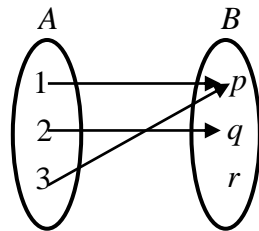


FUNCTIONS

Def 1: A relation f from a set A into a set B is said to be a function or mapping from A into B if for each $x \in A$ there exists a unique $y \in B$ such that $(x, y) \in f$. It is denoted by $f : A \rightarrow B$.

Note: Example of a function may be represented diagrammatically. The above example can be written diagrammatically as follows.



Def 2: A relation f from a set A into a set B is said to be a function or mapping from A into B if
 i) $x \in A \Rightarrow f(x) \in B$ ii) $x_1, x_2 \in A, x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$

Def 3: If $f : A \rightarrow B$ is a function, then A is called domain, B is called codomain and $f(A) = \{f(x) : x \in A\}$ is called range of f .

Def 4: A function $f : A \rightarrow B$ is said to be one one function or injection from A into B if different element in A have different f -images in B .

Note: A function $f : A \rightarrow B$ is one one if $f(x_1, y) \in f, (x_2, y) \in f \Rightarrow x_1 = x_2$.

Note: A function $f : A \rightarrow B$ is one one iff $x_1, x_2 \in A, x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$

Note: A function $f : A \rightarrow B$ is one one iff $x_1, x_2 \in A, f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

Note: A function $f : A \rightarrow B$ which is not one one is called many one function

Note: If $f : A \rightarrow B$ is one one and A, B are finite then $n(A) \leq n(B)$.

Def 5: A function $f : A \rightarrow B$ is said to be onto function or surjection from A onto B if $f(A) = B$.

Note: A function $f : A \rightarrow B$ is onto if $y \in B \Downarrow \exists x \in A \ni f(x) = y$.

Note: A function $f : A \rightarrow B$ which is not onto is called an into function.

Note: If A, B are two finite sets and $f : A \rightarrow B$ is onto then $n(B) \leq n(A)$.

Note: If A, B are two finite sets and $n(B) = 2$, then the number of onto functions that can be defined from A onto B is $2^{n(A)} - 2$.

Def 6: A function $f : A \rightarrow B$ is said to be one one onto function or bijection from A onto B if $f : A \rightarrow B$ is both one one function and onto function.

Theorem: If $f : A \rightarrow B$, $g : B \rightarrow C$ are two functions then the composite relation gof is a function a into C.

Theorem: If $f : A \rightarrow B$, $g : B \rightarrow C$ are two one one onto functions then $gof : A \rightarrow C$ is also one one be onto.

Sol: i) Let $x_1, x_2 \in A$ and $f(x_1) = f(x_2)$.

$$x_1, x_2 \in A, f : A \rightarrow B \Rightarrow f(x_1), f(x_2) \in B$$

$$f(x_1), f(x_2) \in B, \rightarrow C, f(x_2) \Rightarrow g[f(x_1)] = g[f(x_2)] \Rightarrow (gof)(x_1) = (gof)(x_2)$$

$$x_1, x_2 \in A, (gof)(x_1) = (gof)(x_2) \Rightarrow x_1 = x_2$$

$$\therefore x_1, x_2 \in A, f(x_1) = f(x_2) \Rightarrow x_1 = x_2.$$

$$\therefore f : A \rightarrow B \text{ Is one one.}$$

ii) Proof: let $z \in C$, $g : B \rightarrow C$ is onto $\exists y \in B \exists : g(y) = z$ $y \in B$ $f : A \rightarrow B$ is onto

$$\therefore \exists x \in A \exists f(x) = y$$

$$G \{f(x)\} = t$$

$$(g \circ f) x = t$$

$$\forall z \in C \exists x \in A \exists (gof)(x) = z.$$

$$\therefore g \text{ is onto.}$$

Def 7: Two functions $f : A \rightarrow B$, $g : C \rightarrow D$ are said to be equal if

$$\text{i) } A = C, B = D \quad \text{ii) } f(x) = g(x) \forall x \in A. \text{ It is denoted by } f = g$$

Theorem: If $f : A \rightarrow B$, $g : B \rightarrow C$, $h : C \rightarrow D$ are three functions, then $ho(gof) = (hof)of$

Theorem: if A is set, then the identify relation I on A is one one onto.

Def 8: If A is a set, then the function I on A defined by $I(x) = x \forall x \in A$, is called identify function on A. it is denoted by I_A .

Theorem: If $f : A \rightarrow B$ and I_A, I_B are identify functions on A, B respectively then

$$foI_A = I_B of = f.$$

Proof: $I_A : A \rightarrow A$, $f : A \rightarrow B \Rightarrow foI_A : A \rightarrow B$

$$f : A \rightarrow B, I_B : B \rightarrow B \Rightarrow I_B of : A \rightarrow B$$

$$(foI_A)(x) = f\{I_A(x)\} = f(x), \forall x \in A. \quad \therefore foI_A = f$$

$$(I_B of)(x) = I_B\{f(x)\} = f(x), \forall x \in A \quad \therefore I_B of = f$$

$$\therefore foI_A = I_B of = f$$

Def 9: If $f : A \rightarrow B$ is a function then $\{(y, x) \in B \times A : (x, y) \in f\}$ is called inverse of f. It is denoted by f^{-1} .

Def 10: If $f : A \rightarrow B$ is a bijection, then the function $f^{-1} : B \rightarrow A$ defined by $f^{-1}(y) = x$ iff $f(x) = y \forall y \in B$ is called inverse function of f.

Theorem: If $f : A \rightarrow B$ is a bijection, then $f^{-1} \circ f = I_A$, $f \circ f^{-1} = I_B$

Proof: Since $f : A \rightarrow B$ is a bijection $f^{-1} : B \rightarrow A$ is also a bijection and

$$f^{-1}(y) = x \Leftrightarrow f(x) = y \quad \forall y \in B$$

$$f : A \rightarrow B, f^{-1} : B \rightarrow A \Rightarrow f^{-1} \circ f : A \rightarrow A$$

Clearly $I_A : A \rightarrow A$ such that $I_A(x) = x, \forall x \in A$.

Let $x \in A$

$$x \in A, f : A \rightarrow B \Rightarrow f(x) \in B$$

Let $y = f(x)$

$$y = f(x) \Rightarrow f^{-1}(y) = x$$

$$(f^{-1} \circ f)(x) = f^{-1}[f(x)] = f^{-1}(y) = x = I_A(x)$$

$$\therefore (f^{-1} \circ f)(x) = I_A(x) \quad \forall x \in A \quad \therefore f^{-1} \circ f = I_A$$

$$f^{-1} : B \rightarrow A, f : A \rightarrow B \Rightarrow f \circ f^{-1} : B \rightarrow B$$

Clearly $I_B : B \rightarrow B$ such that $I_B(y) = y \quad \forall y \in B$

Let $y \in B$

$$y \in B, f^{-1} : B \rightarrow A \Rightarrow f^{-1}(y) \in A$$

Let $f^{-1}(y) = x$

$$f^{-1}(y) = x \Rightarrow f(x) = y$$

$$(f \circ f^{-1})(y) = f[f^{-1}(y)] = f(x) = y = I_B(y)$$

$$\therefore (f \circ f^{-1})(y) = I_B(y) \quad \forall y \in B \quad \therefore f \circ f^{-1} = I_B$$

Theorem: If $f : A \rightarrow B, g : B \rightarrow C$ are two bijections then $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

Proof: $f : A \rightarrow B, g : B \rightarrow C$ are bijections $\Rightarrow g \circ f : A \rightarrow C$ is bijection $\Rightarrow (g \circ f)^{-1} : C \rightarrow A$ is a bijection.

$f : A \rightarrow B$ is a bijection $\Rightarrow f^{-1} : B \rightarrow A$ is a bijection

$g : B \rightarrow C$ is a bijection $\Rightarrow g^{-1} : C \rightarrow B$ is a bijection

$g^{-1} : C \rightarrow B, f^{-1} : B \rightarrow A$ are bijections $\Rightarrow f^{-1} \circ g^{-1} : C \rightarrow A$ is a bijection

Let $z \in C$

$$z \in C, g : B \rightarrow C \text{ is onto} \Rightarrow \exists y \in B \ni g(y) = z \Rightarrow g^{-1}(z) = y$$

$$y \in B, f : A \rightarrow B \text{ is onto} \Rightarrow \exists x \in A \ni f(x) = y \Rightarrow f^{-1}(y) = x$$

$$(g \circ f)(x) = g[f(x)] = g(y) = z \Rightarrow (g \circ f)^{-1}(z) = x$$

$$\therefore (g \circ f)^{-1}(z) = x = f^{-1}(y) = f^{-1}[g^{-1}(z)] = (f^{-1} \circ g^{-1})(z) \quad \therefore (g \circ f)^{-1} = f^{-1} \circ g^{-1}$$

Theorem: If $f : A \rightarrow B$, $g : B \rightarrow A$ are two functions such that $gof = I_A$ and $fog = I_B$ then $f : A \rightarrow B$ is a bijection and $f^{-1} = g$.

