Indefinite Integration

Basic Concept

Let $F(x)$ be a differentiable function of x such that $\frac{d}{dx}[F(x)] = f(x)$.. Then $F(x)$ is called the integral of $f(x)$.

Symbiotically, it is written as $\int f(x)dx = F(x)$.

$f(x)$, the function to be integrated is called the integrand.
$F(x)$ is also called the anti-derivate (or primitive function) of $f(x)$.

Constant of Integration:

As the differential coefficient of a constant is zero, we have

$$\frac{d}{dx}(F(x)) = f(x)$$

$$\Rightarrow \frac{d}{dx}[F(x) + c] = f(x).$$

Therefore, $\int f(x)\,dx = F(x) + c$.

This constant c is called the constant of integration and can take any real value.

Properties of Indefinite Integration

(*i*) $\int af(x)\,dx = a\int f(x)\,dx$ (Here 'a' is a constant)

(*ii*) $\int (f(x) + g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx$

(*iii*) If $\int f(u)\,du = F(u) + c$, then $\int f(ax + b)\,dx = \frac{1}{a}F(ax + b) + c,\ a \neq 0$.

Basic Formulae

1. If $K \in R$, $\int K dx = Kx + C$
2. $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$
3. $\int \frac{1}{x}\,dx = \log|x| + C$
4. $\int e^x\,.dx = e^x + C$

5. $\int a^x \, dx = \frac{a^x}{\log a} + C$ (for $a > 0, a \neq 1$)

6. $\int \text{Sinx } dx = -\text{Cosx} + C$

7. $\int \text{Cossdx} = \text{Sinx} + C$

8. $\int \text{tanxdx} = \log |\text{Secx}| + C$ or $= -\log|\text{Cosx}| + C$

9. $\int \text{Cotxdx} = \log|\text{Sinx}| + C$

10. $\int \text{Secxdx} = \log|\text{Secx} + \text{tanx}| + C$

$$= \log \left| \tan \left(\frac{\pi}{4} + \frac{x}{2} \right) \right| + C$$

11. $\int \text{Cosxdx} = \log|\text{Cosecx} - \text{Cotx}| + C$

$$= \log \left| \tan \frac{x}{2} \right| + C$$

$= -\log|\text{Cosecx} + \text{Cotx}| + C$

12. $\int \text{Secx tanx } dx = \text{Secx} + C$

13. $\int \text{Cosec x cotx } dx = -\text{Cosecx} + C$

14. $\int \text{Sec}^2 x \, dx = \tan x + C$

15. $\int \text{Cosec}^2 x \, dx = -\text{Cotx} + C$

16. $\int \sinh x \, dx = \text{Cosh } x + C$

17. $\int \text{Cosh } x \, dx = \text{Sinh } x + C$

18. $\int \tanh x \, dx = \log|\text{Cosh } x| + C$

19. $\int \coth x \, dx = \log|\text{Sinh } x| + C$

20. $\int \text{Sech } x \, dx = 2\tan^{-1}(e^x) + C$

21. $\int \text{Co sech } x \, dx = \log\left|\tanh \frac{x}{2}\right| + C$

22. $\int \text{Sech}^2 x \, dx = \tanh x + C$

23. $\int \text{Co sech}^2 x \, dx = -\text{Coth } x + C$

24. $\int \text{Sech } x \tanh x \, dx = -\text{Sech } x + C$

25. $\int \text{Cosech } x \text{ Coth } x \, dx = -\text{Cosech } x + C$

26. $\int \frac{dx}{\sqrt{1-x^2}} = \text{Sin}^{-1} x + C$ or $-\cos^{-1} x + C$

27. $\int \frac{dx}{\sqrt{1+x^2}}$ $\tan^{-1} x + C$ or $-\cot^{-1} x + C$

28. $\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x + C$ or $-\text{cosec}^{-1} x + C$

29. $\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + c$ or $-\cos^{-1}\left(\frac{x}{a}\right) + C$

