

# Random Variables and Probability Distribution

## Random Variable

**Random variable:** If  $S$  is the sample space  $P(S)$  is the power set of the sample space,  $P$  is the probability of the function then  $(S, P(S), P)$  is called the probability space,

In the probability space if  $X : S \rightarrow \mathbb{R}$  is a function then  $x$  is called random variable

**Frequency function of random variable (or) Probability density function:** Function  $f : X(S) \rightarrow \mathbb{R}$  is defined by

$f(r) = p[e/x(e) = r]$  is called the frequency function associated with random variable

Where i)  $0 \leq f(r) \leq 1 \forall r \in x(S)$

ii)  $\sum f(r) = 1 \forall r \in x(S)$

**Arithmetic mean of the random variable:** Arithmetic mean of the random variable  $x$  is denoted by  $\bar{x}$  or all  $E(x)$  expected value of  $X$  and is defined as  $\bar{x} = \sum r f(r)$

**Variance of the random variable:** If ' $x$ ' is a random variable then  $E(x^2)$  is defined such that  $E(x^2) = \sum r^2 f(r) \forall x \in x(S)$ . The variance of random variable ( $\sigma^2$ ) and is defined as  $F(x - \bar{x})^2$

Variance of the random variable

$$\sigma^2 = E(x - \bar{x})^2 = E(x^2 + \bar{x}^2 - 2x\bar{x})$$

$$\sigma^2 = E(x)^2 + \bar{x}^2 - 2\bar{x}E(x)$$

$$\sigma^2 = E(x)^2 + \bar{x}^2$$

Variable of random variable  $\sigma^2 = E(x^2) - \mu^2$

$$\sigma^2 = \sum r^2 f(r) - \mu^2$$

$$\sigma^2 + \mu^2 = \sum r^2 f(r)$$

**Standard deviation:** It is the positive square root of the variance of the standard deviation of the random variable this is denoted by  $\sigma = \sqrt{\text{variance}}$

**Note:** Let  $X$  be a random variable on a sample space  $S$ . If  $x \in R$  then we use the following symbols to denote some events in  $S$ .

i)  $\{a \in S : X(a) = x\} = (X = x)$

$$\text{ii) } \{a \in S : X(a) < x\} = (X < x)$$

$$\text{iii) } \{a \in S : X(a) \leq x\} = (X \leq x)$$

$$\text{iv) } \{a \in S : X(a) > x\} = (X > x)$$

$$\text{v) } \{a \in S : X(a) \geq x\} = (X \geq x)$$

**Def 2:** Let  $S$  be a sample space and  $X : S \rightarrow R$  be a random variable. The function  $F : R \rightarrow R$  defined by  $F(x) = P(X \leq x)$ , is called probability distribution function of the random variable  $X$ .

We now state some properties of probability distribution function for the random variable  $X$  beyond the scope of the book.

**Theorem 2:** Let  $F(x)$  be the probability distribution function for the random variable  $X$ . then

$$\text{i) } 0 \leq F(x) \leq 1, \forall x \in R$$

$$\text{ii) } F(x) \text{ is an increasing function i.e. } x_1, x_2 \in R, x_1 < x_2 \Rightarrow F(x_1) \leq F(x_2)$$

$$\text{iii) } \lim_{x \rightarrow \infty} F(x) = 1, \lim_{x \rightarrow -\infty} F(x) = 0$$

**Theorem 3:** If  $X : S \rightarrow R$  is a discrete random variable with range  $\{x_1, x_2, x_3, \dots\}$  then  $\sum_{r=1}^{\infty} P(X = x_r) = 1$ .

## Mean and Variance

**Def :** Let  $X : S \rightarrow R$  be a discrete random variable with range  $\{x_1, x_2, x_3, \dots\}$ . If  $\sum x_r P(X = x_r)$  exists, then  $\sum x_r P(X = x_r)$  is called the mean of the random variable  $X$ . It is denoted by  $\mu$  or  $\bar{x}$ . If  $\sum (x_r - \mu)^2 P(X = x_r)$  exists, then  $\sum (x_r - \mu)^2 P(X = x_r)$  is called variance of the random variable  $X$ . It is denoted by  $\sigma^2$ . The positive square root of the variance is called the standard deviation of the random variable  $X$ . It is denoted by  $\sigma$ .

**Theorem 4:** Let  $X : S \rightarrow R$  be a discrete random variable with range  $\{x_1, x_2, x_3, \dots\}$ . If  $\mu, \sigma^2$  are the mean and variance of  $X$  then  $\sigma^2 + \mu^2 = \sum x_r^2 P(X = x_r)$ .

**Def:** Let  $n$  be a positive integer and  $p$  be a real number such that  $0 \leq p \leq 1$ . A random variable  $X$  with range  $\{0, 1, 2, \dots, n\}$  is said to follow (or have) binomial distribution or Bernoulli distribution with parameters  $n$  and  $p$  if  $P(X = r) = {}^n C_r p^r q^{n-r}$  for  $r = 0, 1, 2, \dots, n$  where  $q = 1 - p$ .

**Theorem :** If the random variable  $X$  follows a binomial distribution with parameters  $n$  and  $p$  then mean of  $X$  is  $np$  and the variance is  $npq$  where  $q = 1 - p$ .

**Def :** Let  $\lambda > 0$  be a real number. A random variable  $X$  with range  $\{0, 1, 2, \dots\}$  is said to follow (have)

$$\text{Poisson distribution with parameter } \lambda \text{ if } P(X = r) = \frac{e^{-\lambda} \lambda^r}{r!} \text{ for } r = 0, 1, 2, \dots$$

**Theorem :** If a random variable X follows Poisson distribution with parameter  $\lambda$ , then mean of X is  $\lambda$  and variance of X is  $\lambda$ .

