CHAPTER 6
DIRECTION COSINES AND DIRECTION RATIOS

TOPICS:

1.DEFINITION OF D.CS., RELATION BETWEEN D.CS. OF A LINE, CO-ORDINATES OF A POINT WHEN D.CS. ARE GIVEN AND DIRECTION COSINES OF A LINE JOINING TWO POINTS.

2. ANGLE BETWEEN TWO LINES WHEN D.CS ARE GIVEN, FINDING THE ANGLE BETWEEN TWO LINES WHEN THEIR D.CS ARE CONNECTED BY EQUATIONS.

3.DEFINITION OF DIRECTION RATIOS, D.RS. OF A LINE JOINING TWO POINTS

4. RELATION BETWEEN D.CS AND D.RS

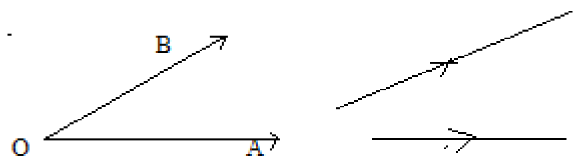
5. CONDITIONS FOR PARALLEL AND PERPENDICULA LINES WHEN D.CS/D.RS ARE GIVEN.

6. ANGLE BETWEEN TWO LINES WHEN D.RS ARE GIVEN, FINDING THE ANGLE BETWEEN TWO LINES WHEN THEIR D.RS ARE CONNECTED BY EQUATIONS.

DIRECTION COSINES & RATIOS (7 MARKS)

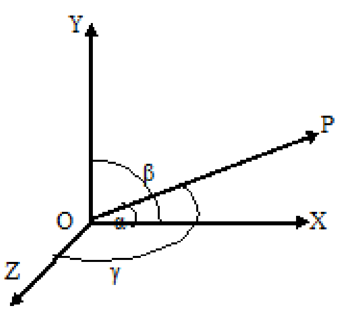
ANGLE BETWEEN TWO LINES:

The angle between two skew lines is the angle between two lines drawn parallel to them through any point in space.



DIRECTION COSINES

If α, β, γ are the angles made by a directed line segment with the positive directions of the coordinate axes respectively, then cos a, cos b, cos g are called the direction cosines of the given line and they are denoted by l, m, n respectively Thus $l = \cos\alpha$, $m = \cos\beta$, $n = \cos\gamma$



The direction cosines of $\overrightarrow{op}$ are

$l = \cos\alpha$, $m = \cos\beta$, $n = \cos\gamma$.

If l, m, n are the d.c's of a line L is one direction then the d.c's of the same line in the opposite direction are –l, –m, –n.

Note : The angles α, β, γ are known as the direction angles and satisfy the condition $0 \leq \alpha, \beta, \gamma \leq \pi$.

Note : The sum of the angles α, β, γ is not equal to 2p because they do not lie in the same plane.

Note: Direction cosines of coordinate axes.

The direction cosines of the x-axis are $\cos 0, \cos\frac{\pi}{2}, \cos\frac{\pi}{2}$ i.e., 1, 0, 0

Similarly the direction cosines of the y-axis are (0,1,0) and z-axis are (0,0,1)

THEOREM

If P(x,y,z) is any point in space such that OP =r and if l, m, n are direction cosines of $\overrightarrow{OP}$ then
x =lr,y =mr, z = nr.

Note: If P(x,y,z) is any point in space such that OP =r then the direction cosines of $\overrightarrow{OP}$ are $\frac{x}{r}, \frac{y}{r}, \frac{z}{r}$

Note: If P is any point in space such that OP =r and direction cosines of $\overrightarrow{OP}$ are l,m,n then the point P =(lr,mr,nr)

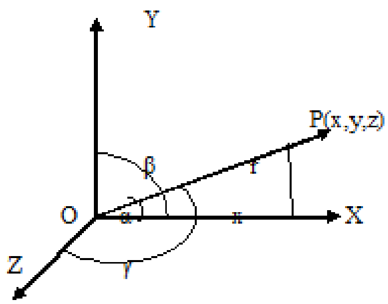
Note: If P(x,y,z) is any point in space then the direction cosines of $\overrightarrow{OP}$ are

$$\frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{y}{\sqrt{x^2+y^2+z^2}}, \frac{z}{\sqrt{x^2+y^2+z^2}}$$

THEOREM

If l, m, n are the direction cosines of a line L then $l^2 + m^2 + n^2 = 1$.

