

## 7. HYPERBOLIC FUNCTIONS

### Some formulae

1.  $\sin hx = \frac{e^x - e^{-x}}{2}$
2.  $\cos hx = \frac{e^x + e^{-x}}{2}$
3.  $\tan hx = \frac{e^x - e^{-x}}{e^x + e^{-x}}$
4.  $\operatorname{cosec} hx = \frac{2}{e^x - e^{-x}}$
5.  $\sec hx = \frac{2}{e^x + e^{-x}}$
6.  $\cot hx = \frac{e^x + e^{-x}}{e^x - e^{-x}}$
7.  $\cos h^2 x - \sin h^2 x = 1$
8.  $1 - \tan h^2 x = \operatorname{sech}^2 x$
9.  $\cot h^2 x - 1 = \operatorname{cosech}^2 x$
10. Prove that  $\sin h^{-1} x = \log \left\{ x + \sqrt{x^2 + 1} \right\}$

Let  $\sin h^{-1} x = y \Rightarrow x = \sinh y$

$$x = \frac{e^y - e^{-y}}{2} = 2x = e^y - \frac{1}{e^y} \Rightarrow 2xe^y = (e^y)^2 - 1$$

$$(e^y)^2 - 2xe^y - 1 = 0 \Rightarrow e^y = \frac{2x \pm \sqrt{4x^2 + 4}}{2} = e^y = x + \sqrt{x^2 + 1} \Rightarrow y = \log_e \left( x + \sqrt{x^2 + 1} \right)$$

$$\boxed{\therefore \sin h^{-1} x = \log_e \left\{ x + \sqrt{x^2 + 1} \right\}}$$

2. **Prove that**  $\cosh^{-1} x = \log_e \left( x + \sqrt{x^2 - 1} \right)$

### Solution:

Let  $\cosh^{-1} x = y \Rightarrow x = \cosh y$

$$x = \frac{e^y + e^{-y}}{2} \Rightarrow 2x = e^y + \frac{1}{e^y}$$

$$2xe^y = (e^y)^2 + 1 \Rightarrow (e^y)^2 - 2xe^y + 1 = 0$$

$$e^y = \frac{2x \pm \sqrt{4x^2 - 4}}{2} \Rightarrow e^y = x \pm \sqrt{x^2 - 1}$$

$$e^y = x + \sqrt{x^2 - 1} \Rightarrow y = \log_e \left( x + \sqrt{x^2 - 1} \right)$$

$$\boxed{\cosh^{-1} x = \log_e \left( x + \sqrt{x^2 - 1} \right)}$$

3. **Prove that**  $\text{Tanh}^{-1}x = \frac{1}{2} \log_e \left( \frac{4x}{1-x} \right)$

Let  $\text{Tanh}^{-1}x = y \Rightarrow x = \tanh y$

$$x = \frac{e^y - e^{-y}}{e^y + e^{-y}} \Rightarrow \frac{1}{x} = \frac{e^y + e^{-y}}{e^y - e^{-y}}$$

Using component and dividend

$$\frac{1+x}{1-x} = \frac{e^y + e^{-y} + e^y - e^{-y}}{e^y + e^{-y} - e^y + e^{-y}}$$

$$\frac{1+x}{1-x} = \frac{2e^y}{2e^{-y}} \Rightarrow \frac{1+x}{1-x} = e^{2y}$$

$$2y = \left\{ \frac{1+x}{1-x} \right\} \Rightarrow y = \frac{1}{2} \log \left( \frac{1+x}{1-x} \right)$$

$$\boxed{\text{cosh}^{-1}x = \frac{1}{2} \log \left( \frac{x+1}{x-1} \right)}$$

$$\rightarrow \text{sech}^{-1}x = \log \left\{ \frac{1+\sqrt{1-x^2}}{x} \right\}$$

$$\rightarrow \text{cosech}^{-1}x = \log \left( \frac{1-\sqrt{1+x^2}}{x} \right) \quad x < 0$$

$$= \log \left\{ \frac{1-\sqrt{1+x^2}}{x} \right\} \quad x > 0$$

$$\rightarrow \sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$$

$$\rightarrow \sinh(x-y) = \sinh x \cosh y - \cosh x \sinh y$$

$$\rightarrow \cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$$

$$\rightarrow \cosh(x-y) = \cosh x \cosh y - \sinh x \sinh y$$

$$\rightarrow \sinh 2x = 2 \sinh x \cosh x = \frac{2 \tanh x}{1 - \tanh^2 x}$$

$$\rightarrow \cosh 2x = \cosh^2 x + \sinh^2 x = 2 \cosh^2 x - 1 = 1 + 2 \sinh^2 x = \frac{1 + \tanh^2 x}{1 - \tanh^2 x}$$

$$\rightarrow \tanh(x+y) = \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$$

