2.MATHEMATICAL INDUCTION

<u>Principle of finite Mathematical Induction</u>:

Let {P(n) / n $\in$ N} be a set of statements. If (i) p(1) is true

(ii) p (m) is true $\Rightarrow$ p (m+1) is true ; then p (n) is true for every n $\in$ N.

Principle of complete induction:

Let {P (n) / n N} be a set of statements. If p (1) is true and p(2), p(3) .... p (m-1) are true => p(m) is true, then p (n) is true for every n $\in$ N.

Note:

(i) The principle of mathematical induction is a method of proof of a statement.

(ii) We often use the finite mathematical induction, hence or otherwise specified the mathematical induction is the finite mathematical induction.

Some important formulae:

1. $\sum n = \frac{n(n+1)}{2}$

2. $\sum n^2 = \frac{n(n+1)(2n+1)}{6}$;

3. $\sum n^3 = \frac{n^2(n+1)^2}{4}$

4. $a, (a+d), (a+2d),$.............. are in a.p

   n th term $t_n = a+(n-1)d$, sum of n terms $S_n = \frac{n}{2}\left[2a+(n-1)d\right] = \frac{n}{2}[a+l]$,

   a = first term, l= last term.

5. $a_, ar, ar^2,$......... is a g.p.

   Nth term $t_n = a.r^{n-1}$. a = 1st term, r = common ratio.

   Sum of n terms $s_n = a\frac{(r^n - 1)}{r-1}$; $r > 1$, $= a\left(\frac{1-r^n}{1-r}\right)$; $r < 1$

Using mathematical induction prove the following

1. $1^2+2^2+3^2+.....+n^2=\dfrac{n(n+1)(2n+1)}{6}$

Sol: Let S_n be the given statement

For n = 1 L.H.S = 1

$R.H.S=\dfrac{1(1+1)(2+1)}{6}=1$

L.H.S = R.H.S

$\therefore S_{(1)}$ is true

Assume S_k is true

$1^1+2^2+3^2+....+k^2=\dfrac{k(k+1)(2k+1)}{6}$

Adding $(k+1)^{th}$ term on both sides i.e. $(k+1)^2$ on both sides

$$1^2+2^2+3^2+....+k^2+(k+1)^2=\frac{k(k+1)(2k+1)}{6}+(k+1)^2$$

$$=(k+1)\left\{\frac{k(2k+1)+6(k+1)}{6}\right\}$$

$$=\frac{(k+1)\{2k^2+7k+6\}}{6}$$

$$=\frac{(k+1)(k+2)(k+3)}{6}$$

$$=\frac{(k+1)\{(k+1)+1\}\{2(k+1)+1\}}{6}$$

$\therefore S_{k+1}$ is true

$\therefore S_n$ is true

2. **Prove that 2.3+ 3.4 + 4.5+ …. Up to n terms** $\dfrac{n(n^2+6n+11)}{3}$

Sol: 2, 3, 4………………… n terms $t_n=2+(n-1)1=n+1$

3, 4, 5……….. n terms $t_n=3+(n-1)1=n+2$

$$2.3+3.4+4.5+.........+(n+1)(n+2)=\frac{n(n^2+6n+11)}{3}$$

Let S_n be the given statement

For n = 1 L.H.S = 2.3 = 6

$R.H.S=\dfrac{1(1+6+11)}{3}=6$

L. H.S = R.H.S

$\therefore S_{(1)}$ is true

Assume S_k is true

$$\therefore 2.3+3.4+4.5+.......+(k+1)(k+2)=\frac{k(k^2+6k+11)}{3}+(k+2)(k+3)$$

$$=\frac{k(k^2+6k+11)+3(k^2+5k+6)}{3}$$

$$= \frac{k^3+9k^2+26k+18}{3}$$

$$= \frac{(k+1)\left\{k^2+8k+18\right\}}{3}$$

$k=-1$	1	9	26	18
	0	−1	−8	−18
	1	8	18	0

$$= \frac{(k+1)\left\{(k+1)^2+6(k+1)+11\right\}}{3}$$

$\therefore S_{k+1}$ is true

Hence $S_{(n)}$ is true for all $n \in N$

3. $\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+.....+\frac{1}{(2n-1)(2n+1)}=\frac{n}{2n+1}$

