DEFINITIONS, CONCEPTS AND FORMULAE

1. $\frac{x+4}{(x-2)(x+2)(x+1)}=\frac{A}{x-2}+\frac{B}{x+2}+\frac{C}{x+1}$.

2. $\frac{x^2+5x+7}{(x-1)^3}$ put x-1 = y.

3. $\frac{1}{x^3(x+2)}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x^3}+\frac{D}{x+2}$.

4. $\frac{x^2+1}{(x^2+4)(x-2)}=\frac{Ax+B}{x^2+4}+\frac{C}{x-2}$.

5. $\frac{x^3}{(x^2+1)^2}=\frac{Ax+B}{x^2+1}+\frac{Cx+D}{(x^2+1)^2}$.

6. $\frac{x^4}{(x-1)(x-2)}=x^2+3x+7+\frac{15x-14}{x^2-3x+2}$

   $\frac{15x-14}{(x-1)(x-2)}=\frac{A}{x-1}+\frac{B}{x-2}$.

7. $\frac{x^3}{(2x-1)(x+2)(x-3)}=\frac{1/2}{1}+\frac{A}{2x-1}+\frac{B}{x+2}+\frac{C}{x-3}$.

LEVEL - I (SAQ)

1. Resolve $\frac{x+4}{(x^2-4)(x+1)}$ into partial fractions.

A: Let $\frac{x+4}{(x-2)(x+2)(x+1)}=\frac{A}{x-2}+\frac{B}{x+2}+\frac{C}{x+1}$

$$\Rightarrow \frac{x+4}{(x-2)(x+2)(x+1)}=\frac{A(x+2)(x+1)+B(x-2)(x+1)+C(x-2)(x+2)}{(x-2)(x+2)(x+1)}$$

$\Rightarrow$ x + 4 = A(x + 2) (x + 1) + B(x - 2) (x + 1) + C(x - 2) (x + 2) ---------- (1)

Put x = 2 in (1), we get

6 = A(4) (3)

$\therefore A=\frac{1}{2}$

Put x = -2 in (1), we get

2 = B(-4) (-1)

4B = 2

$\therefore B=\frac{1}{2}$

Put x = -1 in (1), we get

3 = C(-3) (1)

$\therefore$ C = -1

$$\therefore \frac{x+4}{(x^2-4)(x+1)}=\frac{1}{2(x-2)}+\frac{1}{2(x+2)}-\frac{1}{x+1}.$$

2. Resolve $\frac{x^2+5x+7}{(x-3)^3}$ into partial fractions.

A: Put x - 3 = y

$\Rightarrow$ x = y + 3

$$\text{Now } \frac{x^2+5x+7}{(x-3)^3}=\frac{(y+3)^2+5(y+3)+7}{y^3}$$

$$=\frac{y^2+6y+9+5y+15+7}{y^3}$$

$$=\frac{y^2+11y+31}{y^3}$$

$$=\frac{1}{y}+\frac{11}{y^2}+\frac{31}{y^3}$$

$$=\frac{1}{x-3}+\frac{11}{(x-3)^2}+\frac{31}{(x-3)^3}.$$

3. Resolve $\dfrac{x^2+13x+15}{(2x+3)(x+3)^2}$ into sum of partial fractions.

A: Let $\dfrac{x^2+13x+15}{(2x+3)(x+3)^2} = \dfrac{A}{2x+3} + \dfrac{B}{x+3} + \dfrac{C}{(x+3)^2}$

$$\Rightarrow \frac{x^2+13x+15}{(2x+3)(x+3)^2} = \frac{A(x+3)^2 + B(2x+3)(x+3) + C(2x+3)}{(2x+3)(x+3)^2}$$

$\Rightarrow x^2 + 13x + 15 = A(x+3)^2 + B(2x+3)(x+3) + C(2x+3)$ ------- (1)

