

**DEFINITIONS, CONCEPTS AND FORMULAE**

1. If  $n$  is a positive integer,  $(x + a)^n = {}^nC_0 x^n + {}^nC_1 x^{n-1} \cdot a + {}^nC_2 x^{n-2} \cdot a^2 + \dots + {}^nC_r x^{n-r} \cdot a^r + \dots + {}^nC_n \cdot a^n$ .
2.  ${}^nC_0, {}^nC_1, {}^nC_2, \dots, {}^nC_n$  are called binomial coefficients.
3. No. of terms in  $(x+a)^n$  is  $n + 1$ .
4. No. of terms in  $(x+y+z)^n$  is  $\frac{(n+1)(n+2)}{2}$ .
5. General term in  $(x+a)^n$  is  $T_{r+1} = {}^nC_r \cdot x^{n-r} \cdot a^r$ .
6. In  $(x+a)^n$ ,
  - i) if  $n$  is even then the middle term is  $T_{\frac{n+1}{2}}$
  - ii) If  $n$  is odd, then the two middlemost terms are  $T_{\frac{n+1}{2}}$  &  $T_{\frac{n+3}{2}}$
7. Term independent of 'x' or constant term is the term containing  $x^0$ .
8. In  $(1+x)^n$ ,
  - i) if  $\frac{(n+1)|x|}{|x|+1} = p$  (an integer), then the two numerically greatest terms are  $|T_p|$  &  $|T_{p+1}|$
  - ii) if  $\frac{(n+1)|x|}{|x|+1} = p + f$  where  $p$  is an integer,  $0 < f < 1$  then the only numerically greatest term is  $|T_{p+1}|$
9. i)  $C_0 + C_1 + C_2 + \dots + C_n = 2^n$   
 ii)  $C_0 + C_2 + C_4 + \dots = C_1 + C_3 + C_5 + \dots = 2^{n-1}$
10. i)  $a \cdot C_0 + (a+d)C_1 + (a+2d)C_2 + \dots + (a+nd)C_n = (2a+nd)2^{n-1}$   
 ii)  $a \cdot C_0 - (a+d)C_1 + (a+2d)C_2 - \dots + n+1$  terms  $= 0$
11. i)  $C_1 + 2 \cdot C_2 x + 3 \cdot C_3 x^2 + \dots + n \cdot C_n x^{n-1} = n(1+x)^{n-1}$   
 ii)  $C_1 + 2 \cdot C_2 + 3 \cdot C_3 + \dots + n \cdot C_n = n \cdot 2^{n-1}$   
 iii)  $C_1 - 2 \cdot C_2 + 3 \cdot C_3 - \dots + {}^nC_n (-1)^{n-1} = 0$
12. i)  $C_0 + \frac{C_1}{2} x + \frac{C_2}{3} x^2 + \dots + \frac{C_n}{n+1} x^n = \frac{(1+x)^{n+1} - 1}{(n+1)x}$   
 ii)  $C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \dots + \frac{C_n}{n+1} = \frac{2^{n+1} - 1}{n+1}$   
 iii)  $C_0 - \frac{C_1}{2} + \frac{C_2}{3} - \dots + \frac{C_n}{n+1} (-1)^n = \frac{1}{n+1}$   
 iv)  $C_0 + \frac{C_2}{3} + \frac{C_4}{5} + \dots = \frac{2^n}{n+1}$

- v)  $\frac{C_1}{2} + \frac{C_3}{4} + \frac{C_5}{6} + \dots = \frac{2^n - 1}{n+1}$
13.  $C_0 C_r + C_1 C_{r+1} + C_2 C_{r+2} + \dots + C_{n-r} C_n = {}^{2n}C_{n+r} = \frac{(2n)!}{(n-r)!(n+r)!}$   
 ii)  $C_0^2 + C_1^2 + C_2^2 + \dots + C_n^2 = {}^{2n}C_n$
14. If  $n$  is a rational number and  $|x| < 1$ , then
  - i)  $(1+x)^n = 1 + nx + \frac{n(n-1)}{1 \cdot 2} x^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} x^3 + \dots$
  - ii)  $(1-x)^n = 1 + nx + \frac{n(n+1)}{1 \cdot 2} x^2 + \frac{n(n+1)(n+2)}{1 \cdot 2 \cdot 3} x^3 + \dots$
15. If  $|x| < 1$  and  $p, q$  are positive integers then
  - i)  $(1-x)^{-p/q} = 1 + \frac{p}{1!} \left(\frac{x}{q}\right) + \frac{p(p+q)}{2!} \left(\frac{x}{q}\right)^2 + \dots$
  - ii)  $(1+x)^{-p/q} = 1 - \frac{p}{1!} \left(\frac{x}{q}\right) + \frac{p(p+q)}{2!} \left(\frac{x}{q}\right)^2 - \dots$
16. If  $|x| < 1$ , then
  - i)  $(1-x)^{-1} = 1 + x + x^2 + x^3 + \dots + x^n + \dots$
  - ii)  $(1+x)^{-1} = 1 - x + x^2 - x^3 + \dots + (-1)^n x^n + \dots$
  - iii)  $(1-x)^{-2} = 1 + 2x + 3x^2 + \dots + (n+1)x^n + \dots$
  - iv)  $(1+x)^{-2} = 1 - 2x + 3x^2 - \dots + (-1)^n (n+1)x^n + \dots$
  - v)  $(1-x)^{-3} = 1 + 3x + 6x^2 + 10x^3 + \dots + \frac{(n+1)(n+2)}{2} x^n + \dots$
  - vi)  $(1+x)^{-3} = 1 - 3x + 6x^2 - 10x^3 + \dots + (-1)^n \frac{(n+1)(n+2)}{2} x^n + \dots$
17. If  $|x| < 1$  and  $n$  is a rational number,
  - i)  $(1+x)^n \approx 1 + \frac{n}{1!} x$  if  $x^2$  and higher powers of  $x$  are neglected.
  - ii)  $(1+x)^n \approx 1 + \frac{n}{1!} x + \frac{n(n-1)}{2!} x^2$  if  $x^3$  and higher powers of  $x$  are neglected.
  - iii)  $(1-x)^n \approx 1 + \frac{n}{1!} x$  if  $x^2$  and higher powers of  $x$  are neglected.
  - iv)  $(1-x)^n \approx 1 + \frac{n}{1!} x + \frac{n(n+1)}{2!} x^2$  if  $x^3$  and higher powers of  $x$  are neglected.

### LEVEL - I (VSAQ)

1. Find the number of terms in  $(2a + 3b + c)^5$ .

A: Number of terms in  $(2a + 3b + c)^5$

$$\begin{aligned} &= \frac{(n+1)(n+2)}{2!} \\ &= \frac{(5+1)(5+2)}{2} \\ &= \frac{6 \times 7}{2} \\ &= 21. \end{aligned}$$

2. Find the 3<sup>rd</sup> term from the end in the expansion

of  $\left(x^{\frac{-2}{3}} - \frac{3}{x^2}\right)^8$ .

A: In  $\left(x^{\frac{-2}{3}} - \frac{3}{x^2}\right)^8$ , 3<sup>rd</sup> term from the end

$$\begin{aligned} &= T_7 \\ &= T_{6+1} \\ &= {}^8C_6 (x^{-2/3})^{8-6} (-3/x^2)^6 \\ &= {}^8C_2 x^{-4/3} \times 36/x^{12} \\ &= {}^8C_2 \times 3^6/x^{40/3}. \end{aligned}$$

3. Find the coefficient of  $x^{-6}$  in  $(3x - 4/x)^{10}$

A:  $T_{r+1} = {}^{10}C_r (3x)^{10-r} (-4/x)^r$   
 $= {}^{10}C_r \cdot 3^{10-r} (-4)^r \cdot x^{10-r-r}$

To get the coefficient of  $x^{-6}$ ,

$$10 - 2r = -6$$

$$\Rightarrow 2r = 16$$

$$\Rightarrow r = 8$$

$$\begin{aligned} \text{Coefficient of } x^{-6} &= {}^{10}C_8 \cdot 3^2 \cdot (-4)^8 \\ &= {}^{10}C_2 \cdot 3^2 \cdot 4^8. \end{aligned}$$

4. Find the coefficient of  $x^{-7}$  in  $\left(\frac{2x^2}{3} - \frac{5}{4x^5}\right)^7$ .

A: In  $\left(\frac{2x^2}{3} - \frac{5}{4x^5}\right)^7$ ,

$$\begin{aligned} T_{r+1} &= {}^7C_r \cdot \left(\frac{2x^2}{3}\right)^{7-r} \left(\frac{-5}{4x^5}\right)^r \\ &= {}^7C_r \cdot \left(\frac{2}{3}\right)^{7-r} \left(\frac{-5}{4}\right)^r \cdot x^{14-7r} \end{aligned}$$

To get the coefficient of  $x^{-7}$ ,

$$14 - 7r = 7$$

$$7r = 21$$

$$r = 3$$

$\therefore$  Coefficient of  $x^{-7}$

$$\begin{aligned} &= {}^7C_3 \left(\frac{2}{3}\right)^{7-3} \left(\frac{-5}{4}\right)^3 \\ &= -35 \left(\frac{2^4}{3^4}\right) \left(\frac{5^3}{4^3}\right) \\ &= \frac{-4375}{324}. \end{aligned}$$

5. Find the term independent of  $x$  in  $\left(\frac{\sqrt{x}}{3} - \frac{4}{x^2}\right)^{10}$ .

A: General term  $T_{r+1} = {}^{10}C_r \left(\frac{\sqrt{x}}{3}\right)^{10-r} \left(\frac{-4}{x^2}\right)^r$

$$= {}^{10}C_r \frac{(-4)^r}{3^{10-r}} \cdot x^{\frac{10-r}{2} - 2r}$$

To get the term independent of  $x$ ,  $\frac{10-r}{2} - 2r = 0$ .

$$10 - 5r = 0 \Rightarrow r = 2.$$

$\therefore$  Term independent of  $x$

$$\begin{aligned} &= {}^{10}C_2 \frac{(-4)^2}{3^8} \\ &= \frac{45 \times 16}{3^8} = \frac{80}{729}. \end{aligned}$$

6. Find the numerically greatest terms the expansion of  $(3 + 2a)^{15}$  when  $a = 5/2$ .

A:  $(3 + 2a)^{15} = 3^{15} \left(1 + \frac{2a}{3}\right)^{15}$

$$|x| = \left|\frac{2a}{3}\right| = \left|\frac{2 \cdot 5}{3 \cdot 2}\right| = \frac{5}{3}$$

$$\text{Now } \frac{(n+1)|x|}{|x|+1} = \frac{(15+1) \cdot 5/3}{8/3} = 10$$

$\therefore |T_{10}|$  and  $|T_{11}|$  are numerically greatest.

$$|T_{10}| = {}^{15}C_9 \cdot 3^6 \left(2 \cdot \frac{5}{2}\right)^9 = {}^{15}C_9 \cdot 3^6 \cdot 5^9$$

$$|T_{11}| = {}^{15}C_{10} \cdot 3^5 \left(2 \cdot \frac{5}{2}\right)^{10} = {}^{15}C_0 \cdot 3^5 \cdot 3^{10}$$

and  $|T_{10}| = |T_{11}|$ .

