| CHAPTER 6 DIRECTION COSINES AND DIRECTION RATIOS |
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| TOPICS: <br> 1.DEFINITION OF D.CS., RELATION BETWEEN D.CS. OF A LINE, CO-ORDINATES OF A POINT WHEN D.CS. ARE GIVEN AND DIRECTION COSINES OF A LINE JOINING TWO POINTS. <br> 2. ANGLE BETWEEN TWO LINES WHEN D.CS ARE GIVEN, FINDING THE ANGLE BETWEEN TWO LINES WHEN THEIR D.CS ARE CONNECTED BY EQUATIONS. <br> 3.DEFINITION OF DIRECTION RATIOS, D.RS. OF A LINE JOINING TWO POINTS <br> 4. RELATION BETWEEN D.CS AND D.RS <br> 5. CONDITIONS FOR PARALLEL AND PERPENDICULA LINES WHEN D.CS/D.RS ARE GIVEN. <br> 6. ANGLE BETWEEN TWO LINES WHEN D.RS ARE GIVEN, FINDING THE ANGLE BETWEEN TWO LINES WHEN THEIR D.RS ARE CONNECTED BY EQUATIONS. |

## DIRECTION COSINES \& RATIOS (7 MARKS )

## ANGLE BETWEEN TWO LINES:

The angle between two skew lines is the angle between two lines drawn parallel to them through any point in space.


## DIRECTION COSINES

If $\alpha, \beta, \gamma$ are the angles made by a directed line segment with the positive directions of the coordinate axes respectively, then $\cos a, \cos b, \cos g$ are called the direction cosines of the given line and they are denoted by $1, m, n$ respectively Thus $l=\cos \alpha, m=\cos \beta, n=\cos \gamma$


The direction cosines of $\overrightarrow{o p}$ are
$l=\cos \alpha, m=\cos \beta, n=\cos \gamma$.
If $1, m, n$ are the d.c's of a line $L$ is one direction then the d.c's of the same line in the opposite direction are $-1,-\mathrm{m},-\mathrm{n}$.

Note : The angles $\alpha, \beta, \gamma$ are known as the direction angles and satisfy the condition $0 \leq \alpha, \beta, \gamma \leq \pi$.
Note : The sum of the angles $\alpha, \beta, \gamma$ is not equal to $2 p$ because they do not lie in the same plane.

Note: Direction cosines of coordinate axes.

The direction cosines of the x -axis are $\cos 0, \cos \frac{\pi}{2}, \cos \frac{\pi}{2}$ i.e., $1,0,0$
Similarly the direction cosines of the $y$-axis are $(0,1,0)$ and $z$-axis are $(0,0,1)$

## THEOREM

If $\mathrm{P}(\mathrm{x}, \mathrm{y}, \mathrm{z})$ is any point in space such that $\mathrm{OP}=\mathrm{r}$ and if $1, \mathrm{~m}, \mathrm{n}$ are direction cosines of $\overline{O P}$ then
$\mathrm{x}=\mathrm{lr}, \mathrm{y}=\mathrm{mr}, \mathrm{z}=\mathrm{nr}$.
Note: If $P(x, y, z)$ is any point in space such that $O P=r$ then the direction cosines of $\overrightarrow{O P}$ are $\frac{x}{r}, \frac{y}{r}, \frac{z}{r}$
Note: If P is any point in space such that $\mathrm{OP}=\mathrm{r}$ and direction cosines of $\overrightarrow{O P}$ are $1, \mathrm{~m}, \mathrm{n}$ then the point $\mathrm{P}=(\mathrm{lr}, \mathrm{mr}, \mathrm{nr})$
Note: If $\mathrm{P}(\mathrm{x}, \mathrm{y}, \mathrm{z})$ is any point in space then the direction cosines of $\overline{O P}$ are

$$
\frac{x}{\sqrt{x^{2+} y^{2}+z^{2}}}, \frac{y}{\sqrt{x^{2+} y^{2}+z^{2}}}, \frac{z}{\sqrt{x^{2}+y^{2}+z^{2}}}
$$

## THEOREM

If $1, m, n$ are the direction cosines of a line $L$ then $1^{2}+m^{2}+n^{2}=1$.

## Proof :



From the figure
$l=\cos \alpha=\frac{x}{r}, m=\cos \beta=\frac{y}{r}, n=\cos \gamma=\frac{z}{r} \quad \Rightarrow \cos ^{2} \alpha+\cos ^{2} \beta+\cos ^{2} \gamma=\frac{x^{2}}{r^{2}}+\frac{y^{2}}{r^{2}}+\frac{z^{2}}{r^{2}}$
$=\frac{x^{2}+y^{2}+z^{2}}{r^{2}}=\frac{r^{2}}{r^{2}}=1 . \quad \therefore l^{2}+m^{2}+n^{2}=1$

## THEOREM

The direction cosines of the line joining the points $\boldsymbol{P}\left(x_{1}, y_{1}, z_{1}\right), \boldsymbol{Q}\left(x_{2}, y_{2}, z_{2}\right)$ are $\left(\frac{x_{2}-x_{1}}{r}, \frac{y_{2}-y_{1}}{r}, \frac{z_{2}-z_{1}}{r}\right)$ where $r=\sqrt{\left(x_{2}-x_{1}\right)^{2}+\left(y_{2}-y_{1}\right)^{2}+\left(z_{2}-z_{1}\right)^{2}}$

## Exercise -6A

1. A line makes angle $90^{\circ}, 60^{\circ}$ and $30^{\circ}$ with positive directions of $\mathrm{x}, \mathrm{y}, \mathrm{z}$-axes respectively. Find the direction cosines.