$$\rightarrow \tanh(x-y) = \frac{\tanh x - \tanh y}{1 - \tanh x \tanh y}$$

$$\rightarrow \coth(x+y) = \frac{\coth x \coth y + 1}{\coth y + \coth x}$$

$$\rightarrow \cot h(x-y) = \frac{\cot hx \cot hy - 1}{\cot hy - \cot hx}$$

$$\tan h2x = \frac{2 \tan hx}{1 + \tanh^2 x}$$

$$\rightarrow \cot h2x = \frac{\coth^2 x + 1}{2 \cot hx}$$

### SOME PROBLEMS:

1. If  $\sin hx = \frac{3}{4}$  find  $\cos h2x$  and  $\sin h2x$

**Solution:**

$$\sin hx = \frac{3}{4}$$

$$\cos h^2 x = 1 + \sin h^2 x$$

$$\cos h^2 x = 1 + \frac{9}{16} \Rightarrow \cos h^2 x = \frac{25}{16}$$

$$\cos hx = \frac{5}{4}$$

$$\sin^2 x = 2 \sin hx \cos hx = 2 \times \frac{3}{4} \times \frac{5}{4} = \frac{15}{8}$$

$$\cos^2 x = \cos h^2 x + \sin h^2 x = \frac{25}{16} + \frac{9}{16} = \frac{34}{16} = \frac{17}{8}$$

2. If  $\sin hx = 3$   $x = \log(3 - 1\sqrt{10})$

**Solution:**

$$\sin hx = 3 \Rightarrow x = \sin h^{-1} 3$$

$$\sin h^{-1} y = \cos(y + \sqrt{y^2 + 1})$$

$$\sin h^{-1} 3 = \log\{3 + \sqrt{9+1}\}$$

$$= \log\{3 + \sqrt{10}\}$$

3. Prove that (i)  $\tan h(x-y) = \frac{\tan hx - \tan hy}{1 - \tan hx \tan hy}$

$$\text{R.H.S} = \frac{\tan hx - \tan hy}{1 - \tan hx \tan hy} = \frac{\frac{\sin hx}{\cos hx} - \frac{\sin hy}{\cos hy}}{1 - \frac{\sin hx}{\cos hx} \cdot \frac{\sin hy}{\cos hy}}$$

$$\frac{\sin hx \cos hy - \cos hx \sin hy}{\cos hx \cos hy - \sin hx \sin hy} = \frac{\sin h(x-y)}{\cos h(x-y)} = \tan h(x-y)$$

(ii)  $\cot h(x-y) = \frac{\cot hx \cot hy - 1}{\cot hy - \cot hx}$

$$\text{RHS} \quad \frac{\cot hx \cot hy - 1}{\cot hy - \cot hx} = \frac{\frac{\cos hx}{\sin hx} \cdot \frac{\cos hy}{\sin hy} - 1}{\frac{\cos hy}{\sin hy} - \frac{\cos hx}{\sin hx}}$$

$$\frac{\frac{\cos hx \cos hy - \sin hx \sin hy}{\sin hx \sin hy}}{\frac{\sin hx \cos hy - \cos hx \sin hy}{\sin hx \sin hy}} = \frac{\cos h(x-y)}{\sin h(x-y)} = \cot h(x-y)$$

4. Prove that  $(\cos hx - \sin hx)^n = \cos hnx - \sin hnx$

**Solution:**

$$(\cos hx - \sin hx)^n = \left\{ \frac{e^x + e^{-x}}{2} - \frac{e^x - e^{-x}}{2} \right\}^n$$

$$= \left\{ \frac{e^x + e^{-x} - e^x + e^{-x}}{2} \right\} = (e^{-x}) = e^{-nx}$$

$$\text{RHS} = \cosh nx - \sinh nx = \frac{e^{nx} + e^{-nx}}{2} - \left( \frac{e^{nx} - e^{-nx}}{2} \right)$$

$$= \frac{e^{nx} + e^{-nx} - e^{nx} + e^{-nx}}{2} = e^{-nx}$$

(ii)  $(\cos hx + \sin hx)^n = \cos hnx + \sin hnx$

$$\text{LHS} = (\cos hx + \sin hx)^n = \left( \frac{e^x + e^{-x} + e^x - e^{-x}}{2} \right)^n = (e^x)^n = e^{nx}$$

$$\text{RHS} \quad \cos hnx + \sin hnx = \frac{e^{nx} + e^{-nx} + e^{nx} - e^{-nx}}{2} = e^{nx}$$

$$\text{LHS} = \text{RHS}$$

5. **Prove that**  $\frac{\tan hx}{\sec hx - 1} + \frac{\tan hx}{\sec hx + 1} = -2 \operatorname{cosec} hx$

$$\frac{\tan hx}{\sec hx - 1} + \frac{\tan hx}{\sec hx + 1} = \frac{\tan hx(\sec hx + 1) + \tan hx(\sec hx - 1)}{(\sec hx - 1)(\sec hx + 1)}$$

$$\frac{\tan hx \{ \sec hx + 1 + \sec hx - 1 \}}{\sec^2 hx - 1}$$

$$\frac{2 \tan hx \cdot \sec hx}{-\tanh^2 x} = \frac{\frac{-2}{\cos hx}}{\frac{\sin hx}{\cos hx}} = -\operatorname{cosec} hx$$

