

DEFINITIONS, CONCEPTS AND FORMULAE:

- If n is an integer, $(\cos\theta + i\sin\theta)^n = \cos n\theta + i \sin n\theta$
- If n is a rational number, then one of the values of $(\cos \theta + i \sin \theta)^n$ is $\cos n\theta + i \sin n\theta$.
- If $z = r \operatorname{cis} \theta = r \operatorname{cis} (2k\pi + \theta)$, then

$$z^{1/n} = r^{1/n} \operatorname{cis} \left(\frac{2k\pi + \theta}{n} \right) \text{ where } k = 0, 1, 2, \dots, n - 1.$$

- If $x = \cos\theta + i \sin\theta$ then $\frac{1}{x} = \cos \theta + i \sin \theta$.

and i) $x + \frac{1}{x} = 2\cos\theta$ ii) $x - \frac{1}{x} = 2i\sin\theta$

- If $x = \cos \theta + i \sin \theta$ then $\frac{1}{x} = \cos \theta - i \sin \theta$ and

i) $x^n + \frac{1}{x^n} = 2\cos n\theta$

ii) $x^n - \frac{1}{x^n} = 2i\sin n\theta$.

- Cube roots of unity :
The roots of $x^3 = 1$ are called cube roots of unity which are $1, \omega, \omega^2$ where

$$\omega = \frac{-1 + i\sqrt{3}}{2}, \omega^2 = \frac{-1 - i\sqrt{3}}{2}.$$

- $\operatorname{cis}\theta_1 \cdot \operatorname{cis}\theta_2 = \operatorname{cis}(\theta_1 + \theta_2)$

- $\frac{\operatorname{cis}\theta_1}{\operatorname{cis}\theta_2} = \operatorname{cis}(\theta_1 - \theta_2)$.

- $\operatorname{cis}\theta_1 \cdot \operatorname{cis}\theta_2 \cdot \operatorname{cis}\theta_3 \dots \operatorname{cis}\theta_n$
 $= \operatorname{cis}(\theta_1 + \theta_2 + \theta_3 + \dots + \theta_n)$.

LEVEL - I (VSAQ)

- Find the value of $(1 + i)^{16}$.

A: $(1 + i)^{16} = \left[\sqrt{2} \left(\frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}} \right) \right]^{16}$

$$= (\sqrt{2})^{16} [\cos 45^\circ + i\sin 45^\circ]^{16}$$

$$= 2^8 [\cos 16(45^\circ) - i\sin 16(45^\circ)]$$

$$= 256 [\cos 720^\circ + i\sin 720^\circ] = 256 [1 - i \cdot 0] = 256$$

- Find the value of $(1 + i\sqrt{3})^3$.

A: $(1 + i\sqrt{3})^3 = \left[2 \left(\frac{1}{2} + i \frac{\sqrt{3}}{2} \right) \right]^3 = 8 \left(\frac{1}{2} + i \frac{\sqrt{3}}{2} \right)^3$
 $= 8(\cos 60^\circ + i\sin 60^\circ)^3$

By applying De Moivre's theorem for an integral index.

$$= 8[\cos 3(60^\circ) + i\sin 3(60^\circ)]$$

$$= 8(\cos 180^\circ + i\sin 180^\circ) = 8[-1 + i(0)] = -8.$$

- Find the value of $(1 - i)^8$.



A: $(1 - i)^8 = \left[\sqrt{2} \left(\frac{1}{\sqrt{2}} - i \frac{1}{\sqrt{2}} \right) \right]^8$

$$= (\sqrt{2})^8 (\cos 45^\circ - i \sin 45^\circ)^8$$

By applying De Moivre's theorem for an integral index.

$$= 2^4 [\cos 8(45^\circ) - i\sin 8(45^\circ)]$$

$$= 2^4 [\cos 360^\circ - i\sin 360^\circ] = 16 [1 - i(0)] = 16$$

- Find the value of $\left(\frac{\sqrt{3}}{2} + \frac{i}{2} \right)^5 - \left(\frac{\sqrt{3}}{2} - \frac{i}{2} \right)^5$.

A: $\left(\frac{\sqrt{3}}{2} + \frac{i}{2} \right)^5 - \left(\frac{\sqrt{3}}{2} - \frac{i}{2} \right)^5$

$$= (\cos 30^\circ + i\sin 30^\circ)^5 - (\cos 30^\circ - i\sin 30^\circ)^5$$

$$= \cos 5(30^\circ) + i\sin 5(30^\circ) - [\cos 5(30^\circ) - i\sin 5(30^\circ)]$$

$$=$$

$$\cancel{\cos 150^\circ} + i\sin 150^\circ - [\cancel{\cos 150^\circ} - i\sin 150^\circ]$$

$$= 2i \sin 150^\circ = 2i \left(\frac{1}{2} \right) = i.$$

5. If A, B, C are the angles of a triangle such that $x = \text{cis } A, y = \text{cis } B, z = \text{cis } C$, then find xyz .

A: Given that $x = \text{cis } A, y = \text{cis } B, z = \text{cis } C$
 Now $xyz = \text{cis } A \cdot \text{cis } B \cdot \text{cis } C = \text{cis}(A+B+C)$
 $= \cos(A+B+C) + i \sin(A+B+C)$
 $= \cos 180^\circ + i \sin 180^\circ = -1 + i(0) = -1$.

