LAQ (2× 714)

1. **S.T** $\begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

Sol: L.H.S $\begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}$

$$R_1 \to R_1 + R_2 + R_3$$

$$= \begin{vmatrix} 2(a+b+c) & 2(a+b+c) & 2(a+b+c) \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}$$

$$= 2\begin{vmatrix} (a+b+c) & (a+b+c) & (a+b+c) \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}$$

$R_2 \to R_2 - R_1 : R_3 \to R_3 - R_1$

$$= 2\begin{vmatrix} (a+b+c) & (a+b+c) & (a+b+c) \\ -b & -c & -a \\ -c & -a & -b \end{vmatrix}$$

$R_1 \to R_1 + R_2 + R_3$

$$= 2\begin{vmatrix} a & b & c \\ -b & -c & -a \\ -c & -a & -b \end{vmatrix} = 2(-)(-)\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

$$=2\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \text{ R.H.S}$$

2. $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2 = \begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ac-b^2 & a^2 \\ b^2 & a^2 & 2ab-c^2 \end{vmatrix}$

$$= (a^2+b^2+c^2-3abc)^2$$

L.H.S $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2 = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

$$(R_2 \leftrightarrow R_3)$$

$$= \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}(-)\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$$

$$= \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}\begin{vmatrix} -a & -b & -c \\ c & a & b \\ b & c & a \end{vmatrix}$$

$$= \begin{vmatrix} -a^2+bc+bc & -ab+ab+c^2 & -ac+b^2+ac \\ -ab+c^2+ab & -b^2+ac+ac & -bc+bc+a^2 \\ -ac+ac+b^2 & -bc+a^2+bc & -c^2+ab+ab \end{vmatrix}$$

$$= \begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ac-b^2 & a^2 \\ b^2 & a^2 & 2ab-c^2 \end{vmatrix} \text{....(1)}$$

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2$$

$$= [a(bc-a^2) - b(b^2-ac) + c(ab-c^2)]^2$$

$$= (abc - a^3 - b^3 + abc + abc - c^3)^2$$

$$= (3abc - a^3 - b^3 - c^3)^2$$

$$= (a^3+b^3+c^3-3abc)^2 \text{...... (2)}$$

$from$ (1)& (2)
$L.H.S = R.H.S$

3. **If** $\begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix} = 0$ **and** $\begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} 0,$ **show that** $abc = -1.$

Sol: $\begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix} = 0$

$$\Rightarrow \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} + \begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} + abc \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ c & c & c^2 \end{vmatrix} = 0 \; \{C_1 \leftrightarrow C_2\}$$

$$\Rightarrow \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} - abc \begin{vmatrix} a & 1 & a^2 \\ b & 1 & b^2 \\ c & 1 & c^2 \end{vmatrix} = 0 \quad \{C_2 \leftrightarrow C_3\}$$

$$\Rightarrow \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} + abc \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} (1 + abc) = 0$$

$\therefore (1 + abc) = 0 \Rightarrow abc = -1.$

4. **S.T** $\begin{vmatrix} \mathbf{a+b+2c} & \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{b+c+2a} & \mathbf{b} \\ \mathbf{c} & \mathbf{a} & \mathbf{c+a+2b} \end{vmatrix} = \mathbf{2(a+b+c)^3}$

Sol: $\begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix}$

$C_1 \to C_1 + C_2 + C_3$

$$= \begin{vmatrix} 2(a+b+c) & a & b \\ 2(a+b+c) & b+c+2a & b \\ 2(a+b+c) & a & c+a+2b \end{vmatrix}$$

$$= 2(a+b+c) \begin{vmatrix} 1 & a & b \\ 1 & b+c+2a & b \\ 1 & a & c+a+2b \end{vmatrix}$$

$R_2 \to R_2 - R_1 : R_3 \to R_3 - R_1$

$$= 2(a+b+c) \begin{vmatrix} 1 & a & b \\ 0 & (a+b+c) & 0 \\ 0 & 0 & (a+b+c) \end{vmatrix}$$

$=2(a+b+c)^3 \begin{vmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$ Expanding along C_1

$=2(a+b+c)^3[1(1-0)+0+0]$

$= 2(a+b+c)^3$

5. **S.T** $\begin{vmatrix} \mathbf{a-b-c} & \mathbf{2a} & \mathbf{2b} \\ \mathbf{2b} & \mathbf{b-c-a} & \mathbf{2b} \\ \mathbf{2c} & \mathbf{2c} & \mathbf{c-a-b} \end{vmatrix}$

$= \mathbf{(a+b+c)^3}$

Sol: $\begin{vmatrix} a-b-c & 2a & 2b \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$

s

$R_1 \to R_1 + R_2 + R_3$

$= \begin{vmatrix} (a+b+c) & (a+b+c) & (a+b+c) \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$

$=(a+b+c) \begin{vmatrix} 1 & 1 & 1 \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$

$C_2 \to C_2 - C_1 : C_3 \to C_3 - C_1$

$= (a+b+c) \begin{vmatrix} 1 & 0 & 0 \\ 2b & -(a+b+c) & 0 \\ 2c & 0 & -(a+b+c) \end{vmatrix}$

$=(a+b+c)^3 \begin{vmatrix} 1 & 0 & 0 \\ 2b & 1 & 0 \\ 2c & 0 & 1 \end{vmatrix}$ Expanding along R_1