Put $x = -3$ in (1), we get

$9 - 39 + 15 = C(-3)$

$\Rightarrow -3C = -15$

$\therefore C = 5$

Put $x = \dfrac{-3}{2}$ in (1), we get

$\dfrac{9}{4} - \dfrac{39}{2} + 15 = A\left(\dfrac{9}{4}\right)$

$\dfrac{9A}{4} = \dfrac{-9}{4}$

$\therefore A = -1$

Equating the coefficient of x^2, we get

$A + 2B = 1$

$-1 + 2B = 1$

$B = 1$

$$\therefore \frac{x^2+13x+15}{(2x+3)(x+3)^2} = \frac{-1}{2x+3} + \frac{1}{x+3} + \frac{5}{(x+3)^2}.$$

4. Resolve $\dfrac{3x-18}{x^3(x+3)}$ into partial fractions.

A: Let $\dfrac{3x-18}{x^3(x+3)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x^3} + \dfrac{D}{x+3}$

$$\Rightarrow \frac{3x-18}{x^3(x+3)} = \frac{Ax^2(x+3) + Bx(x+3) + C(x+3) + Dx^3}{x^3(x+3)}$$

$\Rightarrow 3x - 18 = Ax^2(x+3) + Bx(x+3) + C(x+3) + Dx^3$ ----(1)

$3x - 18 = A(x^3 + 3x^2) + B(x^2 + 3x) + C(x+3) + Dx^3$ ------ (2)

Put $x = 0$ in (1), we get

$-18 = C(3)$

$\therefore C = -6$

Put $x = -3$ in (1), we get

$-9 - 18 = D(-27)$

$\therefore D = 1$

Equating the coefficient of x^3 on both sides in (2),

$0 = A + D$

$A + 1 = 0$

$\therefore A = -1$

Equating the coefficient of x^2 on both sides in (2),

$0 = 3A + B$

$B = -3(-1)$

$\therefore B = 3.$

$$\therefore \frac{3x-18}{x^3(x+3)} = \frac{-1}{x} + \frac{3}{x^2} - \frac{6}{x^3} + \frac{1}{x+3}.$$

5. Resolve $\dfrac{x^2-3}{(x+2)(x^2+1)}$ into partial fractions.

A: Let $\dfrac{x^2-3}{(x+2)(x^2+1)} = \dfrac{A}{x+2} + \dfrac{Bx+C}{x^2+1}$

$$\frac{x^2-3}{(x+2)(x^2+1)} = \frac{A(x^2+1) + (Bx+C)(x+2)}{(x+2)(x^2+1)}$$

$x^2 - 3 = A(x^2+1) + (Bx+C)(x+2)$ ------- (1)

$x^2 - 3 = A(x^2+1) + B(x^2+2x) + C(x+2)$ ----------(2)

Put x = -2 in (1), we get

4 - 3 = A(4 + 1)

5A = 1

$\therefore A = \frac{1}{5}$

Equating the coefficient of x^2 on both sides in (2), then

1 = A + B

$\therefore B = \frac{4}{5}$

Equating the coefficient of x on both sides in (2),

0 = 2B + C

$C = -2\left(\frac{4}{5}\right)$

$\therefore C = \frac{-8}{5}$

$$\therefore \frac{x^2 - 3}{(x+2)(x^2+1)} = \frac{1}{5(x+2)} + \frac{4x-8}{5(x^2+1)}.$$

6. Resolve $\frac{2x^2 + 3x + 4}{(x-1)(x^2+2)}$ into partial fractions.

A: Let $\frac{2x^2 + 3x + 4}{(x-1)(x^2+2)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+2}$

$$\Rightarrow \frac{2x^2 + 3x + 4}{(x-1)(x^2+2)} = \frac{A(x^2+2) + (Bx+C)\ (x-1)}{(x-1)(x^2+2)}$$

$\Rightarrow 2x^2 + 3x + 4 = A(x^2 + 2) + (Bx + C)(x - 1)$ ------- (1)

$2x^2 + 3x + 4 = A(x^2 + 2) + B(x^2 - x) + C(x - 1)$ -------- (2)

Put x = 1 in equation (1), we get

2 + 3 + 4= A(1 + 2)

