THEOREMS ON STANDARD LIMITS

THEOREM

If n is a rational number and a > 0 then $\underset{x \to a}{Lt} \dfrac{x^n - a^n}{x - a} = n.a^{n-1}$

Note 1: If n is a positive integer, then for any $a \in R$, $\underset{x \to a}{Lt} \dfrac{x^n - a^n}{x - a} = n.a^{n-1}$

Note 2: If n is a real number and a > 0 then $\underset{x \to a}{Lt} \dfrac{x^n - a^n}{x - a} = n.a^{n-1}$

Note 3: If m and n are any real numbers and a > 0, then $\underset{x \to a}{Lt} \dfrac{x^m - a^m}{x^n - a^n} = \dfrac{m}{n} a^{m-n}$.

THEOREM 2

If 0 < x < $\dfrac{\pi}{2}$ **then sin x < x < tan x.**

Corollary 1:

If $-\dfrac{\pi}{2} < x < 0$ then tan x < x < sin x

Corollary 2:

If $0 < | x | < \dfrac{\pi}{2}$ then $|\sin x| < |x| < |\tan x|$

STANDARD LIMITS

$\underset{x \to 0}{Lt} \dfrac{\sin x}{x} = 1$, $\underset{x \to 0}{Lt} \dfrac{\tan x}{x} = 1$, $\underset{x \to 0}{Lt} \dfrac{e^x - 1}{x} = 1$, $\underset{x \to 0}{Lt} \dfrac{a^x - 1}{x} = \log_e a$, $\underset{x \to 0}{Lt} (1 + x)^{\frac{1}{x}} = e$,

$\underset{x \to \infty}{Lt} \left(1 + \dfrac{1}{x}\right)^x = e$ **and** $\underset{x \to -\infty}{Lt} \left(1 + \dfrac{1}{x}\right)^x = e$

EXERCISE – 8 (c)

I. Compute the following limits

1. $\underset{x\to 1}{Lt}\left[\dfrac{2x+1}{3x^2-4x+5}\right]$

Sol : $\underset{x\to 1}{Lt}\dfrac{2x+1}{3x^2-4x+5}=\dfrac{2.1+1}{3.1^2-4.1+5}=\dfrac{2+1}{8-4}=\dfrac{3}{4}$

2. $\underset{x\to \frac{\pi}{2}}{Lt}\dfrac{\cos x}{\left[x-\frac{\pi}{2}\right]}$

Sol : Let $y=x-\dfrac{\pi}{2}$ so that as $x\to\dfrac{\pi}{2}, y\to 0$ and $x=y+\dfrac{\pi}{2}$

$$\therefore \underset{x\to \frac{\pi}{2}}{Lt}\frac{\cos x}{x-\frac{\pi}{2}}=\underset{y\to 0}{Lt}\frac{\cos\left(\frac{\pi}{2}+y\right)}{y}\ \underset{y\to 0}{Lt}\frac{-\sin y}{y}=-1$$

3. $\underset{x\to 0}{Lt}\dfrac{\sin ax}{x\cos x}$

Sol : $\underset{x\to 0}{Lt}\dfrac{\sin ax}{x.\cos x}=\dfrac{\underset{x\to 0}{Lt}\frac{\sin ax}{ax}.a}{\underset{x\to 0}{Lt}\cos x}=\dfrac{a}{1}=a$

4. $\underset{x\to 1}{Lt}\dfrac{\sin(x-1)}{\left(x^2-1\right)}$

Sol. $\underset{x\to 1}{Lt}\dfrac{\sin(x-1)}{x^2-1}=\underset{x\to 1}{Lt}\dfrac{\sin(x-1)}{(x-1)(x+1)}=\underset{x\to 1}{Lt}\dfrac{\sin(x-1)}{(x-1)}.\underset{x\to 1}{Lt}\dfrac{1}{x+1}$