7. Find the numerically greatest term in the expansion of  $(3x + 5y)^{12}$  when  $x = 1/2, y = 4/3$ .

A:  $(3x + 5y)^{12} = (3x)^{12} \left(1 + \frac{5y}{3x}\right)^{12}$

$$|x| = \left|\frac{5}{3} \cdot \frac{4}{3} \cdot \frac{2}{1}\right| = \frac{40}{9}$$

$$\text{Now } \frac{(n+1)|x|}{|x|+1} = \frac{13x \cdot \frac{40}{9}}{\frac{49}{9}} = \frac{520}{9} = 10.4$$

∴ Numerically greatest term

$$\begin{aligned} &= |T_{10+1}| \\ &= |{}^{12}C_{10} (3 \cdot \frac{1}{2})^{12-10} (5 \cdot \frac{4}{3})^{10}| \\ &= {}^{12}C_2 \cdot (3/2)^2 (20/3)^{10} \end{aligned}$$

8. If the coefficients of  $(2r + 4)^{\text{th}}$  and  $(r - 2)^{\text{nd}}$  terms in the expansion of  $(1 + x)^{18}$  are equal, find r

A:  $\ln(1 + x)^{18}, T_{2r+4} = T_{(2r+3)+1}$   
 $= {}^{18}C_{2r+3}$

$$\begin{aligned} T_{r-2} &= T_{(r-3)+1} \\ &= {}^{18}C_{r-3} \end{aligned}$$

But  ${}^{18}C_{2r+3} = {}^{18}C_{r-3}$

$$\Rightarrow r = s \quad n = r + s$$

$$\Rightarrow 2r + 3 = r - 3 \quad 18 = 2r + 3 + r - 3$$

$$\Rightarrow r = -6 \quad 18 = 3r$$

is not possible  $r = 6$

∴  $r = 6$ .

9. If  ${}^{22}C_r$  is the largest binomial coefficient in the expansion of  $(1 + x)^{22}$  find the value of  ${}^{13}C_r$ .

A:  $= {}^nC_{n/2}$  if n is even  
 $= {}^{22}C_{11}$

$$r = 11.$$

$$\begin{aligned} \text{Now } {}^{13}C_r &= {}^{13}C_{11} \\ &= {}^{13}C_2 \end{aligned}$$

$$= \frac{13 \times 12}{2}$$

$$= 78.$$

then prove that (i)  $a_0 + a_1 + a_2 + \dots + a_{20} = 2^{10}$

(ii)  $a_0 - a_1 + a_2 - \dots + a_{20} = 4^{10}$

A:  $(1 + 3x - 2x^2)^{10} = a_0 + a_1x + a_2x^2 + \dots + a_{20}x^{20}$

Put  $x = 1$  in the above relation,

$$a_0 + a_1 \cdot 1 + a_2 \cdot 1^2 + \dots + a_{20} \cdot 1^{20} = (1 + 3 - 2)^{10}$$

$$\Rightarrow a_0 + a_1 + a_2 + \dots = a_{20} = 2^{10}$$

Put  $x = -1$  in the given relation,

$$a_0 + a_1(-1) + a_2(-1)^2 + \dots + a_{20}(-1)^{20} = (1 - 3 - 2)^{10}$$

$$a_0 - a_1 + a_2 - \dots + a_{20} = 4^{10}$$

11. Obtain the values of x for which the binomial expansion of  $(2 + 3x)^{-2/3}$  is valid.

A:  $(2 + 3x)^{-2/3} = 2^{-2/3} (1 + 3x/x)^{-2/3}$

The above expansion is valid if

$$|3x/x| < 1$$

$$\Rightarrow |x| < 2/3$$

$$\Rightarrow x \in \left(-\frac{2}{3}, \frac{2}{3}\right).$$

12. Find the values of x for which the binomial expansion  $(7 + 3x)^{-5}$  is valid.

A:  $(7 + 3x)^{-5} = 7^{-5} \left(1 + \frac{3x}{7}\right)^{-5}$

The above expansion is valid if  $\left|\frac{3x}{7}\right| < 1$

$$\Rightarrow |x| < \frac{7}{3}$$

$$\Rightarrow x \in \left(-\frac{7}{3}, \frac{7}{3}\right).$$

13. If  $C_r$  denote  ${}^nC_r$ , then prove that

$$a C_0 + (a+d) C_1 + (a+2d) C_2 + \dots + (a+nd) C_n = 2^{n-1}.$$

A: Let

$$S = a C_0 + (a+d) C_1 + (a+2d) C_2 + \dots + (a+nd) C_n \quad (1)$$

$$\therefore C_r = C_{n-r}$$

$$S = (a + nd) C_0 + [a + (n - 1)d] C_1 + [a + (n - 2)d]$$

$$C_2 + \dots + a C_n \quad (2)$$

$$(1) + (2) \Rightarrow 2S = (2a + nd) C_0 + (2a + nd) C_1 +$$

$$(2a + nd) C_2 + \dots + (2a + nd) C_n.$$

$$\Rightarrow 2S = (2a + nd) [C_0 + C_1 + C_2 + \dots + C_n]$$

$$\Rightarrow 2S = (2a + nd) 2^n$$

$$\therefore S = (2a + nd) 2^{n-1}.$$

**1. State and prove 'Binomial theorem' for a positive integral index.**

A: If n is a positive integer, then

$$(x+a)^n = {}^nC_0 x^n + {}^nC_1 x^{n-1} \cdot a + {}^nC_2 x^{n-2} \cdot a^2 + \dots + {}^nC_r x^{n-r} \cdot a^r + \dots + {}^nC_n \cdot a^n.$$

Let S(n) be the given statement.

If n = 1, LHS = (x + a)<sup>1</sup> = x + a

RHS = <sup>1</sup>C<sub>0</sub> x<sup>1</sup> + <sup>1</sup>C<sub>1</sub> x<sup>1-1</sup> . a = x + a

∴ LHS = RHS.

Thus S(1) is true

Assume that S(k) is true.

$$\therefore (x + a)^k = {}^kC_0 x^k + {}^kC_1 x^{k-1} \cdot a + {}^kC_2 x^{k-2} \cdot a^2 + \dots + {}^kC_r x^{k-r} \cdot a^r + \dots + {}^kC_k a^k.$$

Now (x + a)<sup>k+1</sup> = (x + a) (x + a)<sup>k</sup>

$$= (x + a) [{}^kC_0 x^k + {}^kC_1 x^{k-1} \cdot a + {}^kC_2 x^{k-2} \cdot a^2 + \dots + {}^kC_r x^{k-r} \cdot a^r + \dots + {}^kC_k a^k]$$

$$= {}^kC_0 x^{k+1} + {}^kC_1 x^k \cdot a + {}^kC_2 x^{k-1} \cdot a^2 + \dots + {}^kC_r x^{k-r+1} \cdot a^r + \dots + {}^kC_k \cdot x \cdot a^k$$

$$+ {}^kC_0 x^k \cdot a + {}^kC_1 x^{k-1} \cdot a^2 + {}^kC_r x^{k-r} \cdot a^{r+1} + \dots + {}^kC_k a^{k+1}$$

$$= x^{k+1} + ({}^kC_1 + {}^kC_0)x^k \cdot a + ({}^kC_2 + {}^kC_1)x^{k-1} \cdot a^2 + \dots + ({}^kC_r + {}^kC_{r-1})x^{k+1-r} \cdot a^r + \dots + a^{k+1}$$

$$\therefore {}^nC_r + {}^nC_{n-r} = {}^{n+1}C_r$$

$$= {}^{k+1}C_0 x^{k+1} + {}^{k+1}C_1 x^k \cdot a + {}^{k+1}C_2 x^{k-1} \cdot a^2 + \dots + {}^{k+1}C_r \cdot x^{k+1-r} \cdot a^r + \dots + {}^{k+1}C_{k+1} a^{k+1}$$

∴ S(k + 1) is also true.

Hence, by the principle of mathematical induction S(n) is true for all n ∈ N.