6. If $x = \text{cis } \theta$, then find the value of $\left(x^6 + \frac{1}{x^6}\right)$.

A: Given that $x = \cos \theta + i \sin \theta$.
 $\Rightarrow x^6 = (\cos \theta + i \sin \theta)^6 = \cos 6\theta + i \sin 6\theta$.
 Now, $\frac{1}{x^6} = \frac{1}{\cos 6\theta + i \sin 6\theta} = \cos 6\theta - i \sin 6\theta$.
 Hence, $x^6 + \frac{1}{x^6} =$
 $= \cos 6\theta + \cancel{i \sin 6\theta} + \cos 6\theta - \cancel{i \sin 6\theta}$
 $= 2 \cos 6\theta$.

7. Find the cube roots of 8.

A: Let $x = \sqrt[3]{8} \Rightarrow x^3 = 8$
 $x^3 = 2^3 = (2 \cdot 1)^3$
 $\Rightarrow x = 2(1^{1/3})$
 $= 2(1, \omega, \omega^2)$
 $= 2, 2\omega, 2\omega^2$.

8. If α, β are the roots of the equation $x^2 + x + 1 = 0$, then prove that $\alpha^4 + \beta^4 + \alpha^{-1}\beta^{-1} = 0$.

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{3}i}{2} = \omega, \omega^2$
 $\alpha^4 + \beta^4 + \alpha^{-1}\beta^{-1} = \omega^4 + (\omega^2)^4 + \frac{1}{\omega \cdot \omega^2} = \omega + \omega^2 + 1 = 0$

9. Simplify $\frac{(\cos \alpha + i \sin \alpha)^4}{(\sin \beta + i \cos \beta)^8}$.

A: $= \frac{(\cos \alpha + i \sin \alpha)^4}{(-i^2 \sin \beta + i \cos \beta)^8} = \frac{(\cos \alpha + i \sin \alpha)^4}{[i(\cos \beta - i \sin \beta)]^8}$
 $= \frac{(\cos \alpha + i \sin \alpha)^4}{i^8 (\cos \beta - i \sin \beta)^8} = \frac{\cos 4\alpha + i \sin 4\alpha}{\cos 8\beta - i \sin 8\beta}$
 $= (\cos 4\alpha + i \sin 4\alpha)(\cos 8\beta + i \sin 8\beta)$
 $= \cos(4\alpha + 8\beta) + i \sin(4\alpha + 8\beta) = \text{cis}(4\alpha + 8\beta)$

10. Solve $x^4 - 1 = 0$.

A: $x^4 - 1 = 0 \Rightarrow (x^2 + 1)(x^2 - 1) = 0$
 $\Rightarrow x^2 + 1 = 0$ or $x^2 - 1 = 0$
 $\Rightarrow x^2 = -1$ or $x^2 = 1$
 $\Rightarrow x = \sqrt{-1}$ or $x = \sqrt{1}$
 $\Rightarrow x = \pm i$ or $x = \pm 1$.

11. If the cube roots of unity are 1, ω, ω^2 , then find the roots of the equation $(x - 1)^3 + 8 = 0$.

$(x - 1)^3 = -8 = (-2)^3 = -2(1)^{1/3} = -2(1, \omega, \omega^2)$
 $\Rightarrow x - 1 = -2, -2\omega, -2\omega^2$
 $\therefore x = 1 - 2, 1 - 2\omega, 1 - 2\omega^2$
 $= -1, 1 - 2\omega, 1 - 2\omega^2$

LEVEL - I (LAQ)



1. If $\cos \alpha + \cos \beta + \cos \gamma = 0 = \sin \alpha + \sin \beta + \sin \gamma$, then show that

- i) $\cos 3\alpha + \cos 3\beta + \cos 3\gamma = 3 \cos(\alpha + \beta + \gamma)$
- ii) $\sin 3\alpha + \sin 3\beta + \sin 3\gamma = 3 \sin(\alpha + \beta + \gamma)$

A: Given: $\cos \alpha + \cos \beta + \cos \gamma = 0 = \sin \alpha + \sin \beta + \sin \gamma$

Let $a = \cos \alpha + i \sin \alpha,$
 $b = \cos \beta + i \sin \beta$
 $c = \cos \gamma + i \sin \gamma$
 $\therefore a + b + c = (\cos \alpha + \cos \beta + \cos \gamma) + i(\sin \alpha + \sin \beta + \sin \gamma)$
 $= 0 + i(0)$
 $a + b + c = 0$
 $\Rightarrow a^3 + b^3 + c^3 = 3abc$
 $(\cos \alpha + i \sin \alpha)^3 + (\cos \beta + i \sin \beta)^3 + (\cos \gamma + i \sin \gamma)^3$
 $= 3(\text{cis } \alpha)(\text{cis } \beta)(\text{cis } \gamma)$

By applying DeMoivre's Theorem for an integral index, we get

$\cos 3\alpha + i \sin 3\alpha + \cos 3\beta + i \sin 3\beta + \cos 3\gamma + i \sin 3\gamma$
 $= 3 \text{cis}(\alpha + \beta + \gamma)$
 $(\cos 3\alpha + \cos 3\beta + \cos 3\gamma) + i(\sin 3\alpha + \sin 3\beta + \sin 3\gamma)$
 $= 3[\cos(\alpha + \beta + \gamma) + i \sin(\alpha + \beta + \gamma)]$

Equating of the real and imaginary parts on both sides, we get

$\cos 3\alpha + \cos 3\beta + \cos 3\gamma = 3 \cos(\alpha + \beta + \gamma)$
 &
 $\sin 3\alpha + \sin 3\beta + \sin 3\gamma = 3 \sin(\alpha + \beta + \gamma)$.