Proof: Let $x_1, x_2 \in A$, $f(x_1) = f(x_2)$

$$x_1, x_2 \in A, f : A \rightarrow B \Rightarrow f(x_1), f(x_2) \in B$$

$$f(x_1), f(x_2) \in B, f(x_1) = f(x_2), g : B \rightarrow A \Rightarrow g[f(x_1)] = g[f(x_2)]$$

$$\Rightarrow (gof)(x_1) = (gof)(x_2) \Rightarrow I_A(x_2) \Rightarrow x_1 = x_2$$

$$\therefore x_1, x_2 \in A, f(x_1) = f(x_2) \Rightarrow x_1 = x_2 \therefore f : A \rightarrow B \text{ is one one}$$

Let $y \in B$.

$$y \in B, g : B \rightarrow A \Rightarrow g(y) \in A$$

Def 11: A function $f : A \rightarrow B$ is said to be a constant function if the range of f contains only one element i.e., $f(x) = c \forall x \in A$ where c is a fixed element of B

Def 12: A function $f : A \rightarrow B$ is said to be a real variable function if $A \subseteq R$.

Def 13: A function $f : A \rightarrow B$ is said to be a real valued function iff $B \subseteq R$.

Def 14: A function $f : A \rightarrow B$ is said to be a real function if $A \subseteq R, B \subseteq R$.

Def 15: If $f : A \rightarrow R, g : B \rightarrow R$ then $f + g : A \cap B \rightarrow R$ is defined as

$$(f + g)(x) = f(x) + g(x) \forall x \in A \cap B$$

Def 16: If $f : A \rightarrow R$ and $k \in R$ then $kf : A \rightarrow R$ is defined as $(kf)(x) = kf(x), \forall x \in A$

Def 17: If $f : A \rightarrow B, g : B \rightarrow R$ then $fg : A \cap B \rightarrow R$ is defined as

$$(fg)(x) = f(x)g(x) \forall x \in A \cap B.$$

Def 18: If $f : A \rightarrow R, g : B \rightarrow R$ then $\frac{f}{g} : C \rightarrow R$ is defined as $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \forall x \in C$ where

$$C = \{x \in A \cap B : g(x) \neq 0\}.$$

Def 19: If $f : A \rightarrow R$ then $|f|(x) = |f(x)|, \forall x \in A$

Def 20: If $n \in Z, n \geq 0, a_0, a_1, a_2, \dots, a_n \in R, a_n \neq 0$, then the function $f : R \rightarrow R$ defined by $f(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n \forall x \in R$ is called a polynomial function of degree n .

Def 21: If $f : R \rightarrow R, g : R \rightarrow R$ are two polynomial functions, then the quotient f/g is called a rational function.

Def 22: A function $f : A \rightarrow R$ is said to be bounded on A if there exists real numbers k_1, k_2 such that $k_1 \leq f(x) \leq k_2 \forall x \in A$

Def 23: A function $f : A \rightarrow R$ is said to be an even function if $f(-x) = f(x) \forall x \in A$

Def 24: A function $f : A \rightarrow R$ is said to be an odd function if $f(-x) = -f(x) \forall x \in A$.

Def 25: If $a \in \mathbb{R}, a > 0$ then the function $f : \mathbb{R} \rightarrow \mathbb{R}$ defined as $f(x) = a^x$ is called an exponential function.

Def 26: If $a \in \mathbb{R}, a > 0, a \neq 1$ then the function $f : (0, \infty) \rightarrow \mathbb{R}$ defined as $f(x) = \log_a x$ is called a logarithmic function.

Def 27: The function $f : \mathbb{R} \rightarrow \mathbb{R}$ defined as $f(x) = n$ where $n \in \mathbb{Z}$ such that $n \leq x < n+1, \forall x \in \mathbb{R}$ is called step function or greatest integer function. It is denoted by $f(x) = [x]$

Def 28: The functions $f(x) = \sin x, \cos x, \tan x, \cot x, \sec x$ or $\operatorname{cosec} x$ are called trigonometric functions.

Def 29: The functions $f(x) = \sin^{-1} x, \cos^{-1} x, \tan^{-1} x, \cot^{-1} x, \sec^{-1} x$ or $\operatorname{cosec}^{-1} x$ are called inverse trigonometric functions.

Def 30: The functions $f(x) = \sinh x, \cosh x, \coth x, \operatorname{sech} x$ or $\operatorname{cosech} x$ are called hyperbolic functions.

Def 31: The functions $f(x) = \sinh^{-1} x, \cosh^{-1} x, \tanh^{-1} x, \coth^{-1} x, \operatorname{sech}^{-1} x$ or $\operatorname{cosech}^{-1} x$ are called inverse hyperbolic functions

	Function	Domain	Range
1.	a^x	\mathbb{R}	$(0, \infty)$
2.	$\log_a x$	$(0, \infty)$	\mathbb{R}
3.	$[X]$	\mathbb{R}	\mathbb{Z}
4.	$[X]$	\mathbb{R}	$[0, \infty)$
5.	\sqrt{x}	$[0, \infty)$	$[0, \infty)$
6.	$\sin x$	\mathbb{R}	$[-1, 1]$
7.	$\cos x$	\mathbb{R}	$[-1, 1]$
8.	$\tan x$	$\mathbb{R} - \{(2n+1)\frac{\pi}{2} : n \in \mathbb{Z}\}$	\mathbb{R}
9.	$\cot x$	$\mathbb{R} - \{n\pi : n \in \mathbb{Z}\}$	\mathbb{R}
10.	$\sec x$	$\mathbb{R} - \{(2n+1)\frac{\pi}{2} : n \in \mathbb{Z}\}$	$(-\infty, -1] \cup [1, \infty)$
11.	$\operatorname{cosec} x$	$\mathbb{R} - \{n\pi : n \in \mathbb{Z}\}$	$(-\infty, -1] \cup [1, \infty)$
12.	$\sin^{-1} x$	$[-1, 1]$	$[-\pi/2, \pi/2]$
13.	$\cos^{-1} x$	$[-1, 1]$	$[0, \pi]$
14.	$\tan^{-1} x$	\mathbb{R}	$(-\pi/2, \pi/2)$
15.	$\cot^{-1} x$	\mathbb{R}	$(0, \pi)$
16.	$\sec^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$[0, \pi/2) \cup (\pi/2, \pi]$
17.	$\operatorname{cosec}^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$[-\pi/2, 0) \cup (0, \pi/2]$
18.	$\sinh x$	\mathbb{R}	\mathbb{R}
19.	$\cosh x$	\mathbb{R}	$[1, \infty)$
20.	$\tanh x$	\mathbb{R}	$(-1, 1)$
21.	$\coth x$	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, -1) \cup (1, \infty)$
22.	$\operatorname{sech} x$	\mathbb{R}	$(0, 1]$
23.	$\operatorname{cosech} x$	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, 0) \cup (0, \infty)$

24.	$\text{Sinh}^{-1} x$	\mathbf{R}	\mathbf{R}
25.	$\text{Cosh}^{-1} x$	$[1, \infty)$	$[0, \infty)$
26.	$\text{Tanh}^{-1} x$	$(-1, 1)$	\mathbf{R}
27.	$\text{Coth}^{-1} x$	$(-\infty, -1) \cup (1, \infty)$	$(-\infty, 0) \cup (0, \infty)$
28.	$\text{Sech}^{-1} x$	$(0, 1]$	$[0, \infty)$
29.	$\text{Coseh}^{-1} x$	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, 0) \cup (0, \infty)$

PROBLEMS

VSAQ'S

1. If $f : \mathbf{R} - \{0\} \rightarrow \mathbf{R}$ is defined by $f(x) = x^3 - \frac{1}{x^3}$, then show that $f(x) + f\left(\frac{1}{x}\right) = 0$.

Sol. Given that $f(x) = x^3 - \frac{1}{x^3}$

$$f\left(\frac{1}{x}\right) = \frac{1}{x^3} - x^3$$

$$\therefore f(x) + f\left(\frac{1}{x}\right) = x^3 - \frac{1}{x^3} + \frac{1}{x^3} - x^3 = 0$$

2. If $f : \mathbf{R} - [\pm 1] \rightarrow \mathbf{R}$ is defined by $f(x) = \log \left| \frac{1+x}{1-x} \right|$, then show that $f\left(\frac{2x}{1+x^2}\right) = 2f(x)$.

Sol. $f(x) = \log \left| \frac{1+x}{1-x} \right|$

$$f\left(\frac{2x}{1+x^2}\right) = \log \left| \frac{1 + \frac{2x}{1+x^2}}{1 - \frac{2x}{1+x^2}} \right|$$

$$= \log \left| \frac{x^2 + 1 + 2x}{x^2 + 1 - 2x} \right| = \log \left| \frac{(1+x)^2}{(1-x)^2} \right|$$

$$= \log \left| \left(\frac{1+x}{1-x} \right)^2 \right| = 2 \log \left| \frac{1+x}{1-x} \right| = 2f(x)$$

3. If $A = \{-2, -1, 0, 1, 2\}$ and $f : A \rightarrow B$ is a surjection defined by $f(x) = x^2 + x + 1$, then find B .

Sol. Given that

$$f(x) = x^2 + x + 1$$

$$f(-2) = (-2)^2 - 2 + 1 = 4 - 2 + 1 = 3$$

$$f(-1) = (-1)^2 - 1 + 1 = 1 - 1 + 1 = 1$$

$$f(0) = (0)^2 - 0 + 1 = 1$$

$$f(1) = 1^2 + 1 + 1 = 3$$

$$f(2) = 2^2 + 2 + 1 = 7$$

Thus range of f , $f(A) = \{1, 3, 7\}$

Since f is onto, $f(A) = B$

$$\therefore B = \{3, 1, 7\}$$

4. If $A = \{1, 2, 3, 4\}$ and $f : A \rightarrow \mathbb{R}$ is a function defined by $f(x) = \frac{x^2 - x + 1}{x + 1}$ then find the range of f .

