## LENGTH OF THE PERPENDICULAR FROM A POINT TO A STRAIGHT LINE AND DISTANCE BETWEEN TWO PAPALLEL LINES

## THEOREM

The perpendicular distance from a point $P\left(x_{1}, y_{1}\right)$ to the line $a x+b y+c=0$ is $\frac{\left|a x_{1}+b y_{1}+c\right|}{\sqrt{a^{2}+b^{2}}}$
Proof:

Let the axes be translated to the point $\mathrm{P}\left(\mathrm{x}_{1}, \mathrm{y}_{1}\right)$.


Let $(X, Y)$ be the new coordinates of $(x, y)$. Then $x=X+x_{1}, y=Y+y_{1}$ The transformed equation of the given line is

$$
a\left(X+x_{1}\right)+b\left(Y+y_{1}\right)+c=0
$$

$\Rightarrow \mathrm{aX}+\mathrm{bY}+\left(\mathrm{ax}_{1}+\mathrm{by}_{1}+\mathrm{c}\right)=0$
The perpendicular distance from the new origin P to the line is $\frac{\left|a x_{1}+b y_{1}+c\right|}{\sqrt{a^{2}+b^{2}}}$.
(from normal form) The perpendicular distance from a point
$\mathrm{P}\left(\mathrm{x}_{1}, \mathrm{y}_{1}\right)$ to the line $\mathrm{ax}+\mathrm{by}+\mathrm{c}=0$ is $\frac{\left|a x_{1}+b y_{1}+c\right|}{\sqrt{a^{2}+b^{2}}}$.

## DISTANCE BETWEEN PARALLEL LINES THEOREM

The distance between the two parallel lines $a x+b y+c_{1}=0$ and $a x+b y+c_{2}=0$ is $\frac{\left|c_{1}-c_{2}\right|}{\sqrt{a^{2}+b^{2}}}$.
Proof:
Given lines are $a x+b y+c_{1}=0$

$$
\begin{equation*}
a x+b y+c_{2}=0 \tag{1}
\end{equation*}
$$

Let $\mathrm{P}\left(\mathrm{x}_{1}, \mathrm{y}_{1}\right)$ be a point on the line (2).

Then

$$
\begin{aligned}
& a x_{1}+b y_{1}+c_{2}=0 \\
& a x_{1}+b y_{1}=-c_{2}
\end{aligned}
$$



Distance between the parallel lines $=$ Perpendicular distance from P to line (1)

$$
=\frac{\left|a x_{1}+b y_{1}+c_{1}\right|}{\sqrt{a^{2}+b^{2}}}=\frac{\left|c_{1}-c_{2}\right|}{\sqrt{a^{2}+b^{2}}}
$$

## FOOT OF THE PERPENDICULAR

## THEOREM

If $(h, k)$ is the foot of the perpendicular from $\left(x_{1}, y_{1}\right)$ to the line $a x+b y+c=0$
$(\mathbf{a 0}, \mathbf{b 0})$ then $\frac{h-x_{1}}{a}=\frac{k-y_{1}}{b}=\frac{-\left(a x_{1}+b y_{1}+c\right)}{a^{2}+b^{2}}$.

## Proof :



Let $A=\left(x_{1}, y_{1}\right) P=(h, k)$
$P$ lies on $a x+b y+c=0$
$a h+b k+c=0$
$\mathrm{ah}+\mathrm{bk}=-\mathrm{c}$
Slope of $\overline{A P}$ is $\frac{k-y_{1}}{h-x_{1}}$
Slope of given line is $-\frac{a}{b}$
$\overline{A P}$ is perpendicular to the given line
$\Rightarrow\left(\frac{k-y_{1}}{h-x_{1}}\right)\left(-\frac{a}{b}\right)=-1$
$\Rightarrow \frac{k-y_{1}}{b}=\frac{h-x_{1}}{a}$
By the law of multipliers in ratio and proportion
$\frac{h-x_{1}}{a}=\frac{k-y_{1}}{b}=\frac{a\left(h-x_{1}\right)+b\left(k-y_{1}\right)}{a^{2}+b^{2}}$
$=\frac{a h+b k-a x_{1}-b y_{1}}{a^{2}+b^{2}}=\frac{-a x_{1}-b y_{1}-c}{a^{2}+b^{2}}$
$=\frac{-\left(a x_{1}+b y_{1}+c\right)}{a^{2}+b^{2}}$
Hence $\frac{h-x_{1}}{a}=\frac{k-y_{1}}{b}=\frac{-\left(a x_{1}+b y_{1}+c\right)}{a^{2}+b^{2}}$

## IMAGE OF A POINT

## THEOREM

If $(h, k)$ is the image of $\left(\mathbf{x}_{1}, y_{1}\right)$ w.r.t the line $a x+b y+c=0(a \neq 0, b \neq 0)$, then $\frac{h-x_{1}}{a}=\frac{k-y_{1}}{b}=\frac{-2\left(x_{1}+b y_{1}+c\right)}{a^{2}+b^{2}}$.

