

4. THEORY OF EQUATIONS

DEFINITIONS, CONCEPTS AND FORMULAE

1. If α, β, γ are the roots of $ax^3 + bx^2 + cx + d = 0$, then

i) $s_1 = \alpha + \beta + \gamma = \frac{-b}{a}$

ii) $s_2 = \alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a}$

iii) $s_3 = \alpha\beta\gamma = \frac{-d}{a}$

Required cubic equation is $x^3 - s_1x^2 + s_2x - s_3 = 0$

2. If $\alpha, \beta, \gamma, \delta$ are the roots of $ax^4 + bx^3 + cx^2 + dx + e = 0$, then

i) $s_1 = \alpha + \beta + \gamma + \delta = \frac{-b}{a}$

ii) $s_2 = \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = \frac{c}{a}$

iii) $s_3 = \alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = \frac{-d}{a}$

iv) $s_4 = \alpha\beta\gamma\delta = \frac{e}{a}$

Required biquadratic equation is

$$x^4 - s_1x^3 + s_2x^2 - s_3x + s_4 = 0$$

3. For a cubic equation, when the roots are

i) in A.P, then they are taken as $a-d, a, a+d$

ii) in G.P, then they are taken as $a/r, a, ar$.

iii) in H.P, then they are taken as $\frac{1}{a-d}, \frac{1}{a}, \frac{1}{a+d}$

4. For a biquadratic equation, if the roots are

i) in A.P, then they are taken as $a-3d, a-d, a+d, a+3d$

ii) in G.P, then they are taken as $\frac{a}{r^3}, \frac{a}{r}, ar, ar^3$

iii) in H.P, then they are taken as $\frac{1}{a-3d}, \frac{1}{a-d}, \frac{1}{a+d}, \frac{1}{a+3d}$

5. If $f'(\alpha) = 0$ and $f(\alpha) = 0$, then α is a multiple root of order 2 for the algebraic equation $f(x) = 0$.

6. If $f''(\alpha) = 0$ and $f(\alpha) = 0$, then α is a multiple root of order 3 for the algebraic equation $f(x) = 0$

7. In an equation with rational co-efficients, irrational roots occur in pairs of conjugate surds.

8. In an equation with real coefficients, imaginary roots occur in conjugate pairs.

9. Remainder theorem : If $f(x)$ is a polynomial, then the remainder of $f(x)$ when divided by $x - a$ is $f(a)$.

10. Factor theorem: If $f(x)$ is a polynomial and $f(a) = 0$ then $x - a$ is a factor of $f(x)$.

11. Horner's method: If $f(x)$ is a polynomial function of degree 5, then

$$f(x + h) = A_0x^5 + A_1x^4 + A_2x^3 + A_3x^2 + A_4x + A_5$$

where A_1, A_2, A_3, A_4, A_5 are the remainders of $f(x)$ when divided by $(x - h)^5, (x - h)^4, (x - h)^3, (x - h)^2, (x - h)$ respectively and A_0 is the coefficient of x^5 in $f(x)$.

12. For a cubic equation

i) $\sum \alpha^2\beta = s_1s_2 - 3s_3$

ii) $\sum \alpha^3\beta^3 = s_2^3 - 3s_1s_2s_3 + 3s_3^2$

13. For a biquadratic equation:

i) $\sum \alpha^2\beta = s_1s_2 - 3s_3$

ii) $\sum \alpha^2\beta\gamma = s_1s_3 - 4s_4$

14. Consider $f(x) = p_0x^n + p_1x^{n-1} + \dots + p_n = 0$

i) To eliminate the second term, $f(x) = 0$ is transformed

$$\text{to } f(x + h) = 0 \text{ where } h = \frac{-p_1}{n.p_0}$$

ii) To eliminate the third term, $f(x) = 0$ is transformed to $f(x + h) = 0$ where h is given by

$$\frac{n(n-1)}{2} p_0h^2 + (n-1)p_1h + p_2 = 0.$$

15. An equation $f(x) = 0$ is said to be a reciprocal equation if $\frac{1}{\alpha}$ is a root of $f(x) = 0$ whenever α is a root of $f(x) = 0$

16. A R.E. $f(x) = p_0x^n + p_1x^{n-1} + \dots + p_n = 0$ is said to be a reciprocal equation of first class if $p_i = p_{n-i}$ for all i .
17. A R.E. $f(x) = p_0x^n + p_1x^{n-1} + \dots + p_n = 0$ is said to be a reciprocal equation of second class if $p_i = -p_{n-i}$ for all i .
18. If $f(x) = 0$ is a reciprocal equation first class and odd degree, then '-1' is a root of $f(x) = 0$.
19. If $f(x) = 0$ is a reciprocal equation of second class and odd degree, then '1' is a root of $f(x) = 0$.
20. If $f(x) = 0$ is a reciprocal equation of second class and even degree, then 1, -1 are roots of $f(x) = 0$.
21. The transformed equation whose roots are negatives of roots of $f(x) = 0$ is $f(-x) = 0$.
22. The T.E. whose roots are multiplied by $k(\neq 0)$ of the roots of $f(x) = 0$ is $f\left(\frac{x}{k}\right) = 0$.
23. The T.E. whose roots are reciprocals of the roots of $f(x) = 0$ is $f\left(\frac{1}{x}\right) = 0$.
24. The T.E. whose roots exceed by h than those of $f(x) = 0$ is $f(x - h) = 0$.
25. The T.E. whose roots are diminished by h than those of $f(x) = 0$ is $f(x + h) = 0$.
26. The T.E. whose roots are squares of the roots of $f(x) = 0$ is obtained by eliminating square root from $f(\sqrt{x}) = 0$.
27. The T.E. whose roots are cubes of the roots of $f(x) = 0$ is obtained by eliminating cube root from $f(\sqrt[3]{x}) = 0$.

LEVEL - I (VSAQ)

1. Form a polynomial equation of lowest degree, whose roots are 1, -1, 3.

A: Polynomial equation of lowest degree whose roots are 1, -1, 3 is $(x - 1)(x + 1)(x - 3) = 0$

$$\Rightarrow (x^2 - 1)(x + 3) = 0$$

$$\Rightarrow x^3 - 3x^2 - x + 3 = 0.$$
2. Form a polynomial equation with rational coefficients and whose roots are $2 \pm \sqrt{3}, 1 \pm 2i$.

A: Given roots are $2 \pm \sqrt{3}, 1 \pm 2i$.