Sol. Given that

$$f(x) = \frac{x^2 - x + 1}{x + 1}$$

$$f(1) = \frac{1^2 - 1 + 1}{1 + 1} = \frac{1}{2}$$

$$f(2) = \frac{2^2 - 2 + 1}{2 + 1} = \frac{3}{3} = 1$$

$$f(3) = \frac{3^2 - 3 + 1}{3 + 1} = \frac{7}{4}$$

$$f(4) = \frac{4^2 - 4 + 1}{4 + 1} = \frac{13}{5}$$

$$\therefore \text{Range of } f \text{ is } \left\{ \frac{1}{2}, 1, \frac{7}{4}, \frac{13}{5} \right\}$$

5. If $f(x + y) = f(xy) \forall x, y \in \mathbb{R}$ then prove that f is a constant function.

Sol. $f(x + y) = f(xy)$

Let $f(0) = k$

then $f(x) = f(x + 0) = f(x \cdot 0) = f(0) = k$

$\Rightarrow f(x + y) = k$

$\therefore f$ is a constant function.

6. Which of the following are injections or surjections or bijections? Justify your answers.

i) $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = \frac{2x + 1}{3}$

ii) $f : \mathbb{R} \rightarrow (0, \infty)$ defined by $f(x) = 2^x$.

iii) $f : (0, \infty) \rightarrow \mathbb{R}$ defined by $f(x) = \log_e x$

iv) $f : [0, \infty) \rightarrow [0, \infty)$ defined by $f(x) = x^2$

v) $f : \mathbb{R} \rightarrow [0, \infty)$ defined by $f(x) = x^2$

vi) $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = x^2$

i) $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = \frac{2x + 1}{3}$ is a bijection.

Sol. i) $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = \frac{2x + 1}{3}$

a) To prove $f : \mathbb{R} \rightarrow \mathbb{R}$ is injection

Let $x_1, x_2 \in \mathbb{R}$ and $f(x_1) = f(x_2)$

$$\Rightarrow \frac{2x_1+1}{3} = \frac{2x_2+1}{3}$$

$$\Rightarrow 2x_1+1 = 2x_2+1$$

$$\Rightarrow 2x_1 = 2x_2$$

$$\Rightarrow x_1 = x_2$$

$\Rightarrow f : \mathbb{R} \rightarrow \mathbb{R}$ is injection

b) To prove $f : \mathbb{R} \rightarrow \mathbb{R}$ is surjection

Let $y \in \mathbb{R}$ and $f(x) = y$

$$\Rightarrow \frac{2x+1}{3} = y$$

$$\Rightarrow 2x+1 = 3y$$

$$\Rightarrow 2x = 3y - 1$$

$$\Rightarrow x = \frac{3y-1}{2}$$

Thus for every $y \in \mathbb{R}$, \exists an element $\frac{3y-1}{2} \in \mathbb{R}$ such that

$$f\left(\frac{3y-1}{2}\right) = \frac{2\left(\frac{3y-1}{2}\right)+1}{3} = \frac{3y-1+1}{3} = y$$

$\therefore f : \mathbb{R} \rightarrow \mathbb{R}$ is both injection and surjection

$\therefore f : \mathbb{R} \rightarrow \mathbb{R}$ is a bijection.

ii) $f : \mathbb{R} \rightarrow (0, \infty)$ defined by $f(x) = 2^x$.

a) To prove $f : \mathbb{R} \rightarrow \mathbb{R}^+$ is injection

Let $x_1, x_2 \in \mathbb{R}$ and

$$f(x_1) = f(x_2)$$

$$\Rightarrow 2^{x_1} = 2^{x_2}$$

$$\Rightarrow x_1 = x_2$$

$\therefore f : \mathbb{R} \rightarrow \mathbb{R}^+$ is injection.

b) To prove $f : \mathbb{R} \rightarrow \mathbb{R}^+$ is surjection

Let $y \in \mathbb{R}^+$ and $f(x) = y$

$$\Rightarrow 2^x = y$$

$$\Rightarrow x = \log_2 y \in \mathbb{R}$$

Thus for every $y \in \mathbb{R}^+$, \exists an element

$\log_2 y \in \mathbb{R}$ such that

$$f(\log_2 y) = 2^{\log_2 y} = y$$

$\therefore f : \mathbb{R} \rightarrow \mathbb{R}^+$ is a surjection

Thus $f : \mathbb{R} \rightarrow \mathbb{R}^+$ is both injection and surjection.

$\therefore f : \mathbb{R} \rightarrow \mathbb{R}^+$ is a bijection.

iii) $f : (0, \infty) \rightarrow \mathbf{R}$ defined by $f(x) = \log_e x$

Explanation :

a) To prove $f : \mathbf{R}^+ \rightarrow \mathbf{R}$ is injection

Let $x_1, x_2 \in \mathbf{R}^+$ and

$$f(x_1) = f(x_2)$$

$$\Rightarrow \log_e x_1 = \log_e x_2$$

$$\Rightarrow x_1 = x_2$$

$\therefore f : \mathbf{R}^+ \rightarrow \mathbf{R}$ is injection.

b) **To prove $f : \mathbf{R}^+ \rightarrow \mathbf{R}$ is surjection**

Let $y \in \mathbf{R}$ and $f(x) = y$

$$\Rightarrow \log_e x = y$$

$$\Rightarrow x = e^y \in \mathbf{R}^+$$

Thus for every $y \in \mathbf{R}$, \exists an element

$e^y \in \mathbf{R}^+$ such that

$$f(e^y) = \log_e e^y = y \log_e e = y$$

$\therefore f : \mathbf{R}^+ \rightarrow \mathbf{R}$ is surjection

Thus $f : \mathbf{R}^+ \rightarrow \mathbf{R}$ is both injection and surjection.

$\therefore f : \mathbf{R}^+ \rightarrow \mathbf{R}$ is a bijection.

iv) $f : [0, \infty) \rightarrow [0, \infty)$ defined by $f(x) = x^2$

Explanation :

a) **To prove $f : \mathbf{A} \rightarrow \mathbf{A}$ is injection**

Let $x_1, x_2 \in \mathbf{A}$ and

$$f(x_1) = f(x_2)$$

$$\Rightarrow x_1^2 = x_2^2$$

$$\Rightarrow x_1 = x_2 (\because x_1 \geq 0, x_2 \geq 0)$$

$\therefore f : \mathbf{A} \rightarrow \mathbf{A}$ is injection

b) **To prove $f : \mathbf{A} \rightarrow \mathbf{A}$ is surjection**

Let $y \in \mathbf{A}$ and $f(x) = y$

$$\Rightarrow x^2 = y$$

$$\Rightarrow x = \sqrt{y} \in \mathbf{A}$$

Thus for every $y \in \mathbf{A}$, \exists an element $\sqrt{y} \in \mathbf{A}$

$$\text{Such that } f[\sqrt{y}] = (\sqrt{y})^2 = y$$

$\therefore f : \mathbf{A} \rightarrow \mathbf{A}$ is a surjection

Thus $f : \mathbf{A} \rightarrow \mathbf{A}$ is both injection and surjection.

$\therefore f : [0, \infty) \rightarrow [0, \infty)$ is a bijection.

v) $f : \mathbf{R} \rightarrow [0, \infty)$ defined by $f(x) = x^2$

Explanation :

a) To prove $f : \mathbf{R} \rightarrow A$ is not a injection

Since distinct elements have not having distinct f-images

For example :

$$f(2) = 2^2 = 4 = (-2)^2 = f(-2)$$

But $2 \neq -2$

b) To prove $f : \mathbf{R} \rightarrow A$ is surjection

Let $y \in A$ and $f(x) = y$

$$\Rightarrow x^2 = y$$

$$\Rightarrow x = \pm\sqrt{y} \in \mathbf{R}$$

Thus for every $y \in A$, \exists an element $\pm\sqrt{y} \in \mathbf{R}$ such that

$$f(\pm\sqrt{y}) = (\pm\sqrt{y})^2 = y$$

$\therefore f : \mathbf{R} \rightarrow A$ is a surjection

Thus $f : \mathbf{R} \rightarrow A$ is surjection only.

vi) $f : \mathbf{R} \rightarrow \mathbf{R}$ defined by $f(x) = x^2$

a) To prove $f : \mathbf{R} \rightarrow \mathbf{R}$ is not a injection

Since distinct element in set \mathbf{R} are not having distinct f-images in \mathbf{R} .

