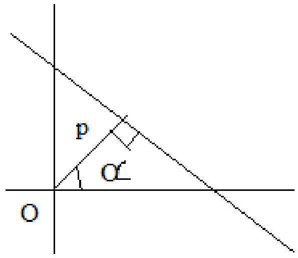
STRAIGHT LINE -2

NORMAL FORM

Let a line be at a distance of p units from the origin and α $(0 \le \alpha < 360^0)$ be the angle made by the normal to the line with positive direction of x - axis. Then the equation of the line is x cosα + y sinα = p.



SYMMETRIC FORM

The equation of the line passing through (x_1, y_1) and having inclination θ

is $\dfrac{x - x_1}{\cos\theta} = \dfrac{y - y_1}{\sin\theta}$, where $\theta \in \left(0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right)$.

NOTE:

A first degree equation in x and y represents a straight line.

The equation ax + by+ c = 0 is called the General Form Of The Equation Of A Line.

Note: The slope of the line ax + by + c = 0 is –a/b.

Theorem

Two lines $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$ are parallel iff $a_1 b_2 = a_2 b_1$ i.e. $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2}$

Theorem

Two equations a1x+b_1y+c_1=0 and $a_2x+b_2y+c_2=0$ represent the same line iff

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

Theorem

Two lines $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$ are perpendicular if $a_1a_2 + b_1b_2 = 0$

Reduction of the equation ax + by + c = 0 of a straight line into various forms.

1. SLOPE INTERCEPT FORM

Equation of the line is ax + by + c = 0

by = –ax – c $\Rightarrow$ y = $-\frac{a}{b}x-\frac{c}{b}$

This equation is of the form $y=mx+K$ .k is constant.

2. INTERCEPTS FORM

Equation of the line is ax + by + c = 0

ax + by = –c

$\Rightarrow \frac{ax}{-c}+\frac{by}{-c}=1 \Rightarrow \frac{x}{-c/a}+\frac{y}{-c/b}=1$ Which is of the form $\frac{x}{A}+\frac{y}{B}=1$

Here x - intercept = $-\frac{c}{a}$, y - intercept = $-\frac{c}{b}$.

3. NORMAL FORM

Case (i) Let c $\geq$ 0

The equation of the line is ax + by + c = 0

ax + by = – c (– c $\geq$ 0) $\Rightarrow \frac{a}{\sqrt{a^2+b^2}}x+\frac{b}{\sqrt{a^2+b^2}}y=\frac{-c}{\sqrt{a^2+b^2}}$

x cos α + y sin α = p, where $\cos\alpha=\frac{a}{\sqrt{a^2+b^2}}, \sin\alpha=\frac{b}{\sqrt{a^2+b^2}}$ and

$p=\frac{-c}{\sqrt{a^2+b^2}}>0$

Case (ii) Let c > 0.

Then ax + by + c = 0

ax + by = – c

(–a)x + (–b)y = c

$$\Rightarrow\left(\frac{-a}{\sqrt{a^2+b^2}}\right)x+\left(\frac{-b}{\sqrt{a^2+b^2}}\right)y=\frac{c}{\sqrt{a^2+b^2}}$$

x cos α + y sin α = p, where $\cos\alpha=\frac{-a}{\sqrt{a^2+b^2}}$, $\sin\alpha=\frac{-b}{\sqrt{a^2+b^2}}$ and $p=\frac{c}{\sqrt{a^2+b^2}}$

Note: The perpendicular distance from origin to the line ax + by + c = 0 is $\frac{|c|}{\sqrt{a^2+b^2}}$.

The area of the triangle formed by the line $\frac{x}{a}+\frac{y}{b}=1$ with the cooridnate axes is $\frac{1}{2}|ab|$.

PARAMETRIC FORM

If P(x, y) is any point on the line passing through A(x_1, y_1) and having inclination θ , then
x = x_1 + r cosθ , y = y_1 + r sinθ where | r | is the distance from A to P. (r is a real parameter)

EXERCISE – 3(B)

I.