Sol: Suppose $l, m, n$ are the direction cosines of the line, then

$$
\begin{aligned}
& l=\cos \alpha=\cos 90^{\circ}=0 \\
& m=\cos \beta=\cos 60^{\circ}=\frac{1}{2} \quad \text { And } n=\cos \gamma=\cos 30^{\circ}=\frac{\sqrt{3}}{2}
\end{aligned}
$$

Direction cosines of the line are $\left(0, \frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
2. If a line makes angles $\alpha, \beta, \gamma$ with the positive direction of $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$ axes, what is the value of $\sin ^{2} \alpha+\sin ^{2} \beta+\sin ^{2} \gamma$
Solution: We know that $\cos ^{2} \alpha+\cos ^{2} \beta+\cos ^{2} \gamma=1$

$$
\begin{aligned}
& 1-\sin ^{2} \alpha+1-\sin ^{2} \beta+1-\sin ^{2} \gamma=1 \\
& \sin ^{2} \alpha+\sin ^{2} \beta+\sin ^{2} \gamma=3-1=2
\end{aligned}
$$

3. If $P(\sqrt{3}, 1,2 \sqrt{3})$ is a point in space, find the direction cosines of $\overrightarrow{O P}$

Solution: Direction ratios of P are $(\sqrt{3}, 1,2 \sqrt{3})=(a, b, c)$

$$
\Rightarrow a^{2}+b^{2}+c^{2}=3+1+12=16 \quad \Rightarrow \sqrt{a^{2}+b^{2}+c^{2}}=4
$$

Direction cosines of $\overrightarrow{O P}$ are

$$
\left(\frac{a}{\sqrt{a^{2}+b^{2}+c^{2}}}, \frac{b}{\sqrt{a^{2}+b^{2}+c^{2}}}, \frac{c}{\sqrt{a^{2}+b^{2}+c^{2}}}\right)=\left(\frac{\sqrt{3}}{4}, \frac{1}{4}, \frac{2 \sqrt{3}}{4}\right)=\left(\frac{\sqrt{3}}{4}, \frac{1}{4}, \frac{\sqrt{3}}{2}\right)
$$

## 4. Find the direction cosines of the line joining the points

$$
(-4,1,7) \text { and }(2,-3,2)
$$

Solution: $A(-4,1,2)$ and $B(2,-3,2)$ are given points
d.rs of PQ are $\left(x_{2}-x_{1}, y_{2}-y_{1}, z_{2}-z_{1}\right)$
i.e. $(2+4,1+3,2-7)$ i.e, $(6,4,-5)=(a, b, c)$
$\Rightarrow \sqrt{a^{2}+b^{2}+c^{2}}=\sqrt{36+16+25}=\sqrt{77}$
Direction cosines of $\overrightarrow{A B}$ are
$\left(\frac{a}{\sqrt{a^{2}+b^{2}+c^{2}}}, \frac{b}{\sqrt{a^{2}+b^{2}+c^{2}}}, \frac{c}{\sqrt{a^{2}+b^{2}+c^{2}}}\right)=\left(\frac{6}{\sqrt{77}}, \frac{4}{\sqrt{77}}, \frac{-5}{\sqrt{77}}\right)$

## II.

## 1. Find the direction cosines of the sides of the triangle whose vertices are

 $(3,5,-4),(-1,1,2)$ and $(-5,-5,-2)$Sol: $A(3,5,-4), B(-1,1,2)$ and $C(-5,-5,-2)$ are the vertices of $\triangle A B C$
d.rs of AB are $(-1-3,1-5,2+4)=(-4,-4,6)$

Dividing with $\sqrt{16+16+36}=\sqrt{68}=2 \sqrt{17}$
d.cs of AB are $\frac{-4}{2 \sqrt{17}}, \frac{-4}{2 \sqrt{17}}, \frac{6}{2 \sqrt{17}}$ i.e., $\frac{-2}{\sqrt{17}}, \frac{-2}{\sqrt{17}}, \frac{3}{\sqrt{17}}$
D.rs of BC are $(-5+1,-5-1,-2-2)$ i.e., $(-4,-6,-4)$

Dividing with $\sqrt{16+16+36}=\sqrt{68}=2 \sqrt{17}$ d.cs of BC are $\frac{-4}{2 \sqrt{17}}, \frac{-6}{2 \sqrt{17}}, \frac{-4}{2 \sqrt{17}}$ i.e., $\frac{-2}{\sqrt{17}}, \frac{-3}{\sqrt{17}}, \frac{-4}{2 \sqrt{17}}$
d.rs of CA are $(3+5,5+5,-4+2)=(8,10,-2)$

Dividing with $\sqrt{64+100+4}=\sqrt{168}=2 \sqrt{42}$
Then d.cs of CA are $\frac{4}{\sqrt{42}}, \frac{5}{\sqrt{42}}, \frac{-1}{\sqrt{42}}$
2. Show that the lines $\overrightarrow{P Q}$ and $\overrightarrow{R S}$ are parallel where $\mathbf{P}, \mathbf{Q}, \mathbf{R}, \mathbf{S}$ are two points ( $2,3,4$ ), (-1, $-2,-1$ ) and (1, 2, 5) respectively

Sol : $P(2,3,4), Q(4,7,8), \mathrm{R}(-1,-2,1)$ and $\mathrm{S}(1,2,5)$ are the given points.
d.rs of PQ are (4-2, 7-3, 8-4 ) i.e.,( $2,4,4$ )
d.rs of $\operatorname{RS}$ are $(1+1,2+2,5-1)$ i.e., $(2,4,4)$
$\therefore$ d.rs of PQ and RS are proportional. Hence, PQ and RS are parallel
III.