Sol: Let $S_{(n)}$ be the given statement

For n = 1 L.H.S $\frac{1}{1.3}=\frac{1}{3}$

R.H.S = $\frac{1}{2+1}=\frac{1}{3}$

Assume S_k is true

$$\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+.....+\frac{1}{(2k-1)(2k+1)}=\frac{k}{2k+1}$$

Adding $(k+1)^{th}$ term i.e. $\frac{1}{(2k+1)(2k+3)}$ on both sides

$$\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+......+\frac{1}{(2k-1)(2k+1)}+\frac{1}{(2k+1)(2k+3)}=\frac{k}{2k+1}+\frac{1}{(2k+1)(2k+3)}$$

$$= \frac{k(2k+3)+1}{(2k+1)(2k+3)}$$

$$= \frac{2k^2+3k+1}{(2k+1)(2k+3)}$$

$$= \frac{k+1}{2k+3}$$

$\therefore S_{k+1}$ is true

Hence $S_{(n)}$ is true for all $n \in N$

4. $4^3+8^3+12^3+....$ **up to n terms =** $16n^2\,(n+1)^2$

Sol: Let $S_{(n)}$ be the given statements

For n = 1 $L.H.S = 4^3 = 64$

R.H.S $=16(1)^2\,(1+1)^2=64$

$\therefore$ L.H.S = R.H.S

Hence $S_{(1)}$ is true

Assume $S_{(k)}$ is true

$\therefore 4^3+8^3+12^3+....+(4k)^3=16k^2\,(k+1)^2$

Adding $\left\{4(k+1)\right\}^3$ on both sides

$$4^3+8^3+12^3+...+(4k)^3+\{4(k+1)\}^3=16k^2(k+1)^2+64(k+1)^3$$
$$4^3+8^3+12^3+.....+\{4(k+1)\}^3=16(k+1)^2\{k^2+4k+4\}$$
$$=16(k+1)^2(k+2)^2$$
$\therefore S_{k+1}$ is true

Hence $S_{(n)}$ is true for all $n\in N$

5. **a + (a +d) +(a +2d) + ….up to n terms** $a=\frac{n}{2}[2a+(n-1)d]$

Sol: $a+(a+d)+(a+2d)+....+[a+(n-1)d]=\frac{n}{2}[2a+(n-1)d]$

Let $S_{(n)}$ be the given statement

For n = 1

L.H.S. = a

R.H.S $=\frac{1}{2}[2a+(1-1)d]=a$

L.H.S. = R.H.S

Assume S_k is true

$$a+(a+d)+(a+2d)+.....+a+(k-1)d+(a+kd)=\frac{k}{2}[2a+(k-1)d]+(a+kd)$$
$$a+(a+d)+....+a+kd=ak+\frac{k}{2}(k-1)d+a+kd$$
$$=a(k+1)+d\left\{\frac{k}{2}(k-1)+k\right\}$$
$$=a(k+1)+kd\left\{\frac{k-1+2}{2}\right\}$$
$$=\left(\frac{k+1}{2}\right)\{2a+(kd)\}$$
$\therefore S_{k+1}$ is true