30. $\int \frac{1}{\sqrt{x^2-a^2}} dx = \text{Cosh}^{-1}\left(\frac{x}{a}\right) + c$ (or) $\log\left|x + \sqrt{x^2-a^2}\right| + c$

31. $\int \frac{1}{\sqrt{x^2+a^2}} dx = \text{Sinh}^{-1}\left(\frac{x}{a}\right) + c$ (or) $\log\left|x + \sqrt{x^2+a^2}\right| + c$

32. $\int \frac{1}{x^2+a^2} dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$

33. $\int \frac{1}{a^2-x^2} dx = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + c$

34. $\int \frac{1}{x^2-a^2} dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c$

35. $\int \sqrt{a^2-x^2}\, dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + c$

36. $\int \sqrt{x^2 - a^2}\, dx = \frac{x}{2}\sqrt{x^2 - a^2} + \frac{a^2}{2} \cosh^{-1}\left(\frac{x}{a}\right) + c$

$= \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2} \log |x + \sqrt{x^2 - a^2}\,| + c$

37. $\int \sqrt{x^2 + a^2}\, dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2} \sinh^{-1}\left(\frac{x}{a}\right) + c$

$= \frac{x}{2}\sqrt{x^2 - a^2} + \frac{a^2}{2} \log |x + \sqrt{x^2 + a^2}\,| + c$

Method of Integration:

If the integrand is not a derivative of a known function, then the corresponding integrals cannot be found directly. In order to find the integral of complex problems, generally three rules of integration are used.

- Integration by substitution or by change of the independent variable.
- Integration by parts.
- Integration by partial fractions.

Integration by substitution

There are following types of substitutions

- **Direct Substitution**

�� If integral is of the form , then put $g(x) = t$, provided exists

e.g. Evaluate $\int \frac{\sin (\ln x)}{x} dx$

Sol. Let $\ln x = t$

Then $dt = \frac{1}{x}$

Hence $I = \int \sin t\, dt = -\cos t + c$

- **Standard Substitutions**
  - For terms of the form $x2 + a2$ or $\sqrt{x^2 + a^2}$, put $x = a \tan \theta$ or $a \cot \theta$
  - For terms of the form $x2 - a2$ or $\sqrt{x^2 - a^2}$, put $x = a \sec \theta$ or $a\ \mathrm{cosec}\ \theta$
  - For terms of the form $a2 - x2$ or $\sqrt{a^2 - x^2}$, put $x = a \sin \theta$ or $a \cos \theta$
  - If both $\sqrt{a + x}$, $\sqrt{a - x}$ are present then put $x = a \cos \theta$
  - For the type $\sqrt{(x - a)(b - x)}$, put $x = a \cos^2\theta + b \sin^2 \theta$
  - For the type $\left(\sqrt{x^2 + a^2} + x\right)^{\prime}$ or $\left(x + \sqrt{x^2 - a^2}\right)^{\prime}$, put the expression within the bracket = t
  - For the type $(x + a)^{-1 - 1/n} (x + b)^{-1 + 1/n}$ or $\left(\frac{x + b}{x + a}\right)^{1/n - 1} \frac{1}{(x + a)^2}$ (n □ N, $n > 1$), put
  - For $\frac{1}{(x + a)\, m\, (x + b)^{n_2}}$, $n_1, n_2 \in$ N (and > 1), again put $(x + a) = t\,(x + b)$

- **Indirect Substitution**

�� If the integrand is of the form $f(x)\, g(x)$, where $g(x)$ is a function of the integral of $f(x)$, then put integral of $f(x) = t$.

e.g. Evaluate $\int \frac{\sqrt{x}}{\sqrt{x^3 + a^3}}\, dx$

Sol. Integral of the numerator = $\frac{x^{3/2}}{3/2}$

Put $x^{3/2} = t$

We get $I = \frac{2}{3}\int \frac{dt}{\sqrt{t^2 + a^3}} = \frac{2}{3}\ln\left| x^{3/2} + \sqrt{x^3 + a^3} \right| + c$

- **Derived Substitution:**

�� Some time it is useful to write the integral as a sum of two related integrals which can be evaluated by making suitable substitutions.