For example :

$$f(2) = 2^2 = 4 = (-2)^2 = f(-2)$$

But $2 \neq -2$

$\therefore f : \mathbf{R} \rightarrow \mathbf{R}$ is not a injection.

b) To prove $f : \mathbf{R} \rightarrow \mathbf{R}$ is not surjection

$-1 \in \mathbf{R}$, suppose $f(x) = -1$

$$x^2 = -1$$

$$x = \sqrt{-1} \notin \mathbf{R}$$

$\therefore f : \mathbf{R} \rightarrow \mathbf{R}$ is not surjection.

7. If $g = \{(1, 1), (2, 3), (3, 5), (4, 7)\}$ is a function from $A = \{1, 2, 3, 4\}$ to $B = \{1, 3, 5, 7\}$? If this is given by the formula $g(x) = ax + b$, then find a and b .

Sol. Given that

$A = \{1, 2, 3, 4\}$ and $B = \{1, 3, 5, 7\}$ and

$g = \{(1, 1), (2, 3), (3, 5), (4, 7)\} \dots(1)$

Clearly every element in set A has unique g -image in set B .

$\therefore g : A \rightarrow B$ is a function.

Consider, $g(x) = ax + b$

$$g(1) = a + b$$

$$g(2) = 2a + b$$

$$g(3) = 3a + b$$

$$g(4) = 4a + b$$

$$\therefore g = \{(1, a + b), (2, 2a + b), (3, 3a + b), (4, 4a + b)\} \dots(2)$$

Comparing (1) and (2)

$$a + b = 1 \Rightarrow a = 1 - b \Rightarrow a = 1 + 1 = 2$$

$$2a + b = 3 \Rightarrow 2[1 - b] + b = 3$$

$$\Rightarrow 2 - 2b + b = 3 \Rightarrow 2 - b = 3 \Rightarrow b = -1$$

8. If $f(x) = 2$, $g(x) = x^2$, $h(x) = 2x$ for all $x \in \mathbf{R}$, then find $[fo(goh)](x)$.

Sol. $fo(goh)(x) = fog [h(x)]$

$$= fog (2x)$$

$$= f [g(2x)]$$

$$= f (4x^2) = 1$$

$$\therefore fo(goh)(x) = 2.$$

9. Find the inverse of the following functions.

i) If $a, b \in \mathbf{R}$, $f : \mathbf{R} \rightarrow \mathbf{R}$ defined by $f(x) = ax + b$ ($a \neq 0$)

ii) $f : \mathbf{R} \rightarrow (0, \infty)$ defined by $f(x) = 5^x$

iii) $f : (0, \infty) \rightarrow \mathbf{R}$ defined by $f(x) = \log_2 x$.

Sol. i) Let $f(x) = ax + b = y$

$$\Rightarrow ax = y - b \Rightarrow x = \frac{y - b}{a}$$

$$\text{Thus } f^{-1}(x) = \frac{x - b}{a}$$

ii) Let $f(x) = 5^x = y$

$$\Rightarrow x = \log_5 y$$

$$\text{Thus } f^{-1}(x) = \log_5 x$$

iii) Let $f(x) = \log_2 x = y$

$$\Rightarrow x = 2^y$$

$$\Rightarrow f^{-1}(x) = 2^x$$

10. If $f(x) = 1 + x + x^2 + \dots$ for $|x| < 1$ then show that $f^{-1}(x) = \frac{x-1}{x}$.

Sol. $f(x) = 1 + x + x^2 + \dots$ for $|x| < 1$

$$= (1 - x)^{-1} \text{ by Binomial theorem for rational index}$$

$$= \frac{1}{1 - x} = y$$

$$1 = y - xy$$

$$xy = y - 1$$

$$x = \frac{y - 1}{y}$$

$$f^{-1}(x) = \frac{x - 1}{x}$$

11. If $f : [1, \infty) \rightarrow [1, \infty)$ defined by $f(x) = 2^{x(x-1)}$ then find $f^{-1}(x)$.

Sol. $f(x) : [1, \dots, \infty) \rightarrow [1, \dots, \infty)$

$$f(x) = 2^{x(x-1)}$$

$$f(x) = 2^{x(x-1)} = y$$

$$x(x-1) = \log_2 y$$

$$x^2 - x - \log_2 y = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{1 \pm \sqrt{1 + 4 \log_2 y}}{2}$$

$$f^{-1}(x) = \frac{1 \pm \sqrt{1 + 4 \log_2 x}}{2}$$

12. $f(x) = 2x - 1$, $g(x) = \frac{x+1}{2}$ for all $x \in \mathbf{R}$, find $\text{gof}(x)$.

Sol. $\text{gof}(x) = g[f(x)] = g(2x - 1)$

$$= \frac{2x - 1 + 1}{2} = \frac{2x}{2} = x$$

$$\therefore \text{gof}(x) = x$$

13. Find the domain of the following real valued functions.

i) $f(x) = \frac{2x^2 - 5x + 7}{(x-1)(x-2)(x-3)}$

ii) $f(x) = \frac{1}{\log(2-x)}$

iii) $f(x) = \sqrt{4x - x^2}$

iv) $f(x) = \frac{1}{\sqrt{1-x^2}}$

v) $f(x) = \sqrt{x^2 - 25}$

vi) $f(x) = \sqrt{x - [x]}$

vii) $f(x) = \sqrt{[x] - x}$

Sol. i) $f(x) = \frac{2x^2 - 5x + 7}{(x-1)(x-2)(x-3)}$

$$(x-1)(x-2)(x-3) \neq 0$$

$$\Rightarrow x-1 \neq 0, x-2 \neq 0, x-3 \neq 0$$

$$\Rightarrow x \neq 1, x \neq 2, x \neq 3$$

$$\Rightarrow x \in \mathbf{R} - \{1, 2, 3\}$$

$$\therefore \text{Domain of } f \text{ is } \mathbf{R} - \{1, 2, 3\}$$

$$\text{ii) } f(x) = \frac{1}{\log(2-x)}$$

$$2-x > 0 \text{ and } 2-x \neq 1$$

$$2 > x \text{ and } 2-1 \neq x$$

$$x < 2 \text{ and } x \neq 1$$

\therefore Domain of f is $(-\infty, 1) \cup (1, 2)$

$$\text{iii) } f(x) = \sqrt{4x-x^2}$$

$$4x-x^2 \geq 0$$

$$x(4-x) \geq 0$$

$$\Rightarrow 0 \leq x \leq 4$$

Since the coefficient of x^2 is -ve

\therefore Domain of f is $[0, 4]$

$$\text{iv) } f(x) = \frac{1}{\sqrt{1-x^2}}$$

$$1-x^2 > 0$$

$$\Rightarrow (1-x)(1+x) > 0$$

$$\Rightarrow -1 < x < 1$$

Since the coefficient of x^2 is -ve

\therefore Domain of f is $(-1, 1)$.

$$\text{v) } f(x) = \sqrt{x^2-25}$$

$$x^2-25 \geq 0$$

$$\Rightarrow (x-5)(x+5) \geq 0$$

$$\Rightarrow x \leq -5 \text{ or } x \geq 5$$

Since the coefficient of x^2 is +ve

\therefore Domain of f is $(-\infty, -5] \cup [5, \infty)$

$$\text{vi) } f(x) = \sqrt{x-[x]}$$

$$x-[x] \geq 0 \Rightarrow x \geq [x]$$

It is true for all $x \in \mathbb{R}$

\therefore Domain of f is \mathbb{R} .

$$\text{vii) } f(x) = \sqrt{[x]-x}$$

$$\Rightarrow [x]-x \geq 0$$

$$\Rightarrow [x] \geq x$$

It is true only when x is an integer

\therefore Domain of f is \mathbb{Z} .

14. Find the ranges of the following real valued functions.

i) $\log|4-x^2|$ ii) $\sqrt{[x]-x}$

iii) $\frac{\sin \pi[x]}{1+[x]^2}$ iv) $\frac{x^2-4}{x-2}$

v) $\sqrt{9+x^2}$

Sol. i) $f(x) = \log|4-x^2|$

Domain of f is $\mathbb{R} - \{-2, 2\}$

\therefore Range = \mathbb{R}

ii) $f(x) = \sqrt{[x]-x}$

Domain of f is \mathbb{Z}

Range of f is $\{0\}$

iii) $\frac{\sin \pi[x]}{1+[x]^2}$

Domain of f is \mathbb{R}

Range of f is $\{0\}$

Since $\sin n\pi = 0, \forall n \in \mathbb{Z}$.