$=(a+b+c)^3[1(1-0)+0+0]$

$= (a+b+c)^3$

6. ***S.T*** $\begin{vmatrix} \mathbf{1} & \boldsymbol{a^2} & \boldsymbol{a^3} \\ \mathbf{1} & \boldsymbol{b^2} & \boldsymbol{b^3} \\ \mathbf{1} & \boldsymbol{c^2} & \boldsymbol{c^3} \end{vmatrix}$ ***=(a-b) (b-c) (c-a)(ab +bc + ca).***

Sol: L.H.S $\begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$

$R_1 \to R_1 - R_2 : R_2 \to R_2 - R_3$

$= \begin{vmatrix} 0 & a^2-b^2 & a^3-b^3 \\ 0 & b^2-c^2 & b^3-c^3 \\ 1 & c^2 & c^3 \end{vmatrix}$

$= \begin{vmatrix} 0 & (a-b)(a+b) & (a-b)(a^2+ab+b^2) \\ 0 & (b-c)(b+c) & (b-c)(b^2+bc+c^2) \\ 1 & c^2 & c^3 \end{vmatrix}$

= (a-b) (b-c) $\begin{vmatrix} 0 & (a+b) & (a^2+ab+b^2) \\ 0 & (b+c) & (b^2+bc+c^2) \\ 1 & c^2 & c^3 \end{vmatrix}$

$R_2 \to R_2 - R_1$:

= (a-b) (b-c) $\begin{vmatrix} 0 & a+b & a^2+ab+b^2 \\ 0 & (c-a) & b^2+bc+c^2-a^2-ab-b^2 \\ 1 & c^2 & c^3 \end{vmatrix}$

$b^2+bc+c^2-a^2-ab-b^2$
$= bc + c^2 - a^2 - ab$
=b(c-a) +(c-a)$(c+a)$
=(c-a)$(a+b+c)$

= (a-b) (b-c) $\begin{vmatrix} 0 & a+b & a^2+ab+b^2 \\ 0 & (c-a) & (c-a)(a+b+c) \\ 1 & c^2 & c^3 \end{vmatrix}$

= (a-b) (b-c)$(c-a)$ $\begin{vmatrix} 0 & a+b & a^2+ab+b^2 \\ 0 & 1 & a+b+c \\ 1 & c^2 & c^3 \end{vmatrix}$

Expanding along C_1

= (a-b) (b-c)$(c-a)1$
$[(a+b)(a+b+c) - a^2 - ab - b^2]$

= (a-b) (b-c)$(c-a)\{a^2+ab+ac+ab+b^2+bc$
$-a^2-ab-b^2\}$

= (a-b) (b-c) (c-a) (ab + bc + ca).

7. S.T $\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix}$ = abc (a-b) (b-c) (c-a)

Sol: $\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix}$

$=abc\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$

$C_1 \to C_1 - C_2 : C_2 \to C_2 - C_3$

$=abc\begin{vmatrix} 0 & 0 & 1 \\ a-b & b-c & c \\ a^2-b^2 & b^2-c^2 & c^2 \end{vmatrix}$

$=abc\begin{vmatrix} 0 & 0 & 1 \\ a-b & b-c & c \\ (a-b)(a+b) & (b-c)(b+c) & c^2 \end{vmatrix}$

$= abc(a-b)(b-c)\begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & c \\ (a+b) & (b+c) & c^2 \end{vmatrix}$

Expanding along R_1

$= abc(a-b)(b-c)1\{b+c-a-b\}$

= abc (a-b) (b-c) (c-a)

8. **find x if** $\begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ x-4 & 2x-9 & 3x-16 \\ x-8 & 2x-27 & 3x-64 \end{vmatrix} = 0.$

sol: $\begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ x-4 & 2x-9 & 3x-16 \\ x-8 & 2x-27 & 3x-64 \end{vmatrix} = 0$

$R_2 \to R_2 - R_1 : R_3 \to R_3 - R_1$

$\Rightarrow \begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ -2 & -6 & -12 \\ -6 & -24 & -60 \end{vmatrix} = 0.$

$\Rightarrow \begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ 1 & 3 & 6 \\ 1 & 4 & 10 \end{vmatrix} = 0.$

$\Rightarrow (x-2)[30-24] - (2x-3)[10-6] + (3x-4)[4-3] = 0$

$\Rightarrow (x-2)[6] - (2x-3)[4] + (3x-4)[1] = 0$

$\Rightarrow 6x - 12 - 8x + 12 + 3x - 4 = 0$

$\Rightarrow x - 4 = 0$

$\therefore$x=4.

9. If $A=\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ ***is a non − singular martix, then prove that A is invertible and*** $A^{-1} = \frac{adjA}{detA}$.

Sol: $A=\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ *is a non singular matrix*

$\therefore detA \neq 0$

and $adjA = \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix}$

Now A. adjA= $\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} . \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix}$

$= \begin{bmatrix} a_1A_1+b_1B_1+c_1C_1 & a_1A_2+b_1B_2+c_1C_2 & a_1A_3+b_1B_3+c_1C_3 \\ a_2A_1+b_2B_1+c_2C_1 & a_2A_2+b_2B_2+c_2C_2 & a_2A_3+b_2B_3+c_2C_3 \\ a_3A_1+b_3B_1+c_3C_1 & a_3A_2+b_3B_2+c_3C_2 & a_3A_3+b_3B_3+c_3C_3 \end{bmatrix}$

$=\begin{bmatrix} detA & 0 & 0 \\ 0 & detA & 0 \\ 0 & 0 & detA \end{bmatrix}$

$=detA\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ = detA.I

$\therefore$A. $\frac{adjA}{\text{detA}}$=I

Similarly we can prove that A. $\frac{adjA}{\text{detA}}$= I

$[AB = I = BA \Rightarrow A \text{ is invirtable}]$

$\therefore A^{-1} = \frac{adjA}{detA}$.