Examples of such integrals are:

A. Algebraic Twins

$$\int \frac{2x^2}{x^4 + 1}\, dx = \int \frac{x^2 + 1}{x^4 + 1}\, dx + \int \frac{x^2 - 1}{x^4 + 1}\, dx$$

$$\int \frac{2}{x^4 + 1}\, dx = \int \frac{x^2 + 1}{x^4 + 1}\, dx - \int \frac{x^2 - 1}{x^4 + 1}\, dx$$

$$\int \frac{2x^2}{\left(x^4 + 1 + kx^2\right)}\, dx, \quad \int \frac{2}{\left(x^4 + 1 + kx^2\right)}\, dx$$

Method:

* Make the integration in the form

$$\int \left(1 + \frac{1}{x^2}\right) f\left(x - \frac{1}{x}\right) dx$$

or $\int \left(1 - \frac{1}{x^2}\right) f\left(x + \frac{1}{x}\right) dx$

*If $\left(1 + \frac{1}{x^2}\right)$ is present then put $x - \frac{1}{x} = t \Rightarrow \left(1 + \frac{1}{x^2}\right) dx = dt$

If $1 - \frac{1}{x^2}$ is present then put $x + \frac{1}{x} = t \Rightarrow \left(1 - \frac{1}{x^2}\right) dx = dt$

B. Trigonometric twins

$$\int \sqrt{\tan x}\, dx, \int \sqrt{\cot x}\, dx$$

$$\int \frac{1}{\left(\sin^4 x + \cos^4 x\right)}\, dx, \quad \int \frac{1}{\left(\sin^6 x + \cos^6 x\right)}\, dx$$

$$\int \frac{\pm \sin x \pm \cos x}{a + b \sin x \cos x}\, dx$$

Integration by Parts

1. If u and v be two function of x, then integral of product of these two functions is given by

$$\int u.v dx = u\int v dx - \int\left[\frac{du}{dx}\int v dx\right]dx$$

Note: In applying the above rule care has to be takne in the selection of the first function (u) and the second function (v). Normally we use the following methods:

(i) If in the product of two functions, one of the functions in not directly integrable ($e.g.$ $\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1} x$ etc.) then we take it as the first function and the remaining function is taken as the second function $e.g.$ In the integration $\int x \tan^{-1}x\, dx$, $\tan^{-1} x$ is taken as the first function and x as the second function.

(ii) If there is no other function, then unity is taken as the second function $e.g.$ In the integration of $\int \tan^{-1}x\, dx$, $\tan^{-1}$ is taken as the first function and 1 as the second function.

(iii) If both of the function are directly integrable then the first function is chosen in such a way that the derivative of the function thus obtained under integral sign is easily integrable. Usually we use the following preference order for the first function (inverse, Logarithmic, Algebraic, Trigonometric, Exponential)

In the aove stated order, the function on the left is always chosen as the first function. This rule is called as ILATE $e.g.$ In the integration of $\int x \sin x\, dx$, x is taken as the first function and $\sin x$ is taken as the second function.

Important Result:

* In the integral , if $g(x)$ ex dx, if g(x) cna be expressed as $g(x) = f(x) + f'(x)$

then $\int e^x\left[f(x)+g(x)\right]dx = e^x f(x)+c$

Some times to solve integral of the form $\int \frac{f(x)}{\{p(x)\}^2}dx$ we write it as $\int \frac{f(x)}{p'(x)}\cdot\frac{p'(x)}{\{p(x)\}^2}dx$ and solve the integral with the help of integration by parts, taking $\frac{f(x)}{p'(x)}$ as the first function.

e.g. $\int \frac{x^2}{(x\sin x+\cos x)^2}dx$ is solved by writting it as

$$\int \underset{(i)}{(x\sec x)}\underset{(ii)}{\frac{(x\cos x)}{(x\sin x+\cos x)^2}}dx$$ and this integral can be solved by parts.