2. If $\cos\alpha + \cos\beta + \cos\gamma = 0 = \sin\alpha + \sin\beta + \sin\gamma$,

show that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{3}{2}$

$= \sin^2\alpha + \sin^2\beta + \sin^2\gamma$.

A: Let $a = \cos\alpha + i \sin\alpha$, $b = \cos\beta + i \sin\beta$, $c = \cos\gamma + i \sin\gamma$

$$\begin{aligned} \Rightarrow a+b+c &= (\cos\alpha + \cos\beta + \cos\gamma) + i(\sin\alpha + \sin\beta + \sin\gamma) \\ &= 0 + i \cdot 0 \\ &= 0. \end{aligned}$$

On squaring, we get

$$(a+b+c)^2 = 0$$

$$\Rightarrow a^2 + b^2 + c^2 + 2(ab + bc + ca) = 0$$

$$\Rightarrow a^2 + b^2 + c^2 = -2(ab + bc + ca)$$

$$\Rightarrow \cos 2\alpha + \cos 2\beta + \cos 2\gamma = -2abc \left(\frac{1}{c} + \frac{1}{a} + \frac{1}{b} \right)$$

$$= -2abc[\cos\gamma - i \sin\gamma + \cos\alpha - i \sin\alpha + \cos\beta - i \sin\beta]$$

$$= -2abc[\cos\alpha + \cos\beta + \cos\gamma - i(\sin\alpha + \sin\beta + \sin\gamma)]$$

$$= -2abc[0 - i(0)]$$

$$= 0.$$

$$\Rightarrow \cos 2\alpha + i \sin 2\alpha + \cos 2\beta + i \sin 2\beta + \cos 2\gamma + i \sin 2\gamma = 0$$

$$\Rightarrow (\cos 2\alpha + \cos 2\beta + \cos 2\gamma) + i(\sin 2\alpha + \sin 2\beta + \sin 2\gamma) = 0 + i(0)$$

Equating the real parts, we get

$$\cos 2\alpha + \cos 2\beta + \cos 2\gamma = 0 \dots\dots\dots(1)$$

$$\Rightarrow 2\cos^2\alpha - 1 + 2\cos^2\beta - 1 + 2\cos^2\gamma - 1 = 0$$

$$\Rightarrow 2(\cos^2\alpha + \cos^2\beta + \cos^2\gamma) = 3$$

$$\therefore \cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{3}{2}$$

Also from (1) $\cos 2\alpha + \cos 2\beta + \cos 2\gamma = 0$

$$\Rightarrow 1 - 2\sin^2\alpha + 1 - 2\sin^2\beta + 1 - 2\sin^2\gamma = 0$$

$$\Rightarrow 3 = 2(\sin^2\alpha + \sin^2\beta + \sin^2\gamma)$$

$$\therefore \sin^2\alpha + \sin^2\beta + \sin^2\gamma = \frac{3}{2}.$$

3. Prove that $(1 + \cos\theta + i \sin\theta)^n + (1 + \cos\theta - i \sin\theta)^n$

$$= 2^{n+1} \cos^n\left(\frac{\theta}{2}\right) \cos\left(\frac{n\theta}{2}\right).$$

A: Now $(1 + \cos\theta + i \sin\theta)^n + (1 + \cos\theta - i \sin\theta)^n$

$$= \left(2\cos^2\frac{\theta}{2} + i 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)^n + \left(2\cos^2\frac{\theta}{2} - i 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)^n$$

$$= \left[2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)\right]^n + \left[2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2} - i\sin\frac{\theta}{2}\right)\right]^n$$

By applying DeMoivre's theorem for an integral index

$$= 2^n \cos^n\left(\frac{\theta}{2}\right) \left[\cos\frac{n\theta}{2} + i\sin\frac{n\theta}{2}\right]$$

$$+ 2^n \cos^n\left(\frac{\theta}{2}\right) \left[\cos\frac{n\theta}{2} - i\sin\frac{n\theta}{2}\right]$$

$$= 2^n \cos^n\left(\frac{\theta}{2}\right) \left[\cos\frac{n\theta}{2} + i\sin\frac{n\theta}{2} + \cos\frac{n\theta}{2} - i\sin\frac{n\theta}{2}\right]$$

$$= 2^n \cos^n\left(\frac{\theta}{2}\right) \cdot 2\cos\left(\frac{n\theta}{2}\right)$$

$$= 2^{n+1} \cos^n\left(\frac{\theta}{2}\right) \cos\left(\frac{n\theta}{2}\right).$$