10. Solve the following equations by using Cramer's rule and matrix inversion method.

(a). $3x + 4y + 5z = 18, 2x - y + 8z = 13$ **and** $5x - 2y + 7z = 20.$

Sol: let A=$\begin{bmatrix} 3 & 4 & 5 \\ 2 & -1 & 8 \\ 5 & -2 & 7 \end{bmatrix}$, X= $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$and B=$\begin{bmatrix} 18 \\ 13 \\ 20 \end{bmatrix}$

$\Delta = \begin{vmatrix} 3 & 4 & 5 \\ 2 & -1 & 8 \\ 5 & -2 & 7 \end{vmatrix}$

$=3\begin{vmatrix} -1 & 8 \\ -2 & 7 \end{vmatrix} - 4\begin{vmatrix} 2 & 8 \\ 5 & 7 \end{vmatrix} + 5\begin{vmatrix} 2 & -1 \\ 5 & -2 \end{vmatrix}$

=3 (- 7 + 16) - 4 (14-40) + 5 (- 4 + 5)

=3 (9) – 4 (- 26) +5 (1)

=27 + 10 + 5

$=136 \neq 0$ *cramer's rule aplicable*

$\Delta_1 = \begin{vmatrix} 18 & 4 & 5 \\ 13 & -1 & 8 \\ 20 & -2 & 7 \end{vmatrix}$

$=18\begin{vmatrix} -1 & 8 \\ -2 & 7 \end{vmatrix} - 4\begin{vmatrix} 13 & 8 \\ 20 & 7 \end{vmatrix} + 5\begin{vmatrix} 13 & -1 \\ 20 & -2 \end{vmatrix}$

=18 (-7+16) - 4 (9 1- 160) +5(-26+20)

=18(9)-4(-69) +5(-6)

=162+276-30 = 408

$\Delta_2 = \begin{vmatrix} 3 & 18 & 5 \\ 2 & 13 & 8 \\ 5 & 20 & 7 \end{vmatrix}$

$=3\begin{vmatrix} 13 & 8 \\ 20 & 7 \end{vmatrix} - 18\begin{vmatrix} 2 & 8 \\ 5 & 7 \end{vmatrix} + 5\begin{vmatrix} 2 & 13 \\ 5 & 20 \end{vmatrix}$

=3(91-160) -18(14-40) +5(40-65)

=3(-69)-18(-26) +5(-25)

=-207+468 -125

=136

$\Delta_3 = \begin{vmatrix} 3 & 4 & 18 \\ 2 & -1 & 13 \\ 5 & -2 & 20 \end{vmatrix}$

$=3\begin{vmatrix} -1 & 13 \\ -2 & 20 \end{vmatrix} - 4\begin{vmatrix} 2 & 13 \\ 5 & 20 \end{vmatrix} + 18\begin{vmatrix} 2 & -1 \\ 5 & -2 \end{vmatrix}$

= 3(-20 + 26) - 4(40 - 65) + 5 (-4+5)

=3(6)-4(-25) + 5(1)

=18 + 100 + 18

=136

$\Rightarrow x = \frac{\Delta_1}{\Delta} = \frac{408}{136}, \; y = \frac{\Delta_2}{\Delta} = \frac{136}{136}, \; z = \frac{\Delta_3}{\Delta} = \frac{136}{136}$

∴x=3, y=1 and z=1.

(b). $2x - y + 3z = 9, \quad x + y + z = 6 \;$ ***and***

$x - y + z = 2$

Sol: let A=$\begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$, **X=** $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$**and B=**$\begin{bmatrix} 9 \\ 6 \\ 2 \end{bmatrix}$

$\Delta = \begin{vmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix} = 2\begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} + 1\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} + 3\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}$

=2(1+1) +1(1-1) +3(-1-1)

=2(2) +1(0) +3(-2)

$=4-6 = -2 \neq$ *cramer'srule aplicable.*

$\Delta_1 = \begin{vmatrix} 9 & -1 & 3 \\ 6 & 1 & 1 \\ 2 & -1 & 1 \end{vmatrix} = 6\begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} + 1\begin{vmatrix} 6 & 1 \\ 2 & 1 \end{vmatrix} + 3\begin{vmatrix} 6 & 1 \\ 2 & -1 \end{vmatrix}$

=9(1+1) +1(6-2) +3(-6-2)

=9(2) +1(4) +3(-8)

=18+4-24=22-24=-2

$\Delta_2 = \begin{vmatrix} 2 & 9 & 3 \\ 1 & 6 & 1 \\ 1 & 2 & 1 \end{vmatrix} = 2\begin{vmatrix} 6 & 1 \\ 2 & 1 \end{vmatrix} - 9\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} + 3\begin{vmatrix} 1 & 6 \\ 1 & 2 \end{vmatrix}$

=2(6-2) -9(1-1) +3(2-6)

=2(4) +1(0) +3(-24)

=8-12=-4

$\Delta_3=\begin{vmatrix}2 & -1 & 9\\1 & 1 & 6\\1 & -1 & 2\end{vmatrix}=2\begin{vmatrix}1 & 6\\-1 & 2\end{vmatrix}+1\begin{vmatrix}1 & 6\\1 & 2\end{vmatrix}+9\begin{vmatrix}1 & 1\\1 & -1\end{vmatrix}$

=2(2+6) +1(2-6) +9(-1-1)

=2(8) +1(-4) +9(-2)

=16-4-18

=16-22=-6

$\Rightarrow x=\frac{\Delta_1}{\Delta}=\frac{-2}{-2}=1,\ y=\frac{\Delta_2}{\Delta}=\frac{-4}{-2}=2,\ z=\frac{\Delta_3}{\Delta}=\frac{-6}{-2}=3$

∴x=1, y=2 and z=3.

