Random Variables and Probability Distribution

Random Variable

Random variable: If S is the sample space P(S) is the power set of the sample space, P is the probability of the function then (S, P(S), P) is called the probability space,

In the probability space if $X : S \rightarrow \mathbb{R}$ is a function then x is called random variable

Frequency function of random variable (or) Probability density function: Function $f : X(S) \rightarrow \mathbb{R}$ is defined by

$f(r) = p[e / x(e) = r]$ is called the frequency function associated with random variable

Where i) $0 \le f(r) \le 1 \ \forall \ r \in x(S)$

ii) $\Sigma f(r) = 1 \ \forall r \in x(S)$

Arithmetic mean of the random variable: Arithmetic mean of the random variable x is denoted by $\bar{x}$ or all E(x) expected value of X and is defined as $\bar{x} = \Sigma r \, f(r)$

Variance of the random variable: If 'x' is a random variable then $E(x^2)$ is defined such that $E(x^2) = \Sigma r^2 \, f(r) \ \forall x \in x(S)$. The variance of random variable (σ^2) and is defined as $F(x - \bar{x})^2$

Variance of the random variable

$$\sigma^2 = E(x - \bar{x})^2 = E(x^2 + \bar{x}^2 - 2x \, \bar{x})$$

$$\sigma^2 = E(x)^2 + \bar{x}^2 - 2\bar{x} E(x)$$

$$\sigma^2 = E(x)^2 + \bar{x}^2$$

Variable of random variable $\sigma^2 = E(x^2) - \mu^2$

$$\sigma^2 = \Sigma r^2 f(r) - \mu^2$$

$$\sigma^2 + \mu^2 = \Sigma r^2 f(r)$$

Standard deviation: It is the positive square root of the variance of the standard deviation of the random variable this is denoted by $\sigma = \sqrt{vairance}$

Note: Let X be a random variable on a sample space S. If $x \in R$ then we use the following symbols to denote some events in S.

i) $\{a \in S : X(a) = x\} = (X = x)$

ii) $\{a \in S : X(a) < x\} = (X < x)$

iii) $\{a \in S : X(a) \leq x\} = (X \leq x)$

iv) $\{a \in S : X(a) > x\} = (X > x)$

v) $\{a \in S : X(a) \geq x\} = (X \geq x)$

Def 2: Let S be a sample space and $X : S \rightarrow R$ be a random variable. The function $F : R \rightarrow R$ defined by $F(x) = P(X \leq x)$, is called probability distribution function of the random variable X.

We now state some properties of probability distribution function for the random yond the scope of the book.

Theorem 2: Let F(x) be the probability distribution function for the random variable X. then

i) $0 \leq F(x) \leq 1, \forall x \in R$

ii) F (x) is an increasing function i.e. $x_1, x_2 \in R, x_1 < x_2 \Rightarrow F(x_1) \, F(x_2)$

iii) $\underset{x \rightarrow \infty}{Lt} \, F(x) = 1, \quad \underset{x \rightarrow -\infty}{Lt} \, F(x) = 0$

Theorem 3: If $X : S \rightarrow R$ is a discrete random variable with range $\{x_1, x_2, x_3....\}$ then $\sum_{r=1}^{\infty} P(X = x_r) = 1$.

Mean and Variance

Def : Let $X : S \rightarrow R$ be a discrete random variable with range $\{x_1, x_2, x_3......\}$. If $\Sigma x_r \, P(X = x_r)$ exists, then $\Sigma x_r \, P(X = x_r)$ is called the mean of the random variable X. It is denoted by μ or $\bar{x}$. If $\Sigma(x_r - \mu)^2 \, P(X = x_r)$ exists, then $\Sigma(x_r - \mu)^2 \, P(X = x_r)$ is called variance of the random variable X. It is denoted by σ^2. The positive square root of the variance is called the standard deviation of the random variable X. It is denoted by σ.