Proof:


Let $\mathrm{A}\left(\mathrm{x}_{1}, \mathrm{y}_{1}\right), \mathrm{B}(\mathrm{h}, \mathrm{k})$
Mid point of is $\mathrm{P}=\left(\frac{x_{1}+h}{2}, \frac{y_{1}+k}{2}\right)$
Since B is the image of A,therefore mid point P lies on $\mathrm{ax}+\mathrm{by}+\mathrm{c}=0$.
$\Rightarrow a\left(\frac{x_{1}+h}{2}\right)+b\left(\frac{y_{1}+k}{2}\right)+\mathrm{c}=0$
$\Rightarrow \mathrm{ax}_{1}+\mathrm{by}_{1}+\mathrm{ah}+\mathrm{bk}+2 \mathrm{c}=0$
$\Rightarrow \mathrm{ah}+\mathrm{bk}=-\mathrm{ax}_{1}+\mathrm{by}_{1}-2 \mathrm{c}$.
Slope of $\overline{A B}$ is $\frac{k-y_{1}}{h-x_{1}}$
And Slope of given line is $-\frac{a}{b}$
$\overline{A B}$ is perpendicular to the given line
$\Rightarrow\left(\frac{k-y_{1}}{h-x_{1}}\right)\left(-\frac{a}{b}\right)=-1$
$\Rightarrow \frac{k-y_{1}}{b}=\frac{h-x_{1}}{a}$
By the law of multipliers in ratio and proportion

$$
\begin{aligned}
& \frac{h-x_{1}}{a}=\frac{k-y_{1}}{b}=\frac{a\left(h-x_{1}\right)+b\left(k-y_{1}\right)}{a^{2}+b^{2}} \\
& =\frac{a h+b k-a x_{1}-b y_{1}}{a^{2}+b^{2}} \\
& =\frac{-a x_{1}-b y_{1}-2 c-a x_{1}-b y_{1}}{a^{2}+b^{2}} \\
& =\frac{-2\left(a x_{1}+b y_{1}+c\right)}{a^{2}+b^{2}}
\end{aligned}
$$

Hence $\frac{h-x_{1}}{a}=\frac{k-y_{1}}{b}=\frac{-2\left(x_{1}+b y_{1}+c\right)}{a^{2}+b^{2}}$
Note 1: The image of $\left(\mathrm{x}_{1}, \mathrm{y}_{1}\right)$ w.r.t the line $\mathrm{x}=\mathrm{y}$ is $\left(\mathrm{y}_{1}, \mathrm{x}_{1}\right)$
Note 2: The image of $\left(x_{1}, y_{1}\right)$ w.r.t the line $x+y=0$ is $\left(-y_{1},-x_{1}\right)$

## THEOREM

If the four straight lines $a x+b y+p=0, a x+b y+q=0, c x+d y+r=0$ and $\mathbf{c x}+\mathrm{dy}+\mathrm{s}=\mathbf{0}$ form a parallelogram. Then the area of the parallelogram so formed is

$$
\left|\frac{(p-q)(r-s)}{b c-a d}\right|
$$

## Proof:

Let $\mathrm{L}_{1}=\mathrm{ax}+\mathrm{by}+\mathrm{p}=0$

$$
\begin{aligned}
& L_{2}=a x+b y+q=0 \\
& L_{3}=c x+d y+r=0 \\
& L_{4}=c x+d y+s=0
\end{aligned}
$$

Clearly
$\mathrm{L}_{1} \| \mathrm{L}_{2}$ and $\mathrm{L}_{3} \| \mathrm{L}_{4}$. So $\mathrm{L}_{1}$ and $\mathrm{L}_{3}$ are nonparallel. Let be the angle between $\mathrm{L}_{1}$ and $\mathrm{L}_{3}$.
Let $\mathrm{d}_{1}=$ distance between $\mathrm{L}_{1}$ and $\mathrm{L}_{2}=\frac{|p-q|}{\sqrt{a^{2}+b^{2}}}$
Let $\mathrm{d}_{2}=$ distance between $\mathrm{L}_{3}$ and $\mathrm{L}_{4}=\frac{|r-s|}{\sqrt{c^{2}+d^{2}}}$
Now $\cos \theta=\frac{|a c+b d|}{\sqrt{\left(a^{2}+b^{2}\right)\left(c^{2}+d^{2}\right)}}$ and $\sin \theta=\sqrt{\frac{\left(a^{2}+b^{2}\right)\left(c^{2}+d^{2}\right)-(a c+b d)^{2}}{\left(a^{2}+b^{2}\right)\left(c^{2}+d^{2}\right)}}$
$=\frac{|b c-a d|}{\sqrt{\left(a^{2}+b^{2}\right)\left(c^{2}+d^{2}\right)}}$
Now area of the parallelogram is $\frac{d_{1} d_{2}}{\sin \theta}=\left|\frac{(p-q)(r-s)}{b c-a d}\right|$

## ANGLE BETWEEN TWO LINES THEOREM

If $\theta$ is an angle between the lines $\mathbf{a}_{1} x+b_{1} y+c_{1}=0, \mathbf{a}_{2} x+b_{2} y+c_{2}=0$ then $\cos \theta= \pm \frac{a_{1} a_{2}+b_{1} b_{2}}{\sqrt{a_{1}^{2}+b_{2}^{3}} \sqrt{a_{2}^{2}+b_{2}^{2}}}$

Proof: The lines passing through the origin and parallel to the given lines are


Let $\theta_{1}, \theta_{2}$ be the inclinations of (1) and (2) respectively $\left(\theta_{1}>\theta_{2}\right)$
Now $\theta$ is an angle between (1) and (2)

$$
\theta=\theta_{1}-\theta_{2}
$$

$P\left(-b_{1}, a_{1}\right)$ satisfies eq(1), the point lies on (1)
Similarly, $\mathrm{Q}\left(-\mathrm{b}_{2}, \mathrm{a}_{2}\right)$ lies on (2)
Let $\mathrm{L}, \mathrm{M}$ be the projection of $\mathrm{P}, \mathrm{Q}$ respectively on the x - axis.