iv) $f(x) = \frac{x^2-4}{x-2}$

Domain of f is $\mathbb{R} - \{2\}$

Range of f is $\mathbb{R} - \{4\}$

v) $f(x) = \sqrt{9+x^2}$

$9+x^2 > 0, \forall x \in \mathbb{R}$

Domain of f is \mathbb{R}

Range of f is $[3, \infty)$

SAQ'S

15. If the function $f: \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = \frac{3^x + 3^{-x}}{2}$, then show that

$$f(x+y) + f(x-y) = 2f(x)f(y).$$

Sol. Given that

$$f(x) = \frac{3^x + 3^{-x}}{2} \text{ and } f(y) = \frac{3^y + 3^{-y}}{2}.$$

$$\text{We have } f(x+y) = \frac{3^{x+y} + 3^{-(x+y)}}{2}$$

$$f(x-y) = \frac{3^{x-y} + 3^{-(x-y)}}{2}$$

$$\text{L.H.S.} = f(x+y) + f(x-y)$$

$$= \frac{3^{x+y} + 3^{-(x+y)}}{2} + \frac{3^{x-y} + 3^{-(x-y)}}{2}$$

$$= \frac{1}{2} [3^{x+y} + 3^{-(x+y)} + 3^{x-y} + 3^{-(x-y)}] \dots (1)$$

$$\text{R.H.S. : } 2 f(x) f(y) = 2 \left[\frac{3^x + 3^{-x}}{2} \cdot \frac{3^y + 3^{-y}}{2} \right]$$

$$= \frac{1}{2} [3^{x+y} + 3^{x-y} + 3^{y-x} + 3^{-x-y}]$$

$$= \frac{1}{2} [3^{x+y} + 3^{-(x-y)} + 3^{x-y} + 3^{-(x+y)}] \dots (2)$$

From (1) and (2)

\therefore L.H.S. = R.H.S.

$$f(x + y) + f(x - y) = 2 f(x) f(y)$$

16. If the function $f : \mathbf{R} \rightarrow \mathbf{R}$ defined by $f(x) = \frac{4^x}{4^x + 2}$, then show that

$f(1 - x) = 1 - f(x)$, and hence deduce the value of $f\left(\frac{1}{4}\right) + 2f\left(\frac{1}{2}\right) + f\left(\frac{3}{4}\right)$.

Sol. Given that $f(x) = \frac{4^x}{4^x + 2}$

We obtain, $f(1 - x) = \frac{4^{1-x}}{4^{1-x} + 2}$

$$= \frac{\frac{4}{4^x}}{\frac{4}{4^x} + 2} = \frac{4}{4 + 2 \cdot 4^x} = \frac{2}{2 + 4^x} \dots (1)$$

$$1 - f(x) = 1 - \frac{4^x}{4^x + 2}$$

$$= \frac{4^x + 2 - 4^x}{4^x + 2} = \frac{2}{2 + 4^x} \dots (2)$$

From (1) and (2) : $f(1 - x) = 1 - f(x)$

We have $f(1 - x) = 1 - f(x)$

Now, $f(1 - x) + f(x) = 1$

Put $x = 1/4$, then $f(1 - 1/4) + f(1/4) = 1$

$$f(3/4) + f(1/4) = 1 \dots \dots \dots (3)$$

$f(1 - x) + f(x) = 1$ put $x = 1/2$ then

$$f(1 - 1/2) + f(1/2) = 1$$

$$f(1/2) + f(1/2) = 1 \Rightarrow 2f(1/2) = 1 \dots \dots \dots (4)$$

$$(3) + (4) \Rightarrow f(3/4) + f(1/4) + 2f(1/2) = 2$$

Therefore, $f\left(\frac{1}{4}\right) + 2f\left(\frac{1}{2}\right) + f\left(\frac{3}{4}\right) = 2$.

17. If the function $f : \{-1, 1\} \rightarrow \{0, 2\}$, defined by $f(x) = ax + b$ is a surjection, then find a and b .

Sol. Domain of f is $\{-1, 1\}$ and

$$f(x) = ax + b$$

$$f(-1) = -a + b$$

$$f(1) = a + b$$

Case I : Suppose $f = \{(-1, 0), (1, 2)\}$... (1)

and $f = \{(-1, (-a + b)), (1, (a + b))\}$... (2)

Comparing (1) and (2)

$$-a + b = 0 \Rightarrow a = b$$

$$a + b = 2 \Rightarrow b + b = 2 \quad (\because a = b)$$

$$\Rightarrow 2b = 2 \Rightarrow b = 1; a = 1.$$

Case II : Suppose $f = \{(-1, 2), (1, 0)\}$... (3)

and $f = \{(-1, (-a + b)), (1, (a + b))\}$... (4)

Comparing (3) and (4) we get

$$-a + b = 2 \Rightarrow a = b - 2$$

$$a + b = 0 \Rightarrow b = -a$$

$$\text{Thus } -a - a = 2$$

$$\Rightarrow -2a = 2 \Rightarrow a = -1$$

$$\Rightarrow b = -(-1) = 1$$

$$\text{Thus } a = -1, b = 1.$$

18. If $f(x) = \cos(\log x)$, then show that $f\left(\frac{1}{x}\right)f\left(\frac{1}{y}\right) - \frac{1}{2}\left[f\left(\frac{x}{y}\right) + f(xy)\right] = 0$.

Sol. Given that $f(x) = \cos(\log x)$

Consider,

$$\begin{aligned} f\left(\frac{1}{x}\right)f\left(\frac{1}{y}\right) &= \cos\left(\log \frac{1}{x}\right)\cos\left(\log \frac{1}{y}\right) \\ &= \cos(\log x^{-1})\cos(\log y^{-1}) \\ &= [-\cos(\log x)][-\cos(\log y)] \\ &= \cos(\log x)\cos(\log y) \end{aligned}$$

$$\therefore f\left(\frac{1}{x}\right)f\left(\frac{1}{y}\right) = \cos(\log x)\cos(\log y) \quad \dots(1)$$

Again

$$\begin{aligned} \frac{1}{2} \left[f\left(\frac{x}{y}\right) + f(xy) \right] &= \frac{1}{2} \left[\cos\left(\log \frac{x}{y}\right) + \cos \log(xy) \right] \\ &= \frac{1}{2} [\cos(\log x - \log y) + \cos(\log x + \log y)] \\ &= \frac{1}{2} \cdot 2 \cos(\log x) \cos(\log y) \\ &= \cos(\log x) \cos(\log y) \quad [\because \cos(A-B) + \cos(A+B) = 2 \cos A \cos B] \\ \therefore \frac{1}{2} \left[f\left(\frac{x}{y}\right) + f(xy) \right] &= \cos(\log x) \cos(\log y) \quad \dots(2) \end{aligned}$$

(1) - (2) :

$$f\left(\frac{1}{x}\right) f\left(\frac{1}{y}\right) - \frac{1}{2} \left[f\left(\frac{x}{y}\right) + f(xy) \right] = 0.$$

19. If $f(y) = \frac{y}{\sqrt{1-y^2}}$ and $g(y) = \frac{y}{\sqrt{1+y^2}}$ then show that $(f \circ g)(y) = y$.

Sol. Given that

$$f(y) = \frac{y}{\sqrt{1-y^2}} \quad \text{and} \quad g(y) = \frac{y}{\sqrt{1+y^2}}$$

$$\therefore f \circ g(y) = f[g(y)] = f\left[\frac{y}{\sqrt{1+y^2}}\right]$$

$$= \frac{y}{\sqrt{1+y^2}} \bigg/ \sqrt{1 - \left(\frac{y}{\sqrt{1+y^2}}\right)^2}$$

$$= \frac{y}{\sqrt{1+y^2}} \times \frac{\sqrt{1+y^2}}{1+y^2 - y^2} = y$$

$$\therefore f \circ g(y) = y$$

20. If $f : \mathbf{R} \rightarrow \mathbf{R}$ and $g : \mathbf{R} \rightarrow \mathbf{R}$ are defined by $f(x) = 2x^2 + 3$ and $g(x) = 3x - 2$ then find

(i) $(f \circ g)(x)$ (ii) $g \circ f(x)$

(iii) $f \circ f(0)$ (iv) $g \circ f \circ f(3)$

Sol. i) $f \circ g(x) = f[g(x)]$

$$= f(3x - 2)$$

$$= 2(3x - 2)^2 + 3$$

$$= 2[9x^2 + 4 - 12x] + 3$$

$$= 18x^2 + 8 - 24x + 3$$

$$= 18x^2 - 24x + 11$$

$$\therefore (f \circ g)(x) = 18x^2 - 24x + 11$$

ii) $g \circ f(x) = g[f(x)]$

$$= g(2x^2 + 3)$$

$$= 3(2x^2 + 3) - 2$$

$$= 6x^2 + 9 - 2$$

$$= 6x^2 + 7$$

$$\therefore (\text{gof})(x) = 6x^2 + 7$$

$$\begin{aligned}\text{iii) } \text{fof}(0) &= f[f(0)] \\ &= f[2(0)^2 + 3] \\ &= f(3) = 2(3)^2 + 3 \\ &= 2 \times 9 + 3 = 18 + 3 = 21\end{aligned}$$

$$\therefore \text{fof}(0) = 21$$

$$\begin{aligned}\text{iv) } \text{go}(\text{fof})(3) &= \text{gof}[f(3)] \\ &= \text{gof}(21) \\ &= g[f(21)] \\ &= g[2(21)^2 + 3] \\ &= g[2(441) + 3] \\ &= g[882 + 3] \\ &= g(885) = 3(885) - 2 \\ &= 2655 - 2 = 2653\end{aligned}$$

$$\therefore \text{go}(\text{fof})(3) = 2653.$$

21. If $f : \mathbf{R} \rightarrow \mathbf{R}$, $g : \mathbf{R} \rightarrow \mathbf{R}$ are defined by

$f(x) = 3x - 1$, $g(x) = x^2 + 1$, then find

(i) $\text{fof}(x^2 + 1)$ (ii) $\text{fog}(2)$, (iii) $\text{gof}(2a - 3)$.