Hence S_n is true for all $n\in N$

6. $a+ar+ar^2+..........$ **upto n terms** $=\frac{a(r^n-1)}{r-1}(r\neq 1)$

Sol: $a+ar+ar^2+.......ar^{n-1}=\frac{a(r^n-1)}{r-1}$

Let $S_{(n)}$ be the given statement

For n = 1

L.H.S = a

R.H.S = $\frac{a(r-1)}{(r-1)}=a$

$\therefore$ L.H.S = R.H.S

$\therefore S_{(1)}$ is true

Assume S_k is true

$$\therefore a+ar+ar^2+.......ar^{k-1}=\frac{a(r^k-1)}{r-1}$$

Adding $a.r^k$ on both sides

$$a+ar+ar^2+...........+ar^{k-1}+ar^k=\frac{a(r^k-1)}{r-1}+ar^k$$

$$a+ar+ar^2+...........+ar^k=\frac{a\left\{r^k-1+r^{k+1}-r^k\right\}}{r-1}$$

$$=\frac{a(r^{k+1}-1)}{r-1}$$

Hence S_{k+1} is true

$\therefore S_n$ is true $\forall n\in N$

7. $2+7+12+........+(5n-3)=\dfrac{n(Sn-1)}{2}$

Sol: Let $S_{(n)}$ be the given statement

For n = 1

L.H.S =2

R.H.S = $\dfrac{1(5-1)}{2}=2$

$\therefore$ L.H.S = R.H.S for n = 1

Hence $S_{(1)}$ is true

Assume $S_{(k)}$ is true

$$2+7+12+......+(5k-3)=\frac{k(5k-1)}{2}$$

$$=\frac{5k^2+9k+4}{2}=\frac{(k+1)\left\{5(k+1)-1\right\}}{2}$$

$\Rightarrow S_{k+1}$ is true

Hence $S_{(n)}$ is true $\forall n\in N$

8. 4^n-3n-1 **is divisible by 9**

Sol: Let $S_{(n)}$ be the given statement i.e. $S_{(n)}=4^n-3n-1$

For $n=1\Rightarrow S_{(1)}=4-3-1=0$ is divisible by 9

Assume $S_{(k)}$ is true i.e. 4^k-3k-1 is divisible by 9

$$S_{k+1}=4^{k+1}-3^{(k+1)}-1=4^k.4-3k-4$$

$$=\left\{9m+3k+1\right\}4-3k-4$$

$$=36m+12k+4-3k-4$$

$$=36m+9k=9\{4m+k\}$$ this is

Divisible by 9

Hence S_{k+1} is divisible by 9

$\therefore S_n$ is divisible by 9 $\forall n\in N$

9. **Show that** $3.5^{2n+1}+2^{3n+1}+2^{3n+1}$ **is divisible by 17**

Sol: Let $S_{(n)}=3.5^{2n+1}+2^{3n+1}$ be the given statement

$S_{(1)}=3.5^{3}+2^{4}=375+16=391=17\times 23$

This is divisible by 17

Assume S_k is true

$S_k=3.5^{2k+1}+2^{3k+1}$ is divisible by 17

Let $3.5^{2k+1}+2^{3k+1}=17m$

$3.5^{2k+1}=17m-2^{3k+1}$

$S_{k+1}=3.5^{2k+3}+2^{3k+4}$

$=3.5^{2k+1}.^{2}+2^{3k+1}.2^{3}$

$=25\left\{17m-2^{3k+1}\right\}+2^{3k+1}.8$

$=25\times 17m-17(2^{3k+1})=17\left\{25m-2^{3k+1}\right\}$ is divisible by 17

Hence S_{k+1} is true

$\therefore S_n$ is true for all $n\in N$

10. **1.23.+2.3.4+3.4.5+……….up to n terms** $=\dfrac{n(n+1)(n+2)(n+3)}{4}$

Sol: $1.2.3+2.3.4+3.4.5+..........+n(n+1)(n+2)=\dfrac{n(n+1)(n+2)(n+3)}{4}$

Let $S_{(n)}$ be the given statement

For n = 1

L.H.S = 1.2.3 = 6

R.H.S = $\dfrac{1(2)(3)(4)}{4}=6$

Assume $S_{(k)}$ is true

$1.2.3+2.3.4+3.4.5+.........+k(k+1)(k+2)=\dfrac{k(k+1)(k+2)(k+3)}{4}$

Adding (k + 1) (k + 2) (k + 3) on both sides

$\therefore 1.2.3+2.3.4+3.4.5+.........+k(k+1)(k+2)+(k+2)(k+2)(k+3)=\dfrac{k(k+1)(k+2)(k+3)}{4}$

S_{k+1} is true

Hence $S_{(n)}$ is true $\forall n\in N$

11. $\dfrac{1^3}{1}+\dfrac{1^3+2^3}{1+3}+\dfrac{1^3+2^3+3^3}{1+3+5}+......$ **upto n terms** $=\dfrac{n}{24}[2n^2+9n+13]$

Sol: $\dfrac{1^3}{1}+\dfrac{1^3+2^3}{1+3}+\dfrac{1^3+2^3+3^3}{1+3+5}+.............+\dfrac{1^3+2^3+3^3+.....+n^3}{1+3+5+....+(2n-1)}=\dfrac{n}{24}\left[2n^2+9n+13\right]$