(c). A= $\begin{bmatrix}1 & 1 & 1\\2 & 5 & 7\\2 & 1 & -1\end{bmatrix}$**, X=** $\begin{bmatrix}x\\y\\z\end{bmatrix}$ **and B=** $\begin{bmatrix}9\\52\\0\end{bmatrix}$

$\Delta=\begin{vmatrix}1 & 1 & 1\\2 & 5 & 7\\2 & 1 & -1\end{vmatrix}=1\begin{vmatrix}5 & 7\\1 & -1\end{vmatrix}-1\begin{vmatrix}2 & 7\\2 & -1\end{vmatrix}+1\begin{vmatrix}2 & 5\\2 & 1\end{vmatrix}$

=1(-5-7) -1(-2-14) +1(2-10)

=1(-12) -1(-16) +1(-8)

=-12+16-8=- 4≠ $cramer's rule\ aplicable.$

$\Delta_1=\begin{vmatrix}9 & 1 & 1\\52 & 5 & 7\\0 & 1 & -1\end{vmatrix}=9\begin{vmatrix}5 & 7\\1 & -1\end{vmatrix}-1\begin{vmatrix}52 & 7\\0 & -1\end{vmatrix}+1\begin{vmatrix}52 & 5\\0 & 1\end{vmatrix}$

=9(-5-7) -1(-52-0) +1(52-0)

=9(-12) -1(-52) +1(52)

=-108+52+52

=-108+104= - 4

$=\Delta_2=\begin{vmatrix}1 & 9 & 1\\2 & 52 & 7\\2 & 0 & -1\end{vmatrix}=1\begin{vmatrix}52 & 7\\0 & -1\end{vmatrix}-9\begin{vmatrix}2 & 7\\2 & -1\end{vmatrix}+1\begin{vmatrix}2 & 52\\2 & 0\end{vmatrix}$

=1(-52-0) -9(-2-14) +1(0-104)

=1(-52) -9(-16) +1(-104)

=-52+144-104

=-156+144=-12

$\Delta_3=\begin{vmatrix}1 & 1 & 9\\2 & 5 & 52\\2 & 1 & 0\end{vmatrix}=1\begin{vmatrix}5 & 52\\1 & 0\end{vmatrix}-1\begin{vmatrix}2 & 52\\2 & 0\end{vmatrix}+9\begin{vmatrix}2 & 5\\2 & 1\end{vmatrix}$

=1(0-52) -1(0-104) +9(2-10)

=1(-52) -1(-104) +9(-8)

=-52+104-72

=-124+104=-20

$\Rightarrow x=\frac{\Delta_1}{\Delta}=\frac{-4}{-4}=1,\ y=\frac{\Delta_2}{\Delta}=\frac{-12}{-4}=3,\ z=\frac{\Delta_3}{\Delta}=\frac{-20}{-4}=5$

∴x=1, y=3 and z=5.

(d). A= $\begin{bmatrix}2 & -1 & 3\\-1 & 2 & 1\\3 & 1 & -4\end{bmatrix}$**, X=** $\begin{bmatrix}x\\y\\z\end{bmatrix}$ **and B=** $\begin{bmatrix}8\\4\\0\end{bmatrix}$

$\Delta=\begin{vmatrix}2 & -1 & 3\\-1 & 2 & 1\\3 & 1 & -4\end{vmatrix}$

$=2\begin{vmatrix}2 & 1\\1 & -4\end{vmatrix}+1\begin{vmatrix}-1 & 1\\3 & -4\end{vmatrix}+3\begin{vmatrix}-1 & 2\\3 & 1\end{vmatrix}$

=2(-8-1) +1(4-3) +3(-1-6)

=2(-9) +1(1) +3(-7)

=--18+1-21 = - 38

$\Delta_1=\begin{vmatrix}8 & -1 & 3\\4 & 2 & 1\\0 & 1 & -4\end{vmatrix}=8\begin{vmatrix}2 & 1\\1 & -4\end{vmatrix}+1\begin{vmatrix}4 & 1\\0 & -4\end{vmatrix}+3\begin{vmatrix}4 & 2\\0 & 1\end{vmatrix}$

=8(-8-1) +1(-16-0) +3(4-0)

=8(-9) +1(-16) +3(4)

=-72-16+12

=-76

$\Delta_2 = \begin{vmatrix} 2 & 8 & 3 \\ -1 & 4 & 1 \\ 3 & 0 & -4 \end{vmatrix} = 2\begin{vmatrix} 4 & 1 \\ 0 & -4 \end{vmatrix} - 8\begin{vmatrix} -1 & 1 \\ 3 & -4 \end{vmatrix} + 3\begin{vmatrix} -1 & 4 \\ 3 & 0 \end{vmatrix}$