$$\begin{aligned}\text{Sol. i) } \text{fof}(x^2 + 1) &= f[f(x^2 + 1)] \\ &= f[3(x^2 + 1) - 1] \\ &= f[3x^2 + 3 - 1] \\ &= f[3x^2 + 2] \\ &= 3(3x^2 + 2) - 1 \\ &= 9x^2 + 6 - 1 \\ &= 9x^2 + 5\end{aligned}$$

$$\text{fof}(x^2 + 1) = 9x^2 + 5$$

$$\begin{aligned}\text{ii) } \text{fog}(2) &= f[g(2)] \\ &= f(2^2 + 1) \\ &= f(5) \\ &= 3(5) - 1 \\ &= 15 - 1 = 14\end{aligned}$$

$$\therefore \text{fog}(2) = 14$$

$$\begin{aligned}\text{iii) } (\text{gof})(2a - 3) &= g[f(2a - 3)] \\ &= g[3(2a - 3) - 1] \\ &= g[6a - 9 - 1] \\ &= g[6a - 10] \\ &= (6a - 10)^2 + 1 \\ &= 36a^2 + 100 - 120a + 1 \\ &= 36a^2 - 120a + 101\end{aligned}$$

$$(\text{gof})(2a - 3) = 36a^2 - 120a + 101$$

22. If $f(x) = \frac{x-1}{x+1}$, $x \neq \pm 1$, show that $\text{fof}^{-1}(x) = x$.

Sol. Given that $f(x) = \frac{x-1}{x+1}$

Let $y = f(x)$

$$\Rightarrow y = \frac{x-1}{x+1} \Rightarrow x = \frac{1+y}{1-y}$$

$$f^{-1}(y) = \frac{1+y}{1-y}$$

$$\therefore f^{-1}(x) = \frac{1+x}{1-x}$$

$$\therefore \text{fof}^{-1}(x) = f[f^{-1}(x)]$$

$$\begin{aligned} &= f\left[\frac{1+x}{1-x}\right] = \frac{\frac{1+x}{1-x} - 1}{\frac{1+x}{1-x} + 1} \\ &= \frac{1+x-1+x}{1+x+1-x} = \frac{2x}{2} = x \end{aligned}$$

$$\therefore \text{fof}^{-1}(x) = x$$

23. If $f : \mathbf{R} \rightarrow \mathbf{R}$, $g : \mathbf{R} \rightarrow \mathbf{R}$ defined by $f(x) = 3x - 2$, $g(x) = x^2 + 1$ then find (i) $\text{gof}^{-1}(2)$, (ii) $\text{gof}(x - 1)$.

Sol. i) Given that $f(x) = 3x - 2$

Let $y = f(x)$

$$y = 3x - 2$$

$$x = \frac{y+2}{3}$$

$$\therefore f^{-1}(x) = \frac{x+2}{3}$$

$$\therefore \text{gof}^{-1}(2) = g[f^{-1}(2)]$$

$$= g\left(\frac{2+2}{3}\right) = g\left(\frac{4}{3}\right)$$

$$= \left(\frac{4}{3}\right)^2 + 1 = \frac{16}{9} + 1 = \frac{25}{9}$$

ii) $\text{gof}(x - 1) = g[f(x - 1)]$

$$= g[3(x-1) - 2]$$

$$= g[3x - 3 - 2]$$

$$= g[3x - 5]$$

$$= (3x - 5)^2 + 1$$

$$= 9x^2 + 25 - 30x + 1$$

$$= 9x^2 - 30x + 26$$

24. Let $f = \{(1, a), (2, c), (4, d), (3, b)\}$ and $g^{-1} = \{(2, a), (4, b), (1, c), (3, d)\}$, then show that $(gof)^{-1} = f^{-1}og^{-1}$.

Sol. Given that,

$$f = \{(1, a), (2, c), (4, d), (3, b)\}$$

$$\Rightarrow f^{-1} = \{(a, 1), (c, 2), (d, 4), (b, 3)\}$$

$$g^{-1} = \{(2, a), (4, b), (1, c), (3, d)\}$$

$$\Rightarrow g = \{(a, 2), (b, 4), (c, 1), (d, 3)\}$$

$$\text{L.H.S. : } gof = \{(1, 2), (2, 1), (4, 3), (3, 4)\}$$

$$(gof)^{-1} = \{(2, 1), (1, 2), (3, 4), (4, 3)\}$$

R.H.S. :

$$f^{-1}og^{-1} = \{(2, 1), (4, 3), (1, 2), (3, 4)\}$$

$$\text{L.H.S.} = \text{R.H.S.}$$

25. Let $f : \mathbf{R} \rightarrow \mathbf{R}$, $g : \mathbf{R} \rightarrow \mathbf{R}$ are defined by $f(x) = 2x - 3$, $g(x) = x^3 + 5$ then find $(fog)^{-1}(x)$.

Sol. Given that,

$$f(x) = 2x - 3 \text{ and } g(x) = x^3 + 5$$

$$fog(x) = f[g(x)]$$

$$= f(x^3 + 5)$$

$$= 2(x^3 + 5) - 3$$

$$= 2x^3 + 10 - 3$$

$$= 2x^3 + 7$$

$$\therefore fog(x) = 2x^3 + 7$$

$$\text{Let } y = fog(x)$$

$$y = 2x^3 + 7$$

$$x^3 = \frac{y-7}{2}$$

$$x = \sqrt[3]{\frac{y-7}{2}}$$

$$\therefore (fog)^{-1}(x) = \sqrt[3]{\frac{x-7}{2}}$$

$$\therefore (fog)^{-1}(x) = \left(\frac{x-7}{2}\right)^{1/3}$$

26. If $f(x) = \frac{x+1}{x-1}$ ($x \neq \pm 1$) then find $(fofof)(x)$ and $(fofofof)(x)$.

Sol. Given that, $f(x) = \frac{x+1}{x-1}$

$$(fofof)(x) = (fof)[f(x)]$$

$$\begin{aligned}
&= \text{fof}\left(\frac{x+1}{x-1}\right) = f\left[f\left(\frac{x+1}{x-1}\right)\right] \\
&= f\left[\frac{\frac{x+1}{x-1}+1}{\frac{x+1}{x-1}-1}\right] = f\left[\frac{x+1+x-1}{x+1-x+1}\right] \\
&= f\left(\frac{2x}{2}\right) = f(x) = \frac{x+1}{x-1}
\end{aligned}$$

$$\therefore (\text{fofof})(x) = \frac{x+1}{x-1}$$

$$(\text{fofofof})(x) = f[(\text{fofof})(x)]$$

$$\begin{aligned}
&= f\left(\frac{1+x}{1-x}\right) \\
&= \frac{\frac{x+1}{x-1}+1}{\frac{x+1}{x-1}-1} = \frac{x+1+x-1}{x+1-x+1} = \frac{2x}{2} = x
\end{aligned}$$

$$\therefore (\text{fofofof})(x) = x$$

27. If f and g are real valued functions defined by $f(x) = 2x - 1$ and $g(x) = x^2$ then find

(i) $(3f - 2g)(x)$ (ii) $(fg)(x)$ (iii) $\left(\frac{\sqrt{f}}{g}\right)(x)$ (iv) $(f + g + 2)(x)$

Sol. Given that $f(x) = 2x - 1$, $g(x) = x^2$

i) $3f = 3(2x - 1) = 6x - 3$

$$g(x) = x^2 \Rightarrow 2g = 2x^2$$

$$\begin{aligned}
\therefore (3f - 2g)(x) &= 3f(x) - 2g(x) \\
&= 6x - 3 - 2x^2 \\
&= -2x^2 + 6x - 3 \\
&= -[2x^2 - 6x + 3]
\end{aligned}$$

ii) $(fg)(x) = f(x)g(x) = (2x - 1)x^2 = 2x^3 - x^2$

iii) $\left(\frac{\sqrt{f}}{g}\right)(x) = \frac{\sqrt{f(x)}}{g(x)} = \frac{\sqrt{2x-1}}{x^2}$

iv) $(f + g + 2)(x) = f(x) + g(x) + 2$

$$\begin{aligned}
&= 2x - 1 + x^2 + 2 \\
&= x^2 + 2x + 1 \\
&= x^2 + x + x + 1 \\
&= x(x+1) + 1(x+1) \\
&= (x+1)(x+1) = (x+1)^2
\end{aligned}$$