1. Find the direction cosines of two lines which are connected by the relation $l-5 m+3 n=0$ and $7 l^{2}+5 m^{2}-3 n^{2}=0$

Sol. Given $l-5 m+3 n=0$

$$
\begin{equation*}
\Rightarrow l=5 m-3 n-----(1) \tag{2}
\end{equation*}
$$

and $\quad 7 l^{2}+5 m^{2}-3 n^{2}=0$
Substituting the value of $l$ in (2)

$$
\begin{aligned}
& 7(5 m-3 n)^{2}+5 m^{2}-3 n^{2}=0 \\
& \Rightarrow 7\left(25^{2}+9 n^{2}-30 m n\right)+5 m^{2}-3 n^{2}=0 \\
& \Rightarrow 175 m^{2}+63 n^{2}-210 m n+5 m^{2}-3 n^{2}=0 \\
& \Rightarrow 180 m^{2}-210 m n+60 n^{2}=0 \\
& \Rightarrow 6 m^{2}-7 m n+2 n^{2}=0 \\
& \Rightarrow(3 m-2 n)(2 m-n)=0
\end{aligned}
$$

Case (i): $3 m_{1}=2 n_{1} \Rightarrow \frac{m_{1}}{2}=\frac{n_{1}}{3}$
Then $m_{1}=\frac{2}{3} n_{1}$
From (1) $l_{1}=5 m_{1}-3 n_{1}=\frac{10}{3} n_{1}-3 n_{1}$
$=\frac{10 n_{1}-9 n_{1}}{3}=\frac{n_{1}}{3}$

$$
\therefore \frac{l_{1}}{1}=\frac{m_{1}}{2}=\frac{n_{1}}{3}
$$

d.rs of the first line are $(1,2,3)$

Dividing with $\sqrt{1+4+9}=\sqrt{14}$
d.cs of the first line are $\left(\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right)$

Case (ii) $2 m_{2}=n_{2}$
From (1) $l_{2}-5 m_{2}+3 n_{2}=0$
$\Rightarrow l_{2}-5 m_{2}+6 m_{2}=0$
$\Rightarrow-l_{2}=m_{2}$
$\therefore \frac{l_{2}}{-1}=\frac{m_{2}}{1}=\frac{n_{2}}{2}$
d.rs of the second line are $-1,1,2$

Dividing with $\sqrt{1+1+4}=\sqrt{6}$
d.cs of the second line are $\left(\frac{-1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right)$

## DIRECTION RATIOS

A set of three numbers a,b,c which are proportional to the direction cosines $1, m, n$ respectively are called DIRECTION RATIOS (d.r's) of a line.
Note : If ( $\mathrm{a}, \mathrm{b}, \mathrm{c}$ ) are the direction ratios of a line then for any non-zero real number $\lambda,\left(\lambda a, \lambda_{b}, \lambda_{c}\right)$ are also the direction ratios of the same line.
Direction cosines of a line in terms of its direction ratios
If ( $a, b, c$ ) are direction ratios of a line then the direction cosines of the line are

$$
\pm\left(\frac{a}{\sqrt{a^{2}+b^{2}+c^{2}}}, \frac{b}{\sqrt{a^{2}+b^{2}+c^{2}}}, \frac{c}{\sqrt{a^{2}+b^{2}+c^{2}}}\right)
$$

## THEOREM

The direction ratios of the line joining the points are $\left(x_{2}-x_{1}, y_{2}-y_{1}, z_{2}-z_{1}\right)$

## Angle between two lines

If $\left(l_{1}, m_{1}, n_{1}\right)$ and $\left(l_{2}, m_{2}, n_{2}\right)$ are the direction cosines of two lines $\theta$ and is the acute angle between them, then $\cos \theta=\left|l_{1} l_{2}+m_{1} m_{2}+n_{1} n_{2}\right|$

## Note.

If $\theta$ is the angle between two lines having d.c's $\left(l_{1}, m_{1}, n_{1}\right)$ and $\left(l_{2}, m_{2}, n_{2}\right)$ then

$$
\sin \theta=\sqrt{\sum\left(l_{1} m_{2}-l_{2} m_{1}\right)^{2}}
$$

$$
\text { and } \tan \theta=\frac{\sqrt{\sum\left(l_{1} m_{2}-l_{2} m_{1}\right)^{2}}}{\left|l_{1} l_{2}+m_{1} m_{2}+n_{1} n_{2}\right|} \text { when } \theta \neq \frac{\pi}{2}
$$

Note 1: The condition for the lines to be perpendicular is $l_{1} l_{2}+m_{1} m_{2}+n_{1} n_{2}=0$
Note 2: The condition for the lines to be parallel is $\frac{l_{1}}{l_{2}}=\frac{m_{1}}{m_{2}}=\frac{n_{1}}{n_{2}}$

## THEOREM

If ( $a_{1}, b_{1}, c_{1}$ ) and ( $a_{2}, b_{2}, c_{2}$ ) are direction ratios of two lines and $\theta$ is the angle between them then $\cos \theta=\frac{a_{1} a_{2}+b_{1} b_{2}+c_{1} c_{2}}{\sqrt{a_{1}^{2}+b_{1}^{2}+c_{1}^{2}} \sqrt{a_{2}^{2}+b_{2}^{2}+c_{2}^{2}}}$

Note 1: If the two lines are perpendicular then $a_{1} a_{2}+b_{1} b_{2}+c_{1} c_{2}=0$
Note 2: If the two lines are parallel then $\frac{a_{1}}{a_{2}}=\frac{b_{1}}{b_{2}}=\frac{c_{1}}{c_{2}}$
Note 3: If one of the angle between the two lines is $\theta$ then other angle is $180^{\circ}-\theta$

## EXERCISE - 6(B)

1. Find the direction ratios of the line joining the points $(3,4,0)$ are $(4,4,4)$

Sol. $A(3,4,0)$ and $B(4,4,4)$ are the given points
d.rs of AB are $\left(x_{2}-x_{1}, y_{2}-y_{1}, z_{2}-z_{1}\right)=(4-3,4-4,4-0) i . e .,(1,0,4)$
2. The direction ratios of a line are $(-6,2,3)$. Find the direction cosines.