1. Find the sum of the square of the intercepts of the line 4x – 3y = 12 on the axes of co-ordinates.

Sol. Given line is 4x – 3y = 12

$\Rightarrow \frac{4x}{12}-\frac{3y}{12}=1 \Rightarrow \frac{x}{3}+\frac{y}{-4}=1$

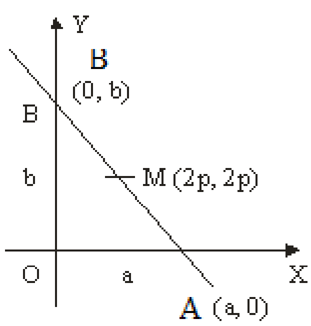
Intercepts are a = 3, b = -4 $\Rightarrow$ Sum of the squares = a^2+b^2 = 9 + 16 = 25

2. If the portion of a straight line intercepted between the axes of co-ordinates is bisected at (2p, 2q), write the equation of the straight line.

Sol. Let a, b be the intercepts of the line and AB be the line segment between the axes.

Then points A =(a, 0) and B = (0, b)

Equation of the line in the intercept form is $\frac{x}{a}+\frac{y}{b}=1$ --- (1)



Mid -point of AB is M= $\left(\frac{a}{2},\frac{b}{2}\right)=(2p,2q)$ given

$\Rightarrow \frac{a}{2}=2p, \frac{b}{2}=2q \Rightarrow a=4p, b=4q$

Substituting in (1), $\frac{x}{4p}+\frac{y}{4q}=1 \Rightarrow \frac{x}{p}+\frac{y}{q}=4$

3. If the linear equation ax + by + c = 0

(a, b, c $\neq 0$**) and** ***lx*** **+ my + n = 0 represent the same line and** $r=\frac{l}{a}=\frac{n}{c}$**, write the value of r in terms of m and b.**

Sol. The equations ax + by + c = 0 and

*l*x + my + n = 0 are representing the same line $\Rightarrow \frac{l}{a}=\frac{m}{b}=\frac{n}{c}=r \Rightarrow \frac{m}{b}=r$

4. Find the angle made by the straight line $y=-\sqrt{3}$ **x + 3 with the positive X-axis measured in the counter clock-wise direction**.

Sol. Equation of the given line is $y=-\sqrt{3}x+3$

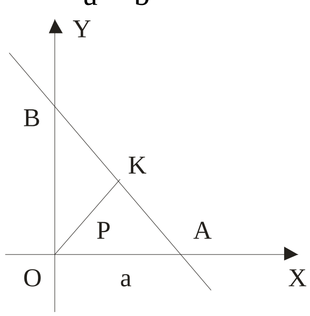
let α be the inclination of the line.

Then $\tan\alpha = -\sqrt{3}=\tan\frac{2\pi}{3} \Rightarrow \alpha=\frac{2\pi}{3}$

5. The intercepts of a straight line on the axes of co-ordinates are a and b. If P is the length of the perpendicular drawn from the origin to this line. Write the value of P in terms of a and b.

Sol. Equation of the line in the intercept form is

$\frac{x}{a}+\frac{y}{b}-1=0$



P = length of the perpendicular from origin

$$p = \frac{|0+0-1|}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}}$$

$$\Rightarrow \frac{1}{p} = \sqrt{\frac{1}{a^2}+\frac{1}{b^2}}$$

Square on both sides

$$\Rightarrow \frac{1}{p^2} = \frac{1}{a^2}+\frac{1}{b^2} = \frac{a^2+b^2}{a^2b^2} \quad \Rightarrow p^2 = \frac{a^2b^2}{a^2+b^2} \Rightarrow p = \frac{|ab|}{\sqrt{a^2+b^2}}$$

II.

1. In what follows, P denotes the distance of the straight line from the origin and α is the angle made by the normal ray drawn from the origin to the straight line with $\overrightarrow{OX}$ measured in the anti-clock wise sense. Find the equations of the straight lines with the following values of P and α.

i) **p = 5,** $\alpha = 60^o$ **ii)** **p = 6,** α **= 150º** **iii)** **p = 1,** $\alpha = \frac{7\pi}{4}$

iv) **p = 4,** α **= 90º** **iv)** **p = 0,** $\alpha = 0$ **v)** **p = 2**$\sqrt{2}, \alpha = \frac{5\pi}{4}$