4. In n is a positive integer, show that

$$(1+i)^n + (1-i)^n = 2^{\frac{n+2}{2}} \cos\left(\frac{n\pi}{4}\right).$$

A: $1+i = \sqrt{2}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)$

$$= \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

$$1-i = \sqrt{2}\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)$$

$$(1+i)^n + (1-i)^n$$

$$= \left[\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right]^n + \left[\sqrt{2}\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)\right]^n$$

$$= \sqrt{2}^n \left(\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}\right) + \sqrt{2}^n \left(\cos\frac{n\pi}{4} - i\sin\frac{n\pi}{4}\right)$$

By applying DeMoivre's theorem for a +ve integer

$$= 2^{\frac{n}{2}} \left[\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4} + \cos\frac{n\pi}{4} - i\sin\frac{n\pi}{4}\right]$$

$$= 2^{\frac{n}{2}} \cdot 2\cos\left(\frac{n\pi}{4}\right)$$

$$= 2^{\frac{n}{2}+1} \cos\left(\frac{n\pi}{4}\right)$$

$$= 2^{\frac{n+2}{2}} \cos\left(\frac{n\pi}{4}\right)$$

5. If α, β are the roots of the equation

$$x^2 - 2x + 4 = 0. \text{ Show that } \alpha^n + \beta^n = 2^{n+1}$$

$$\cos\left(\frac{n\pi}{3}\right).$$

A: Given equation is $x^2 - 2x + 4 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$



$$\begin{aligned}
 &= \frac{2 \pm \sqrt{4-16}}{2} \\
 &= \frac{2 \pm \sqrt{12i^2}}{2} && \because \cos\theta - i\sin\theta \\
 &= \frac{2 \pm 2\sqrt{3}i}{2} && = \frac{1}{\cos\theta + i\sin\theta} \\
 &= 1 \pm \sqrt{3}i
 \end{aligned}$$

Let $\alpha = 1 + \sqrt{3}i$, $\beta = 1 - \sqrt{3}i$

Now $\alpha^n + \beta^n = (1 + \sqrt{3}i)^n + (1 - \sqrt{3}i)^n$

$$\begin{aligned}
 &= \left[2 \left(\frac{1}{2} + i \frac{\sqrt{3}}{2} \right) \right]^n + \left[2 \left(\frac{1}{2} - i \frac{\sqrt{3}}{2} \right) \right]^n \\
 &= \left[2 \left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \right) \right]^n + \left[2 \left(\cos \frac{\pi}{3} - i \sin \frac{\pi}{3} \right) \right]^n \\
 &= 2^n \left(\cos \frac{n\pi}{3} + i \sin \frac{n\pi}{3} \right) + 2^n \left(\cos \frac{n\pi}{3} - i \sin \frac{n\pi}{3} \right) \\
 &= 2^n \left[\cos \frac{n\pi}{3} + i \sin \frac{n\pi}{3} + \cos \frac{n\pi}{3} - i \sin \frac{n\pi}{3} \right] \\
 &= 2^n \cdot 2 \cos \frac{n\pi}{3} \\
 &= 2^{n+1} \cos \frac{n\pi}{3}
 \end{aligned}$$

6. If n is an integer, Show that

$$(1 + i)^{2n} + (1 - i)^{2n} = 2^{n+1} \cos \frac{n\pi}{2}.$$

A: $1 + i = \sqrt{2} \left(\frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}} \right)$

$$= \sqrt{2} \left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right)$$

$$\Rightarrow 1 - i = \sqrt{2} \left(\cos \frac{\pi}{4} - i \sin \frac{\pi}{4} \right)$$

Now $(1 + i)^{2n} + (1 - i)^{2n}$

$$\begin{aligned}
 &= \left[\sqrt{2} \left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right) \right]^{2n} + \left[\sqrt{2} \left(\cos \frac{\pi}{4} - i \sin \frac{\pi}{4} \right) \right]^{2n} \\
 &= (\sqrt{2})^{2n} \left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right)^{2n} + (\sqrt{2})^{2n} \left(\cos \frac{\pi}{4} - i \sin \frac{\pi}{4} \right)^{2n} \\
 &\text{Using the De Moivre's Theorem for an integral index} \\
 &= 2^n \left(\cos 2n \frac{\pi}{4} + i \sin 2n \frac{\pi}{4} \right) + 2^n \left(\cos 2n \frac{\pi}{4} - i \sin 2n \frac{\pi}{4} \right)
 \end{aligned}$$

$$\begin{aligned}
 &= 2^n \left(\cos \frac{n\pi}{2} + i \sin \frac{n\pi}{2} + \cos \frac{n\pi}{2} - i \sin \frac{n\pi}{2} \right) \\
 &= 2^n \cdot 2 \cos \frac{n\pi}{2} \\
 &= 2^{n+1} \cos \frac{n\pi}{2}.
 \end{aligned}$$

7. If n is an integer and z = cisθ, then show that

$$\frac{z^{2n}-1}{z^{2n}+1} = i \tan n\theta.$$

A: Given: $Z = \text{cis}\theta$
 $= \cos\theta + i\sin\theta.$

$$\begin{aligned}
 \text{Now } \frac{z^{2n}-1}{z^{2n}+1} &= \frac{(\cos\theta + i\sin\theta)^{2n} - 1}{(\cos\theta - i\sin\theta)^{2n} + 1} \\
 &= \frac{\cos 2n\theta + i\sin 2n\theta - 1}{\cos 2n\theta - i\sin 2n\theta + 1}
 \end{aligned}$$