**2. If 2<sup>nd</sup>, 3<sup>rd</sup> and 4<sup>th</sup> terms in the expansion of (a + x)<sup>n</sup> are respectively 240, 720, 1080, find a, x, m.**

A: In (a + x)<sup>n</sup>, T<sub>2</sub> = T<sub>1+1</sub>

$$= {}^nC_1 \cdot a^{n-1} \cdot x = 240 \text{ ----- (1)}$$

$$T_3 = T_{2+1}$$

$$= {}^nC_2 \cdot a^{n-2} \cdot x^2 = 720 \text{ ----- (2)}$$

$$T_4 = T_{3+1}$$

$$= {}^nC_3 \cdot a^{n-3} \cdot x^3 = 1080 \text{ ----- (3)}$$

$$\frac{2}{1} \Rightarrow \frac{{}^nC_2 a^{n-2} x^2}{{}^nC_1 a^{n-1} x} = \frac{720}{240}$$

$$\Rightarrow \frac{n(n-1)}{2} \frac{a^n}{a^2} \frac{x^2 a}{a^n x \cdot n} = 3$$

$$\Rightarrow \frac{n(n-1)x}{2a} = 3$$

$$\Rightarrow (n-1)x = 6a \text{ ----- (4)}$$

$$\frac{3}{2} \Rightarrow \frac{{}^nC_3 a^{n-3} x^3}{{}^nC_2 a^{n-2} x^2} = \frac{1080}{72}$$

$$\Rightarrow \frac{n(n-1)(n-2)}{6} \frac{a^n}{a^3} \frac{x^3 \cdot 2 \cdot a^2}{n(n-1)a^n x^2} = \frac{1080}{72}$$

$$\Rightarrow \frac{(n-2)x}{3a} = \frac{3}{2}$$

$$\Rightarrow (n-2)(2x) = 9a \text{ ----- (5)}$$

$$\frac{5}{4} \Rightarrow \frac{(n-2)(2x)}{(n-1)x} = \frac{9a}{6a}$$

$$\Rightarrow 4n - 8 = 3n - 3$$

$$\Rightarrow \therefore n = 5$$

From (4), 4x = 6a

$$x = \frac{3a}{2} \text{ ----- (6)}$$

Now (1) becomes <sup>5</sup>C<sub>1</sub> a<sup>4</sup> x = 240

$$\Rightarrow 5a^4 \cdot \frac{3a}{2} = 240$$

$$\Rightarrow a^5 = 32 = 2^5$$

$$\therefore a = 2$$

$$\text{From (6) } x = \frac{3a}{2} = \frac{3(2)}{2} = 3$$

∴ n = 5, a = 2, x = 3.

**3. If the coefficient of x<sup>10</sup> in the expansion of (ax<sup>2</sup> + 1/bx)<sup>11</sup> is equal to the coefficient of x<sup>-10</sup>**

**in the expansion of (ax - 1/bx<sup>2</sup>)<sup>11</sup>, find the relation between a and b, where a and b**

are real numbers.

A:  $\ln \left( ax^2 + \frac{1}{bx^2} \right)^{11}$ ,

$$T_{r+1} = {}^{11}C_r \cdot (ax^2)^{11-r} \left( \frac{1}{bx^2} \right)^r$$

$$= {}^{11}C_r \cdot \frac{a^{11-r}}{b^r} \cdot x^{22-3r}$$

To get the coefficient of  $x^{10}$ ,

$$22 - 3r = 10$$

$$12 = 3r$$

$$r = 4$$

$\therefore$  Coefficient of  $x^{10}$  in  $\left( ax^2 + \frac{1}{bx^2} \right)^{11} = {}^{11}C_4 \cdot \frac{a^7}{b^4}$

$$\ln \left( ax^2 - \frac{1}{bx^2} \right)^{11}, T_{r+1} = {}^{11}C_r \cdot (ax^2)^{11-r} \left( \frac{-1}{bx^2} \right)^r$$

$$= {}^{11}C_r \cdot (-1)^r \cdot \frac{a^{11-r}}{b^r} \cdot x^{11-3r}$$

To get the coefficient of  $x^{-10}$ ,

$$11 - 3r = -10$$

$$21 = 3r$$

$$r = 7.$$

Thus, coefficient of  $x^{-10}$  in  $\left( ax - \frac{1}{bx^2} \right)^{11}$  is

$$= (-1)^7 {}^{11}C_7 \cdot \frac{a^4}{b^7}$$

But  ${}^{11}C_4 \cdot \frac{a^7}{b^4} = -{}^{11}C_7 \cdot \frac{a^4}{b^7} \quad \therefore {}^nC_r = {}^nC_{n-r}$

$$\Rightarrow a^3 = \frac{-1}{b^3}$$

$$\Rightarrow (ab^3) = -1$$

$$\therefore ab = -1. \quad (\because a, b \in \mathbb{R})$$

4. If  $(7 + 4\sqrt{3})^n = I + f$  where  $I$  and  $n$  are positive integers and  $0 < f < 1$ , then show that  
 i)  $I$  is an odd positive integer  
 ii)  $(I + f)(1 - f) = 1$ .

A: i) Given that  $I, n$  are positive integers.

$$0 < f < 1 \text{ and } (7 + 4\sqrt{3})^n = I + f.$$

Clearly  $0 < 7 - 4\sqrt{3} < 1$

$$\Rightarrow 0 < (7 - 4\sqrt{3})^n < 1$$

$$\text{Let } (7 - 4\sqrt{3})^n = x$$

$$\therefore 0 < x < 1$$

$$\underline{0 < f < 1} \text{ given}$$

on addition  $0 < f + x < 2$  ----- (1)

Now  $(I + f) + x$

$$= (7 + 4\sqrt{3})^n + (7 - 4\sqrt{3})^n$$

$$= {}^nC_0 \cdot 7^n + {}^nC_1 \cdot 7^{n-1} (4\sqrt{3}) + {}^nC_2 \cdot 7^{n-2} \cdot (4\sqrt{3})^2 + \dots$$

$$+ {}^nC_n (4\sqrt{3})^n + {}^nC_0 7^n - {}^nC_1 \cdot 7^{n-1} (4\sqrt{3})$$

$$+ {}^nC_2 \cdot 7^{n-2} (4\sqrt{3})^2 - \dots + {}^nC_2 (-4\sqrt{3})^n.$$

$$= 2[{}^nC_0 \cdot 7^n + {}^nC_2 \cdot 7^{n-2} (4\sqrt{3})^2 + \dots]$$

= 2 (some integer)

$\therefore I + f + x = \text{an even integer}$

$\Rightarrow f + x = \text{an even integer} - (I)$

$\Rightarrow f + x = \text{some integer} \text{ -----}(2)$

Combining (1), (2) the only possibility left is  $f + x = 1$ .

Now  $I + f + x = \text{an even integer}$

$I + 1 = \text{an even integer}$

$\Rightarrow I = \text{an even integer} - 1$

$\Rightarrow I$  is an odd integer

ii)  $(I + f)(1 - f) = (I + f)x$

$$= (7 + 4\sqrt{3})^n + (7 - 4\sqrt{3})^n$$

$$= (49 - 48)^n$$

$$= 1^n$$

$$= 1.$$

5. If  $R, n$  are positive integers,  $n$  is odd,  $0 < F < 1$  and if  $(5\sqrt{5} + 11)^n = R + F$ , then prove that  
 i)  $R$  is an even integer  
 ii)  $(R + F)F = 4^n$ .

A: i) Given that  $R, n$  are positive integers,  $n$  is odd,  
 $0 < F < 1$  and  $R + F = (5\sqrt{5} + 11)^n$ .

Clearly  $0 < 5\sqrt{5} - 11 < 1$

$$\Rightarrow 0 < (5\sqrt{5} - 11)^n < 1$$

$$\text{Let } (5\sqrt{5} - 11)^n = x$$

$$\begin{aligned} \therefore & \Rightarrow 0 < x < 1 \\ & \Rightarrow 0 > -x > -1 \\ & \Rightarrow -1 < -x < 0 \end{aligned}$$

Also  $0 < F < 1$

on addition  $-1 < F - x < 1$  ----- (1)

$$\begin{aligned} \text{Now } (R + F) - x &= (5\sqrt{5} + 11)^n - (5\sqrt{5} - 11)^n \\ &= {}^nC_0 \cdot (5\sqrt{5})^n + {}^nC_1 (5\sqrt{5})^{n-1} (11) + {}^nC_2 (5\sqrt{5})^{n-2} (11)^2 + \dots + {}^nC_n (-11)^n \end{aligned}$$

$$= -\{ {}^nC_0 \cdot (5\sqrt{5})^n - {}^nC_1 (5\sqrt{5})^{n-1} (11) + {}^nC_2 (5\sqrt{5})^{n-2} (11)^2 - \dots + {}^nC_n (-11)^n \}$$

$$= 2\{ {}^nC_1 (5\sqrt{5})^{n-1} (11) + {}^nC_3 (5\sqrt{5})^{n-3} (11^3) + \dots \}$$

= 2 (some integer)

$\therefore R + F - x = \text{an even integer}$

$\Rightarrow F - x = \text{an even integer} - (R)$

$\Rightarrow F - x = \text{some integer} \text{ -----}(2)$

Combing (1), (2) the only possibility left is  $F - x = 0$ .

$$F - x = 0$$

$$\Rightarrow F = x$$

Now  $R + F - x = \text{an even integer}$

$$\Rightarrow R + 0 = \text{an even integer}$$

So R is an even integer

ii)  $(R + F)F = (R + F)x$

$$= (5\sqrt{5} + 11)^n + (5\sqrt{5} - 11)^n$$

$$= (125 - 121)^n$$

$$= 4^n.$$

**6. If P and Q are sum of odd terms and sum of even terms respectively in the expansion of  $(x + a)^n$ , then prove that**

i)  $P^2 - Q^2 = (x^2 - a^2)^n$ .

ii)  $4PQ = (x + a)^{2n} - (x - a)^{2n}$ .