Put $y=x-1$ so that as $x\to 1, y\to 0$

$$= \underset{y \to 0}{Lt} \frac{\sin y}{y} \cdot \frac{1}{1+1} = 1 \cdot \frac{1}{2} = \frac{1}{2}$$

5. $\underset{x \to 0}{Lt} \frac{\sin(a+bx) - \sin(a-bx)}{x}$

Sol :

$$\underset{x \to 0}{Lt} \frac{\sin(a+bx) - \sin(a-bx)}{x}$$

$$= \underset{x \to 0}{Lt} \frac{2\cos\left(\frac{a+bx+a-bx}{2}\right) \cdot \sin\left(\frac{a+bx-a+bx}{2}\right)}{x}$$

$$= \underset{x \to 1}{Lt}\ 2\cos a.\ \underset{x \to 0}{Lt} \frac{\sin bx}{bx} . b = 2\cos a. \quad b = 2b. \cos a.$$

6. $\underset{x \to a}{Lt} \frac{\tan(x-a)}{x^2 - a^2} (a \neq 0)$

Sol :

$$\underset{x \to a}{Lt} \frac{\tan(x-a)}{x^2 - a^2} = \underset{x \to a}{Lt} \frac{\tan(x-a)}{(x-a)(x+a)} = \underset{x \to a}{Lt} \frac{\tan(x-a)}{(x-a)} . \underset{x \to a}{Lt} \frac{1}{(x+a)}$$

In the first limit Put $x - a = h$ so that as $x \to a, h \to 0$

$$\text{G. L} = \underset{h \to 0}{Lt} \frac{\tan h}{h} \cdot \left(\frac{1}{a+a}\right) = 1 \cdot \frac{1}{2a} = \frac{1}{2a}$$

II.

1. $\underset{x \to 1}{Lt} \frac{(2x-1)\left(\sqrt{x}-1\right)}{\left(2x^2+x-3\right)}$

Sol : $\underset{x \to 1}{Lt} \frac{(2x-1)\left(\sqrt{x}-1\right)}{2x^2+x-3} = \underset{x \to 1}{Lt} \frac{(2x-1)\left(\sqrt{x}-1\right)}{(x-1)(2x+3)} = \underset{x \to 1}{Lt} \frac{(2x-1)\left(\sqrt{x}-1\right)}{\left(\sqrt{x}-1\right)\left(\sqrt{x}+1\right)(2x+3)}$

$$= \frac{2.1-1}{\left(\sqrt{1}+1\right)(2.1+3)} = \frac{1}{(1-1)5} = \frac{1}{10}$$

2. $\underset{x\to a}{Lt}\left[\dfrac{x\sin a - a\sin x}{x-a}\right]$

Sol : $\underset{x\to a}{Lt}\dfrac{x\sin a - a\sin x}{x-a} = \underset{x\to a}{Lt}\dfrac{(x\sin a - a\sin a)-(a\sin x - a\sin a)}{(x-a)}$

$$= \underset{x\to a}{Lt}\frac{(x-a)\sin a - a(\sin x - \sin a)}{x-a}$$

$$= \underset{x\to a}{Lt}\frac{(x-a)\sin a}{x-a} - \underset{x\to a}{Lt}\, a\left(\frac{\sin x - \sin a}{x-a}\right)$$

$$= \sin a - a.\ \underset{x\to a}{Lt}\frac{2\cos\frac{x+a}{2}.\sin\frac{x-a}{2}}{x-a}$$

$$= \sin a - a.\ \underset{x\to a}{Lt}\frac{\cos(x+a)}{2}\ \underset{x\to a}{Lt}\frac{\sin\left(\frac{x-a}{2}\right)}{\left(\frac{x-a}{2}\right)}$$

$$= \sin a - a\cos a - 1 = \sin a - a\cos a$$

3. $\underset{x\to 0}{Lt}\left[\dfrac{\cos ax - \cos bx}{x^2}\right]$

Sol : $\underset{x\to 0}{Lt}\dfrac{\cos ax - \cos bx}{x^2} = \underset{x\to 0}{Lt}\dfrac{2\sin\frac{(a+b)x}{2}.\sin\frac{(b-a)x}{2}}{x^2}$