=2(-16-0) -8(4-3) +3(0-12)

=2(-16) -8(1) +3(-12)

=-32-8-36

=-76

$\Delta_3 = \begin{vmatrix} 2 & -1 & 8 \\ -1 & 2 & 4 \\ 3 & 1 & 0 \end{vmatrix} = 2\begin{vmatrix} 2 & 4 \\ 1 & 0 \end{vmatrix} + 1\begin{vmatrix} -1 & 4 \\ 3 & 0 \end{vmatrix} + 8\begin{vmatrix} -1 & 2 \\ 3 & 1 \end{vmatrix}$

=2(0-4+1(0-12+8(-1-6)

=2(-4) +1(-12) +8(-7)

=-8-12-56

=76

$\Rightarrow x = \frac{\Delta_1}{\Delta} = \frac{-76}{-38} = 2,\ y = \frac{\Delta_2}{\Delta} = \frac{-76}{-38} = 2,\ z = \frac{\Delta_3}{\Delta} = \frac{-76}{-38} = 2$

∴x=2, y=2 and z=2.

Matrix inversion method:

***(a).* A=** $\begin{bmatrix} \mathbf{3} & \mathbf{4} & \mathbf{5} \\ \mathbf{2} & \mathbf{-1} & \mathbf{8} \\ \mathbf{5} & \mathbf{-2} & \mathbf{7} \end{bmatrix}$**, X=** $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ **and B=** $\begin{bmatrix} \mathbf{18} \\ \mathbf{13} \\ \mathbf{20} \end{bmatrix}$

AX=B⇒X=A^{-1}.B

And $A^{-1} = \frac{adjA}{detA}$

AdjA= $\begin{bmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{bmatrix}$

$\begin{pmatrix} -1 & 8 & 2 & -1 \\ -2 & 7 & 5 & -2 \\ 4 & 5 & 3 & 4 \\ -1 & 8 & 2 & -1 \end{pmatrix}$

$= \begin{bmatrix} (-7+16) & (40-14) & (-4+5) \\ (-10-28) & (21-25) & (20+6) \\ (32+5) & (10-24) & (-3-8) \end{bmatrix}$

$= \begin{bmatrix} 9 & 26 & 1 \\ -38 & -4 & 26 \\ 37 & -14 & -11 \end{bmatrix}$

AdjA $= \begin{bmatrix} 9 & -38 & 37 \\ 26 & -4 & -14 \\ 1 & 26 & -11 \end{bmatrix}$

DetA$=a_1A_1 + b_1B_1 + c_1C_1$

2(9)-4(-26) +5(1)

=136

$A^{-1} = \frac{adjA}{detA} = \frac{1}{136}\begin{bmatrix} 9 & -38 & 37 \\ 26 & -4 & -14 \\ 1 & 26 & -11 \end{bmatrix}$

⇒X=A^{-1}.B

$= \frac{1}{136}\begin{bmatrix} 9 & -38 & 37 \\ 26 & -4 & -14 \\ 1 & 26 & -11 \end{bmatrix}\begin{bmatrix} 18 \\ 13 \\ 20 \end{bmatrix}$

$= \frac{1}{136}\begin{bmatrix} 162-494+740 \\ 468-52-280 \\ 18+338+220 \end{bmatrix}$

$= \frac{1}{136}\begin{bmatrix} 408 \\ 136 \\ 136 \end{bmatrix} \therefore\ x = 3, y = 1\ and\ z = 1.$

(b). Sol: let A= $\begin{bmatrix} \mathbf{2} & \mathbf{-1} & \mathbf{3} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{-1} & \mathbf{1} \end{bmatrix}$**, X=** $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ **and B=** $\begin{bmatrix} \mathbf{9} \\ \mathbf{6} \\ \mathbf{2} \end{bmatrix}$

AX=B⇒X=A^{-1}.B

And $A^{-1} = \frac{adjA}{detA}$

$detA = \begin{vmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix} = 2\begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} + 1\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} + 3\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}$

=2(1+1) +1(1-1) +3(-1-1)

=2(2) +1(0) +3(-2)

=4-6 = -2≠ 0 $\quad A^{-1} exists.$

AdjA= $\begin{bmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{bmatrix}$

$\begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 3 & 2 & -1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$

$= \begin{bmatrix} (1+1) & (1-1) & (-1-1) \\ (-3+1) & (2-3) & (-1+2) \\ (-1-3) & (3-2) & (2+1) \end{bmatrix}$

$$=\begin{bmatrix} 2 & 0 & -2 \\ -2 & -1 & 1 \\ -4 & 1 & 3 \end{bmatrix}$$

$$\text{AdjA} = \begin{bmatrix} 2 & -2 & -4 \\ 0 & -1 & 1 \\ -2 & 1 & 3 \end{bmatrix}$$

$$A^{-1} = \frac{adjA}{detA} = \frac{1}{-2}\begin{bmatrix} 2 & -2 & -4 \\ 0 & -1 & 1 \\ -2 & 1 & 3 \end{bmatrix}$$