A: Now

$$(x+a)^n = {}^nC_0 x^n + {}^nC_1 x^{n-1} \cdot a + {}^nC_2 x^{n-2} \cdot a^2 + \dots + {}^nC_3 x^{n-3} \cdot a^3 + \dots$$

Given that  $P = T_1 + T_3 + \dots$

$$= {}^nC_0 x^n + {}^nC_2 x^{n-2} \cdot a^2 + \dots$$

$$Q = T_2 + T_4 + \dots$$

$$= {}^nC_1 x^{n-1} \cdot a + {}^nC_3 x^{n-3} \cdot a^3 + \dots$$

Now  $P + Q = (x + a)^n$  and  $P - Q = (x - a)^n$ .

i)  $P^2 - Q^2 = (P + Q)(P - Q)$

$$= (x + a)^n (x - a)^n$$

$$= [(x + a)(x - a)]^n$$

$$= (x^2 - a^2)^n.$$

ii)  $4PQ = (P + Q)^2 - (P - Q)^2$

$$= \{(x + a)^n\}^2 - \{(x - a)^n\}^2$$

$$= (x + a)^{2n} - (x - a)^{2n}.$$

**7. With usual notation, prove that**

$$C_0 + C_2 + C_4 + \dots = C_1 + C_3 + C_5 + \dots = 2^{n-1}.$$

A: We know that

$$(1 + x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + {}^nC_3 x^3 + \dots + {}^nC_n x^n.$$

$$\Rightarrow (1 + x)^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \dots + C_n x^n \text{ ---- (1)}$$

Put  $x = 1$  in (1), we get

$$C_0 + C_1(1) + C_2(1^2) + C_3(1^3) + \dots + C_n(1^n) = (1 + 1)^n$$

$$\therefore C_0 + C_1 + C_2 + \dots + C_n = 2^n \text{ ---- (2)}$$

Put  $x = -1$  in (1), we get

$$C_0 + C_1(-1) + C_2(-1)^2 + C_3(-1)^3 + \dots + C_n(-1)^n = (1 - 1)^n$$

$$\therefore C_0 - C_1 + C_2 - C_3 + \dots + (-1)^n C_n = 0 \text{ ---- (3)}$$

$$(2) + (3) \Rightarrow 2[C_0 + C_2 + C_4 + \dots] = 2^n + 0 = 2^n$$

$$\Rightarrow C_0 + C_2 + C_4 + \dots = \frac{2^n}{2} = 2^{n-1} \text{ ---- (4)}$$

$$(2) - (3) \Rightarrow 2[C_1 + C_3 + C_5 + \dots] = 2^n - 0 = 2^n$$

$$\Rightarrow C_1 + C_3 + C_5 + \dots = \frac{2^n}{2} = 2^{n-1} \text{ ---- (5)}$$

Combing (4) & (5), we get

$$C_0 + C_2 + C_4 + \dots = C_1 + C_3 + C_5 + \dots = 2^{n-1}.$$

**8. With usual notation, prove that**

$$C_1 + 2C_2 x + 3C_3 x^2 + \dots + n C_n x^{n-1} = n(1 + x)^{n-1}$$

Deduce that  $C_1 + 2C_2 + 3C_3 + \dots + n C_n = n \cdot 2^{n-1}$ .

A: We know that

$$(1 + x)^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \dots + C_n x^n.$$

Differentiating w.r.t. x,

$$\frac{d}{dx} [C_0 + C_1 x + C_2 x^2 + \dots + C_n x^n] = \frac{d}{dx} (1 + x)^n.$$

$$\Rightarrow \frac{d}{dx} (C_0) + C_1 \frac{d}{dx} (x) + C_2 \frac{d}{dx} (x^2) + \dots + C_n \cdot n x^{n-1}$$

$$= \frac{d}{dx} (1+x)^n$$

$$\Rightarrow 0 + C_1 \cdot 1 + C_2 \cdot 2x + C_3 \cdot 3x^2 + \dots + C_n \cdot nx^{n-1}$$

$$= n(1+x)^{n-1}$$

$$\Rightarrow C_1 + 2C_2x + 3C_3x^2 + \dots + {}^nC_n x^{n-1}$$

$$= n(1+x)^{n-1} \text{ -----(1)}$$

Put x = 1 in (1), we get

$$C_1 + 2C_2(1) + 3C_3(1^2) + \dots + nC_n(1^{n-1}) = n(1+1)^{n-1}$$

$$\Rightarrow C_1 + 2C_2 + 3C_3 + \dots + nC_n = n(2^{n-1}).$$

9. Prove that  $C_0 + \frac{C_1}{2}x + \frac{C_2}{3}x^2 + \dots + \frac{C_n}{n+1}x^n$

$$= \frac{(1+x)^{n+1} - 1}{(n+1)x}$$

Deduce that  $\frac{C_1}{2} + \frac{C_3}{4} + \frac{C_5}{6} + \dots = \frac{2^n - 1}{n+1}$ .

A: Now  $C_0 + \frac{C_1}{2}x + \frac{C_2}{3}x^2 + \dots + \frac{C_n}{n+1}x^n$

$$= {}^nC_0 + \frac{{}^nC_1}{2}x + \frac{{}^nC_2}{3}x^2 + \dots + \frac{{}^nC_n}{n+1}x^n$$

$$= 1 + \frac{n}{1.2}x + \frac{n(n-1)}{1.2.3}x^2 + \dots + \frac{1}{n+1}x^n$$

$$= \frac{1}{(n+1)x} \left[ (n+1)x + \frac{(n+1)n}{2!}x^2 + \frac{(n+1)n(n-1)}{3!}x^3 + \dots + \frac{n+1}{n+1}x^{n+1} \right]$$

$$= \frac{1}{(n+1)x} [{}^{n+1}C_1x + {}^{n+1}C_2x^2 + {}^{n+1}C_3x^3 + \dots + 1 \cdot x^{n+1}]$$

$$= \frac{1}{(n+1)x} [{}^{n+1}C_0 + {}^{n+1}C_1x + {}^{n+1}C_2x^2 + \dots + {}^{n+1}C_{n+1}x^{n+1} - 1]$$

$$= \frac{(1+x)^{n+1} - 1}{(n+1)x}$$

$$\therefore C_0 + \frac{C_1}{2}x + \frac{C_2}{3}x^2 + \dots + \frac{C_n}{n+1}x^n = \frac{(1+x)^{n+1} - 1}{(n+1)x} \text{ -----(1)}$$

Put x = 1 in (1), we get

$$C_0 + \frac{C_1}{2}(1) + \frac{C_2}{3}(1^2) + \dots + \frac{C_n}{n+1}(1^n) = \frac{(1+1)^{n+1} - 1}{(n+1)(1)}$$

$$\Rightarrow C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \dots + \frac{C_n}{n+1} = \frac{2^{n+1} - 1}{n+1} \text{ ----- (2)}$$

Put x = -1 in (1), we get

$$C_0 + \frac{C_1}{2}(-1) + \frac{C_2}{3}(-1^2) + \dots + \frac{C_n}{n+1}(-1)^n$$

$$= \frac{(1-1)^{n+1} - 1}{(n+1)(-1)}$$

$$C_0 - \frac{C_1}{2} + \frac{C_2}{3} - \dots + (-1)^n \frac{C_n}{n+1} = \frac{1}{n+1} \text{ ----- (3)}$$

$$(2) - (3) \Rightarrow 2 \left[ \frac{C_1}{2} + \frac{C_3}{4} + \frac{C_5}{6} + \dots \right] = \frac{2^{n+1} - 1 - 1}{n+1}$$

$$\Rightarrow 2 \left[ \frac{C_1}{2} + \frac{C_3}{4} + \frac{C_5}{6} + \dots \right] = \frac{2(2^n - 1)}{n+1}$$

$$\therefore \frac{C_1}{2} + \frac{C_3}{4} + \frac{C_5}{6} + \dots = \frac{2^n - 1}{n+1}$$

10. Prove that  $C_0 C_r + C_1 C_{r+1} + C_2 C_{r+2} + \dots + C_{n-r} C_n = 2^n C_{n+r}$

Deduce  $C_0^2 + C_1^2 + C_2^2 + \dots + C_n^2 = 2^n C_n$ .

A: We know that

$$(1+x)^n = C_0 + C_1x + \dots + C_{n-r}x^{n-r} + \dots + C_nx^n \text{ ----- (1)}$$

Also  $(x+1)^n = C_0x^n + C_1x^{n-1} + \dots + C_r x^{n-r} + C_{r+1}x^{n-(r+1)}$

$$+ C_{r+2}x^{n-(r+2)} + \dots + C_n \text{ ----- (2)}$$

Multiplying (1) & (2) and equating the coefficient of  $x^{n-r}$  on both sides,

$$C_0 + C_r + C_1 C_{r+1} + C_2 C_{r+2} + \dots + C_{n-r} C_n$$

$$= \text{coe. of } x^{n-r} \text{ in } (1+x)^n (x+1)^n$$

$$= \text{coe. of } x^{n-r} \text{ in } (1+x)^n (1+x)^n$$

$$= \text{coe. of } x^{n-r} \text{ in } (1+x)^{2n} \quad \ln(1+x)2n$$

$$= {}^{2n}C_{n-r} \quad T_{r+1} = {}^{2n}C_r \cdot x^r$$

$$= {}^{2n}C_{2n-(n-r)}$$

$$= {}^{2n}C_{n+r}$$

Put  $r = 0$  in the above relation, we get

$$C_0 + C_0 + C_1 C_{0+1} + C_2 C_{0+2} + \dots + C_{n-0} C_n = {}^{2n}C_{n+0}$$

$$\therefore C_0^2 + C_1^2 + C_2^2 + \dots + C_n^2 = {}^{2n}C_n.$$

**11. Prove that  $(C_0 + C_1)(C_1 + C_2)(C_2 + C_3) \dots (C_{n-1} + C_n)$**

$$= \frac{(n+1)^n}{n!} C_0 C_1 C_2 \dots C_n.$$

A: Now  $(C_0 + C_1)(C_1 + C_2)(C_2 + C_3) \dots (C_{n-1} + C_n)$

$$= C_0 \left(1 + \frac{C_1}{C_0}\right) C_1 \left(1 + \frac{C_2}{C_1}\right) C_2 \left(1 + \frac{C_3}{C_2}\right) \dots C_{n-1} \left(1 + \frac{C_n}{C_{n-1}}\right)$$

$$= C_0 C_1 C_2 \dots C_{n-1} \left(1 + \frac{n}{1}\right) \left(1 + \frac{n(n-1)}{2} \frac{1}{n}\right)$$

$$\left(1 + \frac{n(n-1)(n-2)}{6} \frac{2}{n(n-1)}\right) \dots \left(1 + \frac{1}{n}\right)$$

$$= C_0 C_1 C_2 \dots C_{n-1} \cdot 1 \left(\frac{1+n}{1}\right) \left(\frac{1+n-1}{2}\right) \left(\frac{1+n-2}{3}\right)$$

$$\dots \left(1 + \frac{1}{n}\right)$$

$$= C_0 C_1 C_2 \dots C_{n-1} C_n \left(\frac{1+n}{1}\right) \left(\frac{1+n}{2}\right) \left(\frac{1+n}{3}\right) \dots$$

$$\left(\frac{1+n}{n}\right)$$

$$= C_0 C_1 C_2 \dots C_{n-1} C_n \frac{(1+n)^n}{n!}.$$

**12. If the coefficients of  $r^{\text{th}}, (r+1)^{\text{th}}, (r+2)^{\text{nd}}$  terms in the expansion of  $(1+x)^n$  are in A.P., then show that  $n^2 - (4r+1)n + 4r^2 - 2 = 0$ .**