Sol: D.rs of the line are $-6,2,3$
Dividing with $\sqrt{36+4+9}=7$
Direction cosines of the line are $-\frac{6}{7}, \frac{2}{7}, \frac{3}{7}$
3. Find the cosine of the angle between the lines, whose direction cosines are
$\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ and $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$
Sol: D.cs of the given lines are $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ and $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$.
Let $\theta$ be the angle between the lines. Then

$$
\cos \theta=l_{1} l_{2}+m_{1} m_{2}+n_{1} n_{2} \quad=\frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}} \cdot 0=\frac{1}{\sqrt{6}}=\frac{2}{\sqrt{6}}=\sqrt{\frac{4}{6}} \quad=\sqrt{\frac{2}{3}}
$$

4. Find the angle between the lines whose direction ratios are $(1,1,2)(\sqrt{3},-\sqrt{3}, 0)$

Sol: D.rs of the given lines are $(1,1,2)$ and $(\sqrt{3},-\sqrt{3}, 0)$
Let $\theta$ be the angle between the lines. Then

$$
\begin{aligned}
& \cos \theta=\frac{a_{1} a_{2}+b_{1} b_{2}+c_{1} c_{2}}{\sqrt{a_{1}^{2}+b_{1}^{2}+c_{1}^{2}} \sqrt{a_{2}^{2}+b_{2}^{2}+c_{2}^{2}}}=\frac{1 \sqrt{3}+1(-\sqrt{3})+2.0}{\sqrt{1+1+4} \sqrt{3+3}}=0 \\
& \Rightarrow \quad \theta=\frac{\pi}{2}
\end{aligned}
$$

5. Show that the lines with direction cosines $\left(\frac{12}{13}, \frac{-3}{13}, \frac{-4}{13}\right)$ and $\left(\frac{4}{13}, \frac{12}{13}, \frac{3}{13}\right)$ are perpendicular to each other.

Sol: Direction cosines of the lines are $\left(\frac{12}{13}, \frac{-3}{13}, \frac{-4}{13}\right)$ and $\left(\frac{4}{13}, \frac{12}{13}, \frac{3}{13}\right)$
Now $l_{1} l_{2}+m_{1} m_{2}+n_{1} n_{2}=\frac{12}{13} \cdot \frac{4}{13}-\frac{3}{13} \cdot \frac{12}{13}-\frac{4}{13} \cdot \frac{3}{13} \quad=\frac{48-36-12}{169}=0$
$\therefore l_{1} l_{2}+m_{1} m_{2}+n_{1} n_{2}=0 \Rightarrow$ the two lines are perpendicular .
6. $O$ is the origin, $\mathbf{P}(2,3,4)$ and $Q(1, k, 1)$ are points such that $\overline{O P} \perp \overline{O Q}$ find $K$

Sol: $\mathrm{O}(0,0,0), \mathrm{P}(2,3,4)$ and $\mathrm{Q}(1, \mathrm{k}, 1)$
d.rs of OP are 2, 3, 4
d.rs of $O Q$ are $1, k, 1$

OP and OQ are perpendicular $\quad \Rightarrow a_{1} a_{2}+b_{1} b_{2}+c_{1} c_{2}=0$
$\Rightarrow \quad 2+3 k+4=0 \Rightarrow 3 k=-6 \quad \Rightarrow \quad k=-2$
II.

1 If the direction ratios of a line are $(\mathbf{3}, 4,0)$ find its direction cosines are also the angles made the co-ordinate axes.

Sol: Direction ratios of the line are $(3,4,0)$
Dividing with $\sqrt{9+16+0}=5$
D.cs of the line are $\left(\frac{3}{5}, \frac{4}{5}, 0\right)$

If $\alpha, \beta, \gamma$ are the angles made by the line with the co-ordinate axes, then
$\cos \alpha=\frac{3}{5} \cos \beta=\frac{4}{5} \cos \gamma=0$
$\alpha=\cos ^{-1}\left(\frac{3}{5}\right), \beta=\cos ^{-1}\left(\frac{4}{5}\right), \gamma=\frac{\pi}{2}$
Angles made with co-ordinate axes are $\cos ^{-1}\left(\frac{3}{5}\right), \cos ^{-1}\left(\frac{4}{5}\right), \frac{\pi}{2}$
2. Show that the line through the points $(1,-1,2)(3,4,-2)$ is perpendicular to the line through the points $(0,3,2)$ and $(3,5,6)$

## Sol:

Given points $A(1,-1,2) B(3,4,-2) C(0,3,2)$ and $D(3,5,6)$
d.rs of AB are (3-1, 4+1,-2-2) i.e., 2, 5, -4 and d.rs of CD are $(3-0,5-3,6-2)$ i.e., $3,2,4$
now $a_{1} a_{2}+b_{1} b_{2}+c_{1} c_{2}=2.3+5.2-4.4=6+10-16=0$
Therefore, AB and CD are perpendicular
3. Find the angle between $\overline{D C}$ and $\overline{A B}$ where
$A=(3,4,5), B=(4,6,3) C=(-1,2,4)$ are $D(1,0,5)$
Sol: $A(3,4,5), B(4,6,3), C(-1,2,4), D(1,0,5)$ are the given points
d.rs of AB are $(4,-3,6-4,3-5)$ i.e., $(1,2,-2)$ and
d.rs of $C D$ are $(1+1,0-2,5-4)$ i.e., $(2,-2,1)$
let $\theta$ be the angle between the lines, then $\cos \theta=\frac{\left|a_{1} a_{2}+b_{1} b_{2}+c_{1} c_{2}\right|}{\sqrt{a_{1}^{2}+b_{1}^{2}+c_{1}^{2}} \sqrt{a_{2}^{2}+b_{2}^{2}+c_{2}^{2}}}$
$=\frac{\mid 1 \cdot 2+2(-2)+(-2) \cdot 1}{\sqrt{1+4+4} \sqrt{4+4}}=\frac{4}{9} \Rightarrow \theta=\cos ^{-1}\left(\frac{4}{9}\right)$
4. Find the direction cosines of a line which is perpendicular to the lines, whose direction ratios are $(1,-1,2)$ and $(2,1,-1)$