Required biquadratic equation is

$$[x^2 - (2 + \sqrt{3} + 2 - \sqrt{3})x + (2 + \sqrt{3})(2 - \sqrt{3})]$$

$$[x^2 - (1 + 2i + 1 - 2i)x + (1 + 2i)(1 - 2i)] = 0$$

$$\Rightarrow (x^2 - 4x + 1)(x^2 - 2x + 5) = 0$$

$$\Rightarrow x^4 - 2x^3 + 5x^2 - 4x^3 + 8x^2 - 20x + x^2 - 2x + 5 = 0$$

$$\Rightarrow x^4 - 6x^3 + 14x^2 - 22x + 5 = 0.$$

3. If -1, 2, α are the roots of $2x^3 + x^2 - 7x - 6 = 0$, then find α .

A: Given that -1, 2, α are the roots of $2x^3 + x^2 - 7x - 6 = 0$

$$\Rightarrow s_1 = \alpha + \beta + \gamma = -b/a$$

$$\Rightarrow -1 + 2 + \alpha = -1/2$$

$$\Rightarrow \alpha = -3/2.$$

4. If the product of the roots of the equation $4x^3 + 16x^2 - 9x - a = 0$ is 9, then find a .

A: Product of the roots of $4x^3 + 16x^2 - 9x - a = 0$ 9

$$\Rightarrow s_3 = \alpha\beta\gamma = -d/a = 9$$

$$\Rightarrow -(-a)/4 = 9$$

$$\Rightarrow a = 36.$$

5. If α, β and 1 are the roots of $x^3 - 2x^2 - 5x + 6 = 0$, then find α and β .

A: Given $\alpha, \beta, 1$ are the roots of $x^3 - 2x^2 - 5x + 6 = 0$

$$\Rightarrow s_1 = \alpha + \beta + 1 = 2$$

$$\Rightarrow \alpha + \beta = 1.$$

$$\text{Also } s_3 = \alpha\beta\gamma = \alpha\beta(1) = -d/a = -6/1 = -6$$

$$\Rightarrow \alpha\beta = -6$$

$$\text{By observation } \alpha = 3, \beta = -2.$$

6. If 1, -2, 3 are the roots of $x^3 - 2x^2 + ax + 6 = 0$, then find a .

A: Given that 1 is a root of $x^3 - 2x^2 + ax + 6 = 0$

$$\Rightarrow 1 - 2 + a + 6 = 0$$

$$\Rightarrow a = -5.$$

7. Solve the equation $x^3 - 3x^2 - 6x + 8 = 0$, given that the roots are in A.P.

A: Let the roots be $a - d, a, a + d$.

$$s_1 = a - d + a + a + d = 3$$

$$\Rightarrow 3a = 3 \Rightarrow a = 1$$

$$s_3 = (a - d)(a)(a + d) = -8$$

$$\Rightarrow (1 - d)(1)(1 + d) = -8$$

$$\Rightarrow 1 - d^2 = -8$$

$$\Rightarrow d^2 = 9$$

$$\therefore d = \pm 3$$

If $d = 3$, the roots are $1 - 3, 1, 1 + 3$

i.e. $-2, 1, 4$.

8. Find s_1, s_2, s_3 and s_4 for the equation $8x^4 - 2x^3 - 27x^2 + 6x + 9 = 0$.

A: Given equation is $8x^4 - 2x^3 - 27x^2 + 6x + 9 = 0$

Comparing this with $ax^4 + bx^3 + cx^2 + dx + e = 0$.

$$s_1 = -\frac{b}{a} = -\frac{2}{8} = -\frac{1}{4}$$

$$s_2 = \frac{c}{a} = \frac{-27}{8}$$

$$s_3 = -\frac{d}{a} = -\frac{-6}{8} = \frac{3}{4}$$

$$s_4 = \frac{e}{a} = \frac{9}{8}$$

9. Solve the equation $x^3 - 3x^2 - 16x + 48 = 0$, one root being 3.

A: Given equation is $x^3 - 3x^2 - 16x + 48 = 0$

$$s_1 = 3 + \beta + \gamma = 3$$

$$\Rightarrow \beta + \gamma = 0$$

$$s_3 = 3(\beta)(\gamma) = -48$$

$$\Rightarrow \beta\gamma = -16$$

The quadratic equation whose roots are β, γ is $x^2 - (\beta + \gamma)x + \beta\gamma = 0$

$$\Rightarrow x^2 - (0)x - 16 = 0$$

$$x = \pm 4.$$

\therefore The other two roots are $4, -4$.

10. If 1, 2, 3, 4 are the roots of $x^4 + ax^3 + bx^2 + cx + d = 0$, then find the values of a, b, c, d .

A: Given that 1, 2, 3, 4 are the roots of

$$x^4 + ax^3 + bx^2 + cx + d = 0.$$

$$\Rightarrow a = -s_1 = -(\alpha + \beta + \gamma + \delta) = -(1 + 2 + 3 + 4) = -10$$

$$b = s_2 = \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta$$

$$= 1(2) + 1(3) + 1(4) + 2(3) + 2(4) + 3(4) = 35$$

$$c = -s_3 = -(\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta)$$

$$= -(1.2.3 + 1.2.4 + 1.3.4 + 2.3.4) = -50$$

$$d = s_4 = \alpha\beta\gamma\delta = 1.2.3.4 = 24$$

$$\therefore a = -10, b = 35, c = -50, d = 24$$

11. If α, β, γ are the roots of $x^3 + px^2 + qx + r = 0$, then find $\alpha^2 + \beta^2 + \gamma^2$.

A: Given that α, β, γ are the roots of $x^3 + px^2 + qx + r = 0$.