A: In  $(1+x)^n$ ,

$$T_r = T_{(r-1)+1} = {}^nC_{r-1} x^{r-1}$$

$$T_{r+1} = {}^nC_r x^r$$

$$T_{r+2} = T_{(r+1)+1} = {}^nC_{r+1} x^{r+1}$$

It is given that  ${}^nC_{r-1}, {}^nC_r, {}^nC_{r+1}$  are in A.P.

$$\Rightarrow 2 \cdot {}^nC_r = {}^nC_{r-1} + {}^nC_{r+1}$$

$$\Rightarrow 2 \cdot \frac{n!}{(n-r)!r!} = \frac{n!}{[n-(r-1)]!(r-1)!} + \frac{n!}{[n-(r+1)]!(r+1)!}$$

$$\Rightarrow \frac{2}{(n-r)[n-(r+1)]!r(r-1)!} = \frac{2}{[n-(r-1)](n-r)[n-(r+1)]!(r-1)!}$$

$$+ \frac{1}{[n-(r+1)]!(r+1)r(r-1)!}$$

$$\Rightarrow \frac{2}{(n-r)r} = \frac{1}{(n-r+1)(n-r)} + \frac{1}{(r+1)r}$$

$$\Rightarrow \frac{2}{(n-r)r} = \frac{(r+1)r + (n-r+1)(n-r)}{(n-r+1)(n-r)(r+1)r}$$

$$\Rightarrow 2(n-r+1)(r+1) = (r+1)r + (n-r+1)(n-r)$$

$$\Rightarrow 2(nr + n - r^2 - r + r + 1) = r^2 + r + n^2 - 2nr + r^2 + n - r$$

$$\Rightarrow 0 = n^2 - 4nr - n + 4r^2 - 2$$

$$\Rightarrow n^2 - (4r+1)n + (4r^2 - 2) = 0.$$

**13. If the coefficients of 4 consecutive terms in the expansion of  $(1+x)^n$  are  $a_1, a_2, a_3, a_4$**

respectively, then show that  $\frac{a_1}{a_1+a_2} + \frac{a_3}{a_3+a_4} = \frac{2a_2}{a_2+a_3}$ .

A: In  $(1+x)^n$ ,

$$T_r = T_{(r-1)+1} = {}^nC_{r-1} \cdot x^{r-1}$$

$$T_{r+1} = {}^nC_r \cdot x^r$$

$$T_{r+2} = T_{(r+1)+1} = {}^nC_{r+1} \cdot x^{r+1}$$

$$T_{r+3} = T_{(r+2)+1} = {}^nC_{r+2} \cdot x^{r+2}$$

Given that  $a_1 = {}^nC_{r-1}, a_2 = {}^nC_r,$

$$a_3 = {}^nC_{r+1}, a_4 = {}^nC_{r+2}$$

$$\text{LHS} = \frac{a_1}{a_1+a_2} + \frac{a_3}{a_3+a_4}$$



$$\begin{aligned}
 &= \frac{{}^n C_{r-1}}{{}^n C_{r-1} + {}^n C_r} + \frac{{}^n C_{r+1}}{{}^n C_{r+1} + {}^n C_{r+2}} \\
 &\qquad \qquad \qquad \because {}^n C_{r-1} + {}^n C_r = {}^{n+1} C_r \\
 &= \frac{{}^n C_{r+1}}{{}^{n+1} C_r} + \frac{{}^n C_{r+1}}{{}^{n+1} C_{r+2}} \\
 &= \frac{n!}{[n-(r-1)]!(r-1)!} \cdot \frac{(n+1-r)!}{(n+1)!} + \frac{n!}{[n-(r+1)]!(r+1)!} \\
 &\qquad \qquad \qquad \cdot \frac{[n+1-(r+2)]!(r+2)!}{(n+1)!} \\
 &= \frac{r}{n+1} + \frac{r+2}{n+1} \\
 &= \frac{2(r+1)}{n+1} \text{ ----- (1)}
 \end{aligned}$$

$$\begin{aligned}
 \text{RHS} &= \frac{2a_2}{a_2 + a_3} \\
 &= \frac{2 \cdot {}^n C_r}{{}^n C_r + {}^n C_{r+1}} \\
 &= \frac{2 \cdot {}^n C_r}{{}^{n+1} C_{r+1}} \\
 &= \frac{2 \cdot n!}{(n-r)!r!} \times \frac{[n+1-(r+1)]!(r+1)!}{(n+1)!} \\
 &= \frac{2(r+1)}{n+1} \text{ ----- (2)}
 \end{aligned}$$

From (1) & (2),

$$\frac{a_1}{a_1 + a_2} + \frac{a_3}{a_3 + a_4} = \frac{2a_2}{a_2 + a_3}$$

14. If n is a positive integer, prove that

$$\sum_{r=1}^n r^3 \left( \frac{{}^n C_r}{{}^n C_{r-1}} \right)^2 = \frac{n(n+1)^2 (n+2)}{12}$$

$$\begin{aligned}
 \text{A: Now } \sum_{r=1}^n r^3 \left( \frac{{}^n C_r}{{}^n C_{r-1}} \right)^2 &= \sum_{r=1}^n r^3 \left[ \frac{n!}{(n-r)!r!} \times \frac{[n-(r-1)]!(r-1)!}{n!} \right]^2 \\
 &= \sum_{r=1}^n r^3 \left( \frac{n-r+1}{r} \right)^2 \\
 &= \sum_{r=1}^n r [(n+1)-r]^2 \\
 &= \sum_{r=1}^n r [(n+1)^2 - 2(n+1)r + r^2] \\
 &= \sum_{r=1}^n [(n+1)^2 r - 2(n+1)r^2 + r^3] \\
 &= (n+1)^2 \sum_{r=1}^n r - 2(n+1) \sum_{r=1}^n r^2 + \sum_{r=1}^n r^3 \\
 &= (n+1)^2 \frac{n(n+1)}{2} - 2(n+1) \frac{n(n+1)(2n+1)}{6} \\
 &\qquad \qquad \qquad + \frac{n^2(n+1)^2}{4} \\
 &= \frac{n(n+1)^2}{12} [6n+1 - 4(2n+1) + 3n] \\
 &= \frac{n(n+1)^2}{12} [6n+6 - 8n - 4 + 3n] \\
 &= \frac{n(n+1)^2}{12} [n+2] \\
 &= \frac{n(n+1)^2 (n+2)}{12}
 \end{aligned}$$

15. Show that for any non - zero rational number

$$\begin{aligned}
 x, 1 + \frac{x}{2} + \frac{x(x-1)}{2.4} + \frac{x(x-1)(x-2)}{2.4.6} + \dots \dots \dots \infty \\
 = 1 + \frac{x}{3} + \frac{x(x+1)}{3.6} + \frac{x(x+1)(x+2)}{3.6.9} + \dots \dots \dots \infty
 \end{aligned}$$

$$\text{A: LHS} = 1 + \frac{x}{2} + \frac{x(x-1)}{2.4} + \frac{x(x-1)(x-2)}{2.4.6} + \dots \dots \dots \infty$$

$$= 1 + \frac{x}{1!} \left(\frac{1}{2}\right) + \frac{x(x-1)}{2!} \left(\frac{1}{2}\right)^2 + \frac{x(x-1)(x-2)}{3!} \left(\frac{1}{2}\right)^3 + \dots$$

$$\therefore (1-x)^{-n} = 1 + \frac{n}{1!}x + \frac{n(n+1)}{2!}x^2 + \dots$$

$$= \left(1 - \frac{1}{3}\right)^{-x}$$

$$= \left(\frac{2}{3}\right)^{-x}$$

$$= \left(\frac{3}{2}\right)^x \text{ ---- (2)}$$

From (1) & (2),

$$1 + \frac{x}{2} + \frac{x(x-1)}{2.4} + \frac{x(x-1)(x-2)}{2.4.6} + \dots$$

$$= 1 + \frac{x}{3} + \frac{x(x+1)}{3.6} + \frac{x(x+1)(x+2)}{3.6.9} + \dots$$

**16. Find the sum to infinite series**

$$\frac{7}{5} \left[ 1 + \frac{1}{10^2} + \frac{1.3}{1.2} + \left(\frac{1}{10^4}\right) + \frac{1.3.5}{1.2.3} + \left(\frac{1}{10^6}\right) + \dots \right]$$

A: Comparing the infinite series with

$$(1-x)^{-p/q} = 1 + \frac{p}{1!} \left(\frac{x}{q}\right) + \frac{p(p+q)}{2!} \left(\frac{x}{q}\right)^2 + \dots$$

$$\frac{7}{5} \left[ 1 + \frac{1}{1!} + \left(\frac{1}{100}\right) + \frac{1.3}{2!} + \left(\frac{1}{100}\right)^2 + \frac{1.3.5}{3!} + \left(\frac{1}{100}\right)^3 + \dots \right]$$

