

DEFINITIONS, CONCEPTS AND FORMULAE

1. $\frac{x+4}{(x-2)(x+2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+2} + \frac{C}{x+1}$.
2. $\frac{x^2+5x+7}{(x-1)^3}$ put $x-1 = y$.
3. $\frac{1}{x^3(x+2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x+2}$.
4. $\frac{x^2+1}{(x^2+4)(x-2)} = \frac{Ax+B}{x^2+4} + \frac{C}{x-2}$.
5. $\frac{x^3}{(x^2+1)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2}$.
6. $\frac{x^4}{(x-1)(x-2)} = x^2+3x+7 + \frac{15x-14}{x^2-3x+2}$
 $\frac{15x-14}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2}$.
7. $\frac{x^3}{(2x-1)(x+2)(x-3)} = \frac{1/2}{1} + \frac{A}{2x-1} + \frac{B}{x+2} + \frac{C}{x-3}$.

LEVEL - I (SAQ)

1. Resolve $\frac{x+4}{(x^2-4)(x+1)}$ into partial fractions.

A: Let $\frac{x+4}{(x-2)(x+2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+2} + \frac{C}{x+1}$

$$\Rightarrow \frac{x+4}{(x-2)(x+2)(x+1)} = \frac{A(x+2)(x+1) + B(x-2)(x+1) + C(x-2)(x+2)}{(x-2)(x+2)(x+1)}$$

$$\Rightarrow x+4 = A(x+2)(x+1) + B(x-2)(x+1) + C(x-2)(x+2) \text{ ----- (1)}$$

Put $x = 2$ in (1), we get

$$6 = A(4) \text{ (3)}$$

$$\therefore A = \frac{1}{2}$$

Put $x = -2$ in (1), we get

$$2 = B(-4)(-1)$$

$$4B = 2$$

$$\therefore B = \frac{1}{2}$$

Put $x = -1$ in (1), we get

$$3 = C(-3) \text{ (1)}$$

$$\therefore C = -1$$

$$\therefore \frac{x+4}{(x^2-4)(x+1)} = \frac{1}{2(x-2)} + \frac{1}{2(x+2)} - \frac{1}{x+1}$$

2. Resolve $\frac{x^2+5x+7}{(x-3)^3}$ into partial fractions.

A: Put $x - 3 = y$

$$\Rightarrow x = y + 3$$

Now $\frac{x^2+5x+7}{(x-3)^3} = \frac{(y+3)^2+5(y+3)+7}{y^3}$

$$= \frac{y^2+6y+9+5y+15+7}{y^3}$$

$$= \frac{y^2+11y+31}{y^3}$$

$$= \frac{1}{y} + \frac{11}{y^2} + \frac{31}{y^3}$$

$$= \frac{1}{x-3} + \frac{11}{(x-3)^2} + \frac{31}{(x-3)^3}$$

3. Resolve $\frac{x^2 + 13x + 15}{(2x + 3)(x + 3)^2}$ into sum of partial fractions.

A: Let
$$\frac{x^2 + 13x + 15}{(2x + 3)(x + 3)^2} = \frac{A}{2x + 3} + \frac{B}{x + 3} + \frac{C}{(x + 3)^2}$$

$$\Rightarrow \frac{x^2 + 13x + 15}{(2x + 3)(x + 3)^2} = \frac{A(x + 3)^2 + B(2x + 3)(x + 3) + C(2x + 3)}{(2x + 3)(x + 3)^2}$$

$$\Rightarrow x^2 + 13x + 15 = A(x + 3)^2 + B(2x + 3)(x + 3) + C(2x + 3) \text{ ----- (1)}$$

Put x = -3 in (1), we get

$$9 - 39 + 15 = C(-3)$$

$$\Rightarrow -3C = -15$$

$$\therefore C = 5$$

Put $x = \frac{-3}{2}$ in (1), we get

$$\frac{9}{4} - \frac{39}{2} + 15 = A\left(\frac{9}{4}\right)$$

$$\frac{9A}{4} = \frac{-9}{4}$$

$$\therefore A = -1$$

Equating the coefficient of x^2 , we get

$$A + 2B = 1$$

$$-1 + 2B = 1$$

$$B = 1$$

$$\therefore \frac{x^2 + 13x + 15}{(2x + 3)(x + 3)^2} = \frac{-1}{2x + 3} + \frac{1}{x + 3} + \frac{5}{(x + 3)^2}$$