$$
\begin{aligned}
& \therefore \cos \theta_{1}=\frac{O L}{O P}=\frac{-b_{1}}{\sqrt{a_{1}^{2}+b_{1}^{2}}}, \sin \theta_{1}=\frac{P L}{O P}=\frac{a_{1}}{\sqrt{a_{1}^{2}+b_{1}^{2}}} \\
& \therefore \cos \theta_{2}=\frac{O M}{O Q}=\frac{-b_{2}}{\sqrt{a_{2}^{2}+b_{2}^{2}}}, \sin \theta_{2}=\frac{O M}{O Q}=\frac{a_{2}}{\sqrt{a_{2}^{2}+b_{2}^{2}}} \\
& \theta=\theta_{1}-\theta_{2} \\
& \cos =\cos \left(\theta_{1}-\theta_{2}\right) \\
& =\cos \theta_{1} \cos \theta_{2}+\sin \theta_{1} \sin \theta_{2} \\
& =\frac{\left(-b_{1}\right)}{\sqrt{a_{1}^{2}+b_{1}^{2}}} \frac{\left(-b_{2}\right)}{\sqrt{a_{1}^{2}+b_{1}^{2}}}+\frac{a_{1}}{\sqrt{a_{1}^{2}+b_{1}^{2}}} \frac{a_{2}}{\sqrt{a_{1}^{2}+b_{1}^{2}}} \\
& =\frac{a_{1} a_{1}+b_{1} b_{2}}{\sqrt{a_{1}^{2}+b_{1}^{2}} \sqrt{a_{2}^{2}+b_{2}^{2}}}
\end{aligned}
$$

Note 1: If is the acute angle between the lines then

$$
\cos \theta=\frac{\left|a_{1} a_{1}+b_{1} b_{2}\right|}{\sqrt{a_{1}^{2}+b_{1}^{2}} \sqrt{a_{2}^{2}+b_{2}^{2}}}
$$

Note 2: If is an angle between two lines, then is another angle between the lines.

Note 3: If is an angle between two lines are not a right angle then the angle between the lines means the acute angle between the lines.

Note 4: If $\theta$ is an angle between the lines $a_{1} x+b_{1} y+c_{1}=0, a_{2} x+b_{2} y+c_{2}=0$ then $\tan \theta=\frac{a_{1} b_{2}-a_{2} b_{1}}{a_{1} a_{2}+b_{1} b_{2}}$

Note 5: If is the acute angle between the lines

$$
\begin{aligned}
& \mathrm{a}_{1} \mathrm{x}+\mathrm{b}_{1} \mathrm{y}+\mathrm{c}_{1}=0, \mathrm{a}_{2} \mathrm{x}+\mathrm{b}_{2} \mathrm{y}+\mathrm{c}_{2}=0 \text { then } \\
& \begin{aligned}
\tan \theta & =\left|\frac{a_{1} b_{2}-a_{2} b_{1}}{a_{1} a_{2}+b_{1} b_{2}}\right|=\left|\frac{a_{1} / b_{1}-a_{2} / b_{2}}{\left(a_{1} a_{2}\right) /\left(b_{1} b_{2}\right)+1}\right| \\
& =\left|\frac{\left(-a_{1} / b_{1}\right)-\left(-a_{2} / b_{2}\right)}{1+\left(-a_{1} / b_{1}\right)\left(-a_{2} / b_{2}\right)}\right| \text { where } \mathrm{m}_{1}, \mathrm{~m}_{2} \text { are the slopes of the lines. }
\end{aligned}
\end{aligned}
$$

## THEOREM

The equation of the line parallel to $a x+b y+c=0$ and passing through $\left(x_{1}, y_{1}\right)$ is $a\left(x-x_{1}\right)+b\left(y-y_{1}\right)=0$.
Proof:
Slope of the given line is $-\mathrm{a} / \mathrm{b}$.
$\Rightarrow$ Slope of the required line is $-\mathrm{a} / \mathrm{b}$.(lines are parallel)
Equation of the required line is

$$
\begin{aligned}
\mathrm{y}-\mathrm{y}_{1} & =-\frac{a}{b}\left(\mathrm{x}-\mathrm{x}_{1}\right) \\
\mathrm{b}\left(\mathrm{y}-\mathrm{y}_{1}\right) & =-\mathrm{a}\left(\mathrm{x}-\mathrm{x}_{1}\right) \\
\mathrm{a}\left(\mathrm{x}-\mathrm{x}_{1}\right) & +\mathrm{b}\left(\mathrm{y}-\mathrm{y}_{1}\right)=0 .
\end{aligned}
$$

Note 1: The equation of a line parallel to $a x+b y+c=0$ may be taken $a s a x+b y+k=0$.

Note 2: The equation of a line parallel to $a x+b y+c=0$ and passing through the origin is $a x+b y=0$.