Let $S_{(n)}$ be the given statement

For n = 1

L.H.S $=\dfrac{1^3}{1}=1$

R.H.S = $=\dfrac{1}{24}[2+9+13]=1$

L.H.S = R.H.S

Hence $S_{(1)}$ is true

Assume S_k is true

$$\therefore \frac{1^3}{1}+\frac{1^3+2^3}{1+3}+\frac{1^3+2^3+3^3}{1+3+5}+....+\frac{1^3+2^3+3^3+.......+k^3}{1+3+5+.....2(k-1)}=\frac{k}{24}\left[2k^2+9k+13\right]$$

$$\frac{1^3}{1}+\frac{1^3+2^3}{1+3}+\frac{1^3+2^3+3^3}{1+3+5}+....+\frac{k^2(k+1)^2}{4k^2}=\frac{k}{24}\left[2k^2+9k+13\right]$$

Adding $\frac{(k+2)^2}{4}$ on both sides

$$\frac{1^3}{1}+\frac{1^3+2^3}{1+3}+\frac{1^3+2^3+3^3}{13+5}+.....+\frac{(k+1)^2}{4}+\frac{(k+2)^2}{4}=\frac{k}{24}\left\{2k^2+9k+13\right\}+\frac{(k+2)^2}{4}$$

$$\frac{1^3}{1}+\frac{1^3+2^3}{1+3}+\frac{1^3+2^3+3^3}{13+5}+....+\frac{(k+2)^2}{4}=\frac{k\left\{2k^2+9k+13\right\}+6(k^2+4k+4)}{24}$$

$$=\frac{2k^2+9k+13+6k^2+24k+24}{24}$$

$$=\frac{2k^2+15k^2+37k+24}{24}$$

$$k=-1\left|\begin{array}{cccc}2 & 15 & 37 & 2\\ 0 & -2 & -13 & -24\\ \hline\end{array}\right.$$

$$\begin{array}{cccc}2 & 13 & 24 & 0\end{array}$$

$$=\frac{(k+1)\left\{(2k^2+13k+24\right\}}{24}$$

$$=\frac{(k+1)\left\{2(k^2+2k+1)+9(k+1)+13\right\}}{24}$$

$$=\frac{(k+1)\left\{2(k+1)^2+9(k+1)+13\right\}}{24}$$

$\therefore S_{k+1}$ is true

$\therefore S_n$ is true $\forall n \in N$

12. $1^2+(1^2+2^2)+(1^2+2^2+3^3)+..........$ **up to n terms** $=\frac{n(n+1)^2(n+2)}{12}$

Sol: $1^2+(1^2+2^2)+(1^2+2^2+3^2)+........+(1^2+2^2+3^2+.....n^2)=\frac{n(n+1)^2(n+2)}{12}$

$$1^2+(1^2+2^2)+(1^2+2^2+3^2)+........=\frac{n(n+1)(2n+1)}{6}=\frac{n(n+1)^2(n+2)}{12}$$

Let $S_{(n)}$ be the given statement

For n = 1

L.H.S = 1

R.H.S = $\frac{1(1+1)^2(1+2)}{12}$

Assume S_k is true

$$1^2+(1^2+2^2)+(1^2+2^2+3^2)+...+\frac{k(k+1)(2k+1)}{6}=\frac{k(k+1)^2(k+2)}{12}$$

Adding $(k+1)^{th}$ term i.e. $=\frac{(k+1)(k+2)(2k+3)}{6}$ on both sides

$$1^2+(1^2+2^2)+(1^2+2^2+3^2)+.....+=\frac{(k+1)(k+2)(2k+3)}{6}=\frac{k(k+1)^2(k+2)}{12}+\frac{(k+1)(k+2)(2k+3)}{6}$$

$$=\frac{k(k+1)^2(k+2)+2(k+1)(k+2)(2k+3)}{12}$$

$$=\frac{(k+1)(k+2)\left\{k^2+k+4k+6\right\}}{12}$$

$$=\frac{(k+1)(k+2)\left\{k^2+5k+6\right\}}{12}$$

$$=\frac{(k+1)(k+2)\,(k+2)\,(k+3)}{12}$$

$$=\frac{(k+1)(k+2)^2\,(k+3)}{12}$$

13. **Using mathematical induction, prove the following statements, for all n $\in$ N.**

$$\left(1+\frac{3}{1}\right)+\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)...\left(1+\frac{2n+1}{n^2}\right)=(n+1)^2$$

Sol. Let S(n): $\left(1+\frac{3}{1}\right)+\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)...\left(1+\frac{2n+1}{n^2}\right)=(n+1)^2$ be the given statement.