$\Rightarrow X = A^{-1}.B$

$$= \frac{1}{-2}\begin{bmatrix} 2 & -2 & -4 \\ 0 & -1 & 1 \\ -2 & 1 & 3 \end{bmatrix}\begin{bmatrix} 9 \\ 6 \\ 2 \end{bmatrix}$$

$$= \frac{1}{-2}\begin{bmatrix} 18-12-8 \\ 0-6+2 \\ -18+6+6 \end{bmatrix}$$

$$= \frac{1}{-2}\begin{bmatrix} -2 \\ -4 \\ -6 \end{bmatrix} \therefore\ x = 1, y = 2 \text{ and } z = 3$$

(c). $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & 7 \\ 2 & 1 & -1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ **and** $B = \begin{bmatrix} 9 \\ 52 \\ 0 \end{bmatrix}$

$$detA = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 5 & 7 \\ 2 & 1 & -1 \end{vmatrix} = 1\begin{vmatrix} 5 & 7 \\ 1 & -1 \end{vmatrix} - 1\begin{vmatrix} 2 & 7 \\ 2 & -1 \end{vmatrix} + 1\begin{vmatrix} 2 & 5 \\ 2 & 1 \end{vmatrix}$$

=1(-5-7) -1(-2-14) +1(2-10)

=1(-12) -1(-16) +1(-8)

=-12+16-8=- 4$\neq$ 0. A^{-1} exists.

$$\text{AdjA} = \begin{bmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{bmatrix} \qquad \begin{pmatrix} 5 & 7 & 2 & 5 \\ 1 & -1 & 2 & 1 \\ 1 & 1 & 1 & 1 \\ 5 & 7 & 2 & 5 \end{pmatrix}$$

$$= \begin{bmatrix} (-5-7) & (14+2) & (2-10) \\ (1+1) & (-1-2) & (2-1) \\ (7-5) & (2-7) & (5-2) \end{bmatrix}$$

$$= \begin{bmatrix} -12 & 16 & -8 \\ 2 & -3 & 1 \\ 2 & -5 & 3 \end{bmatrix}$$

$$\text{AdjA} = \begin{bmatrix} -12 & 2 & 2 \\ 16 & -3 & -5 \\ -8 & 1 & 3 \end{bmatrix}$$

$$A^{-1} = \frac{adjA}{detA} = \frac{1}{-4}\begin{bmatrix} -12 & 2 & 2 \\ 16 & -3 & -5 \\ -8 & 1 & 3 \end{bmatrix}$$

$\Rightarrow X = A^{-1}.B$

$$= \frac{1}{-4}\begin{bmatrix} -12 & 2 & 2 \\ 16 & -3 & -5 \\ -8 & 1 & 3 \end{bmatrix}\begin{bmatrix} 9 \\ 52 \\ 0 \end{bmatrix}$$

$$= \frac{1}{-2}\begin{bmatrix} -108+104+0 \\ 144-156+0 \\ -72+52+0 \end{bmatrix}$$

$$= \frac{1}{-2}\begin{bmatrix} -4 \\ -12 \\ -20 \end{bmatrix} \therefore\ x = 1, y = 3 \text{ and } z = 5.$$

(d).let $A = \begin{bmatrix} 2 & -1 & 3 \\ -1 & 2 & 1 \\ 3 & 1 & -4 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ **and** $B = \begin{bmatrix} 8 \\ 4 \\ 0 \end{bmatrix}$

$$detA = \begin{vmatrix} 2 & -1 & 3 \\ -1 & 2 & 1 \\ 3 & 1 & -4 \end{vmatrix} = 2\begin{vmatrix} 2 & 1 \\ 1 & -4 \end{vmatrix} + 1\begin{vmatrix} -1 & 1 \\ 3 & -4 \end{vmatrix} + 3\begin{vmatrix} -1 & 2 \\ 3 & 1 \end{vmatrix}$$

=2(-8-1) +1(4-3) +3(-1-6)

=2(-9) +1(1) +3(-7)

=--18+1-21

=-38$\neq$ 0. A^{-1} exists.

$$\text{AdjA} = \begin{bmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{bmatrix} \qquad \begin{pmatrix} 2 & 1 & -1 & 2 \\ 1 & -4 & 3 & 1 \\ -1 & 3 & 2 & -1 \\ 2 & 1 & -1 & 2 \end{pmatrix}$$

$$= \begin{bmatrix} (-8-1) & (3-4) & (-1-6) \\ (3-4) & (-8-9) & (-3-2) \\ (-1-6) & (-3-2) & (4-1) \end{bmatrix}$$

$$= \begin{bmatrix} -9 & -1 & -7 \\ -1 & -17 & -5 \\ -7 & -5 & 3 \end{bmatrix}$$

$$\text{AdjA} = \begin{bmatrix} -9 & -1 & -7 \\ -1 & -17 & -5 \\ -7 & -5 & 3 \end{bmatrix}$$

$$A^{-1} = \frac{adjA}{detA} = \frac{1}{-38}\begin{bmatrix} -9 & -1 & -7 \\ -1 & -17 & -5 \\ -7 & -5 & 3 \end{bmatrix}$$

$\Rightarrow X = A^{-1}.B$

$$= \frac{1}{-38}\begin{bmatrix} -9 & -1 & -7 \\ -1 & -17 & -5 \\ -7 & -5 & 3 \end{bmatrix}\begin{bmatrix} 8 \\ 4 \\ 0 \end{bmatrix}$$

$$= \frac{1}{-38}\begin{bmatrix} -72-4+0 \\ -8-68+0 \\ -56-20+0 \end{bmatrix}$$

$$= \frac{1}{-38}\begin{bmatrix} -76 \\ -76 \\ -76 \end{bmatrix} \therefore x = 2, y = 2 \text{ and } z = 2.$$