28. If $f = \{(1, 2), (2, -3), (3, -1)\}$ then find (i) $2f$, (ii) $2 + f$, (iii) f^2 , (iv) \sqrt{f} .

Sol. Given that

$$f = \{(1, 2), (2, -3), (3, -1)\}$$

$$\begin{aligned} \text{i) } 2f &= \{(1, 2 \times 2), (2, -3 \times 2), (3, -1 \times 2)\} \\ &= \{(1, 4), (2, -6), (3, -2)\} \end{aligned}$$

$$\begin{aligned} \text{ii) } 2 + f &= \{(1, 2+2), (2, -3+2), (3, -1+2)\} \\ &= \{(1, 4), (2, -1), (3, 1)\} \end{aligned}$$

$$\begin{aligned} \text{iii) } f^2 &= \{(1, 2^2), (2, (-3)^2), (3, (-1)^2)\} \\ &= \{(1, 4), (2, 9), (3, 1)\} \end{aligned}$$

$$\text{iv) } \sqrt{f} = \{(1, \sqrt{2})\}$$

29. Find the domains at the following real valued functions.

$$\text{i) } f(x) = \sqrt{x^2 - 3x + 2}$$

$$\text{ii) } f(x) = \log(x^2 - 4x + 3)$$

$$\text{iii) } f(x) = \frac{\sqrt{2+x} + \sqrt{2-x}}{x}$$

$$\text{iv) } f(x) = \frac{1}{\sqrt[3]{x-2} \log_{(4-x)} 10}$$

$$\text{v) } f(x) = \sqrt{\frac{4-x^2}{[x]+2}}$$

$$\text{vi) } f(x) = \sqrt{\log_{0.3}(x-x^2)}$$

$$\text{vii) } f(x) = \frac{1}{x+|x|}$$

Sol. i) $f(x) = \sqrt{x^2 - 3x + 2}$

$$x^2 - 3x + 2 \geq 0$$

$$\Rightarrow (x-1)(x-2) \geq 0$$

$$\Rightarrow x \leq 1 \text{ or } x \geq 2$$

Since the coefficient of x^2 is +ve

Domain of f is $(-\infty, 1] \cup [2, \infty)$

ii) $f(x) = \log(x^2 - 4x + 3)$

$$x^2 - 4x + 3 > 0$$

$$(x-1)(x-3) > 0$$

$$x < 1 \text{ or } x > 3$$

Since the coefficient of x^2 is +ve

Domain of f is $\mathbb{R} - [1, 3]$

$$\text{iii) } f(x) = \frac{\sqrt{2+x} + \sqrt{2-x}}{x}$$

$$\begin{array}{l} 2+x \geq 0 \quad | \quad 2-x \geq 0 \quad | \quad x \neq 0 \\ \Rightarrow x \geq -2 \quad | \quad \Rightarrow 2 \geq x \quad | \quad x \neq 0 \\ \quad \quad \quad \quad | \quad \Rightarrow x \leq 2 \quad | \end{array}$$

\therefore Domain of f is $[-2, 2] - \{0\}$

$$\text{iv) } f(x) = \frac{1}{\sqrt[3]{x-2} \log_{(4-x)} 10}$$

$$x-2 \neq 0 \Rightarrow x \neq 2$$

$$4-x > 0 \quad \& \quad 4-x \neq 1 \Rightarrow 4-x \neq 1 \Rightarrow x \neq 3$$

$$\Rightarrow 4 > x \Rightarrow x < 4$$

\therefore Domain of f is $(-\infty, 2) \cup (2, 3) \cup (3, 4)$

or

Domain of f is $(-\infty, 4) - \{2, 3\}$

$$\text{v) } f(x) = \sqrt{\frac{4-x^2}{[x]+2}}$$

Case I :

$$4 - x^2 \geq 0$$

$$(2+x)(2-x) \geq 0$$

$$\Rightarrow x \in [-2, 2] \quad \dots(1)$$

Since the coefficient of x^2 is -ve

Also

$$[x] + 2 > 0$$

$$[x] > -2$$

$$x \in [-1, \infty) \quad \dots(2)$$

From (1) and (2)

$$x \in [-1, 2]$$

Case II :

$$4 - x^2 \leq 0$$

$$x^2 - 4 \leq 0$$

$$(x+2)(x-2) \geq 0$$

$$x \in (-\infty, -2] \cup [2, \infty) \quad \dots(3)$$

Since the coefficient of x^2 is +ve

$$\text{Also } [x] + 2 < 0$$

$$[x] < -2$$

$$x \in (-\infty, -2) \quad \dots(4)$$

From (3) and (4)

$$x \in (-\infty, -2)$$

From case-I and case-II

Domain of f is $(-\infty, -2) \cup [-1, 2]$

$$\text{vi) } f(x) = \sqrt{\log_{0.3}(x - x^2)}$$

$$\log_{0.3}(x - x^2) \geq 0$$

$$\Rightarrow (x - x^2) \leq (0.3)^0$$

$$\Rightarrow x - x^2 \leq 1$$

$$\Rightarrow 0 \leq x^2 - x + 1$$

$$\Rightarrow x^2 - x + 1 \geq 0$$

$$\Rightarrow x^2 - x + 1 > 0, \forall x \in \mathbf{R} \quad \dots(1)$$

$$x - x^2 > 0$$

$$\Rightarrow x^2 - x < 0$$

$$\Rightarrow x(x - 1) < 0$$

$$\Rightarrow 0 < x < 1$$

Since the coefficient of x^2 is +ve

$$\therefore x \in (0, 1) \quad \dots(2)$$

From (1) and (2)

Domain of f is $\mathbf{R} \cap (0, 1) = (0, 1)$

(or) Domain of f is $(0, 1)$

$$\text{vii) } f(x) = \frac{1}{x + |x|}$$

$$x + |x| \neq 0$$

$$x \neq -|x|$$

It is not holds good when $x \in (-\infty, 0]$

$$\therefore \text{Domain of } f \text{ is } (0, \infty) = \mathbf{R}^+$$

30. Prove that the real valued function $f(x) = \frac{x}{e^x - 1} - \frac{x}{2} + 1$ is an even function on $\mathbf{R} - \{0\}$.

$$\text{Sol. } f(x) = \frac{x}{e^x - 1} - \frac{x}{2} + 1 \quad \dots(1)$$

Let $x \in \mathbf{R} - \{0\}$

Consider

$$f(x) = \frac{-x}{e^{-x} - 1} + \frac{x}{2} + 1$$

$$= \frac{-x}{\frac{1}{e^x} - 1} + \frac{x}{2} + 1$$

$$= \frac{-xe^x}{1 - e^x} + \frac{x}{2} + 1 = \frac{-xe^x}{-(e^x - 1)} + \frac{x}{2} + 1$$

$$= \frac{xe^x}{e^x - 1} + \frac{x}{2} + 1 \quad \dots(2)$$

Consider $f(x) - f(-x)$

$$\begin{aligned}
&= \frac{x}{e^x - 1} - \frac{x}{2} + 1 - \frac{xe^x}{e^x - 1} - \frac{x}{2} - 1 \\
&= \frac{x - xe^x}{e^x - 1} - \frac{2x}{2} \\
&= \frac{x(e^x - 1)}{(e^x - 1)} - x \\
&= x - x = 0
\end{aligned}$$

$$f(x) - f(-x) = 0$$

$$\Rightarrow f(-x) = f(x)$$

$\therefore f$ is an even function.

31. Find the domain and range of the following functions.

i) $f(x) = \frac{\tan \pi[x]}{1 + \sin \pi[x] + [x^2]}$

ii) $f(x) = \frac{x}{2 - 3x}$

iii) $f(x) = |x| + |1 + x|$

Sol. i) $f(x) = \frac{\tan \pi[x]}{1 + \sin \pi[x] + [x^2]}$

Domain of f is \mathbb{R} ($\because \tan n\pi = 0, \forall n \in \mathbb{Z}$)

Range of f is $\{0\}$

ii) $f(x) = \frac{x}{2 - 3x}$

$$2 - 3x \neq 0$$

$$2 \neq 3x$$

$$x \neq \frac{2}{3}$$

Domain of f is $\mathbb{R} - \left\{ \frac{2}{3} \right\}$

$$\frac{x}{2 - 3x} = y$$

$$\Rightarrow x = y(2 - 3x)$$

$$\Rightarrow x = 2y - 3yx$$

$$\Rightarrow x + 3yx = 2y$$

$$\Rightarrow x(1 + 3y) = 2y$$

$$\Rightarrow x = \frac{2y}{1 + 3y}$$

$$\Rightarrow 1 + 3y \neq 0$$

$$\Rightarrow 3y \neq -1$$

$$y \neq -\frac{1}{3}$$

$$\therefore \text{Range of } f \text{ is } \mathbb{R} - \left\{ -\frac{1}{3} \right\}.$$

$$\text{iii) } f(x) = |x| + |1+x|$$

Domain of f is \mathbb{R}

Range of f is $[1, \infty)$

32. Determine whether the function $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = \begin{cases} x & \text{if } x > 2 \\ 5x - 2 & \text{if } x \leq 2 \end{cases}$ is an injection or a surjection or a bijection.

Sol. Since $3 > 2$, we have $f(3) = 3$

Since $1 < 2$, we have $f(1) = 5(1) - 2 = 3$

\therefore 1 and 3 have same f image.

Hence f is not an injection.

Let $y \in \mathbb{R}$ then $y > 2$ (or) $y \leq 2$

If $y > 2$ take $x = y \in \mathbb{R}$ so that $f(x) = x = y$

If $y \leq 2$ take

$$x = \frac{y+2}{5} \in \mathbb{R} \text{ and } x = \frac{y+2}{5} < 1$$

$$\therefore f(x) = 5x - 2 = 5\left(\frac{y+2}{5}\right) - 2 = y$$

\therefore f is a surjection

Since f is not an injection, it is not a bijection.

