1. FUNCTIONS

DEFINITIONS, CONCEPTS AND FORMULAE:

1) **Function:** Let A and B be non empty sets and f be a relation from A to B. If for each element $a \in A$, there exists a unique $b \in B$ such that $(a, b) \in f$, then f is called a function from A to B. It is denoted by $f : A \to B$.
The set A is called 'domain of f' and B is called 'codomain of f'and the set of all f images of the elements of A is called the range of f which is denoted by f(A).

2) **One - one function or Injection :-** If $f : A \to B$ is such that distinct elements of A have distinct f - images in B, then f is said to be a one - one function.
$f : A \to B$ is one- one $\Leftrightarrow$ if $a_1, a_2 \in A$ and $f(a_1) = f(a_2)$, then $a_1 = a_2$.

3) **Onto function or Surjection :-** Let $f : A \to B$. If every element of B occurs as the image of atleast one element of A, then we say that f is an onto function.

$f : A \to B$ is onto $\Leftrightarrow$ given $b \in B$, there exists $a \in A$ such that $f(a) = b$.

4) **Bijection :-** If $f : A \to B$ is both one - one and onto, then f is said to be a bijection from A to B. $f(a) = b \Rightarrow a = f^{-1}(b)$.

5) **Constant function :-** A function $f : A \to B$ is said to be a constant function, if the range of f contains only one element. $f(x) = c$ (a constant) for all $x \in$ domain.

6) **Identity function :-** If A is a non - empty set, $f : A \to A$ defined by $f(x) = x$ for all $x \in A$ is called the identity function on A and is denoted by I_A.

7) **Composite function :-** If $f : A \to B$, $g : B \to C$ are two functions, then $gof : A \to C$ is defined by $(gof)(x) = g[f(x)]$ $\forall x \in A$.

8) **Equality of two functions:-** Two functions f and g are said to be equal if
i) they are defined on the same domain A and codomain B
ii) $f(x) = g(x)$ for every $x \in A$.

9) **Domain calculations:**

	Function	Method for finding domain of f
1.	$\frac{f(x)}{g(x)}$	Delete the values of $g(x) = 0$ from R.
2.	$\sqrt{f(x)}$	Solve $f(x) \geq 0$
3.	$\frac{1}{\sqrt{f(x)}}$	Solve $f(x) > 0$
4.	$\log f(x)$	Solve $f(x) > 0$
5.	$\frac{1}{\log f(x)}$	Solve $f(x) > 0$ and $f(x) \neq 1$

LEVEL - I (VSAQ)

1. Define one - one function. Give an example.

A: If $f: A \to B$ is such that distinct elements of A have distinct f - images in B, then f is said to be a one - one function.
Eg: $f : R \to R$ defined by $f(x) = 3x + 2$ is one - one.

2. Define onto function. Give an example.

A: Let $f : A \to B$. If every element of B occurs as the image of atleast one element of A, then f is said to be an onto function.
Eg: $f : R \to R$ defined by $f(x) = 3x + 2$ is onto.

3. $f : N \to N$ is defined as $f(x) = 2x + 3$. Is f onto ? Explain with reason.

A: (Let x_1, x_2 domain N such that $f(x) = f(x_2)$.
$\Rightarrow 2x_1 + 3 = 2x_2 + 3$
$\Rightarrow 2x_1 = 2x_2$
$\Rightarrow x_1 = x_2$
$\therefore f : N \to N$ is an injection)
Here codomain of f = N.
Range of f = { f(1), f(2), f(3),.....∞}
= { 5, 7, 9,.....∞}
$\neq N$
Hence $f : N \to N$ is not a surjection (onto)

4. f : R → R is defined as f(x) = $\frac{2x+1}{3}$, then this function is injection or not? Justify.

A: Let $x_1, x_2 \in$ domain R such that $f(x_1) = f(x_2)$

$\Rightarrow \frac{2x_1+1}{3} = \frac{2x_2+1}{3}$

$\Rightarrow 2x_1 + 1 = 2x_2 + 1$

$\Rightarrow 2x_1 = 2x_2$

$\Rightarrow x_1 = x_2$

Hence f : R → R is an injection.

5. If f : R →R is defined by f(x) = $\frac{1-x^2}{1+x^2}$, then find f (tan θ).

A: Given that f : R →R, f (x) = $\frac{1-x^2}{1+x^2}$

$\therefore$ f (tan θ) = $\frac{1-\tan^2\theta}{1+\tan^2\theta}$ = cos 2θ.

6. If A = { -2, -1, 0, 1, 2} and f : A →B is a surjection defined by f (x) = x^2 + x + 1, then find B.

A: f (-2) = $(-2)^2$ + (-2) + 1 = 3

f (-1) = $(-1)^2$ + (-1) + 1 = 1

f (0) = 0^2 + 0 + 1 = 1

f (1) = 1^2 + 1 + 1 = 3

f (2) = 2^2 + 2 + 1 = 7

Since f : A →B is a surjection,

B = f (A)

= {3, 1, 7}

7. If f : R → R, g : R →R are defined by f (x) = 4x -1 and g (x) = x^2 + 2, then find

(i) (gof)$\left(\frac{a+1}{4}\right)$ (ii) go[fof (0)]

A: f : R →R , g : R →R are given by f (x) = 4x - 1, g(x) = x^2 + 2

(i) (gof) $\left(\frac{a+1}{4}\right)$ = $g\left[f\left(\frac{a+1}{4}\right)\right]$

= $g\left[\frac{4(a+1)}{4}-1\right]$

= g [a + 1 - 1]

= g (a)

= a^2 + 2

(ii) g [(fof) (0)] = g [f{f(0)}]

= g [f(-1)]

= g(-4 - 1)

= g(-5)

= $(-5)^2$ + 2

= 27.

8. If f(x) = $\frac{x+1}{x-1}$, then find (fofof) (x).

A: (fof) (x) = $f\left(\frac{x+1}{x-1}\right)$

$= \frac{\frac{x+1}{x-1}+1}{\frac{x+1}{x-1}-1}$

$= \frac{x+\cancel{1}+x-\cancel{1}}{\cancel{x}+1-\cancel{x}+1}$

$= \frac{2x}{2}$

$\therefore$ (f o f o f) (x) = f [f o f (x)] = f(x).

9. If f : R → R, g : R → R are defined by f(x) = 3x - 2, g(x) = x^2 + 1, then find (g o f^{-1}) (2).

A: Let f(x) = y

$\Rightarrow$ 3x - 2 = y

$\Rightarrow x = \frac{y+2}{3} = f^{-1}(y)$

$\therefore f^{-1}(x) = \frac{x+2}{3}$

$\therefore (gof^{-1})(2) = g[f^{-1}(2)] = g\left[\frac{2+2}{3}\right] = g\left(\frac{4}{3}\right)$

$= \left(\frac{4}{3}\right)^2 + 1 = \frac{16}{9} + 1 = \frac{25}{9}$.