4. Resolve $\frac{3x - 18}{x^3(x + 3)}$ into partial fractions.

A: Let
$$\frac{3x - 18}{x^3(x + 3)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x + 3}$$

$$\Rightarrow \frac{3x - 18}{x^3(x + 3)} = \frac{Ax^2(x + 3) + Bx(x + 3) + C(x + 3) + Dx^3}{x^3(x + 3)}$$

$$\Rightarrow 3x - 18 = Ax^2(x + 3) + Bx(x + 3) + C(x + 3) + Dx^3 \text{ -----(1)}$$

$$3x - 18 = A(x^3 + 3x^2) + B(x^2 + 3x) + C(x + 3) + Dx^3 \text{ ----- (2)}$$

Put x = 0 in (1), we get

$$-18 = C(3)$$

$$\therefore C = -6$$

Put x = -3 in (1), we get

$$-9 - 18 = D(-27)$$

$$\therefore D = 1$$

Equating the coefficient of x^3 on both sides in (2),

$$0 = A + D$$

$$A + 1 = 0$$

$$\therefore A = -1$$

Equating the coefficient of x^2 on both sides in (2),

$$0 = 3A + B$$

$$B = -3(-1)$$

$$\therefore B = 3$$

$$\therefore \frac{3x - 18}{x^3(x + 3)} = \frac{-1}{x} + \frac{3}{x^2} - \frac{6}{x^3} + \frac{1}{x + 3}$$

5. Resolve $\frac{x^2 - 3}{(x + 2)(x^2 + 1)}$ into partial fractions.

A: Let
$$\frac{x^2 - 3}{(x + 2)(x^2 + 1)} = \frac{A}{x + 2} + \frac{Bx + C}{x^2 + 1}$$

$$\frac{x^2 - 3}{(x + 2)(x^2 + 1)} = \frac{A(x^2 + 1) + (Bx + C)(x + 2)}{(x + 2)(x^2 + 1)}$$

$$x^2 - 3 = A(x^2 + 1) + (Bx + C)(x + 2) \text{ ----- (1)}$$

$$x^2 - 3 = A(x^2 + 1) + B(x^2 + 2x) + C(x + 2) \text{ -----(2)}$$

Put $x = -2$ in (1), we get

$$4 - 3 = A(4 + 1)$$

$$5A = 1$$

$$\therefore A = \frac{1}{5}$$

Equating the coefficient of x^2 on both sides in (2), then

$$1 = A + B$$

$$\therefore B = \frac{4}{5}$$

Equating the coefficient of x on both sides in (2),

$$0 = 2B + C$$

$$C = -2 \left(\frac{4}{5} \right)$$

$$\therefore C = \frac{-8}{5}$$

$$\therefore \frac{x^2 - 3}{(x+2)(x^2+1)} = \frac{1}{5(x+2)} + \frac{4x-8}{5(x^2+1)}$$

6. Resolve $\frac{2x^2 + 3x + 4}{(x-1)(x^2+2)}$ into partial fractions.

A: Let
$$\frac{2x^2 + 3x + 4}{(x-1)(x^2+2)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+2}$$

$$\Rightarrow \frac{2x^2 + 3x + 4}{(x-1)(x^2+2)} = \frac{A(x^2+2) + (Bx+C)(x-1)}{(x-1)(x^2+2)}$$

$$\Rightarrow 2x^2 + 3x + 4 = A(x^2+2) + (Bx+C)(x-1) \text{ ---- (1)}$$

$$2x^2 + 3x + 4 = A(x^2+2) + B(x^2-x) + C(x-1) \text{ ---- (2)}$$

Put $x = 1$ in equation (1), we get

$$2 + 3 + 4 = A(1 + 2)$$

$$3A = 9$$

$$\therefore A = 3$$

Equating the coefficient of x^2 on both sides in (2), we get $2 = A + B \Rightarrow B = -1$