3A = 9

$\therefore A = 3$

Equating the coefficient of x^2 on both sides in (2), we get 2 = A + B $\Rightarrow$ B = -1

Equating the coefficient of x on both sides in (2), we get 3 = -B + C

C = 3 - 1

$\therefore C = 2$

$$\therefore \frac{2x^2 + 3x + 4}{(x-1)(x^2+2)} = \frac{3}{x-1} + \frac{(-x+2)}{x^2+2}$$

$$= \frac{3}{x-1} - \frac{(x-2)}{x^2+2}.$$

7.Resolve $\frac{x^3}{(x-a)(x-b)(x-c)}$ into partial fractions .

A: Let $\frac{x^3}{(x-a)(x-b)(x-c)} = \frac{1}{1} + \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}$

$$\frac{x^3}{(x-a)(x-b)(x-c)} = \frac{(x-a)(x-b)(x-c) + A(x-b)(x-c) + B(x-a)(x-c) + C(x-a)(x-b)}{(x-a)(x-b)(x-c)}$$

$\Rightarrow x^3 = (x - a)(x - b)(x - c) + A(x - b)(x - c) + B(x - a)(x - c) + C(x - a)(x - b)$ -------- (1)

Put x = a in (1), we get

$a^3 = A(a - b)(a - c)$

$\therefore A = \frac{a^3}{(a-b)(a-c)}$

Put x = b in (1), we get

$b^3 = B(b - a)(b - c)$

$\therefore B = \frac{b^3}{(b-a)(b-c)}$

Put x = c in (1), we get

$c^3 = C(c - a)(c - b)$

$\therefore C = \frac{c^3}{(c-a)(c-b)}$

$$\therefore \frac{x^3}{(x-a)(x-b)(x-c)} = 1 + \frac{a^3}{(a-b)(a-c)(x-a)} + \frac{b^3}{(b-a)(b-c)(x-b)} + \frac{c^3}{(c-a)(c-b)(x-c)}.$$

8. Find the partial fractions of $\frac{x^3}{(2x-1)(x+2)(x-3)}$.

A: $\frac{x^3}{(2x-1)(x+2)(x-3)}$ is an improper fraction.

Let $\frac{x^3}{(2x-1)(x+2)(x-3)} = \frac{(1/2)}{1} + \frac{A}{2x-1} + \frac{B}{x+2} + \frac{C}{x-3}$

$$\frac{x^3}{(2x-1)(x+2)(x-3)} = \frac{\frac{1}{2}(2x-1)(x+2)(x-3)+A(x+2)(x-3)+B(2x-1)(x-3)+C(2x-1)(x+2)}{(2x-1)(x+2)(x-3)}$$

$\Rightarrow x^3 = \frac{1}{2}(2x-1)(x+2)(x-3) + A(x+2)(x-3) + B(2x-1)(x-3) + C(2x-1)(x+2)$

Put $x = \frac{1}{2}$ in (1), we get

$\frac{1}{8} = A\left(\frac{1}{2}+2\right)\left(\frac{1}{2}-3\right)$

$\frac{1}{8} = A\left(\frac{5}{2}\right)\left(\frac{-5}{2}\right)$

$\therefore A = \frac{-1}{50}$

Put x = -2 in (1), we get

$-8 = B(-4-1)(-5)$

$\therefore B = \frac{-8}{25}$

Put x = 3 in (1), we get

$27 = C(5)(5)$

$\therefore C = \frac{27}{25}$.

$\therefore \frac{x^3}{(2x-1)(x+2)(x-3)} = \frac{1}{2} - \frac{1}{50(2x-1)} - \frac{8}{25(x+2)} + \frac{27}{25(x-3)}$.