## THEOREM

The equation of the line perpendicular to $a x+b y+c=0$ and passing through $\left(x_{1}, y_{1}\right)$ is $b\left(x-x_{1}\right)-a\left(y-y_{1}\right)=0$.
Proof:
Slope of the given line is $-\mathrm{a} / \mathrm{b} . \Rightarrow$ Slope of the required line is $\mathrm{b} / \mathrm{a}$.
(since product of slopes $=-1$ )
Equation of the required line is $\mathrm{y}-\mathrm{y}_{1}=\frac{b}{a}\left(\mathrm{x}-\mathrm{x}_{1}\right)$
$a\left(y-y_{1}\right)=b\left(x-x_{1}\right)$
$b\left(x-x_{1}\right)-a\left(y-y_{1}\right)=0$.

Note 1: The equation of a line perpendicular to $a x+b y+c=0$ may be taken as

$$
b x-a y+k=0
$$

Note 2: The equation of a line perpendicular to $a x+b y+c=0$ and passing through the origin is $b x-a y=0$.

EXERCISE -3 (d)
I. Find the angle between the following straight lines.

1. $y=4-2 x, y=3 x+7$

Sol: given lines are

$$
y=4-2 x \Rightarrow 2 x+y-4=0 \text { and } 3 x-y+7=0
$$

Let $\theta$ be the angle between the lines, then

$$
\begin{aligned}
& \cos \theta=\frac{\left|\mathrm{a}_{1} \mathrm{a}_{2}+\mathrm{b}_{1} \mathrm{~b}_{2}\right|}{\sqrt{\mathrm{a}_{1}^{2}+\mathrm{b}_{1}^{2}} \sqrt{\mathrm{a}_{2}^{2}+\mathrm{b}_{2}^{2}}} \\
& =\frac{|2.3+1(-1)|}{\sqrt{4+1} \sqrt{9+1}}=\frac{5}{\sqrt{5} \sqrt{10}}=\frac{1}{\sqrt{2}} \\
& \therefore \theta=\frac{\pi}{4}
\end{aligned}
$$

2. $3 x+5 y=7,2 x-y+4=0$

Sol. ans : $\theta=\cos ^{-1}\left(\frac{1}{\sqrt{170}}\right)$
3. $\mathbf{y}=-\sqrt{3} \mathrm{x}+5, \mathrm{y}=\frac{1}{\sqrt{3}} \mathrm{x}-\frac{2}{\sqrt{3}}$

Sol. slope of $1^{\text {st }}$ line is $\mathrm{m}_{1}=-\sqrt{3}$
Slope of $2^{\text {nd }}$ line is $m_{2}=\frac{1}{\sqrt{3}}$.
$m_{1} m_{2}=(-\sqrt{3}) \frac{1}{\sqrt{3}}=-1$.
The lines are perpendicular, hence angle between the lines is $\theta=\frac{\pi}{2}$
4. $\quad \mathbf{a x}+\mathbf{b y}=\mathbf{a}+\mathbf{b}, \mathbf{a}(\mathbf{x}-\mathbf{y})+\mathbf{b}(\mathbf{x}+\mathbf{y})=\mathbf{2 b}$

Sol. given lines $a x+b y=a+b,(a+b) x+(-a+b) y=2 b$
let $\theta$ be the angle between the lines, then

$$
\begin{aligned}
& \cos \theta=\frac{|a(a+b)+b(-a+b)|}{\sqrt{a^{2}+b^{2}}+\sqrt{(a+b)^{2}+(-a+b)^{2}}} \\
& =\frac{\left|a^{2}+a b-a b+b^{2}\right|}{\sqrt{a^{2}+b^{2}} \sqrt{2\left(a^{2}+b^{2}\right)}}=\frac{a^{2}+b^{2}}{\sqrt{2}\left(a^{2}+b^{2}\right)}=\frac{1}{\sqrt{2}} \\
& \therefore \theta=\frac{\pi}{4}
\end{aligned}
$$

Find the length of the perpendicular drawn form the point given against the following straight lines.
5. $5 x-2 y+4=0,(-2,-3)$

Sol. Length of the perpendicular $=\frac{\left|\mathrm{ax}_{1}+\mathrm{by}_{1}+\mathrm{c}\right|}{\sqrt{\mathrm{a}^{2}+\mathrm{b}^{2}}}=\frac{|5(-2)-2(-3)+4|}{\sqrt{25+4}}=\frac{|-10+10|}{\sqrt{29}}=0$
6. $3 x-4 y+10=0,(3,4)$

Sol. Length of the perpendicular $=\frac{|3.3-4.4+10|}{\sqrt{9+16}}=\frac{3}{5}$
7. $x-3 y-4=0,(0,0)$

Sol. Ans; $\frac{4}{\sqrt{10}}$
Find the distance between the following parallel lines.
8. $3 x-4 y=12,3 x-4 y=7$

Sol. Given lines are $3 x-4 y=12,3 x-4 y=7$

Distance between parallel lines $=\frac{\left|\mathrm{c}_{1}-\mathrm{c}_{2}\right|}{\sqrt{\mathrm{a}^{2}+\mathrm{b}^{2}}}=\frac{|-12+7|}{\sqrt{9+16}}=\frac{5}{5}=1$
9. $5 \mathrm{x}-3 \mathrm{y}-4=0,10 \mathrm{x}-6 \mathrm{y}-9=0$

