

CHAPTER 4

PAIR OF STRAIGHT LINES

TOPICS:

1. Equation of a pair of lines passing through the origin
2. Angle between pair of lines
3. Bisectors of the angles between two lines.
4. pair of bisectors of angles between the pair of lines.
5. Equation of pair of lines passing through given point and parallel/perpendicular to the given pair of lines.
6. Condition for perpendicular and coincident lines
7. Area of the triangle formed by give pair of lines and a line.
8. pair of lines-second degree general equation
9. Conditions for parallel lines-distance between them.
10. Point of intersection of the pair of lines.
11. Homogenising a second degree equation w.r.t a 1st degree equation in x and y.

PAIR OF STRAIGHT LINES

Let $L_1=0, L_2=0$ be the equations of two straight lines. If $P(x_1, y_1)$ is a point on L_1 then it satisfies the equation $L_1=0$. Similarly, if $P(x_1, y_1)$ is a point on $L_2=0$ then it satisfies the equation.

If $P(x_1, y_1)$ lies on L_1 or L_2 , then $P(x_1, y_1)$ satisfies the equation $L_1 L_2 = 0$.

$\therefore L_1 L_2 = 0$ represents the pair of straight lines $L_1 = 0$ and $L_2 = 0$ and the joint equation of $L_1 = 0$ and $L_2 = 0$ is given by $L_1 \cdot L_2 = 0$.-----(1)

On expanding equation (1) we get an equation of the form $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ which is a second degree (non - homogeneous) equation in x and y .

Definition: If a, b, h are not all zero, then $ax^2 + 2hxy + by^2 = 0$ is the general form of a second degree homogeneous equation in x and y .

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THEOREM

If a, b, h are not all zero and $h^2 \geq ab$ then $ax^2 + 2hxy + by^2 = 0$ represents a pair of straight lines passing through the origin.

Proof:

Case (i) : Suppose $a = 0$.

Given equation $ax^2 + 2hxy + by^2 = 0$ reduces to $2hxy + by^2 = 0 \Rightarrow y(2hx + by) = 0$.

\therefore Given equation represents two straight lines

$y = 0$ -- (1) and $2hx + by = 0$ -- (2) which pass through the origin.

Case (ii): Suppose $a \neq 0$.

Given equation $ax^2 + 2hxy + by^2 = 0$

$$\Rightarrow a^2 x^2 + 2ahxy + aby^2 = 0$$

$$\Rightarrow (ax)^2 + 2(ax)(hy) + (hy)^2 - (h^2 - ab)y^2 = 0$$

$$\Rightarrow (ax + hy)^2 - (y\sqrt{h^2 - ab})^2 = 0$$

$$\left[ax + y\left(h + \sqrt{h^2 - ab}\right) \right] \left[ax + y\left(h - \sqrt{h^2 - ab}\right) \right] = 0$$

\therefore Given equation represents the two lines

$ax + hy + y\sqrt{h^2 - ab} = 0, ax + hy - y\sqrt{h^2 - ab} = 0$ which pass through the origin.

Note 1: If $h^2 > ab$, the two lines are distinct.

Note 2: If $h^2 = ab$, the two lines are coincident.

Note 3: If $h^2 < ab$, the two lines are not real but intersect at a real point (the origin).

Note 4: If the two lines represented by $ax^2 + 2hxy + by^2 = 0$ are taken as $l_1x + m_1y = 0$ and $l_2x + m_2y = 0$ then $ax^2 + 2hxy + by^2 \equiv (l_1x + m_1y)(l_2x + m_2y) \equiv \ell_1\ell_2x^2 + (\ell_1m_2 + \ell_2m_1)xy + m_1m_2y^2$

Equating the co – efficients of x^2 , xy and y^2 on both sides, we get $l_1l_2 = a$, $l_1m_2 + l_2m_1 = 2h$, $m_1m_2 = b$.

THEOREM

If $ax^2 + 2hxy + by^2 = 0$ represent a pair of straight lines, then the sum of slopes of lines is $\frac{-2h}{b}$ and product of the slopes is $\frac{a}{b}$.

Proof:

Let $ax^2 + 2hxy + by^2 = 0$ represent the lines $l_1x + m_1y = 0$ -- (1) and $l_2x + m_2y = 0$ -- (2). Then $l_1l_2 = a$, $l_1m_2 + l_2m_1 = 2h$, $m_1m_2 = b$.