Here  $p = 1, p + q = 3$

$$q = 3 - 1 = 2$$

$$\frac{x}{q} = \frac{1}{100}$$

$$\Rightarrow x = \frac{2}{100} = \frac{1}{50}$$

$$= \frac{7}{5} [1 - x]^{-p/q}$$

$$= \frac{7}{5} \left(1 - \frac{1}{50}\right)^{-1/2}$$

$$= \frac{7}{5} \left(\frac{49}{50}\right)^{-1/2}$$

$$= \frac{7}{5} \sqrt{\frac{50}{49}}$$

$$= \frac{7}{5} \left(\frac{5\sqrt{2}}{7}\right)$$

$$= \sqrt{2}$$

**17. Find the sum of infinite series**

$$\frac{3}{4} + \frac{3.5}{4.8} + \frac{3.5.7}{4.8.12} + \dots$$

A: Now  $\frac{3}{4} + \frac{3.5}{4.8} + \frac{3.5.7}{4.8.12} + \dots$

$$= 1 + \frac{3}{1!} \left(\frac{1}{4}\right) + \frac{3.5}{2!} \left(\frac{1}{4}\right)^2 + \frac{3.5.7}{3!} \left(\frac{1}{4}\right)^3 + \dots$$

Comparing the infinite series with

$$(1-x)^{-p/q} = 1 + \frac{p}{1!} \left(\frac{x}{q}\right) + \frac{p(p+q)}{2!} \left(\frac{x}{q}\right)^2 + \dots$$

Here  $p = 3$

$$p + q = 5 \Rightarrow q = 2$$

$$\frac{x}{q} = \frac{1}{4} \Rightarrow x = \frac{2}{4} = \frac{1}{2}$$

$\therefore$  Sum of the given infinite series

$$= (1-x)^{-p/q} - 1$$

$$= \left(1 - \frac{1}{2}\right)^{-3/2} - 1$$

$$= \left(\frac{1}{2}\right)^{-3/2}$$

$$= 2^{3/2} - 1$$

$$= 2\sqrt{2} - 1$$

18. If  $x = \frac{1}{5} + \frac{1.3}{5.10} + \frac{1.3.5}{5.10.15} + \dots \infty$ , then find the value of  $3x^2 + 6x$ .

A: Given  $x = \frac{1}{5} + \frac{1.3}{5.10} + \frac{1.3.5}{5.10.15} + \dots \infty$

$$\Rightarrow x = \frac{1}{1!} + \left(\frac{1}{5}\right) + \frac{1.3}{2!} + \left(\frac{1}{5}\right)^2 + \frac{1.3.5}{3!} + \left(\frac{1}{5}\right)^3 + \dots \infty$$

$$\Rightarrow x + 1 = 1 + \frac{1}{1!} + \left(\frac{1}{5}\right) + \frac{1.3}{2!} + \left(\frac{1}{5}\right)^2 + \dots \infty$$

Comparing this with

$$(1 - y)^{-p/q} = 1 + \frac{p}{1!} \left(\frac{y}{q}\right) + \frac{p(p+q)}{2!} \left(\frac{y}{q}\right)^2 + \dots \infty$$

Here  $p = 1$

$$p + q = 3 \Rightarrow q = 2$$

$$\frac{y}{q} = \frac{1}{5} \Rightarrow y = \frac{2}{5}$$

$$\therefore x + 1 = (1 - y)^{-p/1}$$

$$= \left(1 - \frac{2}{5}\right)^{-1/2}$$

$$= \left(\frac{3}{5}\right)^{-1/2}$$

$$x + 1 = \sqrt{\frac{5}{3}}$$

Squaring on both sides,

$$x^2 + 2x + 1 = \frac{5}{3}$$

$$\Rightarrow 3x^2 + 6x + 3 = 5$$

$$\Rightarrow 3x^2 + 6x = 5 - 3 = 2.$$

19. If  $x = \frac{1.3}{3.6} + \frac{1.3.5}{3.6.9} + \frac{1.3.5.7}{3.6.9.12} + \dots \infty$ , then prove that  $9x^2 + 24x = 11$ .

A: Given  $x = \frac{1.3}{3.6} + \frac{1.3.5}{3.6.9} + \frac{1.3.5.7}{3.6.9.12} + \dots \infty$

$$\Rightarrow x = \frac{1.3}{2!} + \left(\frac{1}{3}\right)^2 + \frac{1.3.5}{3!} + \left(\frac{1}{3}\right)^3 + \dots \infty$$

$$\Rightarrow x + 1 + \frac{1}{1!} \left(\frac{1}{3}\right) = 1 + \frac{1}{1!} \left(\frac{1}{3}\right) + \frac{1.3}{2!} \left(\frac{1}{3}\right)^2 +$$

$$\frac{1.3.5}{3!} \left(\frac{1}{3}\right)^3 + \dots \infty$$

Comparing RHS with

$$(1 - y)^{-p/q} = 1 + \frac{p}{1!} \left(\frac{y}{q}\right) + \frac{p(p+q)}{2!} \left(\frac{y}{q}\right)^2 + \dots \infty$$

Here  $p = 1$

$$p + q = 3 \Rightarrow q = 2$$

$$\frac{y}{q} = \frac{1}{3} \Rightarrow y = \frac{2}{3}$$

$$\therefore x + \frac{4}{3} = (1 - y)^{-p/q}$$

$$= \left(1 - \frac{2}{3}\right)^{-1/2}$$

$$= \left(\frac{1}{3}\right)^{-1/2}$$

$$\frac{3x + 4}{3} = \sqrt{3}$$

$$\Rightarrow 3x + 4 = 3\sqrt{3}$$

Squaring on both sides,

$$9x^2 + 24x + 16 = 27$$

$$\therefore 9x^2 + 24x = 11.$$

20. Find the sum of the infinite series

$$\frac{3.5}{5.10} + \frac{3.5.7}{5.10.15} + \frac{3.5.7.9}{5.10.15.20} + \dots \infty.$$

A: Now  $\frac{3.5}{5.10} + \frac{3.5.7}{5.10.15} + \frac{3.5.7.9}{5.10.15.20} + \dots \infty$

$$= \frac{3.5}{2!} \left(\frac{1}{5}\right)^2 + \frac{3.5.7}{3!} + \left(\frac{1}{5}\right)^2 + \dots \infty$$

$$= 1 + \frac{3}{1!} \left(\frac{1}{5}\right) + \frac{3 \cdot 5}{2!} + \left(\frac{1}{5}\right)^2 + \dots - \left(1 + \frac{3}{5}\right)$$

Comparing this with

$$(1-x)^{-p/q} = 1 + \frac{p}{1!} \left(\frac{x}{q}\right) + \frac{p(p+q)}{2!} \left(\frac{x}{q}\right)^2 + \dots$$

Here  $p = 3$

$$p + q = 5 \Rightarrow q = 2$$

$$\frac{x}{q} = \frac{1}{5} \Rightarrow x = \frac{2}{5}$$

Sum of the given infinite series

$$= (1-x)^{-p/q} - \frac{1}{5}$$

$$= \left(1 - \frac{2}{5}\right)^{-3/2} - \frac{8}{5}$$

$$= \left(\frac{3}{5}\right)^{-3/2} - \frac{8}{5}$$

$$= \frac{5\sqrt{5}}{3\sqrt{3}} - \frac{8}{5}$$

### LEVEL - II (VSAQ)

1. Find the number of terms with non-zero coefficients in  $(4x - 7y)^{49} + (4x - 7y)^{49}$ .

A: The number of terms in the expansion  $(x + y)^n + (x - y)^n$  when 'n' is odd is

$$\frac{n+1}{2} = \frac{49+1}{2} = 25.$$

2. Write down and simplify 6<sup>th</sup> term in

$$\left(\frac{2x}{3} + \frac{3y}{2}\right)^9.$$

A: 6<sup>th</sup> term =  $T_6 = T_{5+1}$ .

$$= {}^9C_5 \left(\frac{2x}{3}\right)^{9-5} \left(\frac{3y}{2}\right)^5 = {}^9C_5 \left(\frac{2x}{3}\right)^4 \left(\frac{3y}{2}\right)^5$$

$$= 126 \cdot \left[\frac{3}{2}\right] x^4 y^5 = 189 x^4 y^5.$$

3. If A and B are coefficients of  $x^n$  in the expansion of  $(1 + x)^{2n}$  and  $(1 + x)^{2n-1}$

respectively, then find the value of  $\frac{A}{B}$ .

A: A = coefficient of  $x^n$  in  $(1 + x)^{2n} = {}^{2n}C_n$ .

B = Coefficient of  $x^n$  in  $(1 + x)^{2n-1} = {}^{2n-1}C_n$ .

$$\frac{A}{B} = \frac{{}^{2n}C_n}{{}^{2n-1}C_n} = \frac{(2n)!}{(2n-n)!n!} \times \frac{(2n-1-n)!n!}{(2n-1)!}$$

$$= \frac{(2n)!}{(n!)^2} \times \frac{(n-1)!n!}{(2n-1)!} = \frac{(2n)(2n-1)!}{n!n(n-1)!} \times \frac{(n-1)n!}{(2n-1)!}$$

$$= \frac{2n}{n} = 2.$$

4. Find the largest binomial coefficient(s) in the expansion of  $(1 + x)^{24}$ .

A: Here  $n = 24$ , an even integer.

Hence there is only one largest binomial coefficient,

$$\text{that is } {}^nC_{\frac{n}{2}} = {}^{24}C_{12}.$$

5. Find the largest binomial coefficient(s) in the expansion of  $(1 + x)^{19}$ .

A: Here  $n = 19$  (odd).

∴ The largest binomial coefficients are

$${}^nC_{\frac{n-1}{2}}, {}^nC_{\frac{n+1}{2}} = {}^{19}C_9, {}^{19}C_{10}.$$

$$[\text{Note that } {}^{19}C_9 = {}^{19}C_{10}].$$

6. Find the middle terms in the expansion of

$$\left(4a + \frac{3}{2}b\right)^{11}.$$

A: Given expansion is  $\left(4a + \frac{3}{2}b\right)^{11}$ .