$$\text{Now } \alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$$

$$= (-p)^2 - 2(q)$$

$$= p^2 - 2q.$$

12. If α, β, γ are the roots of $x^3 + px^2 + qx + r = 0$, then find the value of $\alpha^3 + \beta^3 + \gamma^3$.

A: We know that

$$\alpha^3 + \beta^3 + \gamma^3 - 3\alpha\beta\gamma = (\alpha + \beta + \gamma)[\alpha^2 + \beta^2 + \gamma^2 - \alpha\beta - \beta\gamma - \gamma\alpha]$$

$$\Rightarrow \alpha^3 + \beta^3 + \gamma^3 = (\alpha + \beta + \gamma)[(\alpha + \beta + \gamma)^2 - 3(\alpha\beta + \beta\gamma + \gamma\alpha)] + 3\alpha\beta\gamma$$

$$= (-p)[(-p)^2 - 3(q)] + 3(-r)$$

$$= 3pq - p^3 - 3r.$$

AIMS

13. If α, β, γ are the roots of $x^3 - 2x^2 + 3x - 4 = 0$, then find $\sum \alpha^2\beta^2$

A: Given that α, β, γ are the roots of $x^3 - 2x^2 + 3x - 4 = 0$

$$\sum \alpha^2\beta^2 = \alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2$$

$$= (\alpha\beta + \beta\gamma + \gamma\alpha)^2 - 2(\alpha\beta \cdot \beta\gamma + \beta\gamma \cdot \gamma\alpha + \gamma\alpha \cdot \alpha\beta)$$

$$= (\alpha\beta + \beta\gamma + \gamma\alpha)^2 - 2\alpha\beta\gamma(\beta + \gamma + \alpha)$$

$$= 3^2 - 2(4)(2)$$

$$= 9 - 16$$

$$= -7.$$

14. Find the quotient and remainder, when $2x^5 - 3x^4 + 5x^3 - 3x^2 + 7x - 9$ divided by $x^2 - x - 3$.

A: By synthetic division, dividing

$$2x^5 - 3x^4 + 5x^3 - 3x^2 + 7x - 9 \text{ is by } x^2 - x - 3$$

	2	-3	5	-3	7	-9
1	0	2	-1	10	4	0
3	0	0	6	-3	30	12
	2	-1	10	4	41	3

Required quotient is $2x^3 - x^2 + 10x + 4$ and the remainder is $41x + 3$.

15. Find the polynomial equation of degree 4 whose roots are negatives of the roots of $x^4 - 6x^3 + 7x^2 - 2x + 1 = 0$

A: Required transformed equation is $f(-x) = 0$
 $\Rightarrow (-x)^4 - 6(-x)^3 + 7(-x)^2 - 2(-x) + 1 = 0$
 $\Rightarrow x^4 + 6x^3 + 7x^2 + 2x + 1 = 0.$

16. Find the algebraic equation whose roots are 3 times the roots of $x^3 + 2x^2 - 4x + 1 = 0$.

A: Given equation is $f(x) = x^3 + 2x^2 - 4x + 1 = 0$.
 Required transformed equation is $f(x/3) = 0$
 $\Rightarrow \frac{x^3}{27} + \frac{2x^2}{9} - \frac{4x}{3} + 1 = 0$
 $\Rightarrow x^3 + 6x^2 - 36x + 27 = 0.$

17. If α, β, γ are the roots of the equation $x^3 + 2x^2 - 4x - 3 = 0$, find the equation whose

roots are $\frac{\alpha}{3}, \frac{\beta}{3}, \frac{\gamma}{3}$.

A: Given equation is $f(x) = x^3 + 2x^2 - 4x - 3 = 0$
 Required transformed equation is $f(3x) = 0$
 $\Rightarrow 27x^3 + 2(9x^2) - 4(3x) - 3 = 0$
 $\Rightarrow 9x^3 + 6x^2 - 4x - 1 = 0.$

18. Find the equation whose roots are squares of the roots of $x^3 + 3x^2 - 7x + 6 = 0$.

A: Given equation is $f(x) = x^3 + 3x^2 - 7x + 6 = 0$.
 Required transformed equation is $f(\sqrt{x}) = 0$
 $\Rightarrow (\sqrt{x})^3 + 3(\sqrt{x})^2 - 7\sqrt{x} + 6 = 0$
 $\Rightarrow x\sqrt{x} + 3x - 7\sqrt{x} + 6 = 0$
 $\Rightarrow \sqrt{x}(x - 7) = -(3x + 6)$
 Squaring on both sides,
 $\Rightarrow x(x^2 - 14x + 49) = 9x^2 + 36x + 36$
 $\Rightarrow x^3 - 14x^2 + 49x - 9x^2 - 36x - 36 = 0$
 $\Rightarrow x^3 - 23x^2 + 13x - 36 = 0.$

LEVEL - I (LAQ)

1. Solve $4x^3 - 24x^2 + 23x + 18 = 0$, given that the roots are in A.P.

A: Given that the roots of $4x^3 - 24x^2 + 23x + 18 = 0$ are in A.P.
 Let the roots be $a - d, a, a + d$.

$$\begin{aligned} \text{Sum of the roots } a - d + a + a + d &= \frac{-b}{a} \\ \Rightarrow 3a &= \frac{24}{4} \\ \Rightarrow a &= \frac{6}{3} = 2. \end{aligned}$$

$$\begin{aligned} \text{Product of the roots } (a - d)(a)(a + d) &= \frac{-d}{a} \\ \Rightarrow (2 - d)(2)(2 + d) &= \frac{-18}{4} \\ \Rightarrow 4 - d^2 &= \frac{-9}{4} \\ \Rightarrow d^2 &= 4 + \frac{9}{4} = \frac{25}{4} \\ \Rightarrow d &= \pm \frac{5}{2} \end{aligned}$$



Hence the roots of the given equation are

$$\begin{aligned} 2 - \frac{5}{2}, 2, 2 + \frac{5}{2} \\ = \frac{-1}{2}, 2, \frac{9}{2}. \end{aligned}$$

2. If the roots of the equation $x^3 + 3px^2 + 3qx + r = 0$ are in A.P., then show that $2p^3 - 3pq + r = 0$.

A: Given that the roots of $x^3 + 3px^2 + 3qx + r = 0$ ---- (1) are in A.P.
 Let the roots be $a - d, a, a + d$.

$$\begin{aligned} \text{Sum of the roots } a - d + a + a + d &= \frac{-b}{a} \\ \Rightarrow 3a &= -3p \\ \Rightarrow a &= -p \end{aligned}$$

Substituting $x = -p$ in (1), we get

$$\begin{aligned} (-p)^3 + 3p(-p)^2 + 3q(-p) + r &= 0 \\ \Rightarrow -p^3 + 3p^3 - 3pq + r &= 0 \\ \Rightarrow 2p^3 - 3pq + r &= 0. \end{aligned}$$

3. Solve the equation $x^3 - 7x^2 + 14x - 8 = 0$, given that the roots are in geometric progression.

A: Given that the roots of $x^3 - 7x^2 + 14x - 8 = 0$ are in G.P.