Proof :



From the figure

$$l = \cos\alpha = \frac{x}{r}, m = \cos\beta = \frac{y}{r}, n = \cos\gamma = \frac{z}{r} \quad \Rightarrow \cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{x^2}{r^2} + \frac{y^2}{r^2} + \frac{z^2}{r^2}$$

$$= \frac{x^2+y^2+z^2}{r^2} = \frac{r^2}{r^2} = 1. \qquad \therefore l^2 + m^2 + n^2 = 1$$

THEOREM

The direction cosines of the line joining the points $P(x_1, y_1, z_1), Q(x_2, y_2, z_2)$ are $\left(\frac{x_2 - x_1}{r}, \frac{y_2 - y_1}{r}, \frac{z_2 - z_1}{r}\right)$ where $r = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

EXERCISE -6A

1. A line makes angle $90^0, 60^0 \, and \, 30^0$ **with positive directions of x, y, z –axes respectively. Find the direction cosines**.

Sol: Suppose l, m, n are the direction cosines of the line, then

$$l = \cos\alpha = \cos 90^0 = 0$$

$$m = \cos\beta = \cos 60^0 = \frac{1}{2} \qquad \text{And} \quad n = \cos\gamma = \cos 30^0 = \frac{\sqrt{3}}{2}$$

Direction cosines of the line are $\left(0, \frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

2. **If a line makes angles** α, β, γ **with the positive direction of X, Y, Z axes, what is the value of** $\sin^2\alpha + \sin^2\beta + \sin^2\gamma$

Solution: We know that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$$1 - \sin^2\alpha + 1 - \sin^2\beta + 1 - \sin^2\gamma = 1$$

$$\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 3 - 1 = 2$$

3. If $P\left(\sqrt{3}, 1, 2\sqrt{3}\right)$ **is a point in space, find the direction cosines of** $\overrightarrow{OP}$

Solution: Direction ratios of P are $\left(\sqrt{3}, 1, 2\sqrt{3}\right) = (a, b, c)$

$$\Rightarrow a^2 + b^2 + c^2 = 3 + 1 + 12 = 16 \qquad \Rightarrow \sqrt{a^2 + b^2 + c^2} = 4$$

Direction cosines of $\overrightarrow{OP}$ are

$$\left(\frac{a}{\sqrt{a^2 + b^2 + c^2}}, \frac{b}{\sqrt{a^2 + b^2 + c^2}}, \frac{c}{\sqrt{a^2 + b^2 + c^2}}\right) = \left(\frac{\sqrt{3}}{4}, \frac{1}{4}, \frac{2\sqrt{3}}{4}\right) = \left(\frac{\sqrt{3}}{4}, \frac{1}{4}, \frac{\sqrt{3}}{2}\right)$$

4. Find the direction cosines of the line joining the points $(-4, 1, 7)$ *and* $(2, -3, 2)$

Solution: $A(-4, 1, 2)$ *and* $B(2, -3, 2)$ are given points

d.rs of PQ are $(x_2 - x_1, y_2 - y_1, z_2 - z_1)$

i.e. $(2+4, 1+3, 2-7)$ i.e, $(6, 4, -5) = (a, b, c)$

$\Rightarrow \sqrt{a^2 + b^2 + c^2} = \sqrt{36+16+25} = \sqrt{77}$

Direction cosines of $\overrightarrow{AB}$ are

$$\left(\frac{a}{\sqrt{a^2+b^2+c^2}}, \frac{b}{\sqrt{a^2+b^2+c^2}}, \frac{c}{\sqrt{a^2+b^2+c^2}}\right) = \left(\frac{6}{\sqrt{77}}, \frac{4}{\sqrt{77}}, \frac{-5}{\sqrt{77}}\right)$$

II.