Sol: Let $l, m, n$ be the d.rs of the required line. This line is perpendicular to the lines with d.rs $(1,-1,2)$ and $(2,1,-1)$
$\therefore l-m+2 n=0 \quad$ and $\quad 2 l+m-n=0$
Solving these two equations

| 1 |  | m |  | n |
| :---: | :---: | :---: | :---: | :---: |
| -1 | 2 | 1 | -1 |  |
| 1 | -1 | 2 | 1 |  |
| $\frac{l}{1-2}=\frac{m}{4+1}=\frac{n}{1+2} \Rightarrow$ |  | $\frac{l}{-1}=\frac{m}{5}=\frac{n}{3}$ |  |  |

d.rs of the line are $-1,5,3$

Dividing with $\sqrt{1+25+9}=\sqrt{35}$
d.cs of the required line are $-\frac{1}{\sqrt{35}}, \frac{5}{\sqrt{35}}, \frac{3}{\sqrt{35}}$
5. Show that the points $(2,3,-4),(1,-2,3)$ and $(3,8,-11)$ are collinear
6. Show that the points $(4,7,8),(2,3,4),(-1,-2,1),(1,2,5)$ are vertices of a parallelogram

Sol: Given points are $A(4,7,8), B(2,3,4), C(-1,-2,1)$ and $D(1,2,5)$ Now

$$
\begin{aligned}
& A B=\sqrt{(4-2)^{2}+(7-3)^{2}+(8-4)^{2}}=\sqrt{4+16+16}=\sqrt{36}=6 \\
& B C=\sqrt{(2+1)^{2}+(3+2)^{2}+(4-1)^{2}}=\sqrt{9+25+9}=\sqrt{43}
\end{aligned}
$$

$$
C D=\sqrt{(-1,-1)^{2}+(-2-2)^{2}+(1-5)^{2}} \quad=\sqrt{4+16+16}=6
$$

and

$$
D A=\sqrt{(1-4)^{2}+(2-7)^{2}+(5-8)^{2}}=\sqrt{9+25+9}=\sqrt{43}
$$

$\therefore A B=C D$ and $B C=D A$
$\therefore A, B, C, D$ are the vertices of parallelogram
III

1. Show that the lines whose direction cosines are given by $l+m+n=0$ $2 m+3 n l-5 l m=0$ are perpendicular to each other

Sol: Given equations are $l+m+n=0-----(1)$

$$
\begin{equation*}
2 m n+3 n l-5 l m=0 \tag{2}
\end{equation*}
$$

From (1), $l=-(m+n)$ Substituting in (2)
$\Rightarrow 2 m n-3 n(m+n)+5 m(m+n)=0$
$\Rightarrow 2 m n-3 m n-3 n^{2}+5 m^{2}+5 m n=0$
$\Rightarrow 5 m^{2}+4 m n-3 n^{2}=0$
$\Rightarrow 5\left(\frac{m}{n}\right)^{2}+4 \frac{m}{n}-3=0$
$\Rightarrow \frac{m_{1} m_{2}}{n_{1} n_{2}}=\frac{-3}{5} \Rightarrow \frac{m_{1} m_{2}}{-3}=\frac{n_{1} n_{2}}{5}-\cdots---$
From (1), $\mathrm{n}=-(l+m)$
Substituting in (2), $-2 m(l+m)-3 l(l+m)-5 l m=0$
$\Rightarrow-2 l m-2 m^{2}-3 l^{2}-3 l m-5 l m=0$
$\Rightarrow 3 l^{2}+10 l m+2 m^{2}=0$

$$
\begin{align*}
& \Rightarrow 3\left(\frac{l}{m}\right)^{2}+10 \frac{l}{m}+2=0 \\
& \Rightarrow \quad \frac{l_{1} l_{2}}{m_{1} m_{2}}=\frac{2}{3} \Rightarrow \frac{l_{1} l_{2}}{2}=\frac{m_{1} m_{2}}{3} . \tag{4}
\end{align*}
$$

Form (3) and (4)

$$
\begin{aligned}
& \frac{l_{1} l_{2}}{2}=\frac{m_{1} m_{2}}{3}=\frac{n_{1} n_{2}}{-5}=k(\text { say }) \Rightarrow l_{1} l_{2}=2 k, m_{1} m_{2}=3 k, n_{1} n_{2}=-5 k \\
& \therefore l_{1} l_{2}+m_{1} m_{2}+n_{1} n_{2}=2 k+3 k-5 k=0
\end{aligned}
$$

The two lines are perpendicular
2.Find the angle between the lines whose direction cosines satisfy the equation $l+m+n=0, l^{2}+m^{2}-n^{2}=0$

Sol: Given equations are

$$
\begin{align*}
& l+m+n=0  \tag{1}\\
& l^{2}+m^{2}-n^{2}=0 \tag{2}
\end{align*}
$$

From (1), $l=-(m+n)$
Substituting in (2)

$$
\begin{aligned}
& (m+n)^{2}+m^{2}-n^{2}=0 \\
& \Rightarrow m^{2}+n^{2}+2 m n+m^{2}-n^{2}=0 \\
& \Rightarrow 2 m^{2}+2 m n=0 \\
& \Rightarrow 2 m(m+n)=0 \\
& \Rightarrow m=0 \text { and } m+n=0
\end{aligned}
$$

Case (i) $\mathrm{m}=0$, Substituting in (1) $l+n=0$

$$
l=-n \Rightarrow \frac{l}{1}=\frac{n}{-1}
$$

D.rs of the first line are $(1,0,-1)$

Case (ii) : $m+n=0 \Rightarrow m=-n \Rightarrow \frac{m}{1}=\frac{n}{-1}$
Substituting in (1) $l=0$
D.rs of the second line are $(0,1,-1)$
let $\theta$ be the angle between the two lines, then

$$
\begin{aligned}
& \cos \theta=\frac{a_{1} a_{2}+b_{1} b_{2}+c_{1} c_{2}}{\sqrt{a_{1}^{2}+b_{1}^{2}+c_{1}^{2}} \sqrt{a_{2}^{2}+b_{2}^{2}+c_{2}^{2}}} \\
& =\frac{|0+0+1|}{\sqrt{2} \cdot \sqrt{2}}=\frac{1}{2} \quad \therefore \theta=\frac{\pi}{3}
\end{aligned}
$$

3. If a ray makes angle $\alpha, \beta, \gamma$ and $\delta$ with the four diagonals of a cube find $\cos ^{2} \alpha+\cos ^{2} \beta+\cos ^{2} \gamma+\cos ^{2} \delta$


## Sol: Z

Let OABC;PQRS be the cube.
Let a be the side of the cube. Let one of the vertices of the cube be the origin $O$ and the co-ordinate axes be along the three edges $\overline{O A}, \overline{O B}$ and $\overline{O C}$ passing through the origin.