Sol. Equation of the line in the normal form is $x\cos\alpha + y\sin\alpha = p$

i) given p = 5, $\alpha = 60^o$

$\cos\alpha = \cos 60^o = \frac{1}{2}$ and $\sin\alpha = \sin 60^o = \sqrt{\frac{3}{2}}$

Equation of the line is $x\cos\alpha + y\sin\alpha = p$

$$\Rightarrow x\frac{1}{2} + y\frac{\sqrt{3}}{2} = 5 \Rightarrow x + \sqrt{3}y = 10$$

ii) ans : $\sqrt{3}x + y + 12 = 0$

iii) p = 4, $\alpha = \frac{7\pi}{4}$

$\cos\alpha = \cos 315^o = \frac{1}{\sqrt{2}}$ and $\sin\alpha = \sin 315^o = -\frac{1}{\sqrt{2}}$

Equation of the line is $x.\frac{1}{\sqrt{2}} - y.\frac{1}{\sqrt{2}} = 1$

$x - y = \sqrt{2} \Rightarrow x - y - \sqrt{2 = 0}$

iv) P = 4, $\alpha = 90^o$

$\cos\alpha = \cos 90^o = 0 \text{and} \sin\alpha = \sin 90^o = 1$

Equation of the line is $x\cos\alpha + y\sin\alpha = p \Rightarrow$ x. 0 +y. 1 = 4 $\Rightarrow$ x = 4

v) ans: $x = 0$

vi) ans: $x + y + 4 = 0$

2. Find the equation of the straight line in the symmetric form, given the slope and a point on the line in each part of the question.

i) $\sqrt{3}$ **,(2, 3)** **ii)** $-\frac{1}{\sqrt{3}}$ **, (-2, 0)** **iii)** **–1, (1, 1)**

Sol. i) point $(x_1, y_1) = (2,3)$

slope $m = \tan\alpha = \sqrt{3} \Rightarrow \alpha = 60^o \Rightarrow \cos\alpha = \cos 60^o \Rightarrow \sin\alpha = \sin 60^o$

Equation of the line in the symmetric form is $\frac{x - x_1}{\cos\alpha} = \frac{y - y_1}{\sin\alpha} = r \Rightarrow \frac{x-2}{\cos\frac{\pi}{3}} = \frac{y-3}{\sin\frac{\pi}{3}}$

ii) $(x_1, y_1) = (-2, 0)$

$\tan\alpha = -\frac{1}{\sqrt{3}} \Rightarrow \alpha = 180^o - 30^o = 150^o$

Equation of the line in the symmetric form is

$$\frac{x - x_1}{\cos\alpha} = \frac{y - y_1}{\sin\alpha} = r \Rightarrow \frac{x-2}{\cos\frac{\pi}{3}} = \frac{y-3}{\sin\frac{\pi}{3}} \qquad \Rightarrow \qquad \frac{x+2}{\cos 150^o} = \frac{y}{\sin 150^o}$$

iii) ans: $\frac{x-1}{\cos\left(\frac{3\pi}{4}\right)} = \frac{y-1}{\sin\left(\frac{3\pi}{4}\right)}$

3. Transform the following equation into a) Slope-intercept form b) Intercept from and c) Normal form

i) **3x + 4y = 5** **ii)** **4x -3y +12 =0** **iii)** $\sqrt{3}x + y = 4$ **iv)** **x + y + 2 = 0**

v) **x + y – 2 = 0** **vi)** $\sqrt{3}x + y + 10 = 0$

Sol. i) equation of the line is $3x + 4y = 5$

1.Slope-intercept form

$4y = -3x + 5 \Rightarrow \quad y = \left(-\frac{3}{4}\right)x + \left(\frac{5}{4}\right))$ which is of the form y =mx+c.

