---- (2)

From (1) and (2) we can tabulate

	X	Y
x	$\cos\theta$	$-\sin\theta$
y	$\sin\theta$	$\cos\theta$

Note:

(i) If the axes are turned through an angle 'θ', then the equation of a curve $f(x, y) = 0$ is transformed to $f(X\cos\theta - Y\sin\theta, X\sin\theta + Y\cos\theta) = 0$

(ii) If $f(X, Y) = 0$ is the transformed equation of a curve when the axes are rotated through an angle 'θ' then the original equation of the curve is

$f(x\cos\theta + y\sin\theta, -x\sin\theta + y\cos\theta) = 0$

Theorem: To find the angle of rotation of the axes to eliminate xy term in the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

Proof: given equation is $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

Since the axes are rotated through an angle θ, then $x = X\cos\theta - Y\sin\theta$, $y = X\sin\theta + Y\cos\theta$

Now the transformed equation is

$a(X\cos\theta - Y\sin\theta)^2 + 2h(X\cos\theta - Y\sin\theta)(X\sin\theta + Y\cos\theta)$

$+b\left(X\sin\theta+Y\cos\theta\right)^2+2g\left(X\cos\theta-Y\sin\theta\right)+2f\left(X\sin\theta+Y\cos\theta\right)+c=0$

$$\Rightarrow a\left(X^2\cos^2\theta+Y^2\sin^2\theta-2XY\cos\theta\sin\theta\right)+$$
$$2h\left[X^2\cos\theta\sin\theta+XY\left(\cos^2\theta-\sin^2\theta\right)-Y^2\sin\theta\cos\theta\right]$$
$$+b\left(X^2\sin^2\theta+Y^2\cos^2\theta+2XY\cos\theta\sin\theta\right)$$

$$+2g\left(X\cos\theta-Y\sin\theta\right)+2f\left(X\sin\theta+Y\cos\theta\right)+c=0$$

Since XY term is to be eliminated, coefficient of XY = 0.

$$2\left(b-a\right)\cos\theta\sin\theta+2h\left(\cos^2\theta-\sin^2\theta\right)=0$$

$$\Rightarrow h\cos 2\theta+\left(b-a\right)\sin 2\theta=0$$

$$\Rightarrow 2h\cos 2\theta=\left(a-b\right)\sin 2\theta$$

$$\Rightarrow \tan 2\theta=\frac{\sin 2\theta}{\cos 2\theta}=\frac{2h}{a-b}$$

$$\Rightarrow \text{Angle of rotation } \left(\theta\right)=\frac{1}{2}Tan^{-1}\left(\frac{2h}{a-b}\right)$$

Note: The angle of rotation of the axes to eliminate xy term in

$ax^2+2hxy+ay^2+2gx+2fy+c=0$ is $\frac{\pi}{4}$

PROBLEMS

1. When the axes are rotated through an angle $30°$, find the new co-ordinates of the following points.

i) (0, 5) **ii) (-2, 4)** **iii) (0, 0)**

Sol. i) Given $\theta=30°$

Old co-ordinates are (0,5)

i.e., x=0, y = 5

$X=x\cos\theta+y\sin\theta$

$= 0.\cos 30° + 5.\sin 30° = \frac{5}{2}$

$Y = -x\sin\theta + y\cos\theta$

$-0.\sin 30° + 5.\cos 30° = \frac{5\sqrt{3}}{2}$

New co-ordinates are $\left(\frac{5}{2}, \frac{5\sqrt{3}}{2}\right)$

ii) Old co-ordinates are (-2,4) $ANS.\left(\sqrt{3}+2,\ 1+2\sqrt{3}\right)$

iii) Given $(x, y) = (0,0)$ **and** $\theta = 30°$

$\Rightarrow X = x.\cos 30° - y\sin 30°$

$= 0.\frac{\sqrt{3}}{2} - 0.\frac{1}{2} = 0$

$Y = x.\sin 30° + y.\cos 30° = 0.\frac{1}{2} + 0.\frac{\sqrt{3}}{2} = 0$

New co-ordinates of the point are (0, 0)