Slopes of the lines (1) and (2) are $-\frac{l_1}{m_1}$ and $-\frac{l_2}{m_2}$.

$$\text{sum of the slopes} = \frac{-l_1}{m_1} + \frac{-l_2}{m_2} = -\frac{l_1m_2 + l_2m_1}{m_1m_2} = -\frac{2h}{b}$$

$$\text{Product of the slopes} = \left(\frac{-l_1}{m_1}\right)\left(-\frac{l_2}{m_2}\right) = \frac{l_1l_2}{m_1m_2} = \frac{a}{b}$$

ANGLE BETWEEN A PAIR OF LINES

THEOREM :

If θ is the angle between the lines represented by $ax^2 + 2hxy + by^2 = 0$, then $\cos \theta = \pm \frac{a+b}{\sqrt{(a-b)^2 + 4h^2}}$

Proof:

Let $ax^2 + 2hxy + by^2 = 0$ represent the lines $l_1x + m_1y = 0$ -- (1) and $l_2x + m_2y = 0$ -- (2). Then $l_1l_2 = a$, $l_1m_2 + l_2m_1 = 2h$, $m_1m_2 = b$.

Let θ be the angle between the lines (1) and (2). Then $\cos \theta = \pm \frac{l_1l_2 + m_1m_2}{\sqrt{(l_1^2 + m_1^2)(l_2^2 + m_2^2)}}$

$$\begin{aligned} &= \pm \frac{l_1l_2 + m_1m_2}{\sqrt{l_1^2l_2^2 + m_1^2m_2^2 + l_1^2m_2^2 + l_2^2m_1^2}} \\ &= \pm \frac{l_1l_2 + m_1m_2}{\sqrt{(l_1l_2 - m_1m_2)^2 + 2l_1l_2m_1m_2 + (l_1m_2 + l_2m_1)^2 - 2l_1m_2l_2m_1}} \\ &= \pm \frac{a+b}{\sqrt{(a-b)^2 + 4h^2}} \end{aligned}$$

Note 1: If θ is the acute angle between the lines $ax^2 + 2hxy + by^2 = 0$ then $\cos \theta = \frac{|a+b|}{\sqrt{(a-b)^2 + 4h^2}}$

Note 2: If θ is the acute angle between the lines $ax^2 + 2hxy + by^2 = 0$ then $\tan \theta = \pm \frac{2\sqrt{h^2 - ab}}{a+b}$ and

$$\sin \theta = \frac{2\sqrt{h^2 - ab}}{\sqrt{(a-b)^2 + 4h^2}}$$

CONDITIONS FOR PERPENDICULAR AND COINCIDENT LINES

1.If the lines $ax^2 + 2hxy + by^2 = 0$ are perpendicular to each other then $\theta = \pi/2$ and $\cos \theta = 0 \Rightarrow a+b=0$.

i.e., co-efficient of x^2 + coefficient of $y^2 = 0$.

2.If the two lines are parallel to each other then $\theta = 0$.

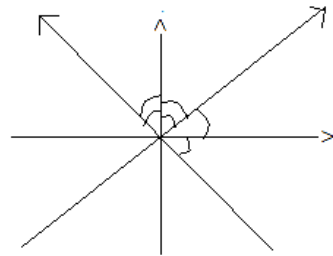
\Rightarrow The two lines are coincident $\Rightarrow h^2 = ab$.

BISECTORS OF ANGLES.

THEOREM

The equations of bisectors of angles between the lines $a_1x + b_1y + c_1 = 0, a_2x + b_2y + c_2 = 0$

$$\text{are } \frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \pm \frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}} .$$



PAIR OF BISECTORS OF ANGLES.

The equation to the pair bisectors of the angle between the pair of lines $ax^2 + 2hxy + by^2 = 0$ is

$$h(x^2 - y^2) = (a - b)xy \quad (\text{or}) \quad \frac{x^2 - y^2}{a - b} = \frac{xy}{h} .$$

Proof:

Let $ax^2 + 2hxy + by^2 = 0$ represent the lines $l_1x + m_1y = 0$ -- (1) and $l_2x + m_2y = 0$ -- (2).

Then $l_1l_2 = a, l_1m_2 + l_2m_1 = 2h, m_1m_2 = b$.