10. Find the inverse of the following functions

(i) If a, b ∈ R, f : R →R defined by f (x) = ax + b (a ≠ 0)

(ii) f : R → (0 , ∞) defined by f (x) = 5^x

(iii) f : (0, ∞)→ R defined by f (x) = $\log_2 x$

(iv) f : Q → Q defined by f(x) = 5x + 4

A: (i) If a, b ∈ R, f : R →R defined by f(x) = ax + b (a ≠ 0)

Let x ∈ domain R and Y ∈ codomain R such that f (x) = y

$\Rightarrow$ ax + b = y

$\Rightarrow$ ax = y - b

$\Rightarrow x = \frac{y-b}{a} = f^{-1}(y)$ $\therefore$ f is bijection

$\Rightarrow f^{-1}(x) = \frac{x-b}{a}$

(ii) f :R → (o, ∞) defined by f(x) = 5^x

Let x ∈ R and y ∈ (0, ∞) such that f(x) = y

$\Rightarrow 5^x = y$

$\Rightarrow x = \log_5 y = f^{-1}(y)$ $\because$ f is bijection

$\Rightarrow f^{-1}(x) = \log_5 x$

(iii) $f : (0, \infty) \to R$ defined by $f(x) = \log_2 x$

let $x \in (0, \infty)$ and $y \in R$ such that $f(x) = y$

$\Rightarrow \log_2 x = y$

$\Rightarrow x = 2^y = f^{-1}(y)$ $\because$ f is bijection

$\Rightarrow f^{-1}(x) = 2^x$.

(iv) $f : Q \to Q$ is defined by $f(x) = 5x + 4$

let $x \in$ domain Q and $y \in$ codomain Q such that $f(x) = y$

$\Rightarrow 5x + 4 = y$

$\Rightarrow 5x = y - 4$

$\Rightarrow x = \dfrac{y-4}{5} = f^{-1}(y)$

$\therefore f^{-1}(x) = \dfrac{x-4}{5}$.

11. Determine whether the following functions are even or odd.

(i) $f(x) = a^x - a^{-x} + \sin x$

(ii) $f(x) = x\left(\dfrac{e^x - 1}{e^x + 1}\right)$

(iii) $f(x) = \log(x + \sqrt{x^2 + 1})$.

A: (i) $f(x) = a^x - a^{-x} + \sin x$

Now $f(-x) = a^{-x} - a^{-(-x)} + \sin(-x)$

$= a^{-x} - a^x - \sin x$

$= -\{a^x - a^{-x} + \sin x\}$

$= -f(x)$

So f (x) is an odd function.

(ii) $f(x) = x\left(\dfrac{e^x - 1}{e^x + 1}\right)$

$f(-x) = (-x)\left(\dfrac{e^{-x} - 1}{e^{-x} + 1}\right)$

$= (-x)\left(\dfrac{\frac{1}{e^x} - 1}{\frac{1}{e^x} + 1}\right)$

$= (-x)\left(\dfrac{1 - e^x}{1 + e^x}\right)$

$= x\left(\dfrac{e^x - 1}{e^x + 1}\right)$

So f (x) is an even function.

(iii) $f(x) = \log\left(x + \sqrt{x^2 + 1}\right)$

$f(-x) = \log\left[-x + \sqrt{(-x)^2 + 1}\right]$

$= \log\left[\sqrt{x^2 + 1} - x\right]$

$= \log\left[\dfrac{(\sqrt{x^2+1} - x)(\sqrt{x^2+1} + x)}{\sqrt{x^2+1} + x}\right]$

$= \log\left(\dfrac{x^2 + 1 - x^2}{x + \sqrt{x^2+1}}\right)$

$= \log(x + \sqrt{x^2+1})^{-1}$

$= -\log(x + \sqrt{x^2+1})$

$= -f(x)$

So f (x) is an odd function.

12. Find the domain of the real valued function

$$f(x) = \frac{1}{\sqrt{x^2 - a^2}} \; (a > 0).$$

A: To get the domain of f, $x^2 - a^2 > 0$.

$\Rightarrow (x + a)(x - a) > 0$.

$\Rightarrow x < -a$ or $x > a$.

$\Rightarrow x \in (-\infty, -a) \cup (a, \infty)$

$\therefore$ Domain of f = $(-\infty, -a) \cup (a, \infty)$.

13. Find the domain of the real valued function

$$f(x) = \sqrt{(x - \alpha)(x - \beta)} \; (0 < \alpha < \beta).$$

A: To get the domain $(x - \alpha)(x - \beta) \geq 0$.

$x \leq \alpha$ or $x \geq \beta$.

$x \in (-\infty, \alpha] \cup [\beta, \infty)$

$\therefore$ Domain of f = $(-\infty, \alpha] \cup [\beta, \infty)$

14. Find the domain of the real valued function

$$f(x) = \frac{2x^2 - 5x + 7}{(x-1)(x-2)(x-3)}.$$

A: To get the domain of f, $(x - 1)(x - 2)(x - 3) \neq 0$.

$\Rightarrow x \neq 1, 2, 3$

$\therefore$ Domain of f = R - {1, 2, 3}

15.Find the domain of the function

(i) $f(x) = \dfrac{1}{(x^2-1)(x+3)}$ (ii) $f(x) = \dfrac{1}{\log(2-x)}$

A: $f(x) = \dfrac{1}{(x^2-1)(x+3)} \in R$

$\Leftrightarrow (x^2 - 1)(x + 3) \neq 0$

$\Leftrightarrow (x + 1)(x - 1)(x + 3) \neq 0$

$\Leftrightarrow x \neq -3, -1, 1$

$\therefore$ Domain of f = R - {-3, -1, 1}

(ii) $f(x) = \dfrac{1}{\log(2-x)}$

$\Leftrightarrow 2 - x > 0$ and $2 - x \neq 1$

$\Leftrightarrow x - 2 < 0$ and $x \neq 1$.

$\Leftrightarrow x < 2$ and $x \neq 1$.

$\Leftrightarrow x \in (-\infty, 2)$ and $x \neq 1$

Domain of f = $(-\infty, 1) \cup (1, 2)$.