The co-ordinate of the vertices of the cube with respect to the frame of reference OABC are as shown in figure are $\mathrm{A}(\mathrm{a}, \mathrm{o}, \mathrm{o}), \mathrm{B}(\mathrm{o}, \mathrm{a}, \mathrm{o}), \mathrm{C}(0, \mathrm{o}, \mathrm{a}) \mathrm{P}(\mathrm{a}, \mathrm{a}, \mathrm{a}) \mathrm{Q}(\mathrm{a}, \mathrm{a}, \mathrm{o})$
$R(o, a, a)$ and $S(a, o, a)$
The diagonals of the cube are $\overline{O P}, \overline{C Q}, \overline{A R}$ and $\overline{B S}$. and their d.rs are respectively $(a, a, a),(a, a,-a),(-a, a, a)$ and $(a,-a, a)$.

Let the direction cosines of the given ray be $(l, m, n)$.

Then $l^{2}+m^{2}+n^{2}=1$

If this ray is making the angles $\alpha, \beta, \gamma$ and $\delta$ with the four diagonals of the cube, then
$\cos \alpha=\frac{|a \times l+a \times m+a \times n|}{\sqrt{a^{2}+a^{2}+a^{2}} .1}=\frac{|l+m+n|}{\sqrt{3}}$
Similarly, $\cos \beta=\frac{|l+m-n|}{\sqrt{3}}$
$\cos \gamma=\frac{|-l+m+n|}{\sqrt{3}}$ and $\cos \delta=\frac{|-l+m+n|}{\sqrt{3}}$
$\cos ^{2} \alpha+\cos ^{2} \beta+\cos ^{2} \gamma+\cos ^{2} \delta=$
$\frac{1}{3}\left\{|l+m+n|^{2}+|l+m-n|^{2}+|-l+m+n|^{2}+|l-m+n|^{2}\right\}$
$\frac{1}{3}\left[(l+m+n)^{2}+(l+m-n)^{2}+(-l+m+n)^{2}+(l-m+n)^{2}\right]$
$\frac{1}{3}\left[4\left(l^{2}+m^{2}+n^{2}\right)\right]=\frac{4}{3} \quad\left(\right.$ since $\left.l^{2}+m^{2}+n^{2}=1\right)$
4. If $\left(l_{1}, m_{1}, n_{1}\right),\left(l_{2}, m_{2}, n_{2}\right)$ and d.cs of two intersecting lines show that d.c.s of two lines, bisecting the angles between them are proportional to $l_{1} \pm l_{2}, m_{1} \pm m_{2} n_{1} \pm n_{2}$

## Sol:



Let OA and OB be the given lines whose d.cs are given by $\left(l_{1}, m_{1}, n_{1}\right),\left(l_{2}, m_{2}, n_{2}\right)$.
Let $\mathrm{OP}=\mathrm{OQ}=1$ unit. Also take a point $\mathrm{P}^{1}$ on AO produced such that $\mathrm{OP}^{1}=\mathrm{OP}=1$.

Join $P Q$ and $P^{1} Q$.
LET $M, M^{1}$ be the mid points of $P Q$ and $P^{1} Q$.
Then $\mathrm{OM} \& \mathrm{OM}^{1}$ are the required bisectors.
Now point $\mathrm{P}=\left(l_{1}, m_{1}, n_{1}\right) \quad \& \mathrm{Q}=\left(l_{2}, m_{2}, n_{2}\right)$
and $P^{1}=\left(-l_{1},-m_{1},-n_{1}\right)$.
And mid points
$\mathrm{M}=\left(\frac{l_{1}+l_{2}}{2}, \frac{m_{1}+m_{2}}{2}, \frac{n_{1}+n l 2}{2}\right)$,
$M^{1}=\left(\frac{l_{1}-l_{2}}{2}, \frac{m_{1}-m_{2}}{2}, \frac{n_{1}-n_{2}}{2}\right)$
Hence the d.cs of the bisector OM are proportional to $\left(\frac{l_{1}+l_{2}}{2}-0, \frac{m_{1}+m_{2}}{2}-0, \frac{n_{1}+n_{2}}{2}-0\right)$ Or $l_{1}+l_{2}, m_{1}+m_{2}, n_{1}+n_{2}$

Similarly, d.cs of the bisector $\mathrm{OM}^{1}$ are proportional to

$$
\left(\frac{l_{1}-l_{2}}{2}-0, \frac{m_{1}-m_{2}}{2}-0, \frac{n_{1}-n_{2}}{2}-0\right) \text { or } l_{1}-l_{2}, m_{1}-m_{2}, n_{1}-n_{2} .
$$

Hence d.cs of bisectors are proportional to $l_{1} \pm l_{2}, m_{1} \pm m_{2}, n_{1} \pm n_{2}$.
5. $A(-1,2,-3), B(5,0,-6), C(0,4,-1)$ are three points. Show that the direction cosines of the bisector of $\angle B A C$ are proportional to $(25,8,5)$ and (-11, 20, 23)

Sol: Given points are $\mathrm{A}(-1,2,-3), \mathrm{B}(5,0,-6)$ and $\mathrm{C}(0,4,-1)$
D.rs of AB are $(5+1,0-2,-6+3)$

$$
\text { i.e.,( } 6,-2,-3)=(a, b, c)
$$

Now $\sqrt{a^{2}+b^{2}+c^{2}}=\sqrt{36+4+9}=7 \quad \therefore$ D.rs of AB are $\frac{6}{7}, \frac{-2}{7}, \frac{-3}{7}$
D.rs of AC are $(0+1,4-2,-1+3)$ i.e., $1,2,2$

$$
\sqrt{a^{2}+b^{2}+c^{2}}=\sqrt{1+4+4}=3 \Rightarrow \text { D.rs of AC are } \frac{1}{3}, \frac{2}{3}, \frac{2}{3}
$$