6. **Prove that**  $\frac{\cos hx}{1 - \tan hx} + \frac{\sin hx}{1 - \cot hx} = \sin hx + \cos hx$

**Solution:**

$$\frac{\cos hx}{1 - \tan hx} + \frac{\sin hx}{1 - \cot hx}$$

$$\frac{\cos hx}{1 - \frac{\sin hx}{\cos hx}} + \frac{\sin hx}{1 - \frac{\cos hx}{\sin hx}} = \frac{\cosh^2 x}{\cos hx - \sin hx} + \frac{\sin h^2 x}{\sin hx - \cos hx}$$

$$= \frac{\cosh^2 x}{\cos hx - \sin hx} - \frac{\sin h^2 x}{\cos hx - \sin hx}$$

$$\frac{\cosh^2 x - \sin h^2 x}{\cos hx - \sin hx} = \frac{(\cos hx + \sin hx)(\cos hx - \sin hx)}{(\cos hx - \sin hx)} = \cos hx + \sin hx$$

7. **Prove that**  $\cos^4 x - \sin^4 x = \cos^2 x$

**Solution:**

$$\cos^4 x - \sin^4 x = (\cos^2 x)^2 - (\sin^2 x)^2$$

$$\cos^4 x - \sin^4 x = (\cos^2 x)^2 - (\sin^2 x)^2$$

$$= (\cos^2 x + \sin^2 x)(\cos^2 x - \sin^2 x)$$

$$= (\cosh^2 x)(1) = \cosh 2x$$

8. If  $\mu = \log_e \left\{ \tan \left( \frac{\pi}{4} + \frac{\pi}{2} \right) \right\}$  the prove that  $\cosh \mu = \sec \theta$

**Solution:**

$$\mu = \log_e \left\{ \tan \left( \frac{\pi}{4} + \frac{\pi}{2} \right) \right\} \Rightarrow e^\mu = \tan \left( \frac{\pi}{4} + \frac{\theta}{2} \right)$$

$$e^{-\mu} = \frac{1}{e^\mu} = \cot \left( \frac{\pi}{4} + \frac{\pi}{2} \right)$$

$$\cosh \mu = \frac{e^\mu + e^{-\mu}}{2} = \frac{1}{2} \left( \tan \left( \frac{\pi}{4} + \frac{\theta}{2} \right) + \cot \left( \frac{\pi}{4} + \frac{\theta}{2} \right) \right)$$

$$= \frac{1}{2} \left( \frac{\sin \left( \frac{\pi}{4} + \frac{\theta}{2} \right)}{\cos \left( \frac{\pi}{4} + \frac{\theta}{2} \right)} + \frac{\cos \left( \frac{\pi}{4} + \frac{\theta}{2} \right)}{\sin \left( \frac{\pi}{4} + \frac{\theta}{2} \right)} \right)$$

$$= \frac{1}{2} \left( \frac{\sin^2 \left( \frac{\pi}{4} + \frac{\theta}{2} \right) \cos^2 \left( \frac{\pi}{4} + \frac{\theta}{2} \right)}{\cos \left( \frac{\pi}{4} + \frac{\theta}{2} \right) \sin \left( \frac{\pi}{4} + \frac{\theta}{2} \right)} \right)$$

$$= \frac{1}{2 \sin \left( \frac{\pi}{4} + \frac{\theta}{2} \right) \cos \left( \frac{\pi}{4} + \frac{\theta}{2} \right)}$$

$$= \frac{1}{\sin \left\{ 2 \left( \frac{\pi}{4} + \frac{\theta}{2} \right) \right\}} = \frac{1}{\sin \left( \frac{\pi}{2} + \theta \right)} = \frac{1}{\cos \theta} = \sec \theta$$

$$\therefore \cosh \mu = \sec \theta$$

## PROBLEMS FOR PRACTICE

1. **If**  $\sin hx = 5$  **show that**  $x = \log(5 + \sqrt{20})$
2. **Show that**  $\tan h^{-1}\left(\frac{1}{2}\right) = \frac{1}{2} \log_e 3$
3. **If**  $x = \log\left\{\cot\left(\frac{\pi}{4} + \theta\right)\right\}$  **then prove that**  $\cos hx = \sec 2\theta$  **and**  $\sin hx = -\tan 2\theta$
4. **If**  $\cos hx = \frac{5}{2}$  **find the values of**  $\cos h2x$  **and**  $\sin h2x$
5. **Prove that**  $\sin h3x = 3\sin hx + 4\sinh^3 x$
6. **Prove that**  $\tan h3x = \frac{3 \tan hx + \tanh^3 x}{1 + 3 \tan h^2 x}$