16. Find the domain of the function

(i) $f(x) = \sqrt{x^2 - 25}$ (ii) $f(x) = \dfrac{1}{\sqrt{1-x^2}}$

(iii) $f(x) = \sqrt{4x - x^2}$

A: $f(x) = \sqrt{x^2 - 25} \in R$

$\Leftrightarrow x^2 - 25 \geq 0$

$\Leftrightarrow (x + 5)(x - 5) \geq 0$

$\Leftrightarrow x \in (-\infty, -5] \cup [5, \infty)$

$\therefore$ Domain of f = $(-\infty, -5] \cup [5, \infty)$

(ii) $f(x) = \dfrac{1}{\sqrt{1-x^2}} \in R$

$\Leftrightarrow 1 - x^2 > 0$

$\Leftrightarrow x^2 - 1 < 0$

$\Leftrightarrow (x + 1)(x - 1) < 0$

$\Leftrightarrow x \in (-1, 1)$

$\therefore$ Domain of f = (-1, 1)

(iii) $f(x) = \sqrt{4x - x^2} \in R$

$\Leftrightarrow 4x - x^2 \geq 0$

$\Leftrightarrow x(4 - x) \geq 0$

$\Leftrightarrow x(x - 4) \leq 0$

$\Leftrightarrow x \in [0, 4]$

$\therefore$ Domain of f = [0 4]

17. Find the domain of the function

(i) $f(x) = \log(x^2 - 4x + 3)$

(ii) $f(x) = \sqrt{\log_{0.3}(x - x^2)}$

A: (i) $f(x) = \log(x^2 - 4x + 3) \in R$

$\Leftrightarrow x^2 - 4x + 3 > 0$

$\Leftrightarrow (x - 1)(x - 3) > 0$

$\Leftrightarrow x \in (-\infty, 1) \cup (3, \infty)$

$\therefore$ Domain of f = $(-\infty, 1) \cup (3, \infty)$.

(ii) $f(x) = \sqrt{\log_{0.3}(x - x^2)} \in R$

$\Leftrightarrow x - x^2 > 0$

$\Leftrightarrow x(1 - x) > 0$

$\Leftrightarrow x(x - 1) < 0$

$\Leftrightarrow x \in (0, 1)$

Domain of f = (0, 1).

18. Find the range of the function

(i) $f(x) = \log|4 - x^2|$

(ii) $f(x) = \sqrt{[x] - x}$

A: (i) $f(x) = \log|4 - x^2| \in R$

Let f (x) = y

$\Rightarrow \log|4 - x^2| = y$

$\Rightarrow |4 - x^2| = e^y > 0 \ \forall y \in R$

$\therefore$ Range of f is R.

(ii) $f(x) = \sqrt{[x] - x} \in R$

$\Leftrightarrow [x] - x \geq 0$

$\Leftrightarrow x \leq [x]$

$\Leftrightarrow x \in Z$

$\therefore$ Domain of f = Z

$\Rightarrow$ Range of f = {0}.

19. Find the range of

(i) $f(x) = \dfrac{x^2 - 4}{x - 2}$ (ii) $f(x) = \sqrt{9 + x^2}$

A: (i) $f(x) = \dfrac{x^2 - 4}{x - 2} \in R$

$\Leftrightarrow x - 2 \neq 0$

$\Leftrightarrow x \neq 2$

$\therefore$ Domain of f = R - {2}

Then y = x + 2 $\quad \because x \neq 2 \Rightarrow y \neq 4$

$\therefore$ Range of f = R - {4}.

(ii) $f(x) = \sqrt{9 + x^2}$

Let $y = f(x) = \sqrt{9 + x^2} \in R$

$\Leftrightarrow$ Domain of f = R

When x = 0, f (0) = $\sqrt{9} = 3$

When $x \in R - \{0\}$, f(x) > 3

$\therefore$ Range of f = $[3, \infty)$.

20. Find the domain and range of

(i) $f(x) = \frac{2+x}{2-x}$ **(ii)** $f(x) = \frac{x}{1+x^2}$

(iii) $f(x) = \sqrt{9-x^2}$ **(iv)** $f(x) = \frac{x}{2-3x}$

A: (i) $f(x) = \frac{2+x}{2-x} \in R$

$2 - x \neq 0$

$\Rightarrow x \neq 2$

Domain of f = R - {2}.

Let f (x) = y

$\Rightarrow \frac{2+x}{2-x} = y$

$\Rightarrow 2 + x = 2y - xy$

$\Rightarrow x(1+y) = 2(y-1)$

$\Rightarrow x = \frac{2(y-1)}{y+1}$

Clearly x is not defined for y + 1 = 0

$\therefore$ Range of f = R - {-1}

.

(ii) $f(x) = \frac{x}{1+x^2}$

$f(x) = \frac{x}{1+x^2} \in R$

$\Leftrightarrow 1 + x^2 \neq 0$

$\Leftrightarrow$ Domain of f = R

Let f(x) = y

$\Rightarrow \frac{x}{1+x^2} = y$

$\Rightarrow x = y + yx^2$

$\Rightarrow yx^2 - x + y = 0$

$\Rightarrow x = \frac{1 \pm \sqrt{1-4y^2}}{2y} \in R$

$\Rightarrow 1 - 4y^2 \geq 0$ and $y \neq 0$

$\Rightarrow (1 + 2y)(1-2y) \geq 0$ and $y \neq 0$

$\Rightarrow (y + ½)(y - ½) \leq 0$ and $y \neq 0$

$\Rightarrow y \in [-½, ½]$ and $y \neq 0$

Also $x = 0 \Rightarrow y = 0$

$\because$ Range of f = [-½, ½]

(iii) $f(x) = \sqrt{9-x^2} \in R$

$\Leftrightarrow 9 - x^2 \geq 0$

$\Leftrightarrow x^2 - 9 \leq 0$

$\Leftrightarrow (x + 3)(x - 3) \leq 0$

$\therefore$ Domain of f = [-3, 3]

Let f (x) = y

$\sqrt{9-x^2} = y$

$9 - x^2 = y^2$

$x = \sqrt{9-y^2}$

$\Rightarrow 9 - y^2 \geq 0$

$\Rightarrow y^2 - 9 \leq 0$

$y \in [-3, 3]$

Since y takes only non negative values

$\therefore$ Range of f = [0, 3].

(iv) $f(x) = \frac{x}{2-3x} \in R$

$\Leftrightarrow 2 - 3x \neq 0$

$\Leftrightarrow x \neq 2/3$

Domain of f is R - {2/3}

Let f (x) = y

$\Rightarrow \frac{x}{2-3x} = y$

$\Rightarrow x = 2y - 3xy$

$\Rightarrow x(1+3y) = 2y$

$\Rightarrow x = \frac{2y}{1+3y}$

$\Rightarrow 1 + 3y \neq 0$

$\Rightarrow y \neq -1/3$

$\therefore$ Range of f = R - {1/3}

21. If a function is defined as

$$f(x) = \begin{cases} x+2, & x > -1 \\ 2, & -1 \leq x \leq 1 \\ x-1, & -3 < x < -1 \end{cases}$$

Find the values of (i) f(0) (ii) f(2) + f(-2)

A: (i) f (0) = 2

(ii) f(2) + f(-2) = {2 + 2} + {-2 -1]

= 4 - 3

= 1.