$\therefore$ D.rs of one of the bisector are proportional to $l_{1}+l_{2}, m_{1}+m_{2}, n_{1}+n_{2}$

$$
=\left(\frac{6}{7}+\frac{1}{3}, \frac{-2}{7}+\frac{2}{3}, \frac{-3}{7}+\frac{2}{3}\right)=\left(\frac{18+7}{21}, \frac{-6+14}{21}, \frac{-9+14}{21}\right)=\left(\frac{25}{21}, \frac{8}{21}, \frac{5}{21}\right)
$$

D.rs of one of the bisector are $(25,8,5)$
D.rs of the other bisectors are proportional to

$$
\begin{aligned}
& l_{1}-l_{2}, m_{1}-m_{2}, n_{1}-n_{2}=\left(\frac{6}{7}-\frac{1}{3}, \frac{-2}{7}-\frac{2}{3}, \frac{-3}{7}-\frac{2}{3}\right)=\left(\frac{18-7}{21}, \frac{-6-14}{21}, \frac{-23}{21}\right) \\
& =\left(\frac{11}{21}, \frac{-20}{21}, \frac{-23}{21}\right)
\end{aligned}
$$

D.rs of the second bisector are $(-11,20,23)$
6. If $(6,10,10),(1,0,-5),(6,-10,0)$ are vertices of a triangle, find the direction ratios of its sides. Determine whether it is right angle or isosceles

## Sol:

Given vertices are $\quad A(6,10,10), B(1,0,-5), C(6,-10,0)$
D.rs of AB are 5, 10, 15 i.e., 1, 2, 3
D.rs of BC are $-5,10,-5$ i.e., $1,-2,1$
D.rs of AC are $0,20,10$, i.e., $0,2,1$
$\cos \underline{A B C}=\frac{[1.1+2(-2)+3.1]}{\sqrt{1+4+9} \sqrt{1+4+1}}=0 \Rightarrow \underline{B}=\frac{\pi}{2}$
Therefore, the triangle is a rt. triangle.
7. The vertices of a triangle are $\mathbf{A}(1,4,2), B(-2,1,2) C(2,3,-4)$. Find $\lfloor$ A, $|\underline{B}| C$,

Sol:


Vertices of the triangle are $\quad A(1,4,2), B(-2,1,2), C(2,3,-4)$
D.rs of AB are 3, 3,0 i.e., 1, 1, 0
D.rs of BC are $-4,-2,5$ i.e., $2,1,-3$
D.rs of AC are $-1,1,6$

$$
\cos \left\lfloor A B C=\frac{|1.2+1.0+0(-3)|}{\sqrt{1+1} \sqrt{4+1+9}}=\frac{3}{\sqrt{28}}=\frac{3}{2 \sqrt{7}} \quad \therefore \underline{B}=\cos ^{-1}\left(\frac{3}{2 \sqrt{7}}\right)\right.
$$

$$
\begin{aligned}
& \cos \left\lfloor B C A=\frac{1(-1)+1.1+(-3) 6}{\sqrt{4+1+9} \sqrt{1+1+36}} \quad=\frac{19}{\sqrt{19} \sqrt{28}}=\sqrt{\frac{19}{28}} \quad \therefore\left\lfloor C=\cos ^{-1}\left(\sqrt{\frac{19}{28}}\right)\right.\right. \\
& \cos \left\lfloor C A B=\frac{|-1.1+1.1+6.0|}{\sqrt{1+1+36} \sqrt{1+1+0}}=0 \quad \Rightarrow\lfloor A=\pi / 2\right.
\end{aligned}
$$

## 8. Find the angle between the lines whose direction cosines are given by

 the equation $3 l+m+5 n=0$ and $6 m n-2 n l+5 l=0$Sol: Given $3 l+m+5 n=0$

$$
6 m n-2 n l+5 l m=0
$$

From (1), $m=-(3 l+5 n)$

Substituting in (2)

$$
\begin{aligned}
& \Rightarrow-6 n(3 l+5 n)-2 n l-5 l(3 l+5 n)=0 \\
& \Rightarrow-18 \ln -30 n^{2}-2 n l-15 l^{2}-25 \ln =0 \\
& \Rightarrow-15 l^{2}-45 \ln -30 n^{2}=0 \\
& \Rightarrow l^{2}+3 \ln +2 n^{2}=0 \\
& \Rightarrow(l+2 n)(l+n)=0 \\
& \Rightarrow l+2 n=0 \text { or } l+n=0
\end{aligned}
$$

Case (i) :

$$
l_{1}+n_{1}=0 \Rightarrow n_{1}=-l_{1} ; \Rightarrow n_{1}=-l_{1} ; \Rightarrow \frac{l_{1}}{1}=\frac{n_{1}}{-1}
$$

But $m_{1}=-\left(3 l_{1}+5 n_{1}\right)=-\left(-3 n_{1}+5 n_{1}\right)=-2 n_{1}$
$\therefore \frac{m_{1}}{+2}=\frac{n_{1}}{-1}$
$\therefore \frac{l_{1}}{1}=\frac{m_{1}}{2}=\frac{n_{1}}{-1}$
D.rs of the first line $l_{1}$ are $(1,2,-1)$

Case (ii) : $l_{2}+2 n_{2}=0$
$\Rightarrow l_{2}=-2 n_{2} \Rightarrow \frac{l_{2}}{-2}=\frac{n_{2}}{1}$
$\Rightarrow m_{2}=-\left(3 l_{2}+5 n_{2}\right)=-\left(-6 n_{2}+5 n_{2}\right)=n_{2}$
$\frac{m_{2}}{1}=\frac{n_{2}}{1}$
$\therefore \frac{l_{2}}{-2}=\frac{m_{2}}{1}=\frac{n_{2}}{1}$
D.rs of the second line $l_{2}$ are $(-2,1,1)$