Algebraic Integrals

I. Integral of the form

$$\int \frac{px+q}{(ax^2+bx+c}dx, \int \frac{px+q}{\sqrt{ax^2+bx+c}}dx, \int (px+q)\sqrt{ax^2+bx+c}dx$$

In these types of integrals we write $px + q = l$ (diff. coefficient of $ax^2 + bx + c$) + m

Find l and m by comparing the coefficient of x and constant term on both sides of the identify. In this way the question will reduce the sum of two integrals which can be integrated easily.

Integral of the type $\frac{ax^2+bx+c}{px^2+qx+r}$ or $\frac{ax^2+bx+c}{\sqrt{px^2+qx+r}}$

In this case substitute $ax^2 + bx + c = M(px^2 + qx + r) + N(2px + q) + R$

Find M, N and R. The integration reduces to integration of three independent functions.

II. Integration of Irrational Algebraic Fractions

1. Rational fucntion of $(ax + b)^{1/n}$ and x can be easily evaluated by the substitution $t^n = ax + b$. Thus

$$\int f(x), (ax+b)^{1/n})dx] \int f\left(\frac{t^n - b}{a}, t\right) \frac{nt^{n-1}}{a} dt.$$

2. In the integration of $\frac{1}{(x-k)} \frac{1}{\sqrt{ax^2+bx+c}}$, the substitution $x - k = 1/t$ reduces the integration $\frac{1}{(x-k)\sqrt{ax^2+bx+c}}$ to the problem of integrating an expression of the form $\frac{1}{\sqrt{At^2+Bt+C}}$.

3. . $\int \frac{dx}{(x-k)^r \sqrt{ax^2+bx+c}}$ Here we substitute, $x - k = 1/t$.

This substitution will reduce the given integral to $\int \frac{t^{r-1}dt}{\sqrt{At^2+Bt+C}}$.

4. To integrate $\frac{1}{(Ax^2+B)\sqrt{Cx^2+D}}$, we first put $x = 1/t$, so that

������������ $\int \frac{dx}{(Ax^2+B)\sqrt{Cx^2+D}} = \int \frac{-1/t^2 dt}{(A/t^2+B)\sqrt{C/t^2+D}} = \int \frac{tdt}{(A+Bt^2)\sqrt{C+Dt^2}}$.

Now the substitution $C + Dt^2 = u^2$ reduces it to the form $\int \frac{du}{u^2 \pm a^2}$.

III. Integration of the function of the type , $\int x^m (a+bx^n)^p \, dx$

Where m, n, p are rational numbers

This integral is expressed through elementary functions only if one of the following conditions is fulfilled:

(1) If p is an integer,

(2) If $\frac{m+1}{n}$ is an intger,

(3) If $\frac{m+1}{n}$ + p is an integer.

1st case :

(a) If p is a positive integer, remove the brackets (a + bxn)p according to the Newton binomial and calculate the integrals of powers.

(b) If p is a negative integer, then the substitution x = tk, where k is the common denominator of the fractions m and n, leads to the integral of a rational fraction;

2nd case :

If $\frac{m+1}{n}$ is an integer, then the substitution $a + bx^n = t^k$ is applied, where k is the denominator of the fraction p;

3rd case :

If $\frac{m+1}{n}$ + p is an integer, then the substitution $a + bx^n = xnt^k$ is applied, where k is the denominator of the fraction p.

Example :

(i) $I = \int \frac{\sqrt{1+\sqrt[3]{x}}}{\sqrt[3]{x^2}} dx$

(ii) $I = \int x^{-11}(1+x^4)^{-\frac{1}{2}} dx$

(iii) $\int \frac{\sqrt[3]{1+\sqrt[4]{x}}}{\sqrt{x}} dx$

(iv) $\dfrac{dx}{\sqrt[4]{1+x^4}}$

Trignometric Integrals

I. Integral of the form $\int R(\sin x, \cos x)\, dx$

Universal substitution tan . $\dfrac{x}{2} = t$

In this case sin x = $\dfrac{2t}{1+t^2}$; cos x = $\dfrac{1-t^2}{1+t^2}$, x = 2 tan- 1 t; dx = $\dfrac{2dt}{1+t^2}$

If R (- sin x, cos x) = - R (sin x, cos x), then the substitution cos x = t is applied.
If R (sin x, - cos x) = - R (sin x, cos x), then the substitution sin x = t is applied.
If R (- sin x, - cos x) = R (sin x, cos x), then the substitution tan x = t is applied.