Equating the coefficient of x on both sides in (2), we get $3 = -B + C$

$$C = 3 - 1$$

$$\therefore C = 2$$

$$\begin{aligned} \therefore \frac{2x^2 + 3x + 4}{(x-1)(x^2+2)} &= \frac{3}{x-1} + \frac{(-x+2)}{x^2+2} \\ &= \frac{3}{x-1} - \frac{(x-2)}{x^2+2} \end{aligned}$$

7. Resolve $\frac{x^3}{(x-a)(x-b)(x-c)}$ into partial fractions .

A: Let
$$\frac{x^3}{(x-a)(x-b)(x-c)} = \frac{1}{1} + \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}$$

$$\begin{aligned} \frac{x^3}{(x-a)(x-b)(x-c)} &= \frac{x^3}{(x-a)(x-b)(x-c) + A(x-b)(x-c) + B(x-a)(x-c) + C(x-a)(x-b)} \\ &= \frac{x^3}{(x-a)(x-b)(x-c)} \end{aligned}$$

$$\Rightarrow x^3 = (x-a)(x-b)(x-c) + A(x-b)(x-c) + B(x-a)(x-c) + C(x-a)(x-b) \text{ ---- (1)}$$

Put $x = a$ in (1), we get

$$a^3 = A(a-b)(a-c)$$

$$\therefore A = \frac{a^3}{(a-b)(a-c)}$$

Put $x = b$ in (1), we get

$$b^3 = B(b-a)(b-c)$$

$$\therefore B = \frac{b^3}{(b-a)(b-c)}$$

Put $x = c$ in (1), we get

$$c^3 = C(c-a)(c-b)$$

$$\therefore C = \frac{c^3}{(c-a)(c-b)}$$

$$\begin{aligned} \therefore \frac{x^3}{(x-a)(x-b)(x-c)} &= 1 + \frac{a^3}{(a-b)(a-c)(x-a)} + \\ &+ \frac{b^3}{(b-a)(b-c)(x-b)} + \frac{c^3}{(c-a)(c-b)(x-c)} \end{aligned}$$

8. Find the partial fractions of

$$\frac{x^3}{(2x-1)(x+2)(x-3)}$$

A: $\frac{x^3}{(2x-1)(x+2)(x-3)}$ is an improper fraction.

Let $\frac{x^3}{(2x-1)(x+2)(x-3)} = \frac{(1/2)}{1} + \frac{A}{2x-1} + \frac{B}{x+2} + \frac{C}{x-3}$

$$\frac{x^3}{(2x-1)(x+2)(x-3)} = \frac{1}{2} + \frac{A(x+2)(x-3) + B(2x-1)(x-3) + C(2x-1)(x+2)}{(2x-1)(x+2)(x-3)}$$

$$\Rightarrow x^3 = \frac{1}{2}(2x-1)(x+2)(x-3) + A(x+2)(x-3) + B(2x-1)(x-3) + C(2x-1)(x+2)$$

Put $x = \frac{1}{2}$ in (1), we get

$$\frac{1}{8} = A \left(\frac{1}{2} + 2 \right) \left(\frac{1}{2} - 3 \right)$$

$$\frac{1}{8} = A \left(\frac{5}{2} \right) \left(\frac{-5}{2} \right)$$

$$\therefore A = \frac{-1}{50}$$

Put $x = -2$ in (1), we get

$$-8 = B(-4-1)(-5)$$

$$\therefore B = \frac{-8}{25}$$

Put $x = 3$ in (1), we get

$$27 = C(5)(5)$$

$$\therefore C = \frac{27}{25}$$

$$\therefore \frac{x^3}{(2x-1)(x+2)(x-3)} = \frac{1}{2} - \frac{1}{50(2x-1)} - \frac{8}{25(x+2)} + \frac{27}{25(x-3)}$$