Suppose ' $\theta$ ' is the angle between the lines $l_{1}$ and $l_{2}$

$$
\begin{aligned}
& \cos \theta=\frac{a_{1} a_{2}+b_{1} b_{2}+c_{1} c_{2}}{\sqrt{a_{1}^{2}+b_{1}^{2}+c_{1}^{2}} \sqrt{a_{2}^{2}+b_{2}^{2}+c_{2}^{2}}} \\
& \frac{|1(-2)+2 \cdot 1+(-1) \cdot 1|}{\sqrt{1+4+1} \sqrt{4+1+1}}=\frac{1}{6} \Rightarrow \theta=\cos ^{-1}(1 / 6)
\end{aligned}
$$

9. If variable line in two adjacent position has direction cosines ( $l, m, n$ ) and $(l+\delta l, m+\delta m, n+\delta n)$, show that the small angle $\delta \theta$ between two position is given by $(\delta \theta)^{2}=(\delta l)^{2}=(\delta m)^{2}+(\delta n)^{2}$

Sol: The d.cs of the line in the two positions are $(l, m, n)$ and $(l+\delta l, m+\delta n, n+n)$.
Therefore, $l^{2}+m^{2}+n^{2}=1$

$$
\begin{equation*}
\operatorname{and}\left(l+\delta l^{2}\right)+(m+\delta m)^{2}+(n+\delta n)^{2}=1 \tag{1}
\end{equation*}
$$

(2) $-(1) \Rightarrow(l+\delta l)^{2}+(m+\delta m)^{2}+(n+\delta n)^{2}-\left(l^{2}+m^{2}+n^{2}\right)=0$
$2(l . \delta l+m . \delta m+n \delta n)=-\left((\delta l)^{2}+(\delta m)^{2}+(\delta n)^{2}\right)$

And $\cos \delta \theta=l(l+\delta l)+m(m+\delta m)+n(n+\delta n)$

$$
\begin{gathered}
=\left(l^{2}+m^{2}+n^{2}\right)+(l . \delta l+m \cdot \delta m+n \cdot \delta n) \\
\cos \delta \theta=1-\frac{1}{2}\left[(\delta l)^{2}+(\delta m)^{2}+(\delta n)^{2}\right] \\
\begin{array}{c}
(\delta l)^{2}+(\delta m)^{2}+(\delta n)^{2}=2(1-\cos \delta \theta) \\
=2.2 \sin ^{2} \frac{\delta \theta}{2}
\end{array}
\end{gathered}
$$

$\delta \theta$ being small, $\sin \frac{\delta \theta}{2}=\frac{\delta \theta}{2}$
$\therefore 4 \sin ^{2} \theta=4\left(\frac{\delta \theta}{2}\right)^{2}=(\delta \theta)^{2}$
$\therefore(\delta \theta)^{2}=(\delta l)^{2}+(\delta m)^{2}+(\delta n)^{2}$

## PROBLEMS FOR PRACTICE

1. If $\mathrm{P}(2,3,-6) \mathrm{Q}(3,-4,5)$ are two points, find the d.c's of $\overrightarrow{O P}, \overrightarrow{Q O}$ and $\overrightarrow{P Q}$ where is the origin

2. Find the d.c's of line that makes equal angles with the axes.?
3. Find the angle between two diagonals of a cube.?
4. Show that the points $\mathrm{A}(1,2,3), \mathrm{B}(4,0,4), \mathrm{C}(-2,4,2)$ are collinear?

## 6. $A(1,8,4), B(0,-11,4), C(2,-3,1)$ are three points and $D$ is the foot of the perpendicular from $A$ to $B C$. Find the coordinates of $D$.

## Solution: -

suppose D divides in the ratio $\mathrm{m}: \mathrm{n}$
Then $D=\frac{\mathfrak{๕}}{m+n} \frac{2 m}{m+n}, \frac{-3 m-11 n}{m+n}, \frac{m+\frac{\ddot{\partial}}{\dot{\grave{\emptyset}}}}{m+}$

Direction ratios of $\overline{B C}:(2,8,-3)$

$$
\begin{aligned}
& 2 \mathrm{~m}-2 \mathrm{n}-88 \mathrm{~m}-152 \mathrm{n}+9 \mathrm{~m}=0 \\
& m=-2 n
\end{aligned}
$$

substituting in (1), $\mathrm{D}=(4,5,-2)$
7. Lines $\overrightarrow{O A}, \overleftrightarrow{O B}$ are drawn from $O$ with direction cosines proportional to (1,-2,-1); $(3,-2,3)$. Find the direction cosines of thenormal to the plane AOB.
Sol : -
Let ( $\mathrm{a}, \mathrm{b}, \mathrm{c}$ ) be the direction ratios of a normal to the plane AOB. since $\overleftrightarrow{O A}, \overleftrightarrow{O B}$ lie on the plane, they are perpendicular to the normal to the plane. Using the condition of perpendicularity
$\mathrm{a} .1+\mathrm{b}(-2)+\mathrm{c}(-1)=0$
$\mathrm{a} .3+\mathrm{b}(-2)+\mathrm{c}(3)=0$.
Solving (1) and (2) $\frac{a}{-8}=\frac{b}{-6}=\frac{c}{4} \operatorname{or} \frac{a}{4}=\frac{b}{3}=\frac{c}{-2}$
The d.c's of the normal are
8. Show that the line whose d.c's are proportional to $(2,1,1),(4, \sqrt{3}-1,-\sqrt{3}-1)$ are inclined to one another at angle .
9. Find the d.r's and d.c's of the line joining the points $(4,-7,3),(6,-5,2)$
10. For what value of $x$ the line joining $A(4,1,2) B(5, x, 0)$ is perpendicular to the line joining $C(1,2,3)$ and $D(3,5,7)$.
11. Find the direction cosines of two lines which are connected by the relations $l+m+n=0$ and $\mathrm{mn}-2 \mathrm{nl}-2 \mathrm{~lm}=0$