2. When the axes are rotated through an angle $60°$ **, the new co-ordinates of three points are the following**

i) (3, 4) ii) (-7, 2) iii) (2, 0) Find their original co-ordinates

Sol. i) Given $\theta = 60°$

New co-ordinates are (3, 4)

$X = 3, Y = 4$

$x = X\cos\theta - Y\sin\theta = 3.\cos 60° - 4.\sin 60°$

$= 3.\frac{1}{2} - \frac{4.\sqrt{3}}{2} = \frac{3 - 4\sqrt{3}}{2}$

$y = X\sin\theta + Y\cos\theta$

$$= 3\sin 60^\circ + 4.\cos 60^\circ = 3.\frac{\sqrt{3}}{2} + 4.\frac{1}{2} = \frac{4+\sqrt{3}}{2}$$

Co-ordinates of P are $\left(\frac{3-4\sqrt{3}}{2}, \frac{4+3\sqrt{3}}{2}\right)$

ii) New coordinates are (-7,2) ANS. $\left(\frac{-7-2\sqrt{3}}{2}, \frac{2-7\sqrt{3}}{2}\right)$

iii) New co-ordinates are (2, 0)

ans. $\left(1, \sqrt{3}\right)$

3. **Find the angle through which the axes are to be rotated so as to remove the xy term in the equation.** $x^2 + 4xy + y^2 - 2x + 2y - 6 = 0$

Sol. Comparing the equation

$x^2 + 4xy + y^2 - 2x + 2y - 6 = 0$ with $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

a = 1, h =2, b=1, g=-1, f=1, c=-6

Let 'θ' be the angle of rotation of axes, then $\theta = \frac{1}{2}\tan^{-1}\left(\frac{2h}{a-b}\right)$

$$= \frac{1}{2}\tan^{-1}\left(\frac{4}{1-1}\right) = \frac{1}{2}\tan^{-1}\left(\frac{4}{0}\right)$$

$$= \frac{1}{2}\tan^{-1}(\infty) = \frac{1}{2} \times \frac{\pi}{2}$$

$$\therefore \quad \theta = \frac{\pi}{4}$$

SHORT ANSWERS QUESTIONS

1. When the axes are rotated through an angle 45°, the transformed equation of a curve is $17x^2 - 16xy + 17y^2 = 225$. Find the original equation of the curve.?

Sol. Angle of rotation = $\theta = 45$

$$X = x\cos\theta + y\sin\theta = x\cos 45 + y\sin 45 \ = \frac{x+y}{\sqrt{2}}$$

$$Y = -x\sin\theta + y\cos\theta = -x\sin 45 + y\cos 45 == \frac{-x+y}{\sqrt{2}}$$

The original equation of

$17X^2 - 16XY + 17Y^2 = 225$ is

$$\Rightarrow 17\left(\frac{x+y}{\sqrt{2}}\right)^2 - 16\left(\frac{x+y}{\sqrt{2}}\right)\left(\frac{-x+y}{\sqrt{2}}\right) + 17\left(\frac{-x+y}{\sqrt{2}}\right)^2 = 225$$

$$\Rightarrow 17\frac{\left(x^2+y^2+2xy\right)}{2} - 16\frac{\left(y^2-x^2\right)}{2} + 17\frac{\left(x^2+y^2-2xy\right)}{2} = 225$$

$$\Rightarrow 17\left[(x+y)^2 + (x-y)^2\right] \text{ -16 } \left(x^2-y^2\right) = 450$$

$$\Rightarrow 17\left[2\left(x^2+y^2\right)\right] \text{ -16 } \left(x^2-y^2\right) = 450$$

$$\Rightarrow 17\left(x^2+y^2\right) \text{ -8 } \left(x^2-y^2\right) = 225$$

$\Rightarrow 9x^2 + 25y^2 = 225$ is the original equation

2. when the axes are rotated through an angle α, find the transformed equation of $x\cos\alpha + y\sin\alpha = p$?