22.If f : R $\to$R and g: R$\to$ R are defined by f (x) = 3x - 1 and g (x) = x^2 + 1, then find (i) (fog)(x) (ii) (gof)(x)

A: Given that f : R $\to$R, g: R$\to$ R are defined by

$f(x) = 3x - 1, g(x) = x^2 + 1$

(i) $(fog)(x) = f[g(x)]$

$= f[x^2 + 1]$

$= 3(x^2 + 1) - 1$

$= 3x^2 + 2.$

(ii) $(gof(x) = g[f(x)]$

$= g[3x - 1]$

$= (3x - 1)^2 + 1$

$= 9x^2 - 6x + 2$

23. If f and g are real valued functions defined by f (x) = 2x - 1 and g (x) = x^2, then find (i) (fg) (x) (ii) (f + g + 2) (x)

A: $f(x) = 2x - 1, g(x) = x^2$

(i) $(fg)(x) = f(x)\, g(x)$

$= (2x - 1)(x^2)$

$= 2x^3 - x^2$

(ii) $(f + g + 2)(x) = f(x) + g(x) + 2$

$= 2x - 1 + x^2 + 2$

$= x^2 + 2x + 1$

$= (x + 1)^2$

LEVEL - I (LAQ)

1. If f : A $\to$ B, g : B $\to$ C are two bijections, then prove that gof: A $\to$ C is also a bijection.

A: Given : f : A $\to$ B, g : B $\to$C are bijections.

Part 1 :- To prove that gof : A $\to$ C is one-one.

Now f : A $\to$ B, g : B $\to$C are one-one functions.

$\Rightarrow$ gof: A$\to$C is a function.

Let $a_1, a_2 \in A \Rightarrow f(a_1), f(a_2) \in B$ and $(gof)(a_1)$, $(gof)(a_2) \in C$.

Suppose that $(gof)(a_1) = (gof)(a_2)$

$\Rightarrow g[f(a_1)] = g[f(a_2)]$

$\Rightarrow f(a_1) = f(a_2)$ $\because$ g is one-one

$\Rightarrow a_1 = a_2$ $\because$ f is one-one

$\therefore$ gof : A $\to$ C is one-one.

Part 2:- To prove that gof: A $\to$C is onto.

Now f : A $\to$B, g : B $\to$ C are onto functions.

$\Rightarrow$ gof : A $\to$C is a function.

Let $c \in C$.

Since g : B $\to$C is onto, there exists atleast one element $b \in B$ such that g(b) = c.

Since f : A $\to$ B is also onto, there exists atleast one element $a \in A$ such that f(a) = b

Now (gof) (a) = g[f(a)]

= g(b)

= c

$\therefore$ For $c \in C$, there is an element $a \in A$ such that (gof) (a) = c.

so gof : A $\to$C is onto.

since gof : A $\to$ C is both one-one and onto, hence gof : A $\to$ C is a bijection.

2. If f: A $\to$ B, g : B $\to$C are bijections, then prove that $(gof)^{-1} = f^{-1}og^{-1}$.

A: Given that f : A $\to$ B, g : B $\to$ C are bijections.

$\Rightarrow$ $f^{-1} : B \to A$, $g^{-1} : C \to B$

Now gof : A $\to$ C is also a bijection.

$\Rightarrow (gof)^{-1} : C \to A$

Also $g^{-1} : C \to B$, $f^{-1} : B \to A \Rightarrow f^{-1}og^{-1} : C \to A$.

Thus $(gof)^{-1}$ and $f^{-1}og^{-1}$ both the functions exist and have the same domain C and the same codomain A.

Let c be any element in C.

Since g : B $\to$ C is onto, there exists atleast one element $b \in B$ such that g(b) = c

$\Rightarrow b = g^{-1}(c)$ $\because$ g is a bijection

Since f : A $\to$ B is onto, there exists atleast one element $a \in A$ such that f(a) = b.

$\Rightarrow a = f^{-1}(b)$ $\because$ f is a bijection

Consider (gof) (a) = g[f(a)]

= g(b)

$\therefore$ (gof) (a) = c.

$\Rightarrow a = (gof)^{-1}(c)$ $\because$ gof is a bijection

Also $(f^{-1}og^{-1})(c) = f^{-1}[g^{-1}(c)]$

$= f^{-1}(b)$

$= a$

$\therefore (gof)^{-1}(c) = (f^{-1}og^{-1})(c)\ \forall\ c \in C$.

Hence $(gof)^{-1} = f^{-1}og^{-1}$

3. If $f : A \to B$ is a bijection, then show that $fof^{-1} = I_B$ and $f^{-1}of = I_A$.

A: Given that $f : A \to B$ is a bijection

$\Rightarrow f^{-1} : B \to A$.

Part 1:- To show that $fof^{-1} = I_B$

Now $f^{-1} : B \to A$, $f : A \to B \Rightarrow fof^{-1} : B \to B$.

Also $I_B : B \to B$

Thus fof^{-1} and I_B have the same domain B and the same codomain B.

Let a be any element in A.

Since $f : A \to B$, there is a unique element $b \in B$. such that $f(a) = b$

$\Rightarrow a = f^{-1}(b)$ $\quad \because$ f is a bijection

Consider $(fof^{-1})(b) = f[f^{-1}(b)]$

$= f(a)$

$= b$

$= I_B(b) \quad \because I_B : B \to B \Rightarrow I_B(b) = b$

$\therefore (fof^{-1})(b) = I_B(b) \ \forall \ b \in B$

Thus $fof^{-1} = I_B$

Part 2:- To prove that $f^{-1}of = I_A$

Now $f : A \to B$, $f^{-1} : B \to A \Rightarrow f^{-1}of : A \to A$

Also $I_A : A \to A$

Thus $f^{-1}of$ and I_A have the same domain A and the same codomain A.

Now $(f^{-1}of)(a) = f^{-1}[f(a)]$

$= f^{-1}(b)$

$= a$

$= I_A(a) \quad \therefore I_A : A \to A \Rightarrow I_A(a) = a$

$(f^{-1}of)(a) = I_A(a) \quad \forall \ a \in A$

$\therefore f^{-1}of = I_A$

Hence $fof^{-1} = I_B$ and $f^{-1}of = I_A$.