Sol. Equations of the lines can be taken as $10 x-6 y-8=0,10 x-6 y-9=0$
Distance between parallel lines $=\frac{|-8+9|}{\sqrt{100+36}}=\frac{1}{2 \sqrt{34}}$
10. Find the equation of the straight line parallel to the line $2 x+3 y+7=0$ and passing through the point $(5,4)$.
Sol. Given line is $2 x+3 y+7=0$
Equation of the parallel to $2 x+3 y+7=0$ is $2 x+3 y=k$.
This line is passing through $\mathrm{P}(5,4)$
$\Rightarrow 10+12=\mathrm{k} \Rightarrow \mathrm{k}=22$
Equation of the required line is $2 \mathrm{x}+3 \mathrm{y}-22=0$
11. Find the equation of the straight line perpendicular to the line $5 x-3 y+1=0$ and passing through the point $(4,-3)$.
Sol. Equation of the given line is $5 x-3 y+1=0$
Equation of the perpendicular to $5 x-3 y+1=0$ is $3 x+5 y+k=0$
This line is passing through $\mathrm{P}(4,-3)$
$\Rightarrow 12-15+\mathrm{k}=0 \Rightarrow \mathrm{k}=3$
Equation of the required line is $3 x+5 y+3=0$
12. Find the value of $k$, if the straight lines $6 x-10 y+3=0$ and $k x-5 y+8=0$ are parallel.
Sol. Given lines are $6 \mathrm{x}-10 \mathrm{y}+3=0$ and $\mathrm{kx}-5 \mathrm{y}+8=0$.
Since the lines are parallel $\Rightarrow \frac{a_{1}}{a_{2}}=\frac{b_{1}}{b_{2}} \Rightarrow-30=-10 \mathrm{k} \Rightarrow \mathrm{k}=3$
13. Fund the value of $P$, if the straight lines $3 x+7 y-1=0$ and $7 x-p y+3=0$ are mutually perpendicular.
Sol. Given lines are $3 x+7 y-1=0,7 x-p y+3=0$
Since the lines are perpendicular

$$
\Rightarrow \mathrm{a}_{1} \mathrm{a}_{2}+\mathrm{b}_{1} \mathrm{~b}_{2}=0 \Rightarrow 3.7+(-\mathrm{p})=0 \Rightarrow 7 \mathrm{p}=21 \Rightarrow \mathrm{p}=3
$$

14. Find the value of $k$, the straight lines $y-3 k x+4=0$ and $(2 k-1) x-(8 k-1) y=6$ are perpendicular.
Sol. Given lines are $-3 \mathrm{kx}+\mathrm{y}+4=0$

$$
(2 k-1) x-(8 k-1) y-6=0
$$

These lines are perpendicular $\Rightarrow a_{1} a_{2}+b_{1} b_{2}=0$
$\Rightarrow-3 \mathrm{k}(2 \mathrm{k}-1)-1(8 \mathrm{k}-1)=0 \Rightarrow-6 \mathrm{k}^{2}+3 \mathrm{k}-8 \mathrm{k}+1=0$
$6 \mathrm{k}^{2}+5 \mathrm{k}-1=0 \Rightarrow(\mathrm{k}+1)(6 \mathrm{k}-1)=0$
$\mathrm{k}=-1$ or $1 / 6$
15. $(-4,5)$ is a vertex of a square and one of its diagonal is $7 x-y+8=0$. Find the equation of the other diagonal.
$(-4,5)$


Sol. let ABCD be the square.
Let the equation of the diagonal AC be $7 x-y+8=0$
The point $(-4,5)$ is not satisfying the equation.
Let $\mathrm{D}=(-4,5)$
The other diagonal BD is perpendicular to AC .
Equation of $B D$ is can be taken as $x+7 y+k=0$
$\mathrm{D}(-4,5)$ is a point on this line $\Rightarrow-4+35+\mathrm{k}=0 \Rightarrow \mathrm{k}=4-35=-31$
Equation of BD is $x+7 y-31=0$

## II.

1. Find the equation of the straight lines passing through $(1,3)$ and
i)parallel to ii) perpendicular to the line passing through the points $(3,-5)$ and $(-6,1)$.

Sol. Given points A $(3,-5)$, B $(-6,1)$
Slope of $A B=\frac{-5-1}{3+6}=\frac{-6}{9}=\frac{-2}{3}$
i) slope of the line parallel to AB is $\frac{-2}{3}$ equation of the line parallel to AB and passing through $(1,3)$ is

$$
y-3=\frac{-2}{3}(x-1) \Rightarrow 3 y-9=-2 x+2 \Rightarrow 2 x+3 y-11=0
$$

ii) slope of the line perpendicular to AB is $3 / 2$.

Equation of the line passing through $(1,3)$ and having slope $3 / 2$ is $y-3=\frac{-2}{3}(x-1)$
$\Rightarrow 2 x+3 y-5=0$.
2. The line $\frac{x}{a}-\frac{y}{b}=1$ meets the $X$ - axis at $P$. Find the equation of the line perpendicular to this line at $P$.
Sol. given line is $\frac{x}{a}-\frac{y}{b}=1$ $\qquad$
On x - axis $\mathrm{y}=0 \Rightarrow \frac{\mathrm{x}}{\mathrm{a}}-\frac{0}{\mathrm{~b}}=1 \Rightarrow \mathrm{x}=\mathrm{a}$
Point $\mathrm{P}=(\mathrm{a}, 0)$
Equation of the line perpendicular to (1) is $\frac{x}{b}+\frac{y}{a}=k$
This line is passing through $P(a, 0) \quad \Rightarrow \frac{a}{b}+0=k \Rightarrow k=a / b$
Equation of the line is $\frac{x}{b}+\frac{y}{a}=\frac{a}{b}$.
3. Find the equation of the line perpendicular to the line $3 x+4 y+6=0$ and making intercept -4 on X - axis .
Sol. Given line is $3 x+4 y+6=0$.
Equation of the perpendicular to $3 x+4 y+6=0$ is $\quad 4 x-3 y=k$
$\Rightarrow \frac{4 \mathrm{x}}{\mathrm{k}}-\frac{3 \mathrm{y}}{\mathrm{k}}=1 \Rightarrow \frac{\mathrm{x}}{\left[\frac{\mathrm{k}}{4}\right]}+\frac{\mathrm{y}}{\left[\frac{-\mathrm{k}}{3}\right]}=1$