9. Resolve $\frac{x^4}{(x-1)(x-2)}$ into partial fractions.

A:

$$x^2 - 3x + 2 \quad x^4 \quad (x^2 + 3x + 7$$

$$\frac{x^4 - 3x^3 + 2x^2}{3x^3 - 2x^2}$$

$$3x^3 - 9x^2 + 6x$$

$$7x^2 - 6x$$

$$7x^2 - 21x + 14$$

$$15x - 14$$

$$\therefore \frac{x^4}{(x-1)(x-2)} = x^2 + 3x + 7 + \frac{15x-14}{(x-1)(x-2)}$$

Let $\frac{15x-14}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2}$

$$\Rightarrow \frac{15x-14}{(x-1)(x-2)} = \frac{A(x-2) + B(x-1)}{(x-1)(x-2)}$$

$$\Rightarrow 15x - 14 = A(x-2) + B(x-1) \text{ ----- (1)}$$

Put $x = 1$ in (1), we get

$$15 - 14 = A(-1)$$

$$\therefore A = -1.$$

Put $x = 2$ in (1), we get

$$30 - 14 = B(2-1)$$

$$B = 16$$

$$\therefore \frac{x^4}{(x-1)(x-2)} = x^2 + 3x + 7 - \frac{1}{x-1} + \frac{16}{x-2}$$

10. Find the coefficient of x^4 in the expansion of

$$\frac{3x}{(x-2)(x+1)}$$

A: Let $\frac{3x}{(x-2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+1}$

$$\Rightarrow \frac{3x}{(x-2)(x+1)} = \frac{A(x+1)+B(x-2)}{(x-2)(x+1)}$$

$$\Rightarrow 3x = A(x+1) + B(x-2) \text{----- (1)}$$

Put x = 2 in (1), we get

$$6 = A(2+1)$$

$$\therefore A = 2$$

Put x = -1 in (1), we get

$$-3 = B(-3)$$

$$\therefore B = 1.$$

$$\begin{aligned} \therefore \frac{3x}{(x-2)(x+1)} &= \frac{2}{x-2} + \frac{1}{x+1} \\ &= \frac{2}{(-2)\left(1-\frac{x}{2}\right)} + \frac{1}{1+x} \\ &= -\left(1-\frac{x}{2}\right)^{-1} + (1+x)^{-1} \end{aligned}$$

$$= \left[1 + \frac{x}{2} + \left(\frac{x}{2}\right)^2 + \left(\frac{x}{2}\right)^3 + \left(\frac{x}{2}\right)^4 + \dots \infty \right] + [1 - x + x^2 - x^3 + x^4 - \dots \infty]$$

Now, the coefficient of x⁴ in the above expansion

$$\begin{aligned} &= \frac{-1}{16} + 1 \\ &= \frac{15}{16}. \end{aligned}$$

11. Find the coefficient of xⁿ in the power series

expansion of $\frac{x-4}{x^2-5x+6}$.

A: Let $\frac{x-4}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3}$

$$\Rightarrow \frac{x-4}{(x-2)(x-3)} = \frac{A(x-3)+B(x-2)}{(x-2)(x-3)}$$

$$\Rightarrow x-4 = A(x-3) + B(x-2) \text{----- (1)}$$

Put x = 2 in (1), we get

$$-2 = -A$$

$$\therefore A = 2$$

Put x = 3 in (1), we get

$$-1 = B$$

$$\therefore B = -1$$

$$\therefore \frac{x-4}{(x-2)(x-3)} = \frac{2}{x-2} - \frac{1}{x-3}$$

$$\begin{aligned} &= \frac{2}{(-2)\left(1-\frac{x}{2}\right)} - \frac{1}{(-3)\left(1-\frac{x}{3}\right)} \\ &= -\left(1-\frac{x}{2}\right)^{-1} + \frac{1}{3}\left(1-\frac{x}{3}\right)^{-1} \\ &= -\left[1 + \frac{x}{2} + \left(\frac{x}{2}\right)^2 + \dots + \left(\frac{x}{2}\right)^n + \dots \infty \right] \\ &\quad + \frac{1}{3}\left[1 + \frac{x}{3} + \left(\frac{x}{3}\right)^2 + \dots + \left(\frac{x}{3}\right)^n + \dots \infty \right] \end{aligned}$$

The coefficient of xⁿ in the above expansion

$$\begin{aligned} &= \frac{-1}{2^n} + \frac{1}{3} \cdot \frac{1}{3^n} \\ &= \frac{1}{3^{n+1}} - \frac{1}{2^n}. \end{aligned}$$

LEVEL - II (SAQ)