II. Integral of the form

(i) $\displaystyle\int \frac{A\sin x}{B\sin x + C\cos x}\, dx$

(ii) $\displaystyle\int \frac{A\cos x}{B\sin x + C\cos x}\, dx$

(iii) $\displaystyle\int \frac{A\sin x + B\cos x}{C\sin x + D\cos x}\, dx$

(iv) $\displaystyle\int \frac{A\sin x + B\cos x + C}{D\sin x + E\cos x + F}\, dx$

This type of integration can be solved by converting the Nr in the form Nr = P(Dr) + Q + R the value of P, Q, R can be findout by comparing the coeffcient of both sides.

III. Integral of the form

(i) $\int \sin ax \sin bx dx$ (ii) $\int \sin ax \cos bx dx$ (iii) $\int \cos ax \cos bx dx$

Transform the product of trigonometric function into a sum or difference, using one of the following formulas:

sin ax sin bx = [cos(a-b) x - cos (a+b)x]
cos ax cos bx = [cos (a - b) x + cos (a + b) x]
sin ax cos bx = [sin (a - b) x + sin (a + b) x]

IV. Integral of the form $\int \sin^m x \cos^n x dx$, where m and n are integers.

(i) If m is an odd positive number, then apply the substitution cos x = t.
(ii) If n is an odd positive number, apply the substitution sin x = t.
(iii) If m and n are even non-negative numbers, use the formulas

$$\sin^2 x = \frac{1-\cos 2x}{2};\ \cos^2 x = \frac{1+\cos 2x}{2}$$

(iv) $\int \sin^p x \cos^q x\, dx$ where (o < x < �/2) and p and q are rational numbers.

Substitute sin x = t

$$\int \sin^p x \cos^q x dx = \int t^p (1-t^2)^{q-1} dt$$

V. Integral of the form

� $\displaystyle\int \frac{dx}{a\sin x + b\cos x} = \frac{1}{\sqrt{a^2+b^2}} \int \mathrm{cosec}(x \pm \theta) dx$ **when** $\theta = \tan^{-1}\left(\dfrac{b}{a}\right)$

VI. Integral of the form

(i) $\int \frac{dx}{a + b\sin^2 x}$ (ii) $\int \frac{dx}{a + b\cos^2 x}$

(iii) $\int \frac{dx}{a\sin^2 x + b\cos^2 x + c\sin x\cos x}$ (iv) $\int \frac{dx}{a\sin^2 x + b\cos^2 x}$

This type of integration can be solved by multiplying $\sec^2 x$, in N^r and D^r and substituting tan x = t, or cot x = t.

VII. Integral of the form

(i) $\int \frac{dx}{a + b\sin x}$ (ii) $\int \frac{dx}{a + b\cos x}$ (iii) $\int \frac{dx}{a\sin x + b\cos x + c}$

To solve this type of integration

1) convert sinx and cosx in terms of tan x/2 by putting.

$$\sin x = \frac{2\tan x/2}{1 + \tan^2 x/2}, \cos x = \frac{1 - \tan^2 x/2}{1 + \tan^2 x/2}$$

2) Write N^r in the form $\sec^2 x/2$ and Dr in the form tan x/2.