$$
x-\text { int ercept }=\frac{k}{4}=-4 \Rightarrow k=-16
$$

Equation of the required line is $4 x-3 y=-16 \Rightarrow 4 x-3 y+16=0$.
4. $\quad A(-1,1), B(5,3)$ are opposite vertices of a square in the $X Y$ plane. Find the equation of the other diagonal (not passing through $A, B$ ) of a square.
Sol. A $(-1,1)$, B $(5,3)$ are opposite vertices of the square.
Slope of $A B=\frac{1-3}{-1-5}=\frac{-2}{-6}=\frac{1}{3}$


The other diagonal is perpendicular to $A B$
Slope of CD $=-\frac{1}{m}=-3$
Let ' O ' is the point of intersection of the diagonals then $\mathrm{O}=\left(\frac{-1+5}{2}, \frac{1+3}{2}\right)=(2,2)$
Diagonal CD is passing through $O(2,2)$, Equation of $C D$ is $y-2=-3(x-2)$ $=-3 x+6 \Rightarrow 3 x+y-8=0$.
5. Find the foot of the perpendicular drawn from $(4,1)$ upon the straight line $3 x-4 y+12=0$.
Sol. Equation of the line is $3 x-4 y+12=0$
If $(h, k)$ is the foot of the perpendicular from $\left(x_{1}, y_{1}\right)$ on the line $a x+b y+c=0$, then
$\frac{\mathrm{h}-\mathrm{x}_{1}}{\mathrm{a}}=\frac{\mathrm{k}-\mathrm{y}_{1}}{\mathrm{~b}}=-\frac{\left(\mathrm{ax}_{1}+\mathrm{by}_{1}+\mathrm{c}\right)}{\mathrm{a}^{2}+\mathrm{b}^{2}}$
$\frac{\mathrm{h}-4}{3}=\frac{\mathrm{k}-1}{-4}=-\frac{(12-4+12)}{9+16}$
$\frac{\mathrm{h}-4}{3}=\frac{\mathrm{k}-1}{-4}=-\frac{20}{25}=-\frac{4}{5}$
$\mathrm{h}-4=-\frac{12}{5} \Rightarrow \mathrm{~h}=4-\frac{12}{5}=\frac{20-12}{5}=\frac{8}{5}$
$\mathrm{k}-1=\frac{16}{5} \Rightarrow \mathrm{k}=1+\frac{16}{5}=\frac{5+16}{5}=\frac{21}{5}$
$\therefore$ Foot of the perpendicular $=\left(\frac{8}{5}, \frac{21}{5}\right)$
6. Find the foot of the perpendicular drawn from $(3,0)$ upon the straight line $5 x+$ 12y-41 $=0$.
Sol. Ans: $\left(\frac{49}{13}, \frac{24}{13}\right)$
7. $x-2 y-5=0$ is the perpendicular bisector of the line segment joining the points $A, B$. If $A=(-1,-3)$, find the co-ordinates of $B$.

Sol. If PQ is the perpendicular bisector of AB , then B is the image of A in the line PQ .
Equation of the line is $x-3 y-5=0$
Given point $\mathrm{A}=(-1,-3)$. Here B is the image of A w.r.t $\mathrm{x}-3 \mathrm{y}-5=0$.
Let $B(h, k)$ be the image of $A(-1,-3)$, then
$\frac{\mathrm{h}-\mathrm{x}_{1}}{\mathrm{a}}=\frac{\mathrm{k}-\mathrm{y}_{1}}{\mathrm{~b}}=\frac{-2\left(\mathrm{ax}_{1}+\mathrm{by}_{1}+\mathrm{c}\right)}{\mathrm{a}^{2}+\mathrm{b}^{2}}$
$\frac{\mathrm{h}+1}{1}=\frac{\mathrm{k}+3}{-3}=\frac{-2(-1+9-5)}{1+9}$
$\frac{\mathrm{h}+1}{1}=\frac{\mathrm{y}_{2}+3}{-3}=-\frac{3}{5}$
$\frac{\mathrm{h}+1}{1}=\frac{-3}{5} \Rightarrow \mathrm{~h}+1=-\frac{3}{5}$
$\mathrm{h}=-1-\frac{3}{5}=-\frac{8}{5} \Rightarrow \frac{\mathrm{y}_{2}+3}{-3}=\frac{-3}{5}$
$\mathrm{k}+3=\frac{9}{5} \Rightarrow \mathrm{k}=\frac{9}{5}-3=-6 / 5$
Co-ordinates of B are $\left(\frac{-8}{5}, \frac{-6}{5}\right)$.
8. Find the image of the point straight line $3 x+4 y-1=0$.