2. Intercept form : $3x + 4y = 5$

$\frac{3x}{5} + \frac{4y}{5} = 5 \Rightarrow \frac{x}{\left(\frac{5}{3}\right)} + \frac{y}{\left(\frac{5}{4}\right)} = 1$ which is of the form $\frac{x}{a} + \frac{y}{b} = 1$

3.Normal form : $3x + 4y = 5$

Dividing with $\sqrt{3^2 + 4^2} = \sqrt{9 + 16} = 5$

$\frac{3}{5}x+\frac{4}{5}y=1 \Rightarrow$ let $\cos\alpha=\frac{3}{5}, \sin\alpha=\frac{4}{5}$ and p=1

Then the above eq. reduces to $x\cos\alpha+y\sin\alpha=p$

ii) ans: $y=\left(\frac{4}{3}\right)x+4$, $\frac{x}{(-3)}+\frac{y}{4}=1$, $\left(\frac{-4}{5}\right)x+\left(\frac{3}{5}\right)y=\frac{12}{5}$

iii) $y=-\sqrt{3}x+4, \frac{x}{\left(\frac{4}{\sqrt{3}}\right)}+\frac{y}{4}=1, x\cos\left(\frac{\pi}{6}\right)+y\sin\left(\frac{\pi}{6}\right)=32$

iv) x + y + 2 = 0

1. Slope-intercept form :

x + y + 2 = 0

$\Rightarrow$ y = -x – 2 = (-1) x + (-2) which is of the form y =mx+c.

Intercept form :

x + y + 2 = 0 $\Rightarrow$ –x – y = 2

$\Rightarrow -\frac{x}{2}-\frac{y}{2}=1 \Rightarrow \frac{x}{-2}+\frac{y}{-2}=1$ which is of the form $\frac{x}{a}+\frac{y}{b}=1$

3.Normal form: x + y + 2 = 0 $\Rightarrow$ –x – y = 2

Dividing with $\sqrt{1+1}=\sqrt{2}$

$\Rightarrow\left(-\frac{1}{\sqrt{2}}\right)x+\left(-\frac{1}{\sqrt{2}}\right)y=\sqrt{2} \Rightarrow x\cos\left(\frac{5\pi}{4}\right)+y\sin\left(\frac{5\pi}{4}\right)=\sqrt{2}$

Which is of the form $x\cos\alpha+y\sin\alpha=p$

v) y = - x + 2, $\frac{x}{2}+\frac{y}{2}=1, x\cos\left(\frac{\pi}{4}\right)+\sin\left(\frac{\pi}{4}\right)=\sqrt{2}$

vi) $y=-\sqrt{3}x-10$, $\frac{x}{\left(\frac{10}{\sqrt{3}}\right)}+\frac{y}{-10}=1$, $x\cos 30^{\circ}+y\sin 30^{\circ}=5$

4. If the product of the intercepts made by the straight line $x\tan\alpha+y\sec\alpha=1\left(0\le\alpha<\frac{\pi}{2}\right)$, **on the co-ordinate axes is equal to** $\sin\alpha$, findα.

Sol. Equation of the line is x tan $\alpha+y\sec\alpha=1$

$\Rightarrow \frac{x}{\cot\alpha}+\frac{y}{\cos\alpha}=1 \Rightarrow a=\cot\alpha, b=\cos\alpha$

Given product of intercepts $=\sin\alpha$

$\Rightarrow \cot\alpha.\cos\alpha = \sin\alpha$

$\Rightarrow \dfrac{\cos^2\alpha}{\sin\alpha} = \sin\alpha \Rightarrow \cos^2\alpha = \sin^2\alpha$

$\Rightarrow \tan^2\alpha = 1 \Rightarrow \tan\alpha - \pm 1 \Rightarrow \alpha = 45^o$

5. If the sum of the reciprocals of the intercept made by a variable straight line on the axes of co-ordinates is a constant then prove that the line always passes through a fixed point.

Sol. Equation of the line in the intercept form is $\dfrac{x}{a}+\dfrac{y}{b}=1$ --- (1)

Sum of the reciprocals of the intercepts

$$\frac{1}{a}+\frac{1}{b}=k \Rightarrow \frac{1}{ak}+\frac{1}{bk}=k \Rightarrow \frac{\left(\frac{1}{k}\right)}{a}+\frac{\left(\frac{1}{k}\right)}{b}=1$$

Comparing this equation with (1),

The line (1) passes through the fixed point $\left(\dfrac{1}{k},\dfrac{1}{k}\right)$

6. Line L has intercepts a and b on the axes of co-ordinates. When the axes are rotated through a given angle, keeping the origin fixed, the same line L has intercepts p and q on the transformed axes. Prove that $\dfrac{1}{a^2}+\dfrac{1}{b^2}=\dfrac{1}{p^2}+\dfrac{1}{q^2}$.