3) Substitute tan x/2 = t, so that $\sec^2 x/2\, dx = 2dt$

VIII. If D^r is in the form K + L sin x cos x, then Nr must be in the form of sin x + cos x, or sin x - cos x.

(1) If N^r has sinx + cos*x* then substitute sin*x* - cos*x* = *t*

$\Rightarrow$ (cos*x* + sin*x*) *dx* = *dt*

(2) *Nr* has sinx - cos*x* then substitute sin*x* + cos*x* = *t*

$\Rightarrow$ (cos*x* - sin*x*)*dx* = *dt*

Note: If sin*x* - cos*x* = $t \Rightarrow$ $1 - \sin 2x = t^2$

$\Rightarrow$ $\sin 2x = 1 - t^2$

If sin*x* + cos*x* = $t \Rightarrow$ $1 + \sin 2x = t^2$

$\Rightarrow$ $\sin 2x = t^2 - 1$

INTEGRATION BY Partial fraction

Let $\frac{P(x)}{Q(x)}, Q(x) \neq 0$ is a proper algebric function.

The partial fractions depend on the nature of the factors of $Q(x)$. We have deal with the following different type when the factors of $Q(x)$ are

(i) Linear and non-repeated
(ii) Linear and repeated
(iii) Quadratic and non-repeated
(iv) Quadratic and repeated

Case I :

When denominator is expressible as the product of non-repeated linear factors :

Let $Q(x) = (x - a_1)(x - a_2)(x - a_3) \dots (x - a_n)$.

Then we assume that ;

$$\frac{P(x)}{Q(x)} = \frac{A_1}{(x - a_1)} + \frac{A_2}{(x - a_2)} + \frac{A_3}{(x - a_3)} + \dots + \frac{A_n}{(x - a_n)}$$

where $A_1, A_2, \dots, A_n$ are constants and can be determined by equating numerator on R.H.S to numerator on L.H.S. and then substituting $x = a_1, a_2, \dots a_n$,

Case II :

When the denominator $Q(x)$ is expressible as the product of the linear factors such that some of them are repeating. (Linear and Repeated)

Let, $Q(x) = (x-a)^k (x-a_1)(x-a_2) \dots (x-a_r)$. Then we assume that

$$\frac{P(x)}{Q(x)} = \frac{A_1}{(x-a)} + \frac{A_2}{(x-a)^2} + \dots \frac{A_k}{(x-a)^k} + \frac{B_1}{(x-a_1)} + \frac{B_2}{(x-a_2)} + \dots + \frac{B_r}{(x-a_r)}$$

Case III :

When some of the factors in denominator are quadratic but non-repeating.

Corresponding to each quadratic factor $ax^2 + bx + c$, we assume the partial fraction of the type $\frac{Ax+B}{ax^2+bx+c}$, where A and B are constants to be determined by comparing coefficients of similar powers of x in numerator of both sides.

Case IV :

When some of the factors of the denominator are quadratic and repeating. For every quadratic repeating factor of the type $(ax^2 + bx + c)^k$, we assume :

$$\frac{A_1x+A_2}{ax^2+bx+c} + \frac{A_3x+A_4}{(ax^2+bx+c)^2} + \dots + \frac{A_{2k-1}x+A_{2k}}{(ax^2+bx+c)^k}$$

Short cut Method of Finding the Constant of a Non-repeated Linear Factor in Denominator

Let

$$\frac{f(x)}{(x-a_1)(x-a_2)\dots(x-a_n)} = \frac{A_1}{x-a_1} + \frac{A_2}{x-a_2} + \dots + \frac{A_n}{x-a_n}$$

$$\therefore A_1 = \lim_{x\to a1}(x-a_1)\left[\frac{f(x)}{(x-a_1)(x-a_2)\dots(x-a_n)}\right]$$

$$= \frac{f(a)}{\prod_{i=2}^{n}(a_1-a_i)}$$

$$A_2 = \lim_{x\to a2}(x-a_2)\left[\frac{f(x)}{(x-a_1)(x-a_2)\dots(x-a_n)}\right]$$

$$\frac{f(x)}{\prod_{i=1}^{n}(a_2-a_i) i \neq 2}$$

$$A_n = \lim_{x\to a_n}(x-a_n)\left[\frac{f(x)}{(x-a_1)(x-a_2)\dots(x-a_n)}\right]$$

$$\frac{f(a_n)}{\prod_{i=1}^{n-1}(a_n-a_i)}$$

Definite Integrals

1. If f and F are two continuous functions defind on [a, b] such that $\frac{d}{dx}[F(x)] = f(x)$. then the number

F(b) - F(a) is called definite integration of f between a and b and is denoted by $\int_a^b f(x)dx$

i.e., $\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$

This is called fundamental theorem of integral calculus. Here 'a' is called lower limit (LL) and 'b' is upper limit (U.L) and $a \le b$ always.