Let the roots be $\frac{a}{r}, a, ar$

$$s_3 = \left(\frac{a}{r}\right) (a) (ar) = \frac{-d}{a}$$

$$\Rightarrow a^3 = 8$$

$$\Rightarrow a = 2$$

$$s_1 = \frac{a}{r} + a + ar = \frac{-b}{a}$$

$$\Rightarrow 2\left(r + \frac{1}{r} + 1\right) = 7$$

$$\Rightarrow r + \frac{1}{r} + 1 = \frac{7}{2}$$

$$\Rightarrow r + \frac{1}{r} = \frac{7}{2} - 1 = \frac{5}{2}$$

$$\Rightarrow r + \frac{1}{r} = 2 + \frac{1}{2} \text{ by observation}$$

$$\therefore r = 2 \text{ or } \frac{1}{2}$$

If $r = 2$, the required roots are $\frac{2}{2}, 2, 2(2)$

$$= 1, 2, 4.$$

4. Show that the condition that the roots of $x^3 + 3px^2 + 3qx + r = 0$ may be in G.P. is $p^3r = q^3$.

A: Given that the roots of $x^3 + 3px^2 + 3qx + r = 0$ ---- (1) are in G.P.

Let the roots be $\frac{\alpha}{\beta}, \alpha, \alpha\beta$.

$$\left(\frac{\alpha}{\beta}\right) (\alpha) (\alpha\beta) = \frac{-d}{a}$$

$$\Rightarrow \alpha^3 = -r$$

$$\Rightarrow \alpha^3 + r = 0 \text{ ----- (2)}$$

substituting $x = \alpha$ in (1),

$$\alpha^3 + 3p\alpha^2 + 3q\alpha + r = 0$$

$$\Rightarrow (\alpha^3 + r) + 3\alpha(p\alpha + q) = 0$$

$$\Rightarrow 0 + 3\alpha(p\alpha + q) = 0$$

$$\Rightarrow p\alpha + q = 0$$

$$\therefore \alpha \neq 0$$

$$\Rightarrow p\alpha = -q$$

Cubing on both sides,

$$p^3\alpha^3 = -q^3$$

$$\Rightarrow p^3(-r) = -q^3$$

$$\Rightarrow p^3r = q^3 \text{ is the required condition.}$$

5. Solve the equation $15x^3 - 23x^2 + 9x - 1 = 0$, given that the roots are in H.P.

A: Given that the roots of

$$f(x) = 15x^3 - 23x^2 + 9x - 1 = 0 \text{ ----- (1) are in H.P.}$$

$$\Rightarrow \text{Roots of } f\left(\frac{1}{x}\right) = 0 \text{ are in A.P.}$$

$$f\left(\frac{1}{x}\right) = \frac{15}{x^3} - \frac{23}{x^2} + \frac{9}{x} - 1 = 0$$

$$\Rightarrow 15 - 23x + 9x^2 - x^3 = 0$$

$$\Rightarrow x^3 - 9x^2 + 23x - 15 = 0 \text{ ----- (2)}$$



Let the roots of (2) be $a - d, a, a + d$.

$$s_1 = a - d + a + a + d = \frac{-b}{a}$$

$$\Rightarrow 3a = 9$$

$$\Rightarrow a = 3$$

$$s_3 = (a - d) (a) (a + d) = \frac{-d}{a}$$

$$\Rightarrow 3(9 - d^2) = 15$$

$$\Rightarrow 9 - d^2 = \frac{15}{3} = 5$$

$$\Rightarrow d^2 = 4$$

$$\Rightarrow d = \pm 2$$

If $d = 2$, the roots of (2), are $3 - 2, 3, 3 + 2$

$$= 1, 3, 5.$$

Hence the roots of the given equation are $1, \frac{1}{3}, \frac{1}{5}$.

6. Given that the roots of $x^3 + 3px^2 + 3qx + r = 0$ are in H.P, show that $2q^3 = 4(3pq - r)$.

A: Let $f(x) = x^3 + 3px^2 + 3qx + r = 0$

Given that roots of $f(x) = 0$ are in H.P.

$$\Rightarrow \text{Roots of } f\left(\frac{1}{x}\right) = 0 \text{ are in A.P.}$$

$$f\left(\frac{1}{x}\right) = 0 \Rightarrow \frac{1}{x^3} + \frac{3p}{x^2} + \frac{3q}{x} + r = 0.$$

$$\Rightarrow rx^3 + 3qx^2 + 3px + 1 = 0$$

Let the roots of this equation be a - d, a, a + d.

$$\text{sum} = a - d + a + a + d = -\frac{3q}{r}.$$

$$\Rightarrow 3a = -\frac{3q}{r}.$$

$$\Rightarrow a = -\frac{q}{r}.$$

Since a is root, $ra^3 + 3qa^2 + 3pa + 1 = 0$.

$$\Rightarrow r\left(-\frac{q}{r}\right)^3 + 3q\left(-\frac{q}{r}\right)^2 + 3p\left(-\frac{q}{r}\right) + 1 = 0$$

$$\Rightarrow \frac{-q^3}{r^2} + \frac{3q^3}{r^2} - \frac{3pq}{r} + 1 = 0$$

$$= \frac{2q^3 - 3pqr + r^2}{r^2} = 0$$

$$\Rightarrow 2q^3 - 3pqr + r^2 = 0$$

$$\Rightarrow 2q^3 = 3pqr - r^2.$$

$$\therefore 2q^3 = r(3pq - r).$$

7. Solve the equation $18x^3 + 81x^2 + 121x + 60 = 0$, given that one root is equal to half the sum of the remaining roots.

A: Let α, β, γ be the roots such that $\beta = \frac{\alpha + \gamma}{2}$

$$\Rightarrow \alpha + \gamma = 2\beta$$

$$\text{Now } s_1 = \alpha + \beta + \gamma = \frac{-b}{a}$$

$$\Rightarrow 2\beta + \beta = \frac{-81}{18}$$

$$\Rightarrow 3\beta = \frac{-9}{2}$$

$$\Rightarrow \beta = \frac{-3}{2}$$

$$\therefore \alpha + \gamma = -3$$

$$\text{Also } s_3 = \alpha\beta\gamma = \frac{-d}{a}$$

$$\Rightarrow \alpha\gamma \left(\frac{-3}{2}\right) = \frac{-60}{18}$$

$$\Rightarrow \alpha\gamma = \frac{20}{9}$$

The quadratic equation, whose roots are α, γ is $x^2 - (\alpha + \gamma)x + \alpha\gamma = 0$.