By applying DeMoivre's Theorem for an integral index

$$\begin{aligned}
 &= \frac{i \sin 2n\theta - (1 - \cos 2n\theta)}{i \sin 2n\theta + (1 + \cos 2n\theta)} \\
 &= \frac{i 2\sin n\theta \cos n\theta - 2\sin^2 n\theta}{i 2\sin n\theta \cos n\theta + 2\cos^2 n\theta} \\
 &= \frac{i 2\sin n\theta \cos n\theta + i^2 2\sin^2 n\theta}{i 2\sin n\theta \cos n\theta + 2\cos^2 n\theta} \\
 &= \frac{2i \sin n\theta [\cos n\theta + i\sin n\theta]}{2\cos n\theta [\cos n\theta + i\sin n\theta]} \\
 &= i \tan n\theta.
 \end{aligned}$$

8. If n is a positive integer, show that
 $(p+iq)^{1/n} + (p-iq)^{1/n}$

$$= 2(p^2+q^2)^{1/2n} \cos \left[\frac{1}{n} \text{arcTan} \frac{q}{p} \right].$$

A: Given: n is a positive integer

Now $(p+iq)^{1/n} + (p-iq)^{1/n}$

$$\begin{aligned}
 &= \left[\sqrt{p^2+q^2} \left\{ \frac{p}{\sqrt{p^2+q^2}} + i \frac{q}{\sqrt{p^2+q^2}} \right\} \right]^{1/n} \\
 &\quad + \left[\sqrt{p^2+q^2} \left\{ \frac{p}{\sqrt{p^2+q^2}} - i \frac{q}{\sqrt{p^2+q^2}} \right\} \right]^{1/n}
 \end{aligned}$$

By applying DeMoivre's theorem for a rational index, we get one value as

$$(p^2+q^2)^{1/2n} [(\cos\alpha + i\sin\alpha)^{1/n} + (\cos\alpha - i\sin\alpha)^{1/n}]$$

Where $\cos\alpha = \frac{p}{\sqrt{p^2+q^2}}$, $\sin\alpha = \frac{q}{\sqrt{p^2+q^2}}$

$$= (p^2+q^2)^{\frac{1}{2n}} \left[\cos\frac{\alpha}{n} + i\sin\frac{\alpha}{n} + \cos\frac{\alpha}{n} - i\sin\frac{\alpha}{n} \right]$$

$$= (p^2+q^2)^{\frac{1}{2n}} 2\cos\left(\frac{\alpha}{n}\right)$$

$$= (p^2+q^2)^{\frac{1}{2n}} 2\cos\left(\frac{1}{n}\tan^{-1}\frac{q}{p}\right)$$

$$= (p^2+q^2)^{\frac{1}{2n}} 2\cos\left(\frac{1}{n}\arctan\left(\frac{q}{p}\right)\right)$$

9. Show that one value of

$$\left[\frac{1 + \sin\frac{\pi}{8} + i\cos\frac{\pi}{8}}{1 + \sin\frac{\pi}{8} - i\cos\frac{\pi}{8}} \right]^{\frac{8}{3}} = -1$$

A: Consider $\frac{1 + \sin\frac{\pi}{8} + i\cos\frac{\pi}{8}}{1 + \sin\frac{\pi}{8} - i\cos\frac{\pi}{8}}$

$$= \frac{1 + \cos\left(\frac{\pi-\pi}{2}\right) + i\sin\left(\frac{\pi-\pi}{2}\right)}{1 + \cos\left(\frac{\pi-\pi}{2}\right) - i\sin\left(\frac{\pi-\pi}{2}\right)}$$

$$= \frac{1 + \cos\frac{3\pi}{8} + i\sin\frac{3\pi}{8}}{1 + \cos\frac{3\pi}{8} - i\sin\frac{3\pi}{8}}$$

$$= \frac{2\cos^2\frac{3\pi}{16} + i2\sin\frac{3\pi}{16}\cos\frac{3\pi}{16}}{2\cos^2\frac{3\pi}{16} - i2\sin\frac{3\pi}{16}\cos\frac{3\pi}{16}}$$

$$= \frac{2\cos\frac{3\pi}{16}\left(\cos\frac{3\pi}{16} + i\sin\frac{3\pi}{16}\right)}{2\cos\frac{3\pi}{16}\left(\cos\frac{3\pi}{16} - i\sin\frac{3\pi}{16}\right)}$$

$$= \left(\cos\frac{3\pi}{16} + i\sin\frac{3\pi}{16}\right)^2$$

By applying DeMoivre's Theorem for an integral

index = $\cos\frac{3\pi}{8} + i\sin\frac{3\pi}{8}$

Now by applying DeMoivre's Theorem for a rational index, then one value of

$$\left[\frac{1 + \sin\frac{\pi}{8} + i\cos\frac{\pi}{8}}{1 + \sin\frac{\pi}{8} - i\cos\frac{\pi}{8}} \right]^{\frac{8}{3}}$$

$$= \text{one value of } \left(\cos\frac{3\pi}{8} + i\sin\frac{3\pi}{8}\right)^{\frac{8}{3}}$$

$$= \cos\pi + i\sin\pi$$

$$= -1 + i(0)$$

$$= -1.$$

10. Solve the equation $x^9 - x^5 + x^4 - 1 = 0$.

A: Given equation is $x^9 - x^5 + x^4 - 1 = 0$.