The equations of bisectors of angles between (1) and (2) are $\frac{l_1x + m_1y}{\sqrt{l_1^2 + m_1^2}} - \frac{l_2x + m_2y}{\sqrt{l_2^2 + m_2^2}} = 0$ AND

$$\frac{l_1x + m_1y}{\sqrt{l_1^2 + m_1^2}} + \frac{l_2x + m_2y}{\sqrt{l_2^2 + m_2^2}} = 0$$

The combined equation of the bisectors is

$$\begin{aligned}
& \left(\frac{l_1x + m_1y}{\sqrt{l_1^2 + m_1^2}} - \frac{l_2x + m_2y}{\sqrt{l_2^2 + m_2^2}} \right) \left(\frac{l_1x + m_1y}{\sqrt{l_1^2 + m_1^2}} + \frac{l_2x + m_2y}{\sqrt{l_2^2 + m_2^2}} \right) = 0 \\
& \Rightarrow \left(\frac{l_1x + m_1y}{\sqrt{l_1^2 + m_1^2}} \right)^2 - \left(\frac{l_2x + m_2y}{\sqrt{l_2^2 + m_2^2}} \right)^2 = 0 \\
& \Rightarrow (l_2^2 + m_2^2)(l_1x + m_1y)^2 - (l_1^2 + m_1^2)(l_2x + m_2y)^2 = 0 \\
& \Rightarrow x^2 [l_1^2(l_2^2 + m_2^2) - l_2^2(l_1^2 + m_1^2)] - y^2 [m_2^2(l_1^2 + m_1^2) - m_1^2(l_2^2 + m_2^2)] - 2xy [l_2m_2(l_1^2 + m_1^2) - l_1m_1(l_2^2 + m_2^2)] = 0 \\
& \Rightarrow x^2 (l_1^2l_2^2 + l_1^2m_2^2 - l_2^2l_1^2 - l_2^2m_1^2) - y^2 (l_1^2m_2^2 + m_1^2m_2^2 - m_1^2l_2^2 - m_1^2m_2^2) - 2xy (l_2m_2l_1^2 + l_2m_2m_1^2 - l_1m_1l_2^2 - l_1m_1m_2^2) = 0 \\
& \Rightarrow x^2 (l_1^2m_2^2 - l_2^2m_1^2) - y^2 (l_1^2m_2^2 - l_2^2m_1^2) = 2xy [l_1l_2(l_1m_2 - l_2m_1) - m_1m_2(l_1m_2 - l_2m_1)] \\
& \Rightarrow (x^2 - y^2)(l_1^2m_2^2 - l_2^2m_1^2) = 2xy(l_1l_2 - m_1m_2)(l_1m_2 - l_2m_1) \\
& \Rightarrow (x^2 - y^2)(l_1m_2 + l_2m_1) = 2xy(l_1l_2 - m_1m_2) \\
& \Rightarrow 2h(x^2 - y^2) = 2xy(a - b) \\
& \therefore h(x^2 - y^2) = (a - b)xy \quad \text{OR} \quad \frac{x^2 - y^2}{a - b} = \frac{xy}{h}
\end{aligned}$$

THEOREM

The equation to the pair of lines passing through (x_0, y_0) and parallel $ax^2 + 2hxy + by^2 = 0$ is $a(x - x_0)^2 + 2h(x - x_0)(y - y_0) + b(y - y_0)^2 = 0$

Proof :

Let $ax^2 + 2hxy + by^2 = 0$ represent the lines $l_1x + m_1y = 0$ -- (1) and $l_2x + m_2y = 0$ -- (2).

Then $l_1l_2 = a, l_1m_2 + l_2m_1 = 2h, m_1m_2 = b$.

The equation of line parallel to (1) and passing through (x_0, y_0) is

$$l_1(x - x_0) + m_1(y - y_0) = 0 \quad \text{-- (3)}$$

The equation of line parallel to (2) and passing through (x_0, y_0) is $l_2(x - x_0) + m_2(y - y_0) = 0$ -- (4)

The combined equation of (3), (4) is

$$[l_1(x - x_0) + m_1(y - y_0)][l_2(x - x_0) + m_2(y - y_0)] = 0$$

$$\Rightarrow l_1l_2(x - x_0)^2 + (l_1m_2 + l_2m_1)(x - x_0)(y - y_0) + m_1m_2(y - y_0)^2 = 0$$

$$\Rightarrow a(x - x_0)^2 + 2h(x - x_0)(y - y_0) + b(y - y_0)^2 = 0$$