Sol. The given equation is $x\cos\alpha + y\sin\alpha = p$

$\because$ The axes are rotated through an angle α

$x = X\cos\alpha - Y\sin\alpha$

$y = X\sin\alpha + Y\cos\alpha$

The given equation transformed to

$$\left(X\cos\theta - Y\sin\theta\right)\cos\theta + \left(X\sin\alpha + Y\cos\alpha\right)\sin\alpha = p$$

$$\Rightarrow X\left(\cos^2\alpha + \sin^2\alpha\right) = p \Rightarrow X = p$$

The equation transformed to x = p

3. When the axes are rotated through an angle $\pi/6$. Find the transformed equation of $x^2 + 2\sqrt{3}xy - y^2 = 2a^2$

Sol. Since $\theta = \dfrac{\pi}{6}, x = X\cos\alpha - Y\sin\alpha$

$$X = X\cos\frac{\pi}{6} - Y\sin\frac{\pi}{6}$$

$$X.\frac{\sqrt{3}}{2} - Y.\frac{1}{2} = \frac{\sqrt{3}X - Y}{2}$$

$$y = X\sin\alpha + Y\cos\theta = X.\sin\frac{\pi}{6} + Y\cos\frac{\pi}{6} \;=\; X.\frac{1}{2} + Y.\frac{\sqrt{3}}{2} = \frac{X + \sqrt{3}Y}{2}$$

Transformed equation is

$$\left(\frac{\sqrt{3}X - Y}{2}\right)^2 + 2\sqrt{3}\left(\frac{\sqrt{3}X - Y}{2}\right)\left(\frac{X + \sqrt{3}Y}{2}\right) - \left(\frac{X + \sqrt{3}Y}{2}\right) = 2a^2$$

$$\Rightarrow \frac{3x^2 - 2\sqrt{3} + Y^2}{4} + \frac{2\sqrt{3}\left[\sqrt{3}X^2 - XY + 3XY - \sqrt{3}Y^2\right]}{4} = \frac{X^2 + 3Y^2 + 2\sqrt{3}XY}{4} = 2a^2$$

$$\Rightarrow 3X^2 - 2\sqrt{3}XY + Y^2 + 2\sqrt{3}\left[\sqrt{3}X^2 + 2XY + \sqrt{3}Y^2\right] - \left(X^2 + 3Y^2 + \sqrt{3}XY\right) = 8a^2$$

$$\Rightarrow 3X^2 - 2\sqrt{3} + Y^2 + 6X^2 + 4\sqrt{3}XY \; - 6Y^2 - X^2 - 3Y^2 - 2\sqrt{3}XY = 8a^2$$

$$\Rightarrow 8X^2 - 8Y^2 = 8a^2 \Rightarrow X^2 - Y^2 = a^2$$

4. **When the axes are rotated through an angle $\frac{\pi}{4}$, find the transformed equation of** $3x^2+10xy+3y^2=9$

Sol. Given equation is $3x^2+10xy+3y-9=0$ .... (1)

Angle of rotation of axes = $\theta=\frac{\pi}{4}$

Let (X,Y) be the new co-ordinates of $(x.y)$

$$x = X\cos\theta - Y\sin\theta$$

$$= X\cos\frac{\pi}{4} - y\sin\frac{\pi}{4} = \frac{X-Y}{\sqrt{2}}$$

$$y = X\sin\theta + Y\cos\theta = X\sin\frac{\pi}{4} + Y\cos\frac{\pi}{4} = \frac{X+Y}{\sqrt{2}}$$

Transformed equation of (1) is

$$3\left(\frac{X-Y}{\sqrt{2}}\right)^2 + 10\left(\frac{X-Y}{\sqrt{2}}\right)\left(\frac{X+Y}{\sqrt{2}}\right) + 3\left(\frac{X+Y}{\sqrt{2}}\right)^2 - 9 = 0$$

$$3\frac{\left(X^2-2XY+Y^2\right)}{2} + 10\frac{\left(X^2-Y^2\right)}{2} + 3\frac{\left(X^2+2XY+Y^2\right)}{2} - 9 = 0$$

$$\Rightarrow 3X^2-6XY+3Y^2+10X^2-10Y^2+3X^2+6XY+3Y^2-18=0$$

$$\Rightarrow 16X^2-4Y^2-18=0$$

$\therefore$ $8X^2-2Y^2=9$ is the transformed equation.