9. Resolve $\frac{x^4}{(x-1)(x-2)}$ into partial fractions.

A:

$$\begin{array}{r|l} x^2-3x+2) & x^4 \quad (x^2+3x+7 \\ & x^4-3x^3+2x^2 \\ \hline & 3x^3-2x^2 \\ & 3x^3-9x^2+6x \\ \hline & 7x^2-6x \\ & 7x^2-21x+14 \\ \hline & 15x-14 \end{array}$$

$\therefore \frac{x^4}{(x-1)(x-2)} = x^2 + 3x + 7 + \frac{15x-14}{(x-1)(x-2)}$

Let $\frac{15x-14}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2}$

$\Rightarrow \frac{15x-14}{(x-1)(x-2)} = \frac{A(x-2)+B(x-1)}{(x-1)(x-2)}$

$\Rightarrow 15x - 14 = A(x-2) + B(x-1)$ --------- (1)

Put x = 1 in (1), we get

$15 - 14 = A(-1)$

$\therefore A = -1$.

Put x = 2 in (1), we get

$30 - 14 = B(2-1)$

$B = 16$

$\therefore \frac{x^4}{(x-1)(x-2)} = x^2 + 3x + 7 - \frac{1}{x-1} + \frac{16}{x-2}$.

10. Find the coefficient of x^4 in the expansion of $\frac{3x}{(x-2)(x+1)}$.

A: Let $\frac{3x}{(x-2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+1}$

$$\Rightarrow \frac{3x}{(x-2)(x+1)} = \frac{A(x+1)+B(x-2)}{(x-2)(x+1)}$$

$\Rightarrow 3x = A(x + 1) + B(x - 2)$ ----------- (1)

Put x = 2 in (1), we get

$6 = A(2 + 1)$

$\therefore A = 2$

Put x = -1 in (1), we get

$-3 = B(-3)$

$\therefore B = 1.$

$$\therefore \frac{3x}{(x-2)(x+1)} = \frac{2}{x-2} + \frac{1}{x+1}$$

$$= \frac{2}{(-2)\left(1-\frac{x}{2}\right)} + \frac{1}{1+x}$$

$$= -\left(1-\frac{x}{2}\right)^{-1} + (1+x)^{-1}$$

$$= -\left[1+\frac{x}{2}+\left(\frac{x}{2}\right)^2+\left(\frac{x}{2}\right)^3+\left(\frac{x}{2}\right)^4+.....\infty\right]$$

$$+ [1 - x + x^2 - x^3 + x^4 - .... \infty]$$

Now, the coefficient of x^4 in the above expansion

$$= \frac{-1}{16} + 1$$

$$= \frac{15}{16}.$$

11.Find the coefficient of x^n in the power series expansion of $\frac{x-4}{x^2-5x+6}$.

A: Let $\frac{x-4}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3}$

$$\Rightarrow \frac{x-4}{(x-2)(x-3)} = \frac{A(x-3)+B(x-2)}{(x-2)(x-3)}$$

$\Rightarrow x - 4 = A(x - 3) + B(x - 2)$ ----------- (1)

Put x = 2 in (1), we get

$-2 = -A$

$\therefore A = 2$

Put x = 3 in (1), we get

$-1 = B$

$\therefore B = -1$

$$\therefore \frac{x-4}{(x-2)(x-3)} = \frac{2}{x-2} - \frac{1}{x-3}$$

$$= \frac{2}{(-2)\left(1-\frac{x}{2}\right)} - \frac{1}{(-3)\left(1-\frac{x}{3}\right)}$$

$$= -\left(1-\frac{x}{2}\right)^{-1} + \frac{1}{3}\left(1-\frac{x}{3}\right)^{-1}$$

$$= -\left[1+\frac{x}{2}+\left(\frac{x}{2}\right)^2+.......+\left(\frac{x}{2}\right)^n+.....\infty\right]$$

$$+\frac{1}{3}\left[1+\frac{x}{3}+\left(\frac{x}{2}\right)^2+.......+\left(\frac{x}{3}\right)^n+.....\infty\right]$$

The coefficient of x^n in the above expansion

$$= \frac{-1}{2^n} + \frac{1}{3}\cdot\frac{1}{3^n}$$

$$= \frac{1}{3^{n+1}} - \frac{1}{2^n}.$$

LEVEL - II (SAQ)