$$\Rightarrow x^2 - (-3)x + \frac{20}{9} = 0$$

$$\Rightarrow 9x^2 + 27x + 20 = 0$$

$$P = 9 \cdot 20$$

$$\Rightarrow 9x^2 + 12x + 15x + 20 = 0$$

$$= 3 \cdot 3 \cdot 4 \cdot 5$$

$$\Rightarrow 3x(3x + 4) + 5(3x + 4) = 0$$

$$= 12 \cdot 15$$

$$\Rightarrow (3x + 4)(3x + 5) = 0$$

$$\Rightarrow x = \frac{-4}{3}, \frac{-5}{3}$$

Hence, the required roots of the given Cubic

$$\text{equation are } \frac{-4}{3}, \frac{-3}{2}, \frac{-5}{3}.$$

8. Solve $x^3 - 9x^2 + 14x + 24 = 0$ given that two of the roots are in the ratio 3 : 2.

A: Given that two of the roots of $x^3 - 9x^2 + 14x + 24 = 0$ are in the ratio 3 : 2. AIMS

Let the roots be $3\alpha, 2\alpha, \gamma$.

$$s_1 = 3\alpha + 2\alpha + \gamma = 9$$

$$\Rightarrow \gamma = 9 - 5\alpha \text{ ----- (1)}$$

$$s_2 = (3\alpha)(2\alpha) + (2\alpha)(\gamma) + \gamma(3\alpha) = 14$$

$$\Rightarrow 6\alpha^2 + 2\alpha\gamma + 3\alpha\gamma = 14$$

$$\Rightarrow 6\alpha^2 + 5\alpha\gamma = 14 \text{ ----- (2)}$$

$$s_3 = (3\alpha)(2\alpha)(\gamma) = -24$$

$$\Rightarrow 6\alpha^2\gamma = -24$$

$$\Rightarrow \alpha^2\gamma = -4 \text{ ----- (3)}$$

From (1) & (2), $6\alpha^2 + 5\alpha(9 - 5\alpha) = 14$

$$\Rightarrow 6\alpha^2 + 45\alpha - 25\alpha^2 - 14 = 0$$

$$\Rightarrow -19\alpha^2 + 45\alpha - 14 = 0 \quad P = 19 \cdot 14$$

$$\Rightarrow 19\alpha^2 - 45\alpha + 14 = 0 \quad = 19 \cdot 2 \cdot 7$$

$$\Rightarrow 19\alpha^2 - 38\alpha - 7\alpha + 14 = 0 \quad = 38 \cdot 7$$

$$\Rightarrow 19\alpha(\alpha - 2) - 7(\alpha - 2) = 0$$

$$\Rightarrow (19\alpha - 7)(\alpha - 2) = 0$$

$$\Rightarrow \alpha = 2, \frac{7}{19}$$

If $\alpha = 2$, then $\gamma = 9 - 5(2) = -1$.

Substituting α, γ values in LHS of (3),
 $\alpha^2\gamma = 2^2(-1)$
 $= -4$
 $= \text{RHS, which is satisfied.}$

Whereas $\alpha = \frac{7}{19}$ does not satisfy equation (3).

Hence, the required roots of given cubic equation are 3(2), 2(2), -1
 $= 6, 4, -1.$

9. Solve the equation $x^4 - 2x^3 + 4x^2 + 6x - 21 = 0$, the sum of two roots being zero.

A: Sum of two roots of $x^4 - 2x^3 + 4x^2 + 6x - 21 = 0$ is zero. Let the roots $\alpha, -\alpha, \gamma, \delta$

$$s_1 = \alpha - \alpha + \gamma + \delta = 2$$

$$\gamma + \delta = 2$$

$$s_3 = \alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = \frac{-d}{a}$$

$$\Rightarrow -\alpha^2\gamma - \alpha^2\delta + \alpha\gamma\delta - \alpha\gamma\delta = -6$$

$$\Rightarrow \alpha^2(\gamma + \delta) = 6$$

$$\Rightarrow \alpha^2 = 3$$

$$\alpha = \pm \sqrt{3}.$$

$$s_4 = (\alpha)(-\alpha)(\gamma)(\delta) = \frac{e}{a}$$

$$\Rightarrow -3\gamma\delta = -21$$

$$\Rightarrow \gamma\delta = 7.$$

The quadratic equation whose roots are γ, δ is

$$x^2 - (\gamma + \delta)x + \gamma\delta = 0$$

$$\Rightarrow x^2 - 2x + 7 = 0$$

$$\Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{2 \pm \sqrt{(-2)^2 - 4(1)(7)}}{2(1)}$$

$$= \frac{2 \pm \sqrt{-24}}{2}$$

$$= \frac{2 \pm 2\sqrt{6}i}{2}$$

$$= 1 \pm \sqrt{6}i$$

Hence, the required roots of the given equation are $\sqrt{3}, -\sqrt{3}, 1 + \sqrt{6}i, 1 - \sqrt{6}i.$

10. Solve the equation $x^4 - 5x^3 + 5x^2 + 5x - 6 = 0$, the product of two roots being 3.

A: Product of two roots of $x^4 - 5x^3 + 5x^2 + 5x - 6 = 0$ is 3. Let $\alpha, \beta, \gamma, \delta$ be the roots with $\alpha\beta = 6$.

$$s_4 = \alpha\beta\gamma\delta = \frac{e}{a}$$

$$3\gamma\delta = -6$$

$$\gamma\delta = -2 \text{ ----- (1)}$$

$$s_1 = \alpha + \beta + \gamma + \delta = \frac{-b}{a} = 5 \text{ ----- (2)}$$

$$s_3 = \alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = \frac{-d}{a}$$

$$\Rightarrow \alpha\beta(\gamma + \delta) + \gamma\delta(\alpha + \beta) = -5$$

$$\Rightarrow 3(\gamma + \delta) - 2(\alpha + \beta) = -5$$

$$\Rightarrow 3(\alpha + \beta + \gamma + \delta) - 5(\alpha + \beta) = -5$$

$$\Rightarrow 3(5) - 5(\alpha + \beta) = -5$$

$$\Rightarrow 15 + 5 = 5(\alpha + \beta)$$

$$\Rightarrow \alpha + \beta = 4, \gamma + \delta = 5 - 4 = 1.$$



$\alpha + \beta = 4, \alpha\beta = 3$	$\gamma + \delta = 1, \gamma\delta = -2$
$x^2 - (\alpha + \beta)x + \alpha\beta = 0$	$x^2 - (\gamma + \delta)x + \gamma\delta = 0$
$x^2 - 4x + 3 = 0.$	$x^2 - x - 2 = 0$
$(x - 1)(x - 3) = 0$	$x^2 - 2x + x - 2 = 0$
$\alpha = 1, \beta = 3$	$x(x - 2) + 1(x - 2) = 0$
	$(x + 1)(x - 2) = 0$
	$\gamma = -1, \delta = 2$

Hence the required roots of the given biquadratic equation are 1, 3, -1, 2.