5. **Find the transformed equation of** $17x^2-16xy+17y^2=225$ **when the axes are rotated through an angle** $45°$

Sol. Let (x, y) the original equation of (X, Y)

Angle of rotation $\theta = 45°$

Now $X = x\cos\theta - y\sin\theta$

$$= x\cos 45^\circ - y\sin 45^\circ = \frac{x-y}{\sqrt{2}}$$

$$Y = x\sin\theta + y\cos\theta$$

$$= x\sin 45^\circ + y\cos 45^\circ = \frac{x+y}{\sqrt{2}}$$

The transformed equation is 45°

$$f\left(\frac{x-y}{\sqrt{2}}, \frac{x+y}{\sqrt{2}}\right) = 0$$

$$\Rightarrow 17\left(\frac{x-y}{\sqrt{2}}\right)^2 - 16\left(\frac{x-y}{\sqrt{2}}\right)\left(\frac{X+Y}{\sqrt{2}}\right) + 17\left(\frac{X+Y}{\sqrt{2}}\right)^2 = 225$$

$$\Rightarrow 17\left(\frac{x^2+y^2-2xy}{2}\right) - 16\left(\frac{x^2-y^2}{2}\right) + 17\left(\frac{x^2+y^2+2xy}{2}\right) = 225$$

$$\Rightarrow 17X^2 + 17Y^2 - 34XY - 16X^2 + 16Y^2 + 17X^2 + 17Y^2 + 34XY = 450$$

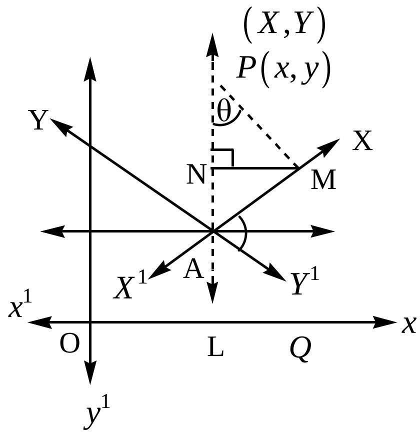
$$\Rightarrow 18X^2 + 50Y^2 = 450$$

$$9X^2 + 25Y^2 = 225$$

GENERAL TRANSFORMATIONS

1. Definition: If the axes are rotated through an angle θ after shifting the origin in the same plane, then the transformation is called "General Transformation"

New origin $A = (x_1, y_1)$, angle of rotation $= \theta$ as in figure



We get the transformed equations as

$$x = x_1 + X\cos\theta - Y\sin\theta$$

$$y = y_1 + X\sin\theta + Y\cos\theta$$

$$X = (x - x_1)\cos\theta + (y - y_1)\sin\theta$$

$$Y = (x - x_1)\sin\theta + (y - y_1)\cos\theta$$

We can easily understand the translation and rotation satisfy commutative property.

PROBLEMS.

1. When the origin is shifted to (-2,-3) and the axes are rotated through an angle 45° **find the transformed of** $2x^2+4xy-5y^2+20x-22y-14=0$ **?**

Sol. Here $(h,k) = (-2,-3), h = -2, k = -3$

$\theta = 45^\circ$

Let $(x^1,\ y^1)$ be the new co-ordinates of any point (x,y) is the plane after transformation