$$= 2.\ \underset{x\to 0}{Lt}\frac{\sin(b+a)\frac{x}{2}}{x}.\ \underset{x\to 0}{Lt}\frac{\sin(b-a)\frac{x}{2}}{x}$$

$$= 2.\ \underset{x\to 0}{Lt}\frac{\sin(b+a)\frac{x}{2}}{(b+a)\frac{x}{2}}\times\frac{(b+a)}{2}.\ \underset{x\to 0}{Lt}\frac{\sin(b-a)\frac{x}{2}}{(b-a)\frac{x}{2}}\times\frac{(b-a)}{2}$$

$$= 2.\left(\frac{b+a}{2}\right)\left(\frac{b-a}{2}\right) \quad = \frac{1}{2}\left(b^2 - a^2\right)$$

4. $$\underset{x\to 2}{Lt} \frac{\left(2x^2 - 7x - 4\right)}{(2x-1)\left(\sqrt{x} - 2\right)}$$

Sol : $$\underset{x\to 2}{Lt} \frac{2x^2 - 7x - 4}{(2x-1)\left(\sqrt{x} - 2\right)} = \underset{x\to 2}{Lt} \frac{(x-4)(2x+1)}{(2x-1)\left(\sqrt{x} - 2\right)} = \underset{x\to 2}{Lt} \frac{\left(\sqrt{x} + 2\right)(2x+1)}{(2x-1)}$$

$$= \frac{\left(\sqrt{2} + 2\right)(4+1)}{(4-1)} = \frac{5\left(2 + \sqrt{2}\right)}{3}$$

III.

1. $$\underset{x\to 0}{Lt}\left[\frac{(1+x)^{\frac{1}{8}} - (1-x)^{\frac{1}{8}}}{x}\right]$$

Sol : $$\underset{x\to 0}{Lt} \frac{(1+x)^{\frac{1}{8}} - (1-x)^{\frac{1}{8}}}{x} = \underset{x\to 0}{Lt} \frac{(1+x)^{\frac{1}{8}} - 1 + 1 - (1-x)^{\frac{1}{8}}}{x}$$

$$= \underset{(1+x)\to 1}{Lt} \frac{(1+x)^{\frac{1}{8}} - 1}{(1+x) - 1} + \underset{(1-x)\to 1}{Lt} \frac{(1-x)^{\frac{1}{8}} - 1^8}{(1-x) - 1}$$

$$= \frac{1}{8}1^{-7/8} + \frac{1}{8}1^{-7/8} = \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$$

2. $\underset{x \to 0}{Lt}\left[\frac{3^x - 1}{\sqrt{1+x} - 1}\right]$

Sol : $\underset{x \to 0}{Lt}\ \frac{3^x - 1}{\sqrt{1+x} - 1} = \underset{x \to 0}{Lt}\ \frac{3^x - 1}{\sqrt{1+x} - 1} \times \frac{\sqrt{1+x}+1}{\sqrt{1+x}+1}$(*rationalising* Dr.)

$$= \underset{x \to 0}{Lt}\ \frac{\left(3^x - 1\right)\left(\sqrt{1+x}+1\right)}{1+x-1} = \underset{x \to 0}{Lt}\ \frac{3^x - 1}{x} . \underset{x \to 0}{Lt}\left(\sqrt{1+x}+1\right)$$

$$= (\log 3)\left(\sqrt{1+0}+1\right) = 2.\log 3$$

3. $\underset{x \to a}{Lt}\left[\frac{\left(\sqrt{a+2x} - \sqrt{3x}\right)}{\left(\sqrt{3a+x} - 2\sqrt{x}\right)}\right]$