11. Solve the equation $x^4 + 4x^3 - 2x^2 - 12x + 9 = 0$, if it has two pairs of equal roots.

A: Given that $x^4 + 4x^3 - 2x^2 - 12x + 9 = 0$ has two pairs of equal roots.

Let the roots be $\alpha, \alpha, \beta, \beta$.

$$s_1 = \alpha + \alpha + \beta + \beta = -4$$

$$\Rightarrow 2(\alpha + \beta) = -4$$

$$\Rightarrow \alpha + \beta = -2$$

$$s_3 = \alpha^2\beta + \alpha^2\beta + \alpha\beta^2 + \alpha\beta^2 = 12$$

$$\Rightarrow 2\alpha\beta(\alpha + \beta) = 12$$

$$\Rightarrow \alpha\beta(\alpha + \beta) = 6$$

$$\Rightarrow \alpha\beta(-2) = 6$$

$$\Rightarrow \alpha\beta = -3$$

The quadratic equation whose roots are α, β is

$$x^2 - (\alpha + \beta)x + \alpha\beta = 0$$

$$\Rightarrow x^2 + 2x - 3 = 0$$

$$\Rightarrow x^2 + 3x - x - 3 = 0$$

$$\Rightarrow x(x + 3) - 1(x + 3) = 0.$$

$$\Rightarrow (x + 3)(x - 1) = 0$$

Hence the required roots of the given biquadratic equation are $-3, -3, 1, 1$.

12. Find the roots of $x^4 - 16x^3 + 86x^2 - 176x + 105 = 0$.

A: Given equation is

$$f(x) = x^4 - 16x^3 + 86x^2 - 176x + 105 = 0$$

$$\text{Now } f(1) = 1 - 16 + 86 - 176 + 105$$

$$= 192 - 192$$

$$= 0.$$

So, 1 is a root of $f(x) = 0$.

By Synthetic division,

1	-16	86	-176	105	
1	0	1	-15	71	-105
1	-15	71	-105	0	

Now, the given equation can be written as

$$(x - 1)(x^3 - 15x^2 + 71x - 105) = 0$$

$$\text{Let } g(x) = x^3 - 15x^2 + 71x - 105$$

$$g(1) \neq 0, g(2) \neq 0, g(3) = 0.$$

So, 3 is a root of $g(x) = 0$.

1	-15	71	-105	
3	0	3	-36	105
1	-12	35	0	

Now $f(x) = 0$ can be written as

$$(x - 1)(x - 3)(x^2 - 12x + 35) = 0$$

$$x^2 - 12x + 35 = 0$$

$$(x - 5)(x - 7) = 0$$

$$x = 5, 7.$$

Hence, the required roots of the given biquadratic equation are $1, 3, 5, 7$.

13. Solve the equation $x^4 + 2x^3 - 5x^2 + 6x + 2 = 0$, given that one root of it is $1 + i$.

A: For the equation $x^4 + 2x^3 - 5x^2 + 6x + 2 = 0$, one root is $1 + i$.

So, the other root is $1 - i$.

The quadratic equation whose roots are

$$1 + i, 1 - i \text{ is } x^2 - (1 + i + 1 - i)x + (1 + i)(1 - i) = 0$$

$$\Rightarrow x^2 - 2x + 2 = 0$$

By synthetic division,

1	2	-5	6	2	
2	0	2	8	2	0
-2	0	0	-2	-8	-2
1	4	1	0	0	

Remainder is 0,

Quotient is $x^2 + 4x + 1$.

Now the given equation can be written as

$$(x^2 - 2x + 2)(x^2 + 4x + 1) = 0$$

$$x^2 + 4x + 1 = 0$$

$$x = \frac{-4 \pm \sqrt{16 - 4}}{2}$$

$$= \frac{-4 \pm 2\sqrt{3}}{2}$$

$$= -2 \pm \sqrt{3}$$

Hence the required roots of the given biquadratic equation are $1 + i, 1 - i, -2 + \sqrt{3}, -2 - \sqrt{3}$.



14. Find the multiple roots of the equation $x^5 - 3x^4 - 5x^3 + 27x^2 - 32x + 12 = 0$.

A: Given equation is

$$f(x) = x^5 - 3x^4 - 5x^3 + 27x^2 - 32x + 12 = 0$$

Differentiating $f(x)$ w.r.t. x ,

$$f'(x) = 5x^4 - 12x^3 - 15x^2 + 54x - 32$$

$$f'(1) = 5 - 12 - 15 + 54 - 32$$

$$= 59 - 59$$

$$= 0.$$

$$\text{Also } f(1) = 1 - 3 - 5 + 27 - 32 + 12$$

$$= 40 - 40$$

$$= 0.$$

Since $f'(1) = 0, f(1) = 0$, so 1 is a multiple root of order 2 for the given equation.

By Synthetic division

	1	-3	-5	27	-32	12
1	0	1	-2	-7	20	-12
	1	-2	-7	20	-12	0
1	0	1	-1	-8	12	
	1	-1	-8	12	0	

Now the given equation can be written as

$$(x - 1)(x - 1)(x^3 - x^2 - 8x + 12) = 0$$

$$\text{Let } g(x) = x^3 - x^2 - 8x + 12$$

differentiating w.r.t. x,

$$g'(x) = 3x^2 - 2x - 8$$

Now $g'(1) \neq 0$

$$g'(2) = 12 - 4 - 8 = 0$$

$$\text{Also } g(2) = 8 - 4 - 16 + 12 = 0.$$

Since $g'(2) = 0, g(2) = 0$, so 2 is a multiple root of order 2 for the cubic equation $x^3 - x^2 - 8x + 12 = 0$

$$s_1 = 2 + 2 + \gamma = -\frac{b}{a} = 1$$

$$\gamma = 1 - 4 = -3$$

Hence the required roots of the given 5th degree equation are 1, 1, 2, 2, -3.

15. Find the algebraic equation of degree 5 whose roots are the translates of roots of $x^5 + 4x^3 - x^2 + 11 = 0$ by -3.