Also $\int_a^b f(x)\ dx$ denotes algebric sum of the area bounded by curve $y = f(x)$, ordinates $x = a$, $x = b$ and *x* -

axis.

2. $\int_a^b f(x)dx$ is always unique.

- $\int_a^b f(x)dx$ is also defined as an infinite limit sum..

4. $\int_a^b f(x)dx = \int_a^b f(\Theta)d\Theta = \int_a^b f(t)dt = \ldots\ldots$

Properties of Definite Integration:

1. $\int_a^a f(x)dx = 0$
2. $\int_a^b f(x)dx = -\int_b^a f(x)dx$
3. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$ where a < c < b
4. $f(x) = g(x) \Rightarrow \int_a^b f(x)dx = \int_a^b g(x)dx$ but converse need not be true
5. $\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$
6. $\int_a^b k\,f(x)dx = k\int_a^b f(x)dx$
7. $\int_a^b 0\,dx = k$
8. $\int_a^b k\,dx = k(b-a)$
9. $\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|dx$
10. $f(x) \ge 0 \Rightarrow \int_a^b f(x)dx \ge 0$ If but converse need not be true.
11. $f(x) \ge g(x) \Rightarrow \int_a^b f(x)dx \ge \int_a^b g(x)dx$ If but converse need not be true.
12. $\int_a^b f[g(x)]g'(x)dx = \int_{g(a)}^{g(b)} f(x)dx$ (change limit Theorem)
13. $\int_0^a f(x)dx = \int_0^a f(a-x)dx$
14. $\int_{-a}^a f(x)dx = \begin{cases} 0, & \text{if } f \text{ is odd } i.e.\ f(-x) = -f(x) \\ 2\int_0^a f(x)dx, & \text{if } f \text{ is even } f(-x) = f(x) \end{cases}$

15. $\int_0^a f(x)dx = \begin{cases} 0, & \text{if } f(a-x) = -f(x) \\ 2\int_0^{a/2} f(x)dx, & \text{if } f(a-x) = f(x) \end{cases}$

16. $\int_0^a f(x)dx = \int_0^{a/2} f(x)dx + \int_0^{a/2} f(a-x)dx$

17. $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

18. $\frac{d}{dx}\int_{\phi(x)}^{\Psi(x)} f(t)dt = f[\Psi(x)]\Psi'(x) - f[\phi(x)].\phi'(x)$ This is called **Leibnitz rule**.

19. $m(b-a) \le \int_a^b f(x)dx \le M(b-a)$ where m and M are respectively the minimum and maximum values of $f(x)$ in $[a, b]$

20. If f is a periodic function with period T, i.e., if $f(x + T) = f(x)$ then

a) $\int_0^{nT} f(x)dx = n\int_0^T f(x)dx$

b) $\int_{mT}^{nT} f(x)dx = (n-m)\int_0^T f(x)dx$

c) $\int_{a+mT}^{b+nT} f(x)dx = \int_0^T f(x)dx$ where m, n are integers

d) $\int_a^{a+T} f(x)dx = \int_0^T f(x)dx$ i.e., it is not dependent on 'a'

21. $\left|\int_a^b f(x)g(x)dx\right| \le \sqrt{\left(\int_a^b f^2(x)dx\right)\left(\int_a^b g^2(x)dx\right)}$ This is called cauchy - schwartz inequality

WALLI'S FORMULAE :