1. Find the direction cosines of the sides of the triangle whose vertices are $(3, 5, -4), (-1, 1, 2)$ **and** $(-5, -5, -2)$

Sol: $A(3, 5, -4), B(-1, 1, 2)$ *and* $C(-5, -5, -2)$ are the vertices of ΔABC

d.rs of AB are $(-1-3, 1-5, 2+4) = (-4, -4, 6)$

Dividing with $\sqrt{16+16+36} = \sqrt{68} = 2\sqrt{17}$

d.cs of AB are $\frac{-4}{2\sqrt{17}}, \frac{-4}{2\sqrt{17}}, \frac{6}{2\sqrt{17}}$ i.e., $\frac{-2}{\sqrt{17}}, \frac{-2}{\sqrt{17}}, \frac{3}{\sqrt{17}}$

D.rs of BC are $(-5+1, -5-1, -2-2)$ i.e., $(-4, -6, -4)$

Dividing with $\sqrt{16+16+36} = \sqrt{68} = 2\sqrt{17}$ d.cs of BC are $\frac{-4}{2\sqrt{17}}, \frac{-6}{2\sqrt{17}}, \frac{-4}{2\sqrt{17}}$

i.e., $\frac{-2}{\sqrt{17}}, \frac{-3}{\sqrt{17}}, \frac{-4}{2\sqrt{17}}$

d.rs of CA are $(3+5, 5+5, -4+2) = (8, 10, -2)$

Dividing with $\sqrt{64+100+4}=\sqrt{168}=2\sqrt{42}$

Then d.cs of CA are $\frac{4}{\sqrt{42}}, \frac{5}{\sqrt{42}}, \frac{-1}{\sqrt{42}}$

2. **Show that the lines $\overleftrightarrow{PQ}$ and $\overleftrightarrow{RS}$ are parallel where P, Q, R, S are two points (2, 3, 4), (-1, -2, -1) and (1, 2, 5) respectively**

Sol : $P(2, 3, 4), Q(4, 7, 8)$, R (-1, -2, 1) and S(1, 2, 5) are the given points.

d.rs of PQ are (4 -2, 7-3, 8-4) i.e.,(2, 4, 4)

d.rs of RS are (1 + 1, 2 + 2, 5 -1) i.e., (2, 4, 4)

$\therefore$ d.rs of PQ and RS are proportional. Hence, PQ and RS are parallel

III.

1. **Find the direction cosines of two lines which are connected by the relation** $l-5m+3n=0$ *and* $7l^2+5m^2-3n^2=0$

Sol. Given $l-5m+3n=0$

$\Rightarrow l=5m-3n-----(1)$

and $7l^2+5m^2-3n^2=0---(2)$

Substituting the value of l in (2)

$7(5m-3n)^2+5m^2-3n^2=0$

$\Rightarrow 7\left(25^2+9n^2-30mn\right)+5m^2-3n^2=0$

$\Rightarrow 175m^2+63n^2-210mn+5m^2-3n^2=0$

$\Rightarrow 180m^2-210mn+60n^2=0$

$\Rightarrow 6m^2-7mn+2n^2=0$

$\Rightarrow (3m-2n)(2m-n)=0$

Case (i): $3m_1 = 2n_1 \Rightarrow \frac{m_1}{2} = \frac{n_1}{3}$

Then $m_1 = \frac{2}{3}n_1$

From $(1)\, l_1 = 5m_1 - 3n_1 = \frac{10}{3}n_1 - 3n_1$

$= \frac{10n_1 - 9n_1}{3} = \frac{n_1}{3}$

$\therefore \frac{l_1}{1} = \frac{m_1}{2} = \frac{n_1}{3}$

d.rs of the first line are (1, 2, 3)

Dividing with $\sqrt{1+4+9} = \sqrt{14}$

d.cs of the first line are $\left(\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right)$

Case (ii) $2m_2 = n_2$

From (1) $l_2 - 5m_2 + 3n_2 = 0$

$\Rightarrow l_2 - 5m_2 + 6m_2 = 0$

$\Rightarrow -l_2 = m_2$

$\therefore \frac{l_2}{-1} = \frac{m_2}{1} = \frac{n_2}{2}$

d.rs of the second line are -1, 1, 2

Dividing with $\sqrt{1+1+4} = \sqrt{6}$

d.cs of the second line are $\left(\frac{-1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right)$

DIRECTION RATIOS

A set of three numbers a,b,c which are proportional to the direction cosines l,m,n respectively are called DIRECTION RATIOS (d.r's) of a line.