$$x^5(x^4 - 1) + 1(x^4 - 1) = 0$$

$$(x^5 + 1)(x^4 - 1) = 0$$

Now $x^5 + 1 = 0$

$$x^5 = -1$$

$$= \cos\pi + i\sin\pi$$

$$= \text{cis } \pi$$

$$= \text{cis}(2k\pi + \pi), k \in \mathbb{Z}$$

$$= \text{cis}(2k + 1)\pi, k \in \mathbb{Z}$$

$$\therefore x = [\text{cis}(2k+1)\pi]^{1/5}$$

$$= \text{cis}(2k+1)\frac{\pi}{5} \text{ where } k = 0, 1, 2, 3, 4.$$

$$= \text{cis}\frac{\pi}{5}, \text{cis}\frac{3\pi}{5}, \text{cis}\pi, \text{cis}\frac{7\pi}{5}, \text{cis}\frac{9\pi}{5}$$

Also $x^4 - 1 = 0$

$$(x^2-1)(x^2+1) = 0$$

$$x = \pm 1, \pm i$$

Hence the required roots are

$$\pm 1, \pm i, \text{cis}\frac{\pi}{5}, \text{cis}\frac{3\pi}{5}, \text{cis}\pi, \text{cis}\frac{7\pi}{5}, \text{cis}\frac{9\pi}{5}.$$



LEVEL - II (VSAQ)

1. If $1, \omega, \omega^2$ are the cube roots of unity, then prove

that $\frac{1}{2+\omega} + \frac{1}{1+2\omega} = \frac{1}{1+\omega}$

$$\frac{1}{2+\omega} + \frac{1}{1+2\omega} = \frac{1+2\omega+2+\omega}{2+4\omega+\omega+2\omega^2}$$

$$= \frac{3(1+\omega)}{2(1+\omega+\omega^2)+3\omega} = \frac{3(1+\omega)}{3\omega} = \frac{(1+\omega)^2}{\omega(1+\omega)} = \frac{(1+\omega+\omega^2)+\omega}{\omega(1+\omega)}$$

$$= \frac{\omega}{\omega(1+\omega)} = \frac{1}{1+\omega}$$

2. Prove that $-\omega$ and $-\omega^2$ are the roots of $z^2 - z + 1 = 0$, where ω and ω^2 are the complex cube roots of unity.

A: $z^2 - z + 1 = 0$.

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{1 \pm \sqrt{(-1)^2 - 4(1)(1)}}{2(1)}$$

$$= \frac{1 \pm \sqrt{3i^2}}{2}$$

$$= \frac{1 - \sqrt{3}i}{2}, \frac{1 + \sqrt{3}i}{2}$$

$$= -\left(\frac{-1 + \sqrt{3}i}{2}\right), -\left(\frac{-1 - \sqrt{3}i}{2}\right)$$

$$= -\omega, -\omega^2.$$

3. If $1, \omega, \omega^2$ are the cube roots of unity, find the value of $(1 - \omega + \omega^2)^3$.

$$(1 - \omega + \omega^2)^3 = \left[(1 + \omega^2) - \omega \right]^3 \because 1 + \omega + \omega^2 = 0$$

$$= (-\omega - \omega)^3$$

$$= (-2\omega)^3$$

$$= -8\omega^3$$

$$= -8.$$

4. If $1, \omega, \omega^2$ are the cube roots of unity, find the value of $(1 - \omega)(1 - \omega^2)(1 - \omega^4)(1 - \omega^8)$.

A: $(1 - \omega)(1 - \omega^2)(1 - \omega^4)(1 - \omega^8)$

$$= (1 - \omega)(1 - \omega^2)(1 - \omega)(1 - \omega^2)$$

$$= [(1 - \omega)(1 - \omega^2)]^2$$

$$= [1 - \omega - \omega^2 + \omega^3]^2$$

$$= [1 - (\omega + \omega^2) + 1]^2$$

$$= [2 - (-1)]^2$$

$$= 3^2$$

$$= 9.$$

LEVEL - II (LAQ)

1. Find all the roots of $x^{11} - x^7 + x^4 - 1 = 0$.

A: Given equation is $x^{11} - x^7 + x^4 - 1 = 0$

$$\Rightarrow x^7(x^4 - 1) + 1(x^4 - 1) = 0$$

$$\Rightarrow (x^4 - 1)(x^7 + 1) = 0$$

Now $x^4 - 1 = 0$.

$$\Rightarrow x^4 = 1 = \text{cis}0 = \text{cis}(0 + 2k\pi) = \text{cis}2k\pi$$

$$\Rightarrow x = (\text{cis}2k\pi)^{\frac{1}{4}} = \text{cis}\frac{2k\pi}{4}, k = 0, 1, 2, 3$$

$$= \text{cis}\frac{k\pi}{2}, k = 0, 1, 2, 3$$

$$= \text{cis}0, \text{cis}\frac{\pi}{2}, \text{cis}\pi, \text{cis}\frac{3\pi}{2}$$