22. $I_n = \int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx = \begin{cases} \frac{n-1}{n}\times\frac{n-3}{n-2}\times\frac{n-5}{n-4}\times.....\times\frac{1}{2}\times\frac{\pi}{2}\,;\ \textit{if } n \textit{ is even} \\ \frac{n-1}{n}\times\frac{n-3}{n-2}\times\frac{n-5}{n-4}\times.....\times\frac{2}{3}\times 1\,;\ \textit{if } n \textit{ is odd} \end{cases}$

23. $I_{m,n} = \int_0^{\pi/2} \sin^m x.\cos^n x\,dx$

Case (1) : If m is odd and n is either even or odd

$I_{m,n} = \frac{m-1}{m+n}\times\frac{m-3}{m+n-2}\times\frac{m-5}{m+n-4}\times.......\times\frac{2}{n+3}\times\frac{1}{n+1}$

Case (2) : If m is even and n is even

$I_{m,n} = \frac{m-1}{m+n}\times\frac{m-3}{m+n-2}\times\frac{m-5}{m+n-4}\times.......\times\frac{1}{n+2}\times\frac{n-1}{n}\times\frac{n-3}{n-2}\times\frac{n-5}{n-4}\times.......\times\frac{1}{2}\times\frac{\pi}{2}$

Case (3) : If m is even and n is odd.

$I_{m,n} = \frac{m-1}{m+n}\times\frac{m-3}{m+n-2}\times\frac{m-5}{m+n-4}\times.......\times\frac{1}{n+2}\times\frac{n-1}{n}\times\frac{n-3}{n-2}\times\frac{n-5}{n-4}\times.......\times\frac{2}{3}\times 1$

Reduction Formulae on Definite Integration :

24. If $I_n = \int_0^{\pi/4} \tan^n x\,dx$ then $I_n + I_{n-1} = \frac{1}{n-1}$ (where n > 2)

25. If $I_n = \int_0^{\pi/4} \sec^n x\,dx$ then $I_n = \frac{\left(\sqrt{2}\right)^{n-2}}{n-1} + \left(\frac{n-2}{n+1}\right) I_{n-2}$

26. If $I_n = \int_{\pi/2}^{\pi/4} \cot^n x\,dx$ then $I_n + I_{n-2} = \frac{1}{1-n}$

27. If $I_n = \int_{\pi/6}^{\pi/4} \operatorname{cosec}^n x\,dx$ then $I_n = \frac{2^{n-2}\sqrt{3}}{n-1} + \frac{n-2}{n-1} I_{n-2}$

28. **By parts Formulae on Definite Integration :**

$$\int_a^b f(x)\,g(x)\,dx = \left[f(x)\int g(x)dx\right]_a^b - \int_a^b \left[f^{|}(x)\int g(x)dx\right]dx$$

29. If a function has finite number of points of discontinuties in [a, b] then the function is definite integrable in that interval.

30. **Definite integration as infinite limit sum :**

1) $\underset{n\to\infty}{Lt} \sum_{r=1}^{n} f\left(\frac{r}{n}\right)\frac{1}{n} = \int_0^1 f(x)dx$

2) **Working rule :**

Step I : First reduce the given infinite limt into the form $\underset{n\to\infty}{Lt} \sum_{r=1}^{n} f\left(\frac{r}{n}\right)\frac{1}{n}$

Step II : Replace r/n with x and 1/n with dx

Step III : Replace $\underset{n\to\infty}{Lt} \sum_{r=1}^{n}$ by $\int_0^1$ \ $\underset{n\to\infty}{Lt} \sum_{r=1}^{n} f\left(\frac{r}{n}\right)\frac{1}{n} = \int_0^1 f(x)dx$

Note : 1) $\underset{n\to\infty}{Lt} \sum_{r=0}^{n} f\left(\frac{r}{n}\right)\frac{1}{n} = \int_0^1 f(x)dx$

2) $\underset{n\to\infty}{Lt} \sum_{r=0}^{n-1} f\left(\frac{r}{n}\right)\frac{1}{n} = \int_0^1 f(x)dx$

3) $\underset{n\to\infty}{Lt} \sum_{r=1}^{kn} f\left(\frac{r}{n}\right)\frac{1}{n} = \int_0^k f(x)dx$