Sol : rationalise both nr. And dr., then

$$\text{G.L.} = \frac{\left(\sqrt{a+2x} - \sqrt{3x}\right)\left(\sqrt{a+2x} + \sqrt{3x}\right)}{\left(\sqrt{a+2x} + \sqrt{3x}\right)} + \underset{x \to a}{Lt}\ \frac{\left(\sqrt{3a+x} + \sqrt{4x}\right)}{\left(\sqrt{3a+x} - \sqrt{4x}\right)\left(\sqrt{3a+x} + \sqrt{4x}\right)}$$

$$= \underset{x \to a}{Lt}\ \frac{a+2x-3x}{\sqrt{a+2x} + \sqrt{3x}} \times \frac{\sqrt{3a+x} + \sqrt{4x}}{3a+x-4x} = \underset{x \to a}{Lt}\ \frac{(a-x)\left(\sqrt{3a+x} + \sqrt{4x}\right)}{\left(\sqrt{a+2x} + \sqrt{3x}\right)3(a-x)}$$

$$= \frac{2(2a)}{2\left(\sqrt{3a}\right)3} = \frac{2}{3\sqrt{3}}$$

4. $\underset{x\to 0}{Lt}\dfrac{\sqrt[3]{1+x}-\sqrt[3]{1-x}}{x}$

Sol : $\underset{x\to 0}{Lt}\dfrac{\sqrt[3]{1+x}-\sqrt[3]{1-x}}{x}=\underset{x\to 0}{Lt}\dfrac{(1+x)^{\frac{1}{3}}-(1-x)^{\frac{1}{3}}}{x}$

$=\underset{x\to 0}{Lt}\dfrac{(1+x)^{\frac{1}{3}}-1+1-(1-x)^{\frac{1}{3}}}{x}$

$=\underset{x\to 0}{Lt}\dfrac{(1+x)^{\frac{1}{3}}-1}{(1+x)-1}+\underset{x\to 0}{Lt}\dfrac{(1-x)^{\frac{1}{3}}-1}{(1-x)-1}=\dfrac{1}{3}.1^{-2/3}+\dfrac{1}{3}.1^{-2/3}=\dfrac{1}{3}+\dfrac{1}{3}=\dfrac{2}{3}$

5. $\underset{x\to a}{Lt}\dfrac{\sin(x-a)\tan^{2}(x-a)}{\left(x^{2}-a^{2}\right)^{2}}$

Sol: $\underset{x\to a}{Lt}\dfrac{\sin(x-a)\tan^{2}(x-a)}{\left(x^{2}-a^{2}\right)^{2}}$

$=\underset{x\to a}{Lt}\dfrac{\sin(x-a)}{x-a}\left(\underset{x\to a}{Lt}\dfrac{\tan(x-a)}{(x-a)}\right)^{2}+\underset{x\to a}{Lt}\dfrac{(x-a)}{(x+a)^{2}}\quad =1.1\dfrac{0}{(2a)^{2}}=0$

6. $\underset{x\to 0}{Lt}\dfrac{1-\cos 2mx}{\sin^{2}nx}(m,n\in -2)$

Sol: $\underset{x\to 0}{Lt}\dfrac{1-\cos 2mx}{\sin^{2}nx}=\underset{x\to 0}{Lt}\dfrac{2\sin^{2}mx}{\sin^{2}nx}$

$=\underset{x\to 0}{Lt}\dfrac{2\sin^{2}mx}{\sin^{2}nx}\times\dfrac{x^{2}}{x^{2}}=\underset{x\to 0}{Lt}\dfrac{2\dfrac{\sin^{2}mx}{x^{2}}}{\dfrac{\sin^{2}nx}{x^{2}}}=2\dfrac{\left(\underset{x\to 0}{Lt}\dfrac{\sin mx}{mx}\right)^{2}}{\left(\underset{x\to 0}{Lt}\dfrac{\sin nx}{nx}\right)^{2}}\times\dfrac{m^{2}}{n^{2}}=\dfrac{2m^{2}}{n^{2}}$