$x = x^1\cos\theta - y^1\sin\theta + h = -2x + x^1\cos 45^\circ - y^1\sin 45^\circ$

$$= -2 + \frac{x^1 - y^1}{\sqrt{2}}$$

$$y = x^1 \sin\theta + y^1 \cos\theta + k = x^1 \sin 45^\circ + y^1 \cos 45^\circ - 3$$

$$-3 + \frac{x^1 + y^1}{\sqrt{2}}$$

The transformed equation is

$$\Rightarrow 2\left(\frac{x^1 - y^1}{\sqrt{2}} - 2\right)^2 + 4\left(\frac{x^1 - y^1}{\sqrt{2}} - 2\right)\left(\frac{x^1 + y^1}{\sqrt{2}} - 3\right)$$

$$-5\left(\frac{x^1 + y^1}{\sqrt{2}} - 3\right)^2 + 20\left(\frac{x^1 - y^1}{\sqrt{2}} - 2\right) - 22\left(\frac{x^1 + y^1}{\sqrt{2}} - 3\right) - 14 = 0$$

$$\Rightarrow 2\left(\frac{\left(x^1 + y^1\right)^2}{2} + 42\sqrt{2}\left(x^1 - y^1\right)\right) + 4$$

$$\left(\frac{x^{1^2} - y^{1^2}}{2} - 3\frac{\left(x^1 - y^1\right)}{\sqrt{2}} - 2\frac{\left(x^1 + y^1\right)}{\sqrt{2}} + 6\right)$$

$$-5\left(\frac{\left(x^1 + y^1\right)^2}{2} + 9 - 3\sqrt{2}\left(x^1 + y^1\right)\right) + 10\sqrt{2}\left[\left(x^1 - y^1\right) - 2\sqrt{2}\right] - 11\sqrt{2}$$

$$\left[\left(x^1 + y^1\right) - 3\sqrt{2}\right] - 14 = 0$$

$$\left(x^1 + y^1\right)^2 + 8 - 4\sqrt{2}\left(x^1 - y^1\right) + 2\left(x^{1^2} - y^{1^2}\right) - 6\sqrt{2}\left(x^1 - y^1\right)$$

$$-4\sqrt{2}\left(x^1 + y^1\right) + 24 = 0 - \frac{5}{2}\left(x^1 + y^1\right)^2 - 45 + 15\sqrt{2}\left(x^1 + y^1\right) + 10\sqrt{2}\left(x^1 - y^1\right)$$

$$-40 - 11\sqrt{2}\left(x^1 + y^1\right) + 66 - 14 = 0$$

$$x^{1^2} + y^{1^2} - 2x^1 y^1 + 2x^{1^2} - 2y^{1^2} - \frac{5}{2}$$

$$\left(x^{1^2} + y^{1^2} + 2x^1 y^1\right) - 1 = 0$$

$$\frac{1}{2}x^{1^2} - \frac{7}{2}y^{1^2} - 7x^1y^1 - 1 = 0$$

i.e, $x^{1^2} - 7y^{1^2} - 14x^1y^1 - 2 = 0$

The transformed equation is (dropping dashes)

$$x^2 - 7y^2 - 14xy - 2 = 0$$

PROBLEMS FOR PRACTICE

1. **Find the transformed equation of** $5x^2 + 4xy + 8y^2 - 12x - 12y = 0$**. When the origin is shifted to** $\left(1, \frac{1}{2}\right)$ **by translation of axes.**

2. **When the origin is shifted to (3,-4) by the translation of axis and the transformed equation is** $x^2 + y^2 = 4$**, find the original equation.**

3. **When the origin is shifted to (2,3) by the translocation of axes, the co-ordinates of a point p are changed as (4,-3). Find the co-ordinates of P in the original system.**

4. **Find the point to which the origin is to be shifted by the translation of axes so as to remove the first degree terms from the equation** $ax^2 + by^2 + 2gx + 2fy + c = 0$**, where** $a \neq 0, \ \ b \neq 0$

5. **If the point P changes to (4,-3) when the axes are rotated through an angle of** $135°$**, find the coordinates of P with respect to the original system.**

6. **Show that the axes are to be rotated through an angle of** $\frac{1}{2}\mathrm{Tan}^{-1}\left(\frac{2h}{a-b}\right)$ **so as to remove the xy term from the equation** $ax^2 + 2hxy + by^2 = 0$**, if** $a \neq b$ **and through the angle** $\frac{\pi}{4}$**, if a = b**