Note : If (a, b, c) are the direction ratios of a line then for any non-zero real number λ , $(\lambda a, \lambda b, \lambda c)$ are also the direction ratios of the same line.

Direction cosines of a line in terms of its direction ratios

If (a, b, c) are direction ratios of a line then the direction cosines of the line are

$$\pm\left(\frac{a}{\sqrt{a^2+b^2+c^2}}, \frac{b}{\sqrt{a^2+b^2+c^2}}, \frac{c}{\sqrt{a^2+b^2+c^2}}\right)$$

THEOREM

The direction ratios of the line joining the points are $(x_2-x_1, y_2-y_1, z_2-z_1)$

ANGLE BETWEEN TWO LINES

If (l_1, m_1, n_1) and (l_2, m_2, n_2) are the direction cosines of two lines θ and is the acute angle between them, then $\cos\theta = |l_1l_2+m_1m_2+n_1n_2|$

Note.

If θ is the angle between two lines having d.c's (l_1, m_1, n_1) and (l_2, m_2, n_2) then

$$\sin\theta = \sqrt{\sum(l_1m_2-l_2m_1)^2}$$

and $\tan\theta = \dfrac{\sqrt{\sum(l_1m_2-l_2m_1)^2}}{|l_1l_2+m_1m_2+n_1n_2|}$ **when** $\theta \neq \dfrac{\pi}{2}$

Note 1 : The condition for the lines to be perpendicular is $l_1l_2+m_1m_2+n_1n_2=0$

Note 2 : The condition for the lines to be parallel is $\dfrac{l_1}{l_2}=\dfrac{m_1}{m_2}=\dfrac{n_1}{n_2}$

THEOREM

If (a_1, b_1, c_1) and (a_2, b_2, c_2) are direction ratios of two lines and θ is the angle between them then $\cos\theta = \dfrac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\,\sqrt{a_2^2+b_2^2+c_2^2}}$

Note 1 : If the two lines are perpendicular then $a_1a_2+b_1b_2+c_1c_2=0$

Note 2 : If the two lines are parallel then $\dfrac{a_1}{a_2}=\dfrac{b_1}{b_2}=\dfrac{c_1}{c_2}$

Note 3 : If one of the angle between the two lines is θ then other angle is $180^0-\theta$

EXERCISE – 6(B)

I

1. **Find the direction ratios of the line joining the points (3, 4, 0) are (4, 4, 4)**

Sol. $A(3,4,0)$ and $B(4,4,4)$ are the given points

d.rs of AB are $(x_2 - x_1, y_2 - y_1, z_2 - z_1) = (4-3, 4-4, 4-0)$ *i.e.,* $(1,0,4)$

2. **The direction ratios of a line are $(-6,2,3)$. Find the direction cosines.**

Sol: D.rs of the line are -6, 2, 3

Dividing with $\sqrt{36+4+9} = 7$

Direction cosines of the line are $-\frac{6}{7}, \frac{2}{7}, \frac{3}{7}$

3. **Find the cosine of the angle between the lines, whose direction cosines are**

$$\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) and \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$$

Sol: D.cs of the given lines are $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) and \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$.

Let θ be the angle between the lines. Then

$$\cos\theta = l_1l_2 + m_1m_2 + n_1n_2 \quad = \frac{1}{\sqrt{3}}.\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}}.\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}}.0 = \frac{1}{\sqrt{6}} = \frac{2}{\sqrt{6}} = \sqrt{\frac{4}{6}} \quad = \sqrt{\frac{2}{3}}$$

4. **Find the angle between the lines whose direction ratios are** $(1,1,2)\left(\sqrt{3}, -\sqrt{3}, 0\right)$

Sol: D.rs of the given lines are $(1,1,2)$ and $\left(\sqrt{3}, -\sqrt{3}, 0\right)$

Let θ be the angle between the lines. Then