11. Show that the following system of equations is consistent and solve it completely.

$x + y + z = 3, 2x + 2y - z = 3, x + y - z = 1;$

sol: the augmented matrix is

$$[AD] = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 2 & 2 & -1 & 3 \\ 1 & 1 & -1 & 1 \end{bmatrix}$$

$on\ applying\ R_2 \to R_2 - 2R_1, R_3 \to R_3 - R_1$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 0 & -3 & -3 \\ 0 & 0 & -2 & -2 \end{bmatrix}$$

$an\ applying\ R_3 \to 3R_3 - 2R_2$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 0 & -3 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{..... (1)}$$

Comparing with echelon form

Number of non-zero rows in A are 2

∴rank (A) =2

Number of non-zero rows in AD are 2

∴rank (AD) =2

Rank (A) =rank (AD) =2

The system is consistent and has infinitely many solutions.

We write equivalent set of equations from … (1)

$x + y + z = 3$ ……. (2)

$-3z = -3$…….. (3)

$Z = 1$

$Hence\ z = 1,\ x + y = 2$

$Let\ x = k \Rightarrow\ y = 2 - k, z = 1$, k∈

$R\ is\ the\ solution\ set.$

12. Apply the test of rank to examine whether the following equations are consistent.

$2x - y + 3z = 8, -x + 2y + z = 4, 3x + y - 4z = 0;$

sol: the augmented matrix is

$$[AD] = \begin{bmatrix} 2 & -1 & 3 & 8 \\ -1 & 2 & 1 & 4 \\ 3 & 1 & -4 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & -1 & 3 & 8 \\ -1 & 2 & 1 & 4 \\ 3 & 1 & -4 & 0 \end{bmatrix} on\ interchanging\ R_1\ and\ R_2$$

$we\ transform\ the\ above\ matrix\ into$

$an\ upper\ triangular\ matrix.$

$on\ applying\ R_2 \to R_2 + 2R_1$

$R_3 \to R_3 + 3R_1$

$$\sim \begin{bmatrix} -1 & 2 & 1 & 4 \\ 0 & 3 & 5 & 16 \\ 0 & 7 & -1 & 12 \end{bmatrix}$$

$an\ applying\ R_3 \to 3R_3 - 7R_2$

$$\sim \begin{bmatrix} -1 & 2 & 1 & 4 \\ 0 & 3 & 5 & 16 \\ 0 & 0 & -38 & -76 \end{bmatrix} \text{...... (1)}$$

Comparing with echelon form

Number of non-zero rows in A are 3 ∴rank (A) =3

Number of non-zero rows in AD are 3 ∴rank (AD) =3

Hence rank (A) =rank [(AD)] =3.

Thus the system has a unique solution.

We write the system of equations from (1)

$-x + 2y + z = 4$… (2)

$3y + 5z = 16$…….. (3)

$-38z = -76$…. (4)

$from\ (4) z = 2\ sub\ in\ (3)$

$\Rightarrow 3y = 16 - 10 = 6$

$y = 2\ \ sub\ y = 2, z = 2\ in\ (1) \Rightarrow x = 2$

$\therefore x = y = z = 2\ is\ the\ solution.$

13. Solve the following system of equations

$x + y + z = 1, 2x + 2y + 3z = 6, x + 4y + 9z = 3;$

Sol: the augmented matrix is

$$[AD] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 3 & 6 \\ 1 & 4 & 9 & 3 \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 & 0 & x \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & z \end{bmatrix}$$

on applying $R_2 \to R_2 - 2R_1, R_3 \to R_3 - R_1$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2-2 & 2-2 & 3-2 & 6-2 \\ 1-1 & 4-1 & 9-1 & 3-1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 4 \\ 0 & 3 & 8 & 2 \end{bmatrix}$$

$R_2 \leftrightarrow R_3$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 3 & 8 & 2 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

$R_1 \to 3R_1 - R_2$

$$= \begin{bmatrix} 3-0 & 3-3 & 3-8 & 3-2 \\ 0 & 3 & 8 & 2 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 & -5 & 1 \\ 0 & 3 & 8 & 2 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

$R_1 \to R_1 + 5R_3, R_2 \to R_2 - 8R_3$

$$= \begin{bmatrix} 3+0 & 0+0 & -5+5 & 1+20 \\ 0-0 & 3-0 & 8-8 & 2-32 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 & 0 & 21 \\ 0 & 3 & 0 & -30 \\ 0 & 0 & 1 & 4 \end{bmatrix} \quad R_1 \leftrightarrow \left[\frac{R_1}{3}\right], R_2 \leftrightarrow \left[\frac{R_2}{3}\right]$$

$$= \begin{bmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & -10 \\ 0 & 0 & 1 & 4 \end{bmatrix} \quad \therefore x = 7, y = -10, z = 4$$

$(ii) x - y + 3z = 5, 4x + 2y - z = 0, -x + 3y + z = 5;$

Sol: the augmented matrix is

$$[AD] = \begin{bmatrix} 1 & -1 & 3 & 5 \\ 4 & 2 & -1 & 0 \\ -1 & 3 & 1 & 5 \end{bmatrix}$$

$$\equiv \begin{bmatrix} 1 & 0 & 0 & x \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & z \end{bmatrix}$$

on applying $R_2 \to R_2 - 4R_1, R_3 \to R_3 + R_1$

$$= \begin{bmatrix} 1 & -1 & 3 & 5 \\ 4-4 & 2+4 & -1-12 & 0-20 \\ -1+1 & 3-1 & 1+3 & 5+5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & 3 & 5 \\ 0 & 6 & -13 & -20 \\ 0 & 2 & 4 & 10 \end{bmatrix}$$