Here  $\boxed{n=11}$ , odd

So, middle terms are  $\frac{T_{11+1}}{2}, \frac{T_{11+3}}{2} = T_6, T_7$ .

$$T_6 = T_{5+1} = {}^{11}C_5 (4a)^{11-5} \left(\frac{3}{2}b\right)^5$$

$$= {}^{11}C_5 4^6 \cdot a^6 \cdot \left(\frac{3}{2}\right)^5 \cdot b^5 = {}^{11}C_5 \cdot 4^6 \cdot \left(\frac{3}{2}\right)^5 a^6 b^5$$

$$T_7 = T_{6+1} = {}^{11}C_6 (4a)^{11-6} \left(\frac{3}{5}b\right)^6$$

$$= {}^{11}C_5 4^5 \cdot a^5 \cdot \left(\frac{3}{2}\right)^6 \cdot b^6 = {}^{11}C_6 \cdot 4^5 \cdot \left(\frac{3}{2}\right)^6 a^5 b^6.$$

7. Find the middle term in the expansion of

$$\left(\frac{3x}{7} - 2y\right)^{10}.$$

A: Here  $n = 10$ , even

So, middle term =  $T_{\frac{10}{2}+1} = T_{5+1}$ .

$$\therefore T_{5+1} = {}^{10}C_5 \cdot \left(\frac{3x}{7}\right)^{10-5} \cdot (-2y)^5$$

$$= {}^{10}C_5 \cdot \left(\frac{3x}{7}\right)^5 \cdot (2y)^5 = {}^{10}C_5 \cdot \left(\frac{6}{7}\right)^5 x^5 y^5.$$

8. Find the coefficient of  $x^7$  in  $\left[\frac{3x^2}{7} + \frac{4}{5x^3}\right]^{11}$ .

A: Given expansion is  $\left[\frac{3x^2}{7} + \frac{4}{5x^3}\right]^{11}$ .

General term  $T_{r+1} = {}^nC_r \cdot x^{n-r} \cdot a^r$ .

$$= {}^{11}C_r \cdot \left(\frac{3x^2}{7}\right)^{11-r} \cdot \left(\frac{4}{5x^3}\right)^r$$

$$= {}^{11}C_r \cdot \left(\frac{3}{7}\right)^{11-r} \cdot x^{22-2r} \cdot \left(\frac{4}{5}\right)^r \cdot x^{-3r}$$

$$= {}^{11}C_r \cdot \left(\frac{3}{7}\right)^{11-r} \cdot \left(\frac{4}{5}\right)^r \cdot x^{22-5r}$$

take  $22 - 5r = 7 \Rightarrow 5r = 15 \Rightarrow r = 3$

Coefficient of  $x^7$

is  ${}^{11}C_3 \cdot \left(\frac{3}{7}\right)^{11-3} \cdot \left(\frac{4}{5}\right)^3 = {}^{11}C_3 \cdot \left(\frac{3}{7}\right)^8 \cdot \left(\frac{4}{5}\right)^3$ .

9. Find the term independent of  $x$  in the

expansion of  $\left(\frac{3}{\sqrt[3]{x}} + 5\sqrt{x}\right)^{25}$ .

A: General term  $T_{r+1} = {}^nC_r x^{n-r} a^r$ .

$$= {}^{25}C_r \left(\frac{3}{\sqrt[3]{x}}\right)^{25-r} (5\sqrt{x})^r = {}^{25}C_r 3^{25-r} \left(\frac{1}{x^{\frac{25-r}{3}}}\right) (5\sqrt{x})^r$$

$$= {}^{25}C_r 3^{25-r} 5^r x^{-\left(\frac{25-r}{3}\right) + \frac{r}{2}}$$

take  $\frac{-25+r}{3} + \frac{r}{2} = 0$  then

$$-50 + 2r + 3r = 0 \Rightarrow r = 10.$$

$\therefore$  The term independent of  $x$  is

$$T_{11} = {}^{25}C_{10} 3^{25-10} 5^{10} = {}^{25}C_{10} 3^{15} 5^{10}.$$

10. Find the term independent of  $x$  in the expansion of  $\left(4x^3 + \frac{7}{x^2}\right)^{14}$ .

A: General term  $T_{r+1} = {}^nC_r x^{n-r} a^r$ .

$$T_{r+1} = {}^{14}C_r (4x^3)^{14-r} \left(\frac{7}{x^2}\right)^r = {}^{14}C_r 4^{14-r} 7^r x^{42-5r}$$

If  $42 - 5r = 0$  then  $r = \frac{42}{5}$ .

Which is not possible.

$\therefore$  The term independent of  $x$  is '0'.

11. Prove that  $C_0 + 2.C_1 + 2^2.C_2 + \dots + 2^n . C_n = 3^n$ .

A: We know that  $C_0 + C_1 \cdot x + C_2 \cdot x^2 + \dots + C_n \cdot x^n = (1+x)^n$ .

Let  $x = 2$

then we get

$$C_0 + 2.C_1 + 2^2.C_2 + \dots + 2^n.C_n = 3^n.$$

12. Find the sum of  $3.C_0 + 6.C_1 + 12.C_2 + \dots + 3.2^n.C_n$ .

A: take  $3.C_0 + 6.C_1 + 12.C_2 + \dots + 3.2^n.C_n$ .

$$= 3.C_0 + 3.2.C_1 + 3.2^2.C_2 + \dots + 3.2^n.C_n$$

$$= 3[C_0 + 2.C_1 + 2^2.C_2 + \dots + 2^n.C_n]$$

$$= 3[(1+2)^n] = 3.3^n = 3^{n+1}.$$

13. Prove that

$$\frac{C_1}{C_0} + 2 \cdot \frac{C_2}{C_1} + 3 \cdot \frac{C_3}{C_2} + \dots + n \cdot \frac{C_n}{C_{n-1}} = \frac{n(n+1)}{2}.$$

$$\begin{aligned}
 \text{A: L.H.S.} &= \frac{C_1}{C_0} + 2 \cdot \frac{C_2}{C_1} + 3 \cdot \frac{C_3}{C_2} + \dots + n \cdot \frac{C_n}{C_{n-1}} \\
 &= \frac{n}{1} + 2 \frac{n(n-1)}{n} + 3 \frac{n(n-1)(n-2)}{n(n-1)} + \dots + n \frac{1}{n} \\
 &= n + (n-1) + (n-2) + \dots + 2 + 1. \\
 &= 1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2} = \text{R.H.S.}
 \end{aligned}$$

**14. Find the range of x for which the binomial expansion  $(3 - 4x)^{3/4}$  is valid.**

$$\begin{aligned}
 \text{A: } (3 - 4x)^{3/4} &= \left[ 3 \left( 1 - \frac{4x}{3} \right) \right]^{3/4} = 3^{3/4} \left( 1 - \frac{4x}{3} \right)^{3/4} \\
 \text{The expansion is valid when } &\left| \frac{-4x}{3} \right| < 1 \\
 \Rightarrow |x| < \frac{3}{4} \Rightarrow &\boxed{-\frac{3}{4} < x < \frac{3}{4}} \text{ (or) } \boxed{x \in \left( -\frac{3}{4}, \frac{3}{4} \right)}.
 \end{aligned}$$

**15. Write the first 3 terms in the expansion of  $(8 - 5x)^{2/3}$ .**

$$\begin{aligned}
 \text{A: } (8 - 5x)^{2/3} &= 8^{2/3} \left( 1 - \frac{5x}{8} \right)^{2/3} = 4 \left( 1 - \frac{5x}{8} \right)^{2/3} \\
 &= 4 \left[ 1 + \frac{2}{3} \left( -\frac{5x}{8} \right) + \frac{2}{3} \left( \frac{2}{3} - 1 \right) \frac{\left( -\frac{5x}{8} \right)^2}{2!} + \dots \right] \\
 &= 4 \left[ 1 - \frac{5x}{12} - \frac{25x^2}{576} + \dots \right]
 \end{aligned}$$

Therefore, the first three terms in the expansion of  $(8 - 5x)^{2/3}$  are  $4, \frac{-5x}{12}, \frac{-25x^2}{576}$ .

**16. Write the first 3 terms in the expansion of  $(2-7x)^{-3/4}$ .**

$$\text{A: } (2-7x)^{-3/4} = 2^{-3/4} \left( 1 - \frac{7x}{2} \right)^{-3/4}$$

$$\begin{aligned}
 &= 2^{-3/4} \left[ 1 + \binom{-3}{4} \left( -\frac{7x}{2} \right) + \frac{\binom{-3}{4} \binom{-3}{4} (-1)}{2!} \left( -\frac{7x}{2} \right)^2 + \dots \right] \\
 &= 2^{-3/4} \left[ 1 + \frac{21x}{8} + \frac{1029x^2}{128} + \dots \right] \\
 \text{The first 3 terms are} &2^{-3/4}, 2^{-3/4} x \frac{21x}{8}, 2^{-3/4} x \frac{1029x^2}{128}.
 \end{aligned}$$

**17. Find an approximate value of the following corrected to 4 decimal places  $\sqrt[3]{1002} - \sqrt[3]{998}$ .**

$$\begin{aligned}
 \text{A: } \sqrt[3]{1002} - \sqrt[3]{998} &= (1002)^{1/3} - (998)^{1/3} \\
 &= (1000+2)^{1/3} - (1000-2)^{1/3} \\
 &= (1000)^{1/3} - (1000-2)^{1/3} \\
 &= (1000)^{1/3} \left[ 1 + \frac{2}{1000} \right]^{1/3} - (1000)^{1/3} \left[ 1 - \frac{2}{1000} \right]^{1/3} \\
 &= 10 \left[ (1+0.002)^{1/3} - (1-0.002)^{1/3} \right] \\
 &= 10 \left[ \left( 1 + \frac{1}{3}(0.002) + \frac{1}{3} \left( \frac{-2}{3} \right) \frac{(0.002)^2}{2!} + \dots \right) \right. \\
 &\quad \left. - \left( 1 - \frac{1}{3}(0.002) + \frac{1}{3} \left( \frac{-2}{3} \right) \frac{(0.002)^2}{2!} + \dots \right) \right] \\
 &\approx \frac{0.04}{3} \approx 0.0133..
 \end{aligned}$$