4. If $f : A \to B$, I_A and I_B are identity functions on A and B respectively, then prove that $foI_A = I_B of = f$.

A: Given that $f : A \to B$

$I_A : A \to A$ is defined by $I_A(a) = a \ \forall \ a \in A$.

$I_B : B \to B$ is defined by $I_B(b) = b \ \forall \ b \in B$.

Part 1:- To prove that $foI_A = f$

Now $I_A : A \to A$, $f : A \to B \Rightarrow foI_A : A \to B$

Also $f : A \to B$

Thus foI_A and f both the functions exist and have the same domain A and the same codomain B.

Let $a \in A$

Since $f : A \to B$, there exists a unique element $b \in B$ such that $f(a) = b$

Consider $(foI_A)(a) = f[I_A(a)]$

$= f(a)$

$\therefore (foI_A)(a) = f(a)$ for all $a \in A$

Hence $foI_A = f$ .....(1)

Part 2:- To show that $I_B of = f$

Now $f : A \to B$, $I_B : B \to B \Rightarrow I_B of : A \to B$

Also $f : A \to B$

Thus $I_B of$ and f both the functions exist and have the same domain A and codomain B.

Consider $(I_B of)(a) = I_B[f(a)]$

$= I_B(b)$

$= b$

$= f(a)$

$\therefore (I_B of)(a) = f(a)$ for all $a \in A$

$\Rightarrow I_B of = f$ .....(2)

From (1) & (2) $foI_A = f = I_B of$.

5. If $f : A \to B$, $g : B \to A$ are two functions such that $gof = I_A$ and $fog = I_B$ then prove that $g = f^{-1}$.

A: Given that $f : A \to B$, $g : B \to A$ are two functions such that $gof = I_A$ and $fog = I_B$.

Part 1:- To prove that f is one-one.

Let $a_1, a_2 \in A \Rightarrow f(a_1), f(a_2) \in B$

Consider $f(a_1) = f(a_2)$

$\Rightarrow g[f(a_1)] = g[f(a_2)]$

$\Rightarrow (gof)(a_1) = (gof)(a_2)$

$\Rightarrow I_A(a_1) = I_A(a_2) \quad \because gof = I_A$

$\Rightarrow a_1 = a_2$

Thus $f : A \to B$ is one-one.

Part 2:- To prove that f is onto.

Let $b \in B$.

$\because g : B \to A$, there exists a unique element $a \in A$ such that $g(b) = a$.

Now $f(a) = f[g(b)]$

$= (fog)(b)$

$= I_B(b) \quad \because fog = I_B$

$= b$

So $f : A \to B$ is onto.
Since f is both one-one and onto, so f is a bijection.
$\Rightarrow f^{-1} : B \to A$
Also $g : B \to A$
Thus both the functions f^{-1} and g have the same domain B and same codomain A.
Part 3:- To show that $g = f^{-1}$
From previous part, $f(a) = b$
$\Rightarrow a = f^{-1}(b)$
Also $g(b) = a$
$\therefore g(b) = f^{-1}(b) \quad \forall \quad b \in B.$

Hence $g = f^{-1}$.

6. If f: A→B, g: B → C, h: C →D are functions, then prove that ho(gof) = (hog)of.

A: Given that $f : A \to B$, $g : B \to C$, $h : C \to D$

Now $f : A \to B$, $g : B \to C \Rightarrow gof : A \to C$

Also $gof : A \to C$, $h: C \to D \Rightarrow ho(gof) : A \to D$

Now $g : B \to C$, $h: C \to D \Rightarrow hog : B \to D$

Also $f : A \to B$, $hog : B \to D \Rightarrow (hog)of : A \to D$

Thus ho(gof) and (hog)of both the functions exist and have the same domain and the same codomain.
Let a be any element in A.
$[ho(gof)] (a) = h[(gof) (a)]$
$= h[g\{f(a)\}]$

Also $[(hog)of] (a) = (hog)[f(a)]$
$= h[g\{f(a)]$

Thus $[ho(gof)](a) = [hog)of] (a)$ for all $a \in A$
Hence ho(gof) = (hog)of.

7. If $f : A \to B$ is a bijection, then prove that $f^{-1} : B \to A$ is a bijection

A: Given that $f : A \to B$ is a bijection
$\Rightarrow f^{-1} : B \to A$ is a function
Part 1: To prove that $f^{-1} : B \to A$ is one-one.
Let $b_1, b_2 \in B$.
$\because f : A \to B$ is onto, there exist $a_1, a_2 \in A$ such that
$f(a_1) = b_1, f(a_2) = b_2$
$\Rightarrow a_1 = f^{-1}(b_1), a_2 = f^{-1}(b_2) \quad \because f: A \to B$ is a bijection

Now, suppose that $f^{-1}(b_1) = f^{-1}(b_2)$
$\Rightarrow a_1 = a_2$
$\Rightarrow f(a_1) = f(a_2) \quad \because f : A \to B$ is a funciton
$\Rightarrow b_1 = b_2$
So $f^{-1}: B \to A$ is a one-one function.
Part 2: To prove that $f^{-1} : B \to A$ is onto.
Let $a \in A$.
Since $f : A \to B$, there exists a unique element $b \in B$ such that $f(a) = b$
$\Rightarrow \quad f^{-1}(b) = a \qquad \because$ f is a bijection

So, for every $a \in A$, there is an element $b \in B$ such that $f^{-1}(b) = a$

So $f^{-1} : B \to A$ is onto
Since $f^{-1}: B \to A$ is both one-one and onto, hence $f^{-1} : B \to A$ is a bijection.

8. Let A = {1, 2, 3}, B = {a, b, c}, C = {p, q, r}. If $f : A \to B$, $g : B \to C$ are defined by f = {(1, a), (2, c), (3, b)}, g = {(a, q), (b, r), (c, p)}, then show that $f^{-1}og^{-1} = (gof)^{-1}$.

A: Given that A = {1, 2, 3}, B = {a, b, c}, C = {p, q, r}
$f : A \to B$, $g : B \to C$ are given by
$f = \{(1, a), (2, c), (3, b)\} \Rightarrow f^{-1} = \{(a, 1), (b, 3), (c, 2)\}$ and
$g = \{(a, q), (b, r), (c, p)\} \Rightarrow g^{-1} = \{(q, a), (r, b), (p, c)\}$

Now $(f^{-1}og^{-1})(p) = f^{-1}[g^{-1}(p)]$
$= f^{-1}(c)$
$= 2$

Similarly $(f^{-1}og^{-1})(q) = 1$, $(f^{-1}og^{-1})(r) = 3$
$\therefore f^{-1}og^{-1} = \{(p, 2), (q, 1), (r, 3)\}$ .....(1)

Also $(gof)(1) = g[f(1)]$
$= g(a)$
$= q$

Similarly $(gof)(2) = p$, $(gof)(3) = r$
$\Rightarrow gof = \{(1, q), (2, p), (3, r)\}$
$\Rightarrow (gof)^{-1} = \{(q, 1), (p, 2), (r, 3)\}$ ..... (2)
From (1) and (2) $f^{-1}og^{-1} = (gof)^{-1}$.