Also $x^7 + 1 = 0$

$$x^7 = -1 = \text{cis}\pi = \text{cis}(\pi + 2k\pi)$$

$$x = [\text{cis}(2k+1)\pi]^{1/7}$$

$$= \text{cis}(2k+1)\frac{\pi}{7}, k = 0, 1, 2, 3, 4, 5, 6$$

$$= \text{cis}\frac{\pi}{7}, \text{cis}\frac{3\pi}{7}, \text{cis}\frac{5\pi}{7}, \text{cis}\frac{7\pi}{7}, \text{cis}\frac{9\pi}{7}, \text{cis}\frac{11\pi}{7}, \text{cis}\frac{13\pi}{7}.$$

Hence the required roots of the given equation are

$$= \text{cis}0, \text{cis}\frac{\pi}{2}, \text{cis}\pi, \text{cis}\frac{3\pi}{2}, \text{cis}\frac{\pi}{7}, \text{cis}\frac{3\pi}{7}, \text{cis}\frac{5\pi}{7},$$

$$\text{cis}\frac{7\pi}{7}, \text{cis}\frac{9\pi}{7}, \text{cis}\frac{11\pi}{7}, \text{cis}\frac{13\pi}{7}$$

AIMS

2. If $(1+x)^n = a_0 + a_1x + a_2x^2 + \dots + a_nx^n$, then show that

i) $a_0 - a_2 + a_4 - \dots = 2^{\frac{n}{2}} \cos \frac{n\pi}{4}$.

ii) $a_1 - a_3 + a_5 - \dots = 2^{\frac{n}{2}} \sin \frac{n\pi}{4}$.

Now $(1+x)^n = a_0 + a_1x + a_2x^2 + \dots + a_nx^n$.

Put $x = i$, then

$$a_0 + a_1i + a_2i^2 + \dots + a_ni^n = (1+i)^n$$

$$= \left[\sqrt{2} \left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right) \right]^n.$$

By applying De Moivre's theorem for an integral index

$$a_0 + a_1i - a_2 - a_3i + a_4 + \dots = 2^{\frac{n}{2}} \left(\cos \frac{n\pi}{4} + i \sin \frac{n\pi}{4} \right)$$

$$(a_0 - a_2 + a_4 \dots) + i(a_1 - a_3 + a_5 \dots)$$

$$= 2^{\frac{n}{2}} \cos \frac{n\pi}{4} + i 2^{\frac{n}{2}} \sin \frac{n\pi}{4}$$

Equating real and imaginary parts both sides.

$$\therefore a_0 - a_2 + a_4 \dots = 2^{\frac{n}{2}} \cos \frac{n\pi}{4}$$

$$a_1 - a_3 + a_5 \dots = 2^{\frac{n}{2}} \sin \frac{n\pi}{4}$$

3. If $z^2 + z + 1 = 0$, where z is a complex number,

prove that $\left(z + \frac{1}{z}\right)^2 + \left(z^2 + \frac{1}{z^2}\right)^2 + \left(z^3 + \frac{1}{z^3}\right)^2 +$

$$\left(z^4 + \frac{1}{z^4}\right)^2 + \left(z^5 + \frac{1}{z^5}\right)^2 + \left(z^6 + \frac{1}{z^6}\right)^2 = 12.$$

Now $z^2 + z + 1 = 0$.

$$\Rightarrow z = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{3}i}{2} = \omega, \omega^2$$

Taking $z = \omega$, we get

$$\left(z + \frac{1}{z}\right)^2 + \left(z^2 + \frac{1}{z^2}\right)^2 + \left(z^3 + \frac{1}{z^3}\right)^2 +$$

$$\left(z^4 + \frac{1}{z^4}\right)^2 + \left(z^5 + \frac{1}{z^5}\right)^2 + \left(z^6 + \frac{1}{z^6}\right)^2$$

$$= \left(\omega + \frac{1}{\omega}\right)^2 + \left(\omega^2 + \frac{1}{\omega^2}\right)^2 + \left(\omega^3 + \frac{1}{\omega^3}\right)^2 +$$

$$\left(\omega^4 + \frac{1}{\omega^4}\right)^2 + \left(\omega^5 + \frac{1}{\omega^5}\right)^2 + \left(\omega^6 + \frac{1}{\omega^6}\right)^2$$

$$= \left(\omega + \frac{\omega^3}{\omega}\right)^2 + \left(\omega^2 + \frac{\omega^3}{\omega^2}\right)^2 + \left(1 + \frac{1}{1}\right)^2 + \left(\omega + \frac{\omega^3}{\omega}\right)^2 + \left(\omega^2 + \frac{\omega^3}{\omega^2}\right)^2 + \left(1 + \frac{1}{1}\right)^2$$

$$= (\omega + \omega^2)^2 + (\omega^2 + \omega)^2 + 2^2 + (\omega + \omega^2)^2 + (\omega^2 + \omega)^2 + 2^2$$

$$= (-1)^2 + (-1)^2 + 4 + (-1)^2 + (-1)^2 + 4$$

$$= 12.$$

4. State and prove De Moivre's Theorem for an integral index.

A: De Moivre's Theorem for an integral index:

For any real number θ and any integer n ,

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i \sin n\theta.$$

Part1: Let n be a positive integer. We prove the theorem by using the principle of mathematical induction.