**18. Expand  $5\sqrt{5}$  in increasing powers of  $\frac{4}{5}$ .**

$$\text{A: } 5\sqrt{5} = 5.5^{\frac{1}{2}} = 5.5^{\frac{3}{2}} = \left( \frac{1}{5} \right)^{-3/2} = \left( 1 - \frac{4}{5} \right)^{-3/2}$$

Formula :

$$\begin{aligned}
 (1-x)^{-n} &= 1 + nx + \frac{n(n+1)}{2!} x^2 + \dots \\
 &= 1 + \frac{3}{2} \left( \frac{4}{5} \right) + \frac{3 \left( \frac{3}{2} + 1 \right)}{2!} \left( \frac{4}{5} \right)^2 + \dots \\
 &= 1 + \frac{3}{2} \left( \frac{4}{5} \right) + \frac{3.5}{2 \cdot 2!} \left( \frac{4}{5} \right)^2 + \dots
 \end{aligned}$$

**LEVEL - II (LAQ)**

1. If the coefficients of  $x^9, x^{10}, x^{11}$  in the expansion of  $(1+x)^n$  are in A. P. Then prove that  $n^2 - 41n + 398 = 0$

A. Given expansion  $(1+x)^n$

The coefficient of  $x^r$  in the expansion of  $(1+x)^n$  is  ${}^nC_r$ . Given that coefficients of  $x^9, x^{10}, x^{11}$  are in A. P.

$$\Rightarrow 2 \cdot {}^nC_{10} = {}^nC_9 + {}^nC_{11} \quad \because a, b, c \text{ are in A. P.}$$

$$\Rightarrow 2b = a + c$$

$$\Rightarrow 2 \frac{n!}{(n-10)!10!} = \frac{n!}{(n-9)!9!} + \frac{n!}{(n-11)!11!}$$

$$\Rightarrow 2 \frac{2}{(n-10)10} = \frac{1}{(n-9)(n-10)} + \frac{1}{11 \times 10}$$

$$\Rightarrow \frac{2}{(n-10)10} = \frac{110 + (n-9)(n-10)}{(n-9)(n-10) \times 11 \times 10}$$

$$\Rightarrow 22(n-9) = 110 + n^2 + 19n + 90$$

$$\Rightarrow 22n - 198 = 200 + n^2 - 19n$$

$$\Rightarrow n^2 - 41n + 318 = 0.$$

2. Show that the middle term in the expansion of  $(1+x)^{2n}$  is  $\frac{1 \cdot 3 \cdot 5 \dots (2n-1)}{n!} (2x)^n$ .

A: Given expansion is  $(1+x)^{2n}$

Middle term  $T_{\frac{2n}{2}+1} = T_{n+1}$

$$T_{n+1} = {}^{2n}C_n \cdot 1^{2n-x} \cdot x^n$$

$$= \frac{(2n)!}{(2n-n)!n!} x^n$$

$$= \frac{(2n)!}{n!n!} x^n$$

$$= \frac{1 \cdot 2 \cdot 3 \cdot 4 \dots (2n-2)(2n-1)(2n)}{1 \cdot 2 \dots (n-1)n(n!)} \cdot x^n$$

$$= \frac{1 \cdot 3 \cdot 5 \dots (2n-1)}{n!} \cdot 2^n \cdot x^n$$

$$= \frac{1 \cdot 3 \cdot 5 \dots (2n-1)}{n!} (2x)^n.$$

3. Prove that

$$({}^{2n}C_0)^2 - ({}^{2n}C_1)^2 + ({}^{2n}C_2)^2 - \dots + ({}^{2n}C_{2n})^2 = (-1)^n {}^{2n}C_n.$$

A: We know that

$$(1+x)^{2n} = {}^{2n}C_0 + {}^{2n}C_1x + {}^{2n}C_2x^2 + \dots + {}^{2n}C_{2n}x^{2n}$$

Also  $(x+1)^{2n} = {}^{2n}C_0x^{2n} - {}^{2n}C_1x^{2n-1} + {}^{2n}C_2x^{2n-2} - \dots + {}^{2n}C_{2n}$

Multiplying the above two expansions and equating the coefficient of  $x^{2n}$ , we get

$$({}^{2n}C_0)^2 - ({}^{2n}C_1)^2 + ({}^{2n}C_2)^2 - \dots + ({}^{2n}C_{2n})^2$$

$$= \text{coe. of } x^{2n} \text{ in } (1+x)^{2n} (x-1)^{2n}$$

$$= \text{coe. of } x^{2n} \text{ in } (x+1)^{2n} (x-1)^{2n}$$

$$= \text{coe. of } x^{2n} \text{ in } (x^2-1)^{2n} \quad \ln(x^2-1)2n,$$

$$T_{r+1} = {}^{2n}C_r (x^2)^{2n-r} (-1)^r$$

Here  $r = n$

$$= {}^{2n}C_n (-1)^n.$$

4. Find the sum of the infinite series

$$1 + \frac{2}{3} \cdot \frac{1}{2} + \frac{2}{3} \cdot \frac{5}{6} \left(\frac{1}{2}\right)^2 + \frac{2}{3} \cdot \frac{5}{6} \cdot \frac{8}{9} \left(\frac{1}{2}\right)^3 + \dots \infty.$$

A: Now  $1 + \frac{2}{3} \cdot \frac{1}{2} + \frac{2}{3} \cdot \frac{5}{6} \left(\frac{1}{2}\right)^2 + \frac{2}{3} \cdot \frac{5}{6} \cdot \frac{8}{9} \left(\frac{1}{2}\right)^3 + \dots \infty$

$$= 1 + 2 \left(\frac{1}{6}\right) + \frac{2 \cdot 5}{2!} \left(\frac{1}{6}\right)^2 + \frac{2 \cdot 5 \cdot 8}{3!} \left(\frac{1}{6}\right)^3 + \dots \infty$$

Comparing this with

$$(1-x)^{-p/q} = 1 + \frac{p}{1!} \left(\frac{x}{q}\right) + \frac{p(p+q)}{2!} \left(\frac{x}{q}\right)^2 + \dots \infty$$

Here  $p = s$

$$p + q = 5 \Rightarrow q = 2$$

$$\frac{x}{q} = \frac{1}{4} \Rightarrow x = \frac{2}{4} = \frac{1}{2}$$

$\therefore$  Sum of the given infinite series

$$= (1-x)^{-p/q}$$

$$= \left(1 - \frac{1}{2}\right)^{-2/3}$$

$$= \left(\frac{1}{2}\right)^{-2/3}$$

$$= 2^{2/3}$$

$$= \sqrt[3]{4}.$$

5. If  $x = \frac{5}{2!3} + \frac{5.7}{3!3^2} + \frac{5.7.9}{4!3^3} + \dots\infty$ , then find the value of  $x^2 + 4x$ .

A: Given  $x = \frac{5}{2!3} + \frac{5.7}{3!3^2} + \frac{5.7.9}{4!3^3} + \dots\infty$

$$\Rightarrow x = \frac{3.5}{2!}\left(\frac{1}{3}\right)^2 + \frac{3.5.7}{3!}\left(\frac{1}{3}\right)^3 + \dots\infty$$

$$\Rightarrow x + 2 = 1 + \frac{3}{1!}\left(\frac{1}{3}\right) + \frac{3.5}{2!}\left(\frac{1}{3}\right)^2 + \frac{3.5.7}{3!}\left(\frac{1}{3}\right)^3 + \dots\infty$$

Comparing RHS with

$$(1 - y)^{-p/q} = 1 + \frac{p}{1!}\left(\frac{y}{q}\right) + \frac{p(p+q)}{2!}\left(\frac{y}{q}\right)^2 + \dots\infty$$

Here  $p = 3$

$$p + q = 5 \Rightarrow q = 2$$

$$\frac{y}{q} = \frac{1}{3} \Rightarrow y = \frac{2}{3}$$

$$\therefore x + 2 = (1 - y)^{-p/q}$$

$$= \left(1 - \frac{2}{3}\right)^{-3/2}$$

$$= \left(\frac{1}{3}\right)^{-3/2}$$

$$x + 2 = 3^{3/2}$$

Squaring on both sides

$$x^2 + 4x + 4 = 27$$

$$\therefore x^2 + 4x = 23.$$

6. Find the sum of the infinite series

$$\frac{3}{4.8} - \frac{3.5}{4.8.12} + \frac{3.5.7}{4.8.12.16} - \dots\infty.$$

A: Given  $\frac{3}{4.8} - \frac{3.5}{4.8.12} + \frac{3.5.7}{4.8.12.16} - \dots\infty$

$$= \frac{1.3}{4.8} - \frac{1.3.5}{4.8.12} + \frac{1.3.5.7}{4.8.12.16} - \dots\infty$$

$$= \frac{1.3}{2!}\left(\frac{1}{4}\right)^2 - \frac{1.3.5}{3!}\left(\frac{1}{4}\right)^3 + \dots\infty$$

$$= 1 - \frac{1}{1!}\left(\frac{1}{4}\right) + \frac{1.3}{2!}\left(\frac{1}{4}\right)^2 - \frac{1.3.5}{3!}\left(\frac{1}{4}\right)^3 + \dots\infty$$

$$\left(1 - \frac{1}{4}\right)$$

Comparing the infinite series with

$$(1 + x)^{-p/q} = 1 - \frac{p}{1!}\left(\frac{x}{q}\right) + \frac{p(p+q)}{2!}\left(\frac{x}{q}\right)^2 - \dots\infty$$

Here  $p = 1$

$$p + q = 3 \Rightarrow q = 2$$

$$\frac{x}{q} = \frac{1}{4} \Rightarrow x = \frac{2}{4} = \frac{1}{2}$$

Sum of the given infinite series

$$= (1 + x)^{-p/q} - \frac{3}{4}$$

$$= \left(1 + \frac{1}{2}\right)^{-1/2} - \frac{3}{4}$$

$$= \left(\frac{3}{2}\right)^{-1/2} - \frac{3}{4}$$

$$= \sqrt{\frac{2}{3}} - \frac{3}{4}$$