**EXERCISE – 9(a)**

**1. A p.d.f of a discrete random variable is zero except at the points  $x = 0, 1, 2$ . At these points it has the value  $p(0) = 3c^3, p(1) = 4c - 10c^2, p(2) = 5c - 1$  for some  $c > 0$ . Find the value of c.**

**Sol.**  $P(x = 0) + p(x = 1) + p(x = 2) = 1$

$$3c^3 + 4c - 10c^2 + 5c - 1 = 1$$

$$3c^3 - 10c^2 + 9c - 2 = 0$$

$$\text{Put } c = 1, \text{ then } 3 - 10 + 9 - 2 = 12 - 12 = 0$$

$C = 1$  satisfy the above equation

$C = 1 \Rightarrow p(x = 0) = 3$  which is not possible dividing with  $c - 1$ , we get

$$3c^2 - 7c + 2 = 0$$

$$(c - 2)(3c - 1) = 0$$

$$c = 2 \text{ or } c = 1/3$$

$c = 2 \Rightarrow p(x = 0) = 3.2^3 = 24$  which is not possible

$$\therefore c = 1/3$$

**2. Find the constant C, so that  $F(x) = C\left(\frac{2}{3}\right)^x, x = 1, 2, 3, \dots$  is the p.d.f of a discrete random variable X.**

**Sol.** Given  $F(x) = C\left(\frac{2}{3}\right)^x, x = 1, 2, 3$

We know that  $p(x) = C\left(\frac{2}{3}\right)^x, x = 1, 2, 3, \dots$

$$\therefore \sum_{x=1}^{\infty} p(x) = 1$$

$$\Rightarrow \sum_{x=1}^{\infty} c\left(\frac{2}{3}\right)^x = 1$$

$$\Rightarrow c \left[ \left(\frac{2}{3}\right) + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + \dots \infty \right] = 1$$

$$\Rightarrow C \frac{2}{3} \left[ 1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \dots \infty \right] = 1$$

$$\Rightarrow \frac{2c}{3} \left( \frac{1}{1 - \frac{2}{3}} \right) = 1$$

$$\left[ \therefore a + ar + ar^2 + \dots \infty = \frac{a}{1-r}, \text{ if } |r| < 1 \right]$$

$$\Rightarrow \frac{2c}{3} \times 3 = 1 \Rightarrow c = \frac{1}{2}$$

3.

<b>X=x</b>	-2	-1	0	1	2	3
<b>P(X=x)</b>	0.1	k	0.2	k	0.3	k

is the probability distribution of a random variable x. find the value of K and the variance of x.

**Sol.** Sum of the probabilities = 1

$$0.1 + k + 0.2 + 2k + 0.3 + k = 1$$

$$4k + 0.6 = 1$$

$$4k = 1 - 0.6 = 0.4$$

$$k = \frac{0.4}{4} = 0.1$$

$$\text{Mean} = (-2)(0.1) + (-1)k + 0(0.2) + 1(2k) + 2(0.3) + 3k$$

$$= -0.2 - k + 0 + 2k + 0.6 + 3k$$

$$= 4k + 0.4 = 4(0.1) + 0.4 = 0.4 + 0.4 = 0.8$$

$$\mu = 0.8$$

$$\text{Variance } (\sigma^2) = \sum_{i=1}^n x^2 p(x = x_i) - \mu^2$$

$$\therefore \text{Variance} = 4(0.1) + 1(k) + 0(0.2) + 1(2k) + 4(0.3) + 9k - \mu^2$$

$$= 0.4 + k + 0 + 2k + 4(0.3) + 9k - \mu^2$$

$$= 12k + 0.4 + 1.2 - (0.8)^2$$

$$= 12(0.1) + 1.6 - 0.64$$

$$= 1.2 + 1.6 - 0.64$$

$$\therefore \sigma^2 = 2.8 - 0.64 = 2.16$$

4.

<b>X=x</b>	-3	-2	-1	0	1	2	3
<b>P(X=x)</b>	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$

is the probability distribution of a random variable x. find the variance of x.

$$\text{Sol Mean } (\mu) = -3\left(\frac{1}{9}\right) - 2\left(\frac{1}{9}\right) - 1\left(\frac{1}{9}\right) + 0\left(\frac{1}{9}\right) + 1\left(\frac{1}{9}\right) + 2\left(\frac{1}{9}\right) + 3\left(\frac{1}{9}\right)$$

$$= -\frac{3}{9} - \frac{2}{9} - \frac{1}{9} + 0 + \frac{1}{9} + \frac{2}{9} + \frac{3}{9} = (\mu) = 0$$

$$\text{Variance } (\sigma^2) = (-3)^2 \frac{1}{9} + (-2)^2 \frac{1}{9} + (-1)^2 \frac{1}{9} + (0)^2 \left(\frac{1}{9}\right) + (1)^2 \frac{1}{9} + (2)^2 \frac{1}{9} + (3)^2 \frac{1}{9} - \mu^2$$

$$= \frac{9}{9} + \frac{4}{9} + \frac{1}{9} + 0 + \frac{1}{9} + \frac{4}{9} + \frac{9}{9} - 0^2 = \frac{28}{9} - 0$$

$$\sigma^2 = \frac{28}{9}$$

5. A random variable x has the following probability distribution.

<b>X=x</b>	0	1	2	3	4	5	6	7
<b>P(X=x)</b>	0	k	2k	2k	3k	K <sup>2</sup>	2k <sup>2</sup>	7k <sup>2</sup> +k

Find i) k ii) the mean and iii) p(0 < x < 5).

**Sol.** Sum of the probabilities =

$$0 + k + 2k + 2k + 3k + K^2 + 2k^2 + 7k^2 + k = 1$$

