1. Resolve $\frac{2x^2+2x+1}{x^3+x^2}$ into partial fractions.

A: Let $\frac{2x^2+2x+1}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$

$$\Rightarrow \frac{2x^2+2x+1}{x^2(x+1)} = \frac{Ax(x+1)+B(x+1)+Cx^2}{x^2(x+1)}$$

$$\Rightarrow 2x^2+2x+1 = Ax(x+1) + B(x+1) + Cx^2 \text{----- (1)}$$

Put x = 0 in (1), we get

$$1 = B$$

$$\therefore B = 1$$

Put x = -1 in (1), we get

$$2 - 2 + 1 = C(1)$$

$$\therefore C = 1$$

Put x = 1 in (1), we get

$$2 + 2 + 1 = 2A + 2B + C$$

$$5 = 2A + 2 + 1$$

$$2A = 2$$

$$\therefore A = 1.$$

$$\therefore \frac{2x^2+2x+1}{x^3+x^2} = \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x+1}.$$

2. Resolve $\frac{3x-1}{(1-x+x^2)(x+2)}$ into partial fractions.

A: $\frac{3x-1}{(1-x+x^2)(x+2)} = \frac{Ax+B}{1-x+x^2} + \frac{C}{x+2}$.

$$\Rightarrow \frac{3x-1}{(1-x+x^2)(x+2)} = \frac{(Ax+B)(x+2)+C(1-x+x^2)}{(1-x+x^2)(x+2)}$$

$$\Rightarrow 3x-1 = (Ax+B)(x+2) + C(1-x+x^2) \rightarrow \textcircled{1}$$

$$3x-1 = A(x^2+2x) + B(x+2) + C(1-x+x^2) \rightarrow \textcircled{2}$$

put $x = -2$ in (1), we get

$$3(-2) - 1 = 0 + C(1 + 2 + 4)$$

$$7C = -7$$

$$C = -1$$

Equating the coefficient of x^2 in (2)

$$0 = A + C.$$

$$\therefore A = 1.$$

Equating the coefficient of x in (2)

$$3 = 2A + B - C$$

$$3 = 2 + B + 1$$

$$B = 0.$$

\therefore

$$\frac{3x-1}{(1-x+x^2)(x+2)} = \frac{1 \cdot x + 0}{1-x+x^2} - \frac{1}{x+2}.$$

$$= \frac{x}{1-x+x^2} - \frac{1}{x+2}.$$

3. Resolve $\frac{3x^3 - 8x^2 + 10}{(x-1)^4}$ into partial fractions.

A. Given $\frac{3x^3 - 8x^2 + 10}{(x-1)^4}$

put $x - 1 = y \Rightarrow x = y + 1$

Now $\frac{3x^3 - 8x^2 + 10}{(x-1)^4} = \frac{3(y+1)^3 - 8(y+1)^2 + 10}{y^4}$

$$= \frac{3(y^3 + 3y^2 + 3y + 1) - 8(y^2 + 1 + 2y) + 10}{y^4}$$

$$= \frac{3y^3 + 9y^2 + 9y + 3 - 8y^2 - 8 - 16y + 10}{y^4}$$

$$= \frac{3y^3 + y^2 - 7y + 5}{y^4}$$

$$= \frac{3}{y} + \frac{1}{y^2} - \frac{7}{y^3} + \frac{5}{y^4}$$

$$= \frac{3}{x-1} + \frac{1}{(x-1)^2} - \frac{7}{(x-1)^3} + \frac{5}{(x-1)^4}.$$

4. Resolve $\frac{1}{(x-1)^2(x-2)}$ into partial fractions.

A. Let $\frac{1}{(x-1)^2(x-2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x-2} \rightarrow (1)$

$$\Rightarrow 1 = A(x-1)(x-2) + B(x-2) + C(x-1)^2$$

| | |
|-------------|-------------|
| Put $x = 1$ | Put $x = 2$ |
|-------------|-------------|

| | |
|--------------|----------------|
| $1 = B(1-2)$ | $1 = C(2-1)^2$ |
| $B = -1$ | $C = 1$ |

comparing coefficient of x^2 on both sides

$$\begin{aligned} 0 &= A + C \\ 0 &= A + 1 \\ A &= -1 \end{aligned}$$

$$\therefore \frac{1}{(x-1)^2(x-2)} = \frac{-1}{x-1} - \frac{1}{(x-1)^2} + \frac{1}{x-2}.$$