Sol. Ans: $\left(-\frac{7}{5},-\frac{6}{5}\right)$
9. Show that the distance of the point $(6,-2)$ from the line $4 x+3 y=12$ is half of the distance of the point $(3,4)$ from the line $4 x-3 y=12$.
10. Find the locus of foot of the perpendicular from the origin to a variable straight line which always passes through a fixed point (a,b).

Sol.


Let $\mathrm{P}(\mathrm{x}, \mathrm{y})$ be the foot of the perpendicular from $\mathrm{O}(0,0)$ to the line.

Slope of OP $=\frac{y}{x}$
Line is passing through $A(a, b)$.
Slope of $A P=\frac{y-a}{x-b}$
Line AP is perpendicular to $\mathrm{OP}, \Rightarrow$ product of slopes $=-1$
$\Rightarrow \frac{y}{x} \cdot \frac{y-b}{x-a}=-1 \Rightarrow y^{2}-b y=-\left(x^{2}-a x\right)$
$\Rightarrow x^{2}+y^{2}-a x-b y=0$
III.

1. Show that the lines $x-7 y-22=0,3 x+4 y+9=0$ and $7 x+y-54=0$ form a right angled isosceles triangle.
Sol. Given lines $x-7 y-22=0$

$$
\begin{align*}
& 3 x+4 y+9=0  \tag{1}\\
& 7 x+y-54=0
\end{align*}
$$

(1)
(3)

A (2) B
Let ' A ' be the angle between (1),(2)
$\cos \mathrm{A}=\frac{\left|\mathrm{a}_{1} \mathrm{a}_{2}+\mathrm{b}_{1} \mathrm{~b}_{2}\right|}{\sqrt{\mathrm{a}_{1}{ }^{2}+\mathrm{b}_{1}{ }^{2}} \sqrt{\mathrm{a}_{2}{ }^{2}+\mathrm{b}_{2}{ }^{2}}}=\frac{|3-28|}{\sqrt{1+49} \sqrt{9+16}}=\frac{25}{5 \sqrt{2.5}}=\frac{25}{25 \sqrt{2}}=\frac{1}{\sqrt{2}}$
$\Rightarrow \mathrm{A}=45^{\circ}$
Let $B$ be the angle between (2), (3) then $\cos B=\frac{21+4}{\sqrt{9+16} \sqrt{49+1}}=\frac{25}{25 \sqrt{2}}=\frac{1}{\sqrt{2}}$
$\Rightarrow \mathrm{B}=45^{\circ}$
Let ' C ' be the angle between (3), (1)
$\cos C=\frac{7-7}{\sqrt{1+49} \sqrt{49+1}}=0 \Rightarrow C=90^{\circ}$
Since $\left\lfloor\underline{A}=\underline{B}=45^{\circ}\right.$ and $\mid C=90^{\circ}$
$\therefore$ Given lines form a right angled isosceles triangle.
2. Find the equation of the straight lines passing through the point (-3,2) and making an angle of $45^{\circ}$ with the straight line $\mathbf{3 x}-\mathbf{y}+\mathbf{4}=\mathbf{0}$.
Sol. Given point $\mathrm{P}(-3,2)$
Given line $3 x-y+4=0$

Slope $m_{1}=-\frac{a}{b}=3$


Let $m$ be the slope of the required line.
Then $\tan 45^{\circ}=\frac{m-3}{1+3 m}$
$\Rightarrow\left|\frac{\mathrm{m}-3}{1+3 \mathrm{~m}}\right|=1 \Rightarrow \frac{\mathrm{~m}-3}{1+3 \mathrm{~m}}=1$
$\Rightarrow \mathrm{m}-3=1+3 \mathrm{~m} \Rightarrow 2 \mathrm{~m}=-4$ or $\mathrm{m}=-2$
$\Rightarrow \frac{\mathrm{m}-3}{1+3 \mathrm{~m}}=-\Rightarrow \mathrm{m}-3=-1-3 \mathrm{~m}$
$\Rightarrow 4 \mathrm{~m}=2 \Rightarrow \mathrm{~m}=1 / 2$
case (1) $m=-2$ and point $(-3,2)$
Equation of the line is

$$
y-2=-2(x+3)=-2 x-6 \quad \Rightarrow 2 x+y+4=0
$$

case (2) $\mathrm{m}=\frac{1}{2}$, point ( $-3,2$ )
Equation of the line is

$$
y-2=\frac{1}{2}(x+3) \Rightarrow 2 y-4=x+3 \quad \Rightarrow x-2 y+7=0
$$

3. Find the angle of the triangle whose sides are $x+y-4=0$,

$$
2 x+y-6=0,5 x+3 y-15=0 .
$$

Sol. Ans: $\cos ^{-1}=\left(\frac{4}{\sqrt{17}}\right), \cos ^{-1}\left(\frac{13}{\sqrt{170}}\right), \cos ^{-1}\left(\frac{3}{\sqrt{10}}\right)$
4. Prove that the foot of the perpendiculars from the origin on the lines $x+y=4$, $x+5 y=26$ and $15 x-27 y=424$ are collinear.
5. Find the equation of the lines passing through the point of intersection of the lines $3 x+2 y+4=0,2 x+5 y=1$ and whose distance form $(2,-1)$ is 2 .