A: Given equation is $f(x) = x^5 + 4x^3 - x^2 + 11 = 0$.

Here the roots are translated by '-3', so the transformed equation is $f(x + 3) = 0$.

By Horner's method,

	1	0	4	-1	0	11
3	0	3	9	39	114	342
	1	3	13	38	114	353 = A ₅
3	0	3	18	93	393	
	1	6	31	131	507 = A ₄	
3	0	3	27	174		
	1	9	58	305 = A ₃		
3	0	3	36			
	1	12	94 = A ₂			
3	0	3				
	1 = A ₀	15 = A ₁				

∴ Required transformed equation is

$$x^5 + 15x^4 + 94x^3 + 305x^2 + 507x + 353 = 0.$$

16. Transform $x^4 + 4x^3 + 2x^2 - 4x - 2 = 0$ into another equation in which the coefficient of second highest power of x is zero and find the transformed equation.

A: Given equation is $f(x) = x^4 + 4x^3 + 2x^2 - 4x - 2 = 0$.

Comparing this with $p_0x^4 + p_1x^3 + p_2x^2 + p_3x + p_4 = 0$.

Here $p_0 = 1, p_1 = 4, n = 4$ (degree)

To eliminate the second term, $f(x) = 0$ is transformed to $f(x + h) = 0$ where h is given by

$$h = \frac{-p_1}{n.p_0} = \frac{-4}{4(1)} = -1$$

∴ Required transformed equation is $f(x - 1) = 0$.

By Horner's method

$$f(x - 1) = A_0x^4 + A_1x^3 + A_2x^2 + A_3x + A_4$$

	1	4	2	-4	-2
-1	0	-1	-3	1	3
	1	3	-1	-1	1 = A ₄
-1	0	-1	-2	3	
	1	2	-3	0 = A ₃	
-1	0	-1	-1		
	1	1	-4 = A ₂		
-1	0	-1			
	1 = A ₀	0 = A ₁			

∴ Required transformed equation is $x^4 - 4x^2 + 1 = 0$.

17. Solve $2x^5 + x^4 - 12x^3 - 12x^2 + x + 2 = 0$.

A: Given equation is $2x^5 + x^4 - 12x^3 - 12x^2 + x + 2 = 0$.

It is a reciprocal equation of first class and odd degree. So -1 is a root of it.

By Synthetic division,

	2	1	-12	-12	1	2
-1	0	-2	1	11	1	-2
	2	-1	-11	-1	2	0

Now the given equation can be written as $(x + 1)(2x^4 - x^3 - 11x^2 - x + 2) = 0$.

Consider $2x^4 - x^3 - 11x^2 - x + 2 = 0. \quad \div x^2$

$$\Rightarrow 2x^2 - x - 11 - \frac{1}{x} + \frac{2}{x^2} = 0.$$

$$\Rightarrow 2 \left(x^2 + \frac{1}{x^2} \right) - \left(x + \frac{1}{x} \right) - 11 = 0.$$

Put $x + \frac{1}{x} = z \Rightarrow x^2 + \frac{1}{x^2} = z^2 - 2$.

Now the above equation becomes,

$$\Rightarrow 2(z^2 - 2) - z - 11 = 0.$$

$$\Rightarrow 2z^2 - z - 15 = 0.$$

$$\Rightarrow 2z^2 - 6z + 5z - 15 = 0.$$

$$\Rightarrow 2z(z - 3) + 5(z - 3) = 0.$$

$$z - 3 = 0$$

$$x + \frac{1}{x} - 3 = 0.$$

$$x^2 - 3x + 1 = 0.$$

$$x = \frac{3 \pm \sqrt{9-4}}{2}$$

$$= \frac{3 \pm \sqrt{5}}{2}$$

$$2z + 5 = 0.$$

$$2\left(x + \frac{1}{x}\right) + 5 = 0.$$

$$2x^2 + 5x + 2 = 0.$$

$$2x^2 + 4x + x + 2 = 0.$$

$$2x(x + 2) + 1(x + 2) = 0.$$

$$(x + 2)(2x + 1) = 0.$$

$$x = \frac{-1}{2}, -2$$

Hence the roots of the given 5th degree equation

are $-1, \frac{3 \pm \sqrt{5}}{2}, \frac{-1}{2}, -2$.

18. Solve the equation

$$6x^6 - 25x^5 + 31x^4 - 31x^2 + 25x - 6 = 0.$$

A: This is a reciprocal equation of 2nd class and even degree. So 1, -1 are the roots of it.

By Synthetic division,

	6	-25	31	0	-31	25	-6
1	0	6	-19	12	12	-19	6
	6	-19	12	12	-19	6	0
-1	0	-6	25	-37	25	-6	
	6	-25	37	-25	6	0	

Now the given equation can be written as $(x - 1)(x + 1)(6x^4 - 25x^3 + 37x^2 - 25x + 6) = 0$.

Consider $6x^4 - 25x^3 + 37x^2 - 25x + 6 = 0 \quad \div x^2$

$$\Rightarrow 6x^2 - 25x + 37 - \frac{25}{x} + \frac{6}{x^2} = 0.$$

$$6\left(x^2 + \frac{1}{x^2}\right) - 25\left(x + \frac{1}{x}\right) + 37 = 0.$$

Put $x + \frac{1}{x} = z \Rightarrow x^2 + \frac{1}{x^2} = z^2 - 2$.

Now the above equation becomes,

$$\Rightarrow 6(z^2 - 2) - 25z + 37 = 0.$$

$$\Rightarrow 6z^2 - 25z + 25 = 0.$$

$$\Rightarrow 6z^2 - 15z - 10z + 25 = 0.$$

$$\Rightarrow 3z(2z - 5) - 5(2z - 5) = 0.$$

$$\Rightarrow (3z - 5)(2z - 5) = 0.$$

$$3z - 5 = 0.$$

$$3\left(x + \frac{1}{x}\right) - 5 = 0.$$

$$3x^2 - 5x + 3 = 0.$$

$$x = \frac{5 \pm \sqrt{25-36}}{2(3)}$$

$$= \frac{5 \pm \sqrt{11}i}{6}$$

$$2z - 5 = 0.$$

$$2\left(x + \frac{1}{x}\right) - 5 = 0.$$

$$2x^2 - 5x + 2 = 0.$$

$$2x^2 - 4x - x + 2 = 0.$$

$$2x(x - 2) - 1(x - 2) = 0.$$

$$(x - 2)(2x - 1) = 0.$$

$$x = 2, \frac{1}{2}$$



Hence the required roots of the given 6th degree

reciprocal equation are $-1, 1, \frac{5 \pm \sqrt{11}i}{6}, 2, \frac{1}{2}$.