9. If f : Q $\rightarrow$ Q defined by f(x) = 5x+4 for all x $\in$ Q, show that f is a bijection and find f^{-1}.

Given : f : Q $\rightarrow$ Q is defined by f(x) = 5x + 4

Part 1:- To prove that f is one-one

Let $x_1, x_2 \in$ Q (domain) and

$f(x_1) = f(x_2)$

$\Rightarrow 5x_1 + 4 = 5x_2 + 4$

$\Rightarrow 5x_1 = 5x_2$

$\Rightarrow x_1 = x_2$

$\therefore$ f : Q $\rightarrow$ Q is one-one.

Part 2:- To prove that f is onto

Let y $\in$ the codomain Q and x $\in$ domain Q such that

f(x) = y

$\Rightarrow 5x + 4 = y$

$\Rightarrow x = \frac{y-4}{5}$

So for every y $\in$ codomain Q, there is a preimage

$\frac{y-4}{5} \in$ domain Q such that

$f\left(\frac{y-4}{5}\right) = y$

Thus f : Q $\rightarrow$ Q is onto.

Part 3:- To find $f^{-1}(x)$

Since f is both one-one, onto, so it is a bijection.

$f(x) = y \Rightarrow x = f^{-1}(y)$

$5x + 4 = y \Rightarrow x = \frac{y-4}{5} = f^{-1}(y)$

$\therefore f^{-1}(x) = \frac{x-4}{5}$.

LEVEL - II (VSAQ)

1. If f : R - {0} $\rightarrow$ R is defined by f(x) = $x^3 - 1/x^3$, then show that f(x) + f(1/x) = 0.

A: $f(x) = x^3 - 1/x^3$

Now $f(x) + f(1/x) = x^3 - 1/x^3 + 1/x^3 - x^3 = 0$.

2. If f(x) = $\begin{cases} 3x-2, & x \geq 1 \\ x^2-2, & -2 \leq x \leq 2 \\ 2x+1, & x < -3 \end{cases}$ then find f(4), f(2.5), f(-2), f(-4), f(0), f(-7)

A: i) f(4) = 3(4) - 2 = 10

ii) f(2.5) is not defined

iii) $f(-2) = (-2)^2 - 2 = 4 - 2 = 2$

iv) f(- 4) = 2 (- 4) + 1 = - 8 + 1 = - 7

v) $f(0) = 0^2 - 2 = -2$

vi) f(- 7) = 2 (- 7) + 1 = - 14 + 1 = - 13

3. If f(x) = x + $\frac{1}{x}$ then prove that $[f(x)]^2 = f(x^2) + f(1)$.

A: Given $f(x) = x + \frac{1}{x}$

$[f(x)]^2 = \left(x + \frac{1}{x}\right)^2 = x^2 + \frac{1}{x^2} + 2 \times \frac{1}{\cancel{x}} = x^2 + \frac{1}{x^2} + 2.$

and $f(x^2) = x^2 + \frac{1}{x^2}$, $f(1) = 1 + \frac{1}{1^2} = 2.$

$f(x^2) + f(1) = x^2 + \frac{1}{x^2} + 2 = [f(x)]^2$

Hence proved.

4. If f : R - {$\pm$1} $\rightarrow$ R is defined by f(x) = log $\left|\frac{1+x}{1-x}\right|$, then show that $f\left(\frac{2x}{1+x^2}\right) = 2f(x)$.

A: Given $f(x) = \log\left|\frac{1+x}{1-x}\right|$

Now, $f\left(\frac{2x}{1+x^2}\right) = \log\left|\frac{1+\frac{2x}{1+x^2}}{1-\frac{2x}{1+x^2}}\right| = \log$

$\left|\frac{\frac{1+x^2+2x}{\cancel{1+x^2}}}{\frac{1+x^2-2x}{\cancel{1+x^2}}}\right|$

$= \log\left|\frac{(1+x)^2}{(1-x)^2}\right| = \log\left|\frac{1+x}{1-x}\right|^2 = 2\log\left|\frac{1+x}{1-x}\right| = 2f(x).$

5. If f(x) = $\frac{\cos^2 x + \sin^4 x}{\sin^2 x + \cos^4 x}$ x $\forall$ R then show that f(2012) = 1

A: Given that f(x) = $\frac{\cos^2 x + \sin^4 x}{\sin^2 x + \cos^4 x}$

$$f(x) = \frac{1 - \sin^2 x + \sin^4 x}{1 - \cos^2 x + \cos^4 x} = \frac{1 - \sin^2 x\,(1 - \sin^2 x)}{1 - \cos^2 x\,(1 - \cos^2 x)}$$

$$= \frac{1 - \sin^2 x \cos^2 x}{1 - \cos^2 x \sin^2 x}.$$

$$= 1$$

$$\Rightarrow \boxed{f(2012) = 1}$$

6. If f(x + y) = f(xy) $\forall$ x,y $\in$ R then prove that 'f' is a constant function.

A: Let f(0) = k

Given that f(x + y) = f(xy)

Now, f(x) = f(x + 0) = f(x.0) = f(0) = k.

which is a constant, $\forall$ x $\in$ R

Hence, f(x) is a constant function.

7. If the function f : {-1, 1} $\rightarrow$ {0, 2} is defined by f(x) = ax + b is a surjection, then find a, b.

A: Here f is a surjection, so two cases arise.

case i) f (-1) = 0, f(1) = 2

$\Rightarrow$ - a + b = 0, a + b = 2 $\Rightarrow$ a = 1, b = 1

case ii) f(1) = 0, f(-1) = 2

$\Rightarrow$ a + b = 0, - a + b = 2 $\Rightarrow$ a = - 1, b = 1

Hence, $\boxed{a = \pm 1, b = 1}$

8. If f: R $\rightarrow$ R is defined by f(x) = $2x^2$ + 3 and g(x) = 3x - 2 then find i) fog(x) ii) gof (x), iii) fof (0), iv) [go(fof)](3)

A: Given that f(x) = $2x^2$ + 3 and g(x) = 3x - 2

i) fog(x) = f[g(x)] = f[3x - 2] = $2(3x - 2)^2$ + 3

= $2(9x^2 + 4 - 12x) + 3 = 18x^2 - 24x + 11$

ii) gof(x) = g[f(x)] = g[$2x^2$ + 3] = $3(2x^2 + 3) - 2$

= $6x^2 + 9 - 2 = 6x^2 + 7$

iii) fof(0) = f[f(0)] = f[3] = $2(3)^2 + 3 = 21$

iv) [go(fof)](3) = g[fof(3)] = g[f{f(3)}]

= g[f(21)] = g[$2(21)^2$ + 3]

= g[885] = 3(885) - 2 = 2653.