Let n=1

L.H.S. = 1 + 3 = 4

R.H.S = $(1 + 1)^2 = 2^2 = 4$

L.H.S. = R.H.S.

Therefore, S(1) is true.

Let us assume that S(k) is true.

i.e.$\left(1+\frac{3}{1}\right)+\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right).........\left(1+\frac{2k+1}{k^2}\right)=(k+1)^2$

To prove S(k + 1) is true.

We know that (k + 1)th factor is :

$$1+\frac{2(k+1)+1}{(k+1)^2}=\left[1+\frac{2k+3}{(k+1)^2}\right]$$

Consider$\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right).........\left(1+\frac{2k+1}{k^2}\right)=(k+1)^2$

Multiplying both sides by $\left(1+\frac{2k+3}{(k+1)^2}\right)$

$$\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)\ldots\ldots\ldots\left(1+\frac{2k+1}{k^2}\right)\left(1+\frac{2k+3}{(k+1)^2}\right)$$

$$=(k+1)^2\left(1+\frac{2k+3}{(k+1)^2}\right)$$

$$=(k+1)^2\,\frac{\left[(k+1)^2+2k+3\right]}{(k+1)^2}$$

$$=k^2+1+2k+2k+3$$

$$=k^2+4k+4$$

$$=(k+2)^2=\left[(\overline{k+1})+1\right]^2$$

Thus S(k + 1) is true

By the principle of mathematical induction S(n) is true for all n $\in$ N.

14. **If x and y are natural numbers and x ≠ y, using mathematical induction show that $x^n - y^n$ is divisible by x – y for all n $\in$ N.**

Sol. Let S(n): $x^n - y^n$ is divisible by x – y be the given statement.

Put n = 1, $x^1 - y^1 = x - y$

$x^n - y^n$ is divisible by x – y

$\therefore$ S(1) is true for n = 1

Put n = k, $x^k - y^k = (x - y)p$ ($\because$ p is an integer)

$\therefore$ $x^k - y^k = (x - y)p$ …(1)

S(k) is true for n = k

We know that,

$$x^{k+1}-y^{k+1}=x^{k+1}-x^k y+x^k y-y^{k+1}$$

$$=x^k\cdot x-x^k y+x^k y-y^k\cdot y$$

$$=x^k(x-y)+y(x^k-y^k)$$

$$=x^k(x-y)+y(x-y)p \qquad (\because \text{from}(1))$$

$$=(x-y)(x^k+yp)$$

$$=(x-y)q \qquad (\because q \text{ is an integer})$$

Since p is a polynomial in x and y, so is q.

Hence $x^{k+1} - y^{k+1}$ is divisible by (x – y)

$\therefore$ S(k + 1) is true for n = k + 1

By the principle of mathematical induction, S(n) is true for all n $\in$ N.

15. **Show that $49^n + 16n - 1$ is divisible by 64 for all positive integers n.**

Sol. Let S(n): $49^n + 16n - 1$ is divisible by 64 be the statement.

Since $49^1 + 16.1 - 1 = 64$ is divisible by 64.

$\therefore$ S(n) is true for n = 1

Assume that the statement S(n) is true for n = k

i.e. $49^n + 16n - 1$ is divisible by 64

Then $49^k + 16k - 1 = 64\ M$ …(1) ($\because$ M is an integer)

We show that the statement S(n) is true for n = k + 1

i.e. we show that $49^{k+1} + 16(k+1) - 1$ is divisible by 64.

From (1), we have

$49^k + 16k - 1 = 64\ M$

$49^k = 64\ M - 16k + 1$

$49^k \times 49 = (64\ M - 16k + 1) \times 49$

$49^{k+1} + 16(k + 1) - 1 = (64M - 16k + 1)49 + 16(k + 1) - 1$

$= 64 \times 49\ M - 49 \times 16k + 49 + 16k + 16 - 1$

$= 64 \times 49\ M - 48 \times 16k + 64$

$= 64 \times 49\ M - 64 \times 12k + 64$

$= 64(49\ M - 12k + 1)$

$= 64\ N$ [$\because$ N is an integer]

$\therefore$ S(n) is true for n = k + 1

$\therefore$ By the principle of mathematical induction, S(n) is true for all $n \in N$.