LEVEL - II (VSAQ)

1. Form the monic polynomial equation of degree 3 whose roots are 2, 3 and 6.

A: The polynomial equation whose roots 2, 3, 6 is $(x - 2)(x - 3)(x - 6) = 0$.

$$\Rightarrow (x^2 - 5x + 6)(x - 6) = 0$$

$$\Rightarrow x^3 - 6x^2 - 5x^2 + 30x + 6x - 36 = 0$$

$$\Rightarrow x^3 - 11x^2 + 36x - 36 = 0.$$

2. If 1, 1, α are the roots of $x^3 - 6x^2 + 9x - 4 = 0$ then find 'α'.

A: Given equation is $x^3 - 6x^2 + 9x - 4 = 0$

Given that 1, 1, α are the roots of given equation

$$\text{then } s_1 = 1 + 1 + \alpha = -p_1$$

$$\Rightarrow 2 + \alpha = -(-6)$$

$$\Rightarrow \alpha = 6 - 2$$

$$\Rightarrow \boxed{\alpha = 4}$$

3. If a, b, c are the roots of $x^3 - px^2 + qx - r = 0$

then find $\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}$.

A: Given equation is $x^3 - px^2 + qx - r = 0$(1).

Given that 'a, b, c' are the roots of (1), then

$$a + b + c = p, ab + bc + ca = q, abc = r.$$

$$\begin{aligned} \therefore \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} &= \frac{b^2c^2 + a^2c^2 + a^2b^2}{a^2b^2c^2} \\ &= \frac{(bc + ac + ab)^2 - 2abc(a + b + c)}{(abc)^2} = \frac{q^2 - 2rp}{r^2}. \end{aligned}$$

4. If α, β, γ are the roots of $x^3 - ax^2 + bx + c = 0$, then find $\Sigma\alpha^2\beta + \Sigma\alpha\beta^2$.

A: Given that α, β, γ are the roots of $x^3 + ax^2 + bx + c = 0$ then

$$s_1 \Rightarrow \alpha + \beta + \gamma = -a$$

$$s_2 \Rightarrow \alpha\beta + \beta\gamma + \gamma\alpha = b$$

$$s_3 \Rightarrow \alpha\beta\gamma = -c$$

$$\Sigma\alpha^2\beta + \Sigma\alpha\beta^2 = \alpha^2\beta + \beta^2\gamma + \gamma^2\alpha + \alpha\beta^2 + \beta\gamma^2 + \gamma\alpha^2.$$

$$= (\alpha + \beta + \gamma)(\alpha\beta + \beta\gamma + \gamma\alpha) - 3\alpha\beta\gamma$$

$$= (-a)(b) - 3(-c)$$

$$= s_1s_2 - 3s_3 = 3c - ab.$$

5. Find the equation whose roots are the cubes of the roots of $x^3 + 3x^2 + 2 = 0$.

A: Let $f(x) = x^3 + 3x^2 + 2$.

Formula : The equation whose roots are the cubes of the roots of $f(x) = 0$ is $f(\sqrt[3]{x}) = 0$.

$$\Rightarrow (\sqrt[3]{x})^3 + 3(\sqrt[3]{x})^2 + 2 = 0 \Rightarrow x + 3(x^{2/3}) + 2 = 0.$$

$$\Rightarrow (x + 2) = -3x^{2/3} \text{ [cubing on both sides]}$$

$$\Rightarrow (x + 2)^3 = (-3x^{2/3})^3$$

$$= x^3 + 2^3 + 3(x^2)(2) + 3x(2^2) = -27x^2$$

$$= x^3 + 8 + 6x^2 + 12x + 27x^2 = 0$$

$$\Rightarrow \boxed{x^3 + 33x^2 + 12x + 8 = 0}$$

LEVEL - II (LAQ)

1. Solve the equation $x^5 - 5x^4 + 9x^3 - 9x^2 + 5x - 1 = 0$

A: Given equation $x^5 - 5x^4 + 9x^3 - 9x^2 + 5x - 1 = 0$. This is a reciprocal equation of 2nd class and odd degree.

So, 1 is a root of this equation.

Therefore 'x - 1' is a factor of it

By synthetic division.

$$\begin{array}{r|rrrrrr} 1 & 1 & -5 & 9 & -9 & 5 & -1 \\ & & 0 & 1 & -4 & 5 & -4 & 1 \\ \hline & 1 & -4 & 5 & -4 & 1 & 0 \end{array}$$

Now the given equation can be written as

$$(x - 1)(x^4 - 4x^3 + 5x^2 - 4x + 1) = 0$$

consider, $x^4 - 4x^3 + 5x^2 - 4x + 1 = 0$

On dividing this equation by x^2 , we get

$$x^2 - 4x + 5 - \frac{4}{x} + \frac{1}{x^2} = 0$$

AIMS

$$\text{i.e. } \left(x^2 + \frac{1}{x^2}\right) - 4\left(x + \frac{1}{x}\right) + 5 = 0.$$

$$\text{Put } x + \frac{1}{x} = z \Rightarrow x^2 + \frac{1}{x^2} = z^2 - 2$$

Now the above equation becomes

$$\Rightarrow z^2 - 2 - 4z + 5 = 0$$

$$\Rightarrow z^2 - 4z + 3 = 0$$

$$\Rightarrow (z - 3)(z - 1) = 0$$

$$\Rightarrow z = 1 \text{ or } 3$$

Case: 1. If $z = 1$

Case: 2 If $z = 3$

$$\Rightarrow x + \frac{1}{x} = 1$$

$$\Rightarrow x + \frac{1}{x} = 3$$

$$\Rightarrow x^2 - x + 1 = 0$$

$$\Rightarrow x^2 - 3x + 1 = 0$$

$$\Rightarrow x = \frac{1 \pm \sqrt{1 - 4}}{2}$$

$$\therefore x = \frac{3 \pm \sqrt{9 - 4}}{2}$$

$$\Rightarrow \frac{1 \pm \sqrt{-3}}{2}$$

$$\Rightarrow \frac{3 \pm \sqrt{5}}{2}$$

$$\Rightarrow \frac{1 \pm i\sqrt{3}}{2}$$