Let $P(n)$ be the statement:

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i \sin n\theta$$

If $n = 1$, LHS = $(\cos\theta + i\sin\theta)^1$

$$= \cos\theta + i\sin\theta$$

$$\text{RHS} = \cos 1\theta + i\sin 1\theta$$

$$= \cos\theta + i\sin\theta$$

\therefore LHS = RHS

Thus $P(1)$ is TRUE.

Assume that $P(k)$ is true.

$$\Rightarrow (\cos\theta + i\sin\theta)^k = \cos k\theta + i \sin k\theta$$

Multiplying both sides by $\cos\theta + i\sin\theta$, we get

$$(\cos\theta + i\sin\theta)^{k+1} = (\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta)$$

$$= \cos k\theta \cos\theta + i \sin k\theta \cos\theta + i \cos k\theta$$

$$\sin\theta + i^2 \sin k\theta \sin\theta$$

$$= \cos(k\theta + \theta) + i \sin(k\theta + \theta)$$

$$= \cos(k+1)\theta + i \sin(k+1)\theta$$

$\therefore P(k+1)$ is TRUE

By induction, $P(n)$ is true for all positive integers n .

i.e. $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ for all $n \in \mathbb{Z}^+$.

Part 2: If $n = 0$, LHS = $(\cos\theta + i\sin\theta)^0$

$$= 1$$

$$\text{RHS} = \cos 0\theta + i\sin 0\theta$$

$$= 1$$

\therefore LHS = RHS

If $n = 1$, the statement is TRUE.

Part 3: Let n be a negative integer and $n = -m$,

where $m \in \mathbb{Z}^+$

So for m , part 1 is applicable.

Now $(\cos\theta + i\sin\theta)^n = (\cos\theta + i\sin\theta)^{-m}$

$$= \frac{1}{(\cos\theta + i\sin\theta)^m}$$

$$= \frac{1}{\cos m\theta + i\sin m\theta} \text{ from Part 1}$$

$$= \cos m\theta - i\sin m\theta$$

$$= \cos(-m)\theta + i\sin(-m)\theta$$

$$= \cos n\theta + i\sin n\theta.$$

5. If $1, \omega, \omega^2$ are the cube roots of unity, prove that

i) $(1-\omega+\omega^2)^6 + (1-\omega^2+\omega)^6 = 128 = (1-\omega+\omega^2)^7 + (1+\omega-\omega^2)^7$

ii) $(a+b)(a\omega+b\omega^2)(a\omega^2+b\omega) = a^3 + b^3$.

A: Given that $1, \omega, \omega^2$ are the cube roots of unity,

then $1 + \omega + \omega^2 = 0$ and $\omega^3 = 1$

i) $(1 - \omega + \omega^2)^6 + (1 - \omega + \omega^2)^6$

$$= (-\omega - \omega)^6 + (-\omega^2 - \omega^2)^6$$

$$= (-2\omega)^6 + (-2\omega^2)^6$$

$$= (-2)^6 [\omega^6 + \omega^{12}]$$

$$= 64(1 + 1)$$

$$= 128$$

$$(1 - \omega + \omega^2)^7 + (1 + \omega - \omega^2)^7$$

$$= (-\omega - \omega)^7 + (-\omega^2 - \omega^2)^7$$

$$= (-2\omega)^7 + (-2\omega^2)^7$$

$$= (-2)^7 [\omega^7 + \omega^{14}]$$

$$= (-128)(\omega + \omega^2)$$

$$= (-128)(-1)$$

$$= 128.$$

ii) $(a+b)(a\omega+b\omega^2)(a\omega^2+b\omega)$

$$= (a+b)(a^2\omega^3 + ab\omega^2 + ab\omega^4 + b^2\omega^3)$$

$$= (a+b)[a^2 + ab(\omega^2 + \omega) + b^2]$$

$$= (a+b)[a^2 + ab(-1) + b^2]$$

$$= (a+b)(a^2 - ab + b^2)$$

$$= a^3 + b^3$$

6. Find all the values of $(1+i)^{2/3}$.

A: Now $1+i = \sqrt{2} \left(\frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}} \right)$

$$= \sqrt{2} \left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right)$$

$$= \sqrt{2} \operatorname{cis} \frac{\pi}{4}$$

$$= \sqrt{2} \operatorname{cis} \left(2k\pi + \frac{\pi}{4} \right), \quad k \in \mathbb{Z}$$

$$\therefore (1+i)^{\frac{2}{3}} = \sqrt{2}^{\frac{2}{3}} \left[\operatorname{cis} \left(8k + 1 \right) \frac{\pi}{4} \right]^{\frac{2}{3}}$$

$$= 2^{\frac{1}{3}} \operatorname{cis} \left(8k + 1 \right) \cdot \frac{2}{3} \cdot \frac{\pi}{4}, \quad k = 0, 1, 2$$

$$= 2^{\frac{1}{3}} \operatorname{cis} \left(8k + 1 \right) \frac{\pi}{6}, \quad k = 0, 1, 2$$