9. If f(x) = 4x - 1, g(x) = x^2 + 2 then find i) gof (x) ii) fof (x)

A: Given f(x) = 4x - 1, g(x) = x^2 + 2

i) gof(x) = g[f(x)] = g[4x - 1] = $(4x - 1)^2$ + 2

= $16x^2 + 1 - 8x + 2 = 16x^2 - 8x + 3$

ii) (fof (x) = f[f(x)] = f(4x - 1)

= 4(4x - 1) - 1 = 16x - 5

10. If f(x) = 2, g(x) = x^2, h(x) = 2x, then find fo(goh) (x).

A: Given f(x) = 2, g(x) = x^2, h(x) = 2x

fo(goh)(x) = f[g(h(x)] = f[g(2x)]

= f[$(2x)^2$] = f[$4x^2$]

= 2

11. If f(x) = x^2, g(x) = 2^x then solve the equation fog(x) = gof(x)

A: fog(x) = f[g(x)] = f(2^x) = $\left(2^x\right)^2 = 2^{2x}$

and gof(x) = g[f(x)] = g[x^2] = 2^{x^2}

Since, fog(x) = gof(x) $\Rightarrow$ $2^{2x} = 2^{x^2}$

$\Rightarrow$ 2x = x^2 $\Rightarrow$ x^2 - 2x = 0 $\Rightarrow$ x(x - 2) = 0

$\Rightarrow$ x = 0 or 2

12. If f, g: R $\rightarrow$ R are defined by f(x) = $\begin{cases} 0, & \text{if } x \in Q \\ 1, & \text{if } x \in Q \end{cases}$

and g (x) = $\begin{cases} -1, & \text{if } x \in Q \\ 0, & \text{if } x \in Q \end{cases}$ = then find (fog) (π) + (gof) (e)

A: (fog) (π) = f[g(π)] = f(0) = 0

(gof) (e) = g[f(e)] = g(1) = - 1 [$\because \pi \notin Q$]

$\therefore$ (fog) (π) + (gof) (e) = 0 - 1 = - 1

13. If f(x) = e^x and g(x) = $\log_e x$ then show that fog = gof and find f^{-1}, g^{-1}

A: Given that f(x) = e^x and g(x) = $\log_e x$

take fog(x) = f[g(x)] = f[$\log_e x$] = $e^{\log_e x}$ = x

gof(x) = g[f(x)] = g(e^x) = $\log_e e^x$ = x $\log_e x$ = x

Clearly, fog(x) = gof(x)

Hence, $f^{-1}(x)$ = g(x) = $\log_e x$ and g^{-1} (x) = f(x) = e^x.

14. If $f(x) = 1 + x + x^2 + ...$ for $|x| < 1$ then show that $f^{-1}(x) = \frac{x-1}{x}$.

A: Given that $f(x) = 1 + x + x^2 + ... = \frac{1}{1-x}$

$\because a + ar + ar^2 + ... = \frac{a}{1-r}, r < 1$

Let $f(x) = y \Rightarrow x = f^{-1}(y)$

$\therefore \frac{1}{1-x} = y \Rightarrow 1 - x = \frac{1}{y} \Rightarrow x = 1 - \frac{1}{y}$.

$\Rightarrow x = \frac{y-1}{y} \Rightarrow f^{-1}(y) = \frac{y-1}{y} \Rightarrow \boxed{f^{-1}(x) = \frac{x-1}{x}}$.

15. If f = {(1,2) (2,-3) (3,-1)} then find (i) 2f (ii) f^2 (iii) 2 +f (iv) $\sqrt{f}$

A: Given f = {(1, 2)(2, -3)(3, -1)}

i) take 2f(1) = 2[f(1)] = 2(2) = 4

2f(2) = 2[f(2)] = 2(-3) = - 6

2f(3) = 2[f(3)] = 2(-1) = -2

$\therefore$ 2f = {(1, 4)(2, -6) (3, -2)}

ii) take $f^2(1) = [f(1)]^2 = (2)^2 = 4$

$f^2(2) = [f(2)]^2 = (-3)^2 = 9$

$f^2(3) = [f(3)]^2 = (-1)^2 = 1$

$\therefore f^2 = \{(1, 4)(2, 9) (3, 1)\}$

iii) take (2 + f) (1) = 2 + f(1) = 2 + 2 = 4

(2 + f) (2) = 2 + f(2) = 2 - 3 = - 1

(2 + f) (3) = 2 + f(3) = 2 - 1 = 1

$\therefore$ 2 + f = {(1, 4)(2, -1) (3, 1)}

iv) take $\sqrt{f}(1) = \sqrt{f(1)} = \sqrt{2}$

$\sqrt{f}(2) = \sqrt{f(2)} = \sqrt{-3}$ (not valid)

$\sqrt{f}(3) = \sqrt{f(3)} = \sqrt{-1}$ (not valid)

$\therefore \sqrt{f} = \left\{\left(1, \sqrt{2}\right)\right\}$

16. If f = {(4, 5), (5, 6), (6,- 4)} g = {(4, -4), (6, 5), (8, 5) then find (i) f + 4 (ii) fg (iii) f/g (iv) f + g (v) 2f + 4g (vi) |f| (vii) $\sqrt{f}$ (viii) f^2

A: Given f = {(4,5), (5, 6), (6, -4)}, g = {(4, -4), (6, 5), (8, 5)

Here Domain of $f \cap g$ = {4, 6}

i) take (f + 4) (4) = f(4) + 4 = 5 + 4 = 9

(f + 4) (5) = f(5) + 4 = 6 + 4 = 10

(f + 4) (6) = f(6) + 4 = - 4 + 4 = 10

$\therefore$ f + 4 = {(4, 9), (5, 10), (6, 0)}

ii) take (fg) (4) = [f(4)] [g(4)] = (5) (-4) = - 20

(fg) (6) = [f(6)] [g(6)] = (- 4) (5) = - 20

$\therefore$ fg = {(4, - 20), (6, - 20)}

iii) take $\left(\frac{f}{g}\right)(4) = \frac{f(4)}{g(4)} = \frac{5}{-4} = \frac{-5}{4}$ and

$\left(\frac{f}{g}\right)(6) = \frac{f(6)}{g(6)} = \frac{-4}{5} \therefore \frac{f}{g} = \left\{\left(4, \frac{-5}{4}\right)\left(6, \frac{-4}{5}\right)\right\}$

(iv) {(4, 1), (6, 1) (v) {4, - 6), (6, 12)}

(vi) {(4, 5), (5, 6), (6, 4)} (vii) $\left\{\left(4, \sqrt{5}\right)\left(5, \sqrt{6}\right)\right\}$

(viii) {(4, 25), (5, 36), (6, 16)}

17. On what domain the function $f(x) = x^2 - 2x$ and $g(x) = -x + 6$ are equal?

A: Take $f(x) = g(x) \Rightarrow x^2 - 2x = -x + 6$

$\Rightarrow x^2 - x - 6 = 0 \Rightarrow (x - 3)(x + 2) = 0$ x = 3, -2

f(x) and g(x) are equal on the domain {-2,3}

18. Find the domain of definition of the function y(x), given by the equation $2^x + 2^y = 2$.

A: Given equation is $2^x + 2^y = 2$.