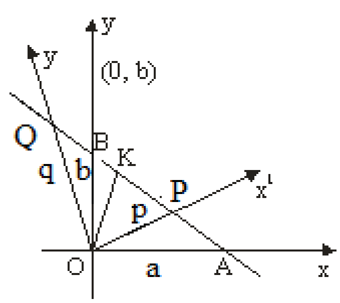
Sol. Equation of the line in the old system in intercept form is $\dfrac{x}{a}+\dfrac{y}{b}=1 \Rightarrow \dfrac{x}{a}+\dfrac{y}{b}-1=0$

Length of the perpendicular form origin $=\dfrac{|0+0-1|}{\sqrt{\dfrac{1}{a^2}+\dfrac{1}{b^2}}}$ ---(1)

Equation of the line in the new system in intercept form

is $\dfrac{X}{p}+\dfrac{Y}{q}=1 \Rightarrow \dfrac{X}{p}+\dfrac{Y}{q}-1=0$

Length of the perpendicular form origin $=\dfrac{|0+0-1|}{\sqrt{\dfrac{1}{p^2}+\dfrac{1}{q^2}}}$ ---

(2) 

Since the position of the origin and the given line remain unchanged ,perpendicular distances in both the systems are same.

$$\frac{|-1|}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}}=\frac{|-1|}{\sqrt{\frac{1}{p^2}+\frac{1}{q^2}}}\Rightarrow\frac{1}{\left(\frac{1}{a^2}+\frac{1}{b^2}\right)}=\frac{1}{\left(\frac{1}{p^2}+\frac{1}{q^2}\right)}\quad\Rightarrow\frac{1}{a^2}+\frac{1}{b^2}=\frac{1}{p^2}+\frac{1}{q^2}$$

7. Transform the equation $\frac{x}{a}+\frac{y}{b}=1$ into the normal form when a > 0 and b > 0. If the perpendicular distance of the straight line form the origin is p, deduce that $\frac{1}{p^2}=\frac{1}{a^2}+\frac{1}{b^2}$.

Sol. Equation of the line is $\frac{x}{a}+\frac{y}{b}=1$

⇨ $bx+ay=ab$

Dividing with $\sqrt{a^2+b^2}$

$$\frac{b}{\sqrt{a^2+b^2}}.x+\frac{a}{\sqrt{a^2+b^2}}.y=\frac{ab}{\sqrt{a^2+b^2}} \text{ ----(1)}$$

Let $\cos\alpha=\frac{b}{\sqrt{a^2+b^2}},\sin\alpha=\frac{a}{\sqrt{a^2+b^2}}$ and $p=\frac{ab}{\sqrt{a^2+b^2}}$

Now eq. (1) reduces to $x\cos\alpha+y\sin\alpha=p$ which is normal form.

The perpendicular distance from O to the line is $p=\frac{ab}{\sqrt{a^2+b^2}}\Rightarrow p^2=\frac{a^2b^2}{a^2+b^2}$

$$\Rightarrow\frac{1}{p^2}=\frac{a^2+b^2}{a^2b^2}=\frac{a^2}{a^2b^2}+\frac{b^2}{a^2b^2}=\frac{1}{b^2}+\frac{1}{a^2}\quad\therefore\frac{1}{p^2}=\frac{1}{a^2}+\frac{1}{b^2}$$

III. **1. A straight line passing through A(-2, 1) makes an angle 30º with $\overrightarrow{OX}$ in the positive direction. Find the points on the straight line whose distance form A is 4 units**.