1. Resolve $\frac{2x^2+2x+1}{x^3+x^2}$ into partial fractions.

A: Let $\frac{2x^2+2x+1}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$

$$\Rightarrow \frac{2x^2+2x+1}{x^2(x+1)} = \frac{Ax(x+1)+B(x+1)+Cx^2}{x^2(x+1)}$$

$\Rightarrow 2x^2 + 2x + 1 = Ax(x + 1) + B(x + 1) + Cx^2$ ------------(1)

Put x = 0 in (1), we get

$1 = B$

$\therefore B = 1$

Put x = -1 in (1), we get

$2 - 2 + 1 = C(1)$

$\therefore C = 1$

Put x = 1 in (1), we get

$2 + 2 + 1 = 2A + 2B + C$

$5 = 2A + 2 + 1$

$2A = 2$

$\therefore A = 1.$

$$\therefore \frac{2x^2+2x+1}{x^3+x^2} = \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x+1}.$$

2. Resolve $\dfrac{3x-1}{(1-x+x^2)(x+2)}$ into partial fractions.

A: $\dfrac{3x-1}{(1-x+x^2)(x+2)} = \dfrac{Ax+B}{1-x+x^2} + \dfrac{C}{x+2}$.

$$\Rightarrow \frac{3x-1}{(1-x+x^2)(x+2)} = \frac{(Ax+B)(x+2)+C(1-x+x^2)}{(1-x+x^2)(x+2)}$$

$$\Rightarrow 3x-1 = (Ax+B)(x+2)+C(1-x+x^2) \rightarrow ①$$

$$3x-1 = A(x^2+2x)+B(x+2)+C(1-x+x^2) \rightarrow ②$$

put x = -2 in (1), we get

3(- 2) - 1 = 0 + C (1 + 2 + 4)

7C = - 7

C = - 1

Equating the coefficient of x^2 in (2)

0 = A + C.

$\therefore$ A = 1.

Equating the coefficient of x in (2)

3 = 2A + B - C

3 = 2 + B + 1

B = 0.

$\therefore$

$$\frac{3x-1}{(1-x+x^2)(x+2)} = \frac{1.x+0}{1-x+x^2} - \frac{1}{x+2}.$$

$$= \frac{x}{1-x+x^2} - \frac{1}{x+2}.$$

3. Resolve $\dfrac{3x^3-8x^2+10}{(x-1)^4}$ into partial fractions .

A. Given $\dfrac{3x^3-8x^2+10}{(x-1)^4}$

put x - 1 = y $\Rightarrow$ x = y + 1

Now $\dfrac{3x^3-8x^2+10}{(x-1)^4} = \dfrac{3(y+1)^3-8(y+1)^2+10}{y^4}$

$$= \frac{3(y^3+3y^2+3y+1)-8(y^2+1+2y)+10}{y^4}$$

$$= \frac{3y^3+9y^2+9y+3-8y^2-8-16y+10}{y^4}$$

$$= \frac{3y^3+y^2-7y+5}{y^4}$$

$$= \frac{3}{y}+\frac{1}{y^2}-\frac{7}{y^3}+\frac{5}{y^4}$$

$$= \frac{3}{x-1}+\frac{1}{(x-1)^2}-\frac{7}{(x-1)^3}+\frac{5}{(x-1)^4}.$$

4. Resolve $\dfrac{1}{(x-1)^2(x-2)}$ into partial fractions.

A. Let $\dfrac{1}{(x-1)^2(x-2)} = \dfrac{A}{(x-1)}+\dfrac{B}{(x-1)^2}+\dfrac{C}{x-2} \rightarrow (1)$

$\Rightarrow$ 1 = A(x - 1)(x - 2) + B(x - 2) + C(x - 1)^2

Put x = 1	Put x = 2
1 = B(1 - 2)	1 = C(2 - 1)^2
B = - 1	C = 1

<u>comparing coefficient of x^2 on both sides</u>

0 = A + C
0 = A + 1
A = -1

$$\therefore \frac{1}{(x-1)^2(x-2)} = \frac{-1}{x-1} - \frac{1}{(x-1)^2} + \frac{1}{x-2}.$$

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