$\Rightarrow 2^x = 2 - 2^y$

$\Rightarrow 2^x < 2$

$\Rightarrow \log 2^x < \log 2$

$\Rightarrow x\log 2 < \log 2$

$\Rightarrow x < 1$

$\therefore x \in (-\infty, 1)$

$\therefore$ Domain = $(-\infty, 1)$

19. Find the domain of $\frac{\sqrt{2+x} + \sqrt{2-x}}{x}$.

A: Let $f(x) = \frac{\sqrt{2+x} + \sqrt{2-x}}{x}$

The function f(x) is defined for

$2 + x \geq 0 \Rightarrow x \geq -2 \rightarrow (1)$ and

$2 - x \geq 0 \Rightarrow x \leq 2 \rightarrow (2)$ and

$x \neq 0 \rightarrow (3)$

from (1) and (2) and (3)

$x \in [-2,2] - \{0\}$ (or) $\boxed{x \in [-2,0) \cup (0,2]}$

20. Find the domain of $\sqrt{x+2} + \frac{1}{\log_{10}(1-x)}$.

A: The function is defined for
$x + 2 \geq 0 \Rightarrow x \geq -2 \rightarrow (1)$ and
$1 - x > 0$ and $1 - x \neq 1$
$x - 1 < 0$ and $x \neq 0$.
$x \in [-2,1) - \{0\}$
or) $x \in [-2, 0) \cup (0,1)$.

21. Find the domain of the function

(i) $f(x) = \frac{1}{\sqrt{|x| - x}}$ (ii) $f(x) = \sqrt{|x| - x}$

A: (i) $f(x) = \frac{1}{\sqrt{|x| - x}} \in R$
$\Leftrightarrow |x| - x > 0$.
$\Leftrightarrow |x| > x$
$\Leftrightarrow x \in (-\infty, 0)$
$\therefore$ Domain of $f = (-\infty, 0)$

(ii) $f(x) = \sqrt{|x| - x}$
$\Leftrightarrow |x| - x \geq 0$
$\Leftrightarrow |x| \geq x$
$\Leftrightarrow x \in R$
$\therefore$ Domain of $f = R$ or $(-\infty, \infty)$

22. Find the domain of the function

(i) $f(x) = \sqrt{x - [x]}$

(ii) $f(x) = \sqrt{[x] - x}$

A: (i) $f(x) = \sqrt{x - [x]} \in R$
$\Leftrightarrow x - [x] \geq 0$
$\Leftrightarrow x \geq [x]$
$\Leftrightarrow x \in R$
$\therefore$ Domain of $f = R$ or $(-\infty, \infty)$

(ii) $f(x) = \sqrt{[x] - x} \in R$
$\Leftrightarrow [x] - x \geq 0$
$\Leftrightarrow [x] \geq x$
$\Leftrightarrow x \leq [x]$
$\Leftrightarrow x \in Z$
$\therefore$ Domain of $f = Z$.

23. Find the range of $\frac{\sin\pi[x]}{1+[x]^2}$.

A: The function is defined for $1 + [x]^2 \neq 0$
Which is true $\forall\ x \in R$ Hence, Domain = R
If $x \in R$ then $[x] \in Z \Rightarrow \sin \pi[x] = 0$
$\Rightarrow \frac{\sin\pi[x]}{1+[x]^2} = 0\ \forall\ x \in R$ Hence, Range = {0}.

24. Determine the function $f(x) = \frac{x}{e^x - 1} + \frac{x}{2} + 1$ is even or odd.

A: take $f(-x) = \frac{-x}{e^{-x} - 1} + \frac{-x}{2} + 1$

$$= \frac{-x}{\frac{1}{e^x} - 1} - \frac{x}{2} + 1 = \frac{-xe^x}{1 - e^x} - \frac{x}{2} + 1$$

$$= \frac{-xe^x + x - x}{1 - e^x} - \frac{x}{2} + 1$$

$$= \frac{x(1 - e^x) - x}{1 - e^x} - \frac{x}{2} + 1$$

$$= x - \frac{x}{1 - e^x} - \frac{x}{2} + 1 = \frac{x}{e^x - 1} + \frac{x}{2} + 1 = f(x)$$

Hence, f(x) is an even function.

LEVEL - II (LAQ)

1. **If A = {1, 2, 3}, B = {a, b, c}, C = {p, q, r} and $f : A \rightarrow B$, $g : B \rightarrow C$ are defined by**

$f = \{(1, a)(2, c)(3, b)\}$,

$g = \{(a, q), (b, r), (c, p)\}$ **then**

show that f^{-1} o $g^{-1} = (gof)^{-1}$.

A: f = { (1, a), (2, c), (3, b)} g = {(a, q), (b, r), (c, p)} then gof = {(1, q) (2, p) (3, r)}
$\Rightarrow (gof)^{-1} = \{(q,1)(p, 2)(r,3)\} \rightarrow ①$
g^{-1} = {(q, a) (r, b), (p, c)}
f^{-1} = {(a, 1) (c, 2) (b, 3)}
$\Rightarrow f^{-1} o g^{-1} = \{(q,1)(r,3)(p,2)\} \rightarrow ②$
from ① and ②
$(gof)^{-1} = f^{-1}$ o g^{-1}

2. **If the function $f : R \rightarrow R$ defined by $f(x) = \frac{3^x + 3^{-x}}{2}$, then show that $f(x + y) + f(x - y) = 2f(x) f(y)$.**

A: Given that $f : R \rightarrow R$ and $f(x) = \frac{3^x + 3^{-x}}{2}$

$\therefore f(x+y) = \frac{3^{x+y} + 3^{-(x+y)}}{2}$, $f(x-y) = \frac{3^{x-y} + 3^{-(x-y)}}{2}$