$R_2 \leftrightarrow \left[\frac{R_3}{2}\right]$

$$= \begin{bmatrix} 1 & -1 & 3 & 5 \\ 0 & 6 & -13 & -20 \\ 0 & 1 & 2 & 5 \end{bmatrix}$$

$R_2 \to R_2 - 5R_3$

$$= \begin{bmatrix} 1 & -1 & 3 & 5 \\ 0-0 & 6-5 & -13-10 & -20-25 \\ 0 & 1 & 2 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & 3 & 5 \\ 0 & 1 & -23 & -45 \\ 0 & 1 & 2 & 5 \end{bmatrix}$$

$R_1 \to R_1 + R_2, R_3 \to R_3 - R_2$

$$= \begin{bmatrix} 1+0 & -1+1 & 3-23 & 5-45 \\ 0 & 1 & -23 & -45 \\ 0-0 & 1-1 & 2+23 & 5+45 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & -20 & -40 \\ 0 & 1 & -23 & -45 \\ 0 & 0 & 25 & 50 \end{bmatrix} \quad R_3 \leftrightarrow \left[\frac{R_3}{25}\right]$$

$$=\begin{bmatrix}1 & 0 & -20 & -40\\0 & 1 & -23 & -45\\0 & 0 & 1 & 2\end{bmatrix}$$

$R_1 \to R_1 + 20R_3$; $R_2 \to R_2 + 23R_3$

$$=\begin{bmatrix}1+0 & 0+0 & -20+20 & -40+40\\0 & 1 & -23+23 & -45+46\\0 & 0 & 1 & 2\end{bmatrix}$$

$$=\begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 1\\0 & 0 & 1 & 2\end{bmatrix} \quad \therefore x = 0, y = 1, z = 2$$

(*iii*) $\boldsymbol{2x - y + 3z = 9, x + y + z = 6, x - y + z = 2;}$
Sol: the augmented matrix is

$$[AD] = \begin{bmatrix}\mathbf{2} & \mathbf{-1} & \mathbf{3} & \mathbf{9}\\\mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{6}\\\mathbf{1} & \mathbf{-1} & \mathbf{1} & \mathbf{2}\end{bmatrix}$$

$$\equiv \begin{bmatrix}\mathbf{1} & \mathbf{0} & \mathbf{0} & \boldsymbol{x}\\\mathbf{0} & \mathbf{1} & \mathbf{0} & \boldsymbol{y}\\\mathbf{0} & \mathbf{0} & \mathbf{1} & \boldsymbol{z}\end{bmatrix}$$

$R_1 \leftrightarrow R_{1-} R_2$

$$=\begin{bmatrix}2-1 & -1-1 & 3-1 & 9-6\\1 & 1 & 1 & 6\\1 & -1 & 1 & 2\end{bmatrix}$$

$$=\begin{bmatrix}1 & -2 & 2 & 3\\1 & 1 & 1 & 6\\1 & -1 & 1 & 2\end{bmatrix}$$

on applying $R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$

$$=\begin{bmatrix}1 & -2 & 2 & 3\\1-1 & 1+2 & 1-2 & 6-3\\1-1 & -1+2 & 1-2 & 2-3\end{bmatrix}$$

$$=\begin{bmatrix}1 & -2 & 2 & 3\\0 & 3 & -1 & 3\\0 & 1 & -1 & -1\end{bmatrix}$$

$R_2 \to R_2 - 2R_3$

$$=\begin{bmatrix}1 & -2 & 2 & 3\\0-0 & 3-2 & -1+2 & 3+2\\0 & 1 & -1 & -1\end{bmatrix}$$

$$=\begin{bmatrix}1 & -2 & 2 & 3\\0 & 1 & 1 & 5\\0 & 1 & -1 & -1\end{bmatrix}$$

$R_1 \to R_1 + 2R_2, R_3 \to R_3 - R_2$

$$=\begin{bmatrix}1+0 & -2+2 & 2+2 & 3+10\\0 & 1 & 1 & 5\\0-0 & 1-1 & -1-1 & -1-5\end{bmatrix}$$

$$=\begin{bmatrix}1 & 0 & 4 & 13\\0 & 1 & 1 & 5\\0 & 0 & -2 & -6\end{bmatrix}$$

$R_3 \leftrightarrow \left[\frac{R_3}{2}\right]$

$$=\begin{bmatrix}1 & 0 & 4 & 13\\0 & 1 & 1 & 5\\0 & 0 & 1 & 3\end{bmatrix}$$

$R_1 \to R_1 - 4R_3$; $R_2 \to R_2 - R_3$

$$=\begin{bmatrix}1 & 0 & 4-4 & 13-12\\0 & 1 & 1-1 & 5-3\\0 & 0 & 1 & 3\end{bmatrix}$$

$$=\begin{bmatrix}1 & 0 & 0 & 1\\0 & 1 & 0 & 2\\0 & 0 & 1 & 3\end{bmatrix} \therefore x = 1, y = 2, z = 3$$

14. $x + y + z = 1,$

$2x + 2y + 3z = 6,$

$x + 4y + 9z = 3$