33. If $f : \mathbb{R} \rightarrow \mathbb{R}$, $g : \mathbb{R} \rightarrow \mathbb{R}$ are defined by $f(x) = 4x - 1$ and $g(x) = x^2 + 2$ then find

$$\text{(i) } (g \circ f)(x) \quad \text{(ii) } (g \circ f)\left(\frac{a+1}{4}\right) \quad \text{(iii) } f \circ f(x) \quad \text{(iv) } g \circ (f \circ f)(0)$$

Sol. i) $(g \circ f)(x) = g(f(x))$

$$= g(4x - 1)$$

$$= (4x - 1)^2 + 2$$

$$= 16x^2 + 1 - 8x + 2$$

$$= 16x^2 - 8x + 3$$

$$\text{ii) } (g \circ f)\left(\frac{a+1}{4}\right) = g\left[f\left(\frac{a+1}{4}\right)\right]$$

$$= g\left[4\left(\frac{a+1}{4}\right) - 1\right]$$

$$= g(a) = a^2 + 2$$

$$\text{iii) } f \circ f(x) = f[f(x)]$$

$$= f(4x - 1) = 4[4x - 1] - 1$$

$$= 16x - 4 - 1$$

$$= 16x - 5$$

$$\begin{aligned}
\text{iv) } g \circ (f \circ f)(0) &= g \circ (f \circ f) \\
&= g[16 \times 0 - 5] \\
&= g[-5] \\
&= (-5)^2 + 2 \\
&= 25 + 2 = 27
\end{aligned}$$

34. If $f : \mathbb{Q} \rightarrow \mathbb{Q}$ is defined by $f(x) = 5x + 4$ for all $x \in \mathbb{Q}$, show that f is a bijection and find f^{-1} .

Sol. Let $x_1, x_2 \in \mathbb{Q}$, $f(x_1) = f(x_2)$

$$\Rightarrow 5x_1 + 4 = 5x_2 + 4$$

$$\Rightarrow 5x_1 = 5x_2$$

$$\Rightarrow x_1 = x_2$$

$\therefore f$ is an injection.

Let $y \in \mathbb{Q}$, then $x = \frac{y-4}{5} \in \mathbb{Q}$ and

$$f(x) = f\left(\frac{y-4}{5}\right) = 5\left(\frac{y-4}{5}\right) + 4 = y$$

$\therefore f$ is a surjection, f is a bijection.

$\therefore f^{-1} : \mathbb{Q} \rightarrow \mathbb{Q}$ is a bijection.

We have $f \circ f^{-1}(x) = 1(x)$

$$f[f^{-1}(x)] = x$$

$$5f^{-1}(x) + 4 = x$$

$$f^{-1}(x) = \frac{x-4}{5} \text{ for all } x \in \mathbb{Q}$$

35. Find the domains of the following real valued functions.

i) $f(x) = \frac{1}{\sqrt{|x| - x}}$ ii) $f(x) = \sqrt{|x| - x}$

Sol. i) $f(x) = \frac{1}{\sqrt{|x| - x}} \in \mathbb{R}$

$$\Rightarrow |x| - x > 0$$

$$\Rightarrow |x| > x$$

$$\Rightarrow x \in (-\infty, 0)$$

\therefore Domain of f is $(-\infty, 0)$

ii) $f(x) = \sqrt{|x| - x}$

$$\Rightarrow |x| - x \geq 0, \text{ which is true } \forall x \in \mathbb{R}$$

\therefore Domain of f is \mathbb{R} .

36. If $f = \{(4, 5), (5, 6), (6, -4)\}$ and $g = \{(4, -4), (6, 5), (8, 5)\}$ then find

(i) $f + g$ (ii) $f - g$ (iii) $2f + 4g$ (iv) $f + 4$ (v) fg (vi) $\frac{f}{g}$

(vii) $|f|$ (viii) \sqrt{f} (ix) f^2 (x) f^3 .

Sol. Domain of $f = A = \{4, 5, 6\}$

Domain of $g = B = \{4, 6, 8\}$

Domain of $f \pm g = A \cap B = \{4, 6\}$

i) $f + g = \{(4, 5 - 4), (6, -4 + 5)\}$
 $= \{(4, 1), (6, 1)\}$

(ii) $f - g = \{(4, 5 + 4), (6, -4 - 5)\}$
 $= \{(4, 9), (6, -9)\}$

(iii) $2f = \{(4, 2 \times 5), (5, 6 \times 2), (6, -4 \times 2)\}$
 $= \{(4, 10), (5, 12), (6, -8)\}$

$4g = \{(4, -4 \times 4), (6, 5 \times 4), (8, 5 \times 4)\}$
 $= \{(4, -16), (6, 20), (8, 20)\}$

Domain of $2f + 4g = \{4, 6\}$

$\therefore 2f + 4g = \{(4, 10, -16), (6, -8 + 20)\}$
 $= \{(4, -6), (6, 12)\}$

(iv) $f + 4 = \{(4, 5 + 4), (5, 6 + 4), (6, -4 + 4)\}$
 $= \{(4, 9), (5, 10), (6, 0)\}$

(v) $fg = \{(4, (5 \times -4)), (6, -4 \times 5)\}$
 $= \{(4, -20), (6, -20)\}$

(vi) $\frac{f}{g} = \left\{ \left(4, \frac{-5}{4} \right), \left(6, \frac{-4}{5} \right) \right\}$

(vii) $|f| = \{(4, |5|), (5, |6|), (6, |-4|)\}$
 $= \{(4, 5), (5, 6), (6, 4)\}$

(viii) $\sqrt{f} = \{(4, \sqrt{5}), (5, \sqrt{6})\}$

(ix) $f^2 = \{(4, 25), (5, 36), (6, 16)\}$

(x) $f^3 = \{(4, 125), (5, 216), (6, -64)\}$

37. Find the domain of the following real valued functions.

i) $f(x) = \frac{1}{\sqrt{[x]^2 - [x] - 2}}$ ii) $f(x) = \log(x - |x|)$

iii) $f(x) = \sqrt{\log_{10}\left(\frac{3-x}{x}\right)}$ iv) $f(x) = \sqrt{x+2} + \frac{1}{\log(1-x)}$

v) $f(x) = \frac{\sqrt{3+x} + \sqrt{3-x}}{x}$

Sol. i) $f(x) = \frac{1}{\sqrt{[x]^2 - [x] - 2}} \in \mathbb{R}$

$$\Leftrightarrow [x]^2 - [x] - 2 > 0$$

$$\Rightarrow ([x]+1)([x]-2) > 0$$

$$\Rightarrow [x] < -1 \text{ (or) } [x] > 2$$

But $[x] < -1$

$$\Rightarrow [x] = -2, -3, -4, \dots$$

$$\Rightarrow x < -1$$

$$[x] > 2 \Rightarrow [x] = 3, 4, \dots$$

$$\Rightarrow x \geq 3$$

\therefore Domain of $f = (-\infty, -1) \cup [3, \infty) = \mathbb{R} - [-1, 3)$

ii) $f(x) = \log(x - [x]) \in \mathbb{R}$

$$\Leftrightarrow x - [x] > 0 \Leftrightarrow x > [x]$$

$\Rightarrow x$ is a non-integer \therefore Domain of f is $\mathbb{R} - \mathbb{Z}$.

iii) $f(x) = \sqrt{\log_{10}\left(\frac{3-x}{x}\right)} \in \mathbb{R}$

$$\log_{10}\left(\frac{3-x}{x}\right) \geq 0 \text{ and } \frac{3-x}{x} > 0$$

$$\Rightarrow \frac{3-x}{x} \geq 10^0 = 1 \text{ and } 3-x > 0, x > 0$$

$$\Rightarrow 3-x \geq x \text{ and } 0 < x < 3$$

$$\Rightarrow x \leq \frac{3}{2} \text{ and } 0 < x < 3$$

$$\Rightarrow x \in \left(-\infty, \frac{3}{2}\right] \cap (0, 3) = \left(0, \frac{3}{2}\right]$$

\therefore Domain of f is $\left(0, \frac{3}{2}\right]$.

iv) $f(x) = \sqrt{x+2} + \frac{1}{\log(1-x)} \in \mathbb{R}$

$$x+2 \geq 0 \text{ and } 1-x > 0 \text{ and } 1-x \neq 1$$

$$\Rightarrow x \geq -2 \text{ and } 1 > x \text{ and } x \neq 0$$

$$\Rightarrow x \in [-2, \infty) \cap (-\infty, 1) - \{0\}$$

$$\Rightarrow x \in [-2, 1) - \{0\}$$

\therefore Domain of f is $[-2, 1) - \{0\}$.

v) $f(x) = \frac{\sqrt{3+x} + \sqrt{3-x}}{x} \in \mathbb{R}$

$$\Leftrightarrow 3+x \geq 0, 3-x \geq 0, x \neq 0$$

$$\Rightarrow -3 \leq x \leq 3, x \neq 0$$

$$\Rightarrow x \in [-3, 3] - \{0\}$$

\therefore Domain of f is $[-3, 3] - \{0\}$.