Sol. Equation of the lines passing through the point of intersection of the line

$$
\mathrm{L}_{1} \equiv 3 \mathrm{x}+2 \mathrm{y}+4=0, \mathrm{~L}_{2} \equiv 2 \mathrm{x}+5 \mathrm{y}-1=0 \text { is }
$$

$$
\mathrm{L}_{1}+\lambda \mathrm{L}_{1}=0
$$

$$
(3 x+2 y+4)+\lambda(2 x+5 y-1)=0
$$

$$
\Rightarrow(3+2 \lambda) x+(2+5 \lambda) y+(4-\lambda)=0----(1)
$$

Given distance from $(2,-1)$ to $(1)=2$

$$
\begin{aligned}
& \frac{\mid 3+2 \lambda) 2+(2+5 \lambda)(-1)+(4-\lambda) \mid}{\sqrt{(3+2 \lambda)^{2}+(2+5 \lambda)^{2}}}=2 \\
& \Rightarrow \frac{|-2 \lambda+8|}{\sqrt{(3+2 \lambda)^{2}+(2+5 \lambda)^{2}}}=2 \\
& \Rightarrow(-\lambda+4)^{2}=9+4 \lambda^{2}+12 \lambda+4+25 \lambda^{2}+20 \lambda \\
& \Rightarrow 28 \lambda^{2}+40 \lambda-3=0 \\
& \Rightarrow 28 \lambda^{2}-2 \lambda+42 \lambda-3=0 \\
& \Rightarrow(2 \lambda+3)(14 \lambda-1)=0 \Rightarrow \lambda=\frac{1}{14}, \lambda=-\frac{3}{2}
\end{aligned}
$$

From (1)
If $\lambda=\frac{1}{14}$, then equation of the line is $4 x+3 y+5=0$
If $\lambda=-\frac{3}{2}$, then equation of the line is
$\mathrm{y}-1=0$.
6. Each side of a square is of length 4 units. The center of the square is $(3,7)$ and one of its diagonals is parallel to $y=x$. Find the Co-ordinates of its vertices.
Sol. Let ABCD be the square. Side $\mathrm{AB}=4$
Point of intersection of the diagonals is the center $\mathrm{P}(3,7)$


From $P$ drawn $P M \perp A B$. Then $M$ is midpoint of $A B$
$\therefore \mathrm{AM}=\mathrm{MB}=\mathrm{PM}=2$
Since a diagonal is parallel to $\mathrm{y}=\mathrm{x}$, its sides are parallel to the co-ordinate axes.

$$
\mathrm{M}(3,5)
$$

$\Rightarrow \mathrm{A}(3-2,5), \mathrm{B}(3+2,5), \mathrm{C}(3+5,7+2), \mathrm{D}(3-2,7+2)$
$\Rightarrow \mathrm{A}(1,5), \mathrm{B}(5,5), \mathrm{C}(5,9), \mathrm{D}(1,9)$
7. If $\mathrm{ab}>0$, find the area of the rhombus enclosed by the four straight lines $a x \pm b y \pm c=0$.

Sol. let the Equation of AB be $\mathrm{ax}+\mathrm{by}+\mathrm{c}=0$
Equation of CD be $\mathrm{ax}+\mathrm{by}-\mathrm{c}=0$


Equation of BC be ax - by $+c=0$
Equation of AD be ax $-b y-c=0$
Solving (1) and (3), $B=\left(-\frac{c}{a}, 0\right)$
Solving (1) and (4), $\mathrm{A}=\left(0,-\frac{\mathrm{c}}{\mathrm{b}}\right)$
Solving (2) and (3), $\mathrm{C}=\left(0, \frac{\mathrm{c}}{\mathrm{b}}\right)$
Solving (2) and (4), D $=\left(\frac{\mathrm{c}}{\mathrm{a}}, 0\right)$
Area of rhombus $\mathrm{ABCD}=\frac{1}{2}\left|\sum \mathrm{x}_{1}\left(\mathrm{y}_{2}-\mathrm{y}_{4}\right)\right|$
$=\frac{1}{2\left|0(0-0)-\frac{c}{a}\left(\frac{c}{b}+\frac{c}{b}\right)+0(0-0)+\frac{c}{a}\left(\frac{-c}{b} \frac{c}{b}\right)\right|}$
$=\frac{1}{2} \cdot \frac{4 \mathrm{c}^{2}}{|\mathrm{ab}|}=\frac{2 \mathrm{c}^{2}}{|\mathrm{ab}|}$ sq. units
8. Find the area of the parallelogram whose sides are $3 x+4 y+5=0,3 x+4 y-2=0$, $2 x+3 y+1=0$ and $2 x+3 y-7=0$
Sol. Given sides are

$$
\begin{align*}
& 3 x+4 y+5=0  \tag{1}\\
& 3 x+4 y-2=0  \tag{2}\\
& 2 x+3 y+1=0  \tag{3}\\
& 2 x+3 y-7=0 \tag{4}
\end{align*}
$$

Area of parallelogram formed by (1), (2), (3), (4)
$=\left|\frac{\left(c_{1}-c_{2}\right)\left(d_{1}-d_{2}\right)}{a_{1} b_{2}-a_{2} b_{1}}\right|=\left|\frac{(5+2)(1+7)}{3(3)-2(4)}\right|$
$=\left|\frac{7 \times 8}{9-8}\right|=\frac{56}{1}=56$ sq. units