Sol. Let $(x_1,y_1)=(-2,1)$, r =4 and $\alpha=30^\circ$

Co-ordinates of any point on the given line are

$(x_1\pm r\cos\alpha, y_1\pm r\sin\alpha)$ $\cos\alpha=\cos30^\circ=\frac{\sqrt{3}}{2},\sin\alpha=\sin30^\circ=\frac{1}{2}$

Require points are

$$\left(-2+4.\frac{\sqrt{3}}{2},1+4.\frac{1}{2}\right)=\left(-2+2\sqrt{3},3\right)$$ And

$$\left(-2+4.\frac{\sqrt{3}}{2},1-4.\frac{1}{2}\right)=\left(-2-2\sqrt{3,-1}\right)$$

2. Find the points on the line 3x – 4y – 1 = 0 which are at a distance of 5 units form the point (3, 2).

Sol. Equation of the line is 3x – 4y – 1 = 0 $\Rightarrow$ slope of the line is tanθ = ¾

$\Rightarrow$ sinθ = 3/5 and cosθ =4/5

Given Point is (3,2) $=(x_{1,}y_1)$ and r = 5.

Co-ordinates of any point on the given line are $(x_1 \pm r\cos\theta, y_1 \pm r\sin\theta)$

Co-ordinates of the points are $\left(3+5.\frac{4}{5},2+5.\frac{3}{5}\right)=(7,5)$ And

$$\left(3-5.\frac{4}{5},2-5.\frac{3}{5}\right)=(-1,-1)$$

3. A straight line whose inclination with the positive direction of the X – axis measured in the anti clock wise sense is $\pi/3$ makes positive intercept on the Y – axis. If the straight line is at a distance of 4 form the origin, find its equation.

Sol. Let $\alpha=\pi/3, P=4$

Slope m = $\tan\alpha=\tan 60^{o}=\sqrt{3}$

Equation of the line in the slope – intercept

form is $y=\sqrt{3}x+c \Rightarrow \sqrt{3}x-y+c=0$

Distance from the origin = 4

$$\frac{|0-0+c|}{\sqrt{3+1}}=4 \Rightarrow |c|=4\times 2=8 \Rightarrow c=\pm 8$$

Given c > 0 $\therefore c=8$

Equation of the line $\sqrt{3}x-y+8=0$

4. A straight line L is drawn through the point A(2, 1) such that its point of intersection with the straight line x + y = 9 is at a distance of 3$\sqrt{2}$ form A. Find the angle which the line L makes with the positive direction of the X – axis.

Sol. Suppose α is the angle made by L with the positive X – axis

Given point **(2, 1) =(x_1 ,y_1)** **and r =3**$\sqrt{2}$

Any point on the line is $(x_1+r\cos\theta, y_1+\sin\theta)=(2+3\sqrt{2}\cos\theta+3\sqrt{2}\sin\theta)$

This is a point on the line x + y = 9

$$\Rightarrow 2+3\sqrt{2}\cos\alpha+1+3\sqrt{2}\sin\alpha=9$$

$$\Rightarrow 3\sqrt{2}(\cos\alpha+\sin\alpha)=6\Rightarrow\frac{1}{\sqrt{2}}.\cos\alpha+\frac{1}{\sqrt{2}}\sin\alpha=1\Rightarrow\cos\alpha.\cos 45^{o}+\sin\alpha.\sin 45^{o}=1$$

$$\Rightarrow\cos(\alpha-45^{o})=\cos 0^{o}\Rightarrow\alpha-45^{o}=0\Rightarrow\alpha=45^{o}=\frac{\pi}{4}$$

5. A straight line L with negative slope passes through the point (8, 2) and cuts positive co-ordinates axes at the points P and Q. Find the minimum value of OP + OQ as L varies, when O is the origin.

Sol. Let –m be the slope of the line where m>0.

Equation of the line passing through A (8, 2)

With negative slope '-m' is

y – 2= – m(x – 8)

$$\Rightarrow mx+y-(2+8m)=0\Rightarrow mx+y=2+8m$$

$$\Rightarrow\frac{m}{2+8m}x+\frac{y}{2+8m}=1\Rightarrow\frac{x}{\frac{2+8m}{m}}+\frac{y}{2+8m}=1$$

OP = x Intercept = $\frac{2+8m}{m}$

OQ= Y-Intercept = 2 + 8m

$$OP+OQ=\frac{8m^{2}+10m+2}{m}$$

$$=8m+\frac{2}{m}+10\geq 10+2\sqrt{8m.\frac{2}{m}}\geq 10+2.4=18$$

Therefore the minimum value is 18.