Theorem 4: Let $X : S \rightarrow R$ be a discrete random variable with range $\{x_1, x_2, x_3,......\}$. If μ, σ^2 are the mean and variance of X then $\sigma^2 + \mu^2 = \Sigma x_r^2 \, P(X = x_r)$.

Def: Let n be a positive integer and p be a real number such that $0 \leq p \leq 1$. A random variable X with range $\{0, 1, 2, .n\}$ is said to follows (or have) binomial distribution or Bernoulli distribution with parameters n and p if $P(X = r) = {}^n C_r \, p^r \, q^{n-r}$ for r = 0, 1, 2... n where q = 1 - p.

Theorem : If the random variable X follows a binomial distribution with parameters n and p then mean of X is np and the variance is npq where q = 1- p.

Def : Let $\lambda > 0$ be are real number. A random variable X with range $\{0, 1, 2....\}$ is said to follows (have) Poisson distribution with parameter λ if $P(X = r) = \dfrac{e^{-\lambda} \lambda^r}{r!}$ for r = 0, 1, 2,.......

Theorem : If a random variable X follows Poisson distribution with parameter λ, then mean of X is λ and variance of X is λ.

<p style="text-align:center"><u>**EXERCISE – 9(a)**</u></p>

1. **A p.d.f of a discrete random variable is zero except at the points x = 0, 1, 2. At these points it has the value p (0) = 3c³, p(1) = 4c – 10c², p(0) = 5c – 1 for some c > 0. Find the value of c.**

Sol. $P(x = 0) + p(x = 1) + p(x = 2) = 1$
$3c^3 + 4c - 10c^2 + 5c - 1 = 1$
$3c^3 - 10^2 + 9c - 2 = 0$
Put $c = 1$, then $3 - 10 + 9 - 2 = 12 - 12 = 0$

$C = 1$ satisfy the above equation

$C = 1 \Rightarrow p(x = 0) = 3$ which is not possible dividing with $c - 1$, we get
$3c^2 - 7c + 2 = 0$
$(c - 2)(3c - 1) = 0$
$c = 2$ or $c = 1/3$
$c = 2 \Rightarrow p(x = 0) = 3.2^3 = 24$ which is not possible
$\therefore c = 1/3$

2. **Find the constant C, so that** $F(x) = C\left(\dfrac{2}{3}\right)^x$ **, x = 1, 2, 3........... is the p.d.f of a discrete random variable X.**

Sol. Given $F(x) = C\left(\dfrac{2}{3}\right)^x$, $x = 1, 2, 3$

We know that $p(x) = C\left(\dfrac{2}{3}\right)^x$, $x = 1, 2, 3 \dots$

$\because \displaystyle\sum_{x=1}^{\infty} p(x) = 1$

$\Rightarrow \displaystyle\sum_{x=1}^{\infty} c\left(\dfrac{2}{3}\right)^x = 1$

$\Rightarrow c\left[\left(\dfrac{2}{3}\right) + \left(\dfrac{2}{3}\right)^2 + \left(\dfrac{2}{3}\right)^3 + \dots \infty\right] = 1$

$\Rightarrow C\dfrac{2}{3}\left[1 + \dfrac{2}{3} + \left(\dfrac{2}{3}\right)^2 + \dots \infty\right] = 1$

$\Rightarrow \dfrac{2c}{3}\left(\dfrac{1}{1 - \dfrac{2}{3}}\right) = 1$

$\left[\because a + ar + ar^2 + \dots \infty = \dfrac{a}{1-r}, \text{if } |r| <\right]$

$\Rightarrow \dfrac{2c}{3} \times 3 = 1 \Rightarrow c = \dfrac{1}{2}$

3.

X=x	−2	−1	0	1	2	3
P(X=x)	0.1	k	0.2	k	0.3	k

is the probability distribution of a random variable x. find the value of K and the variance of x.

Sol. Sum of the probabilities = 1

$0.1 + k + 0.2 + 2k + 0.3 + k = 1$

$4k + 0.6 = 1$

$4k = 1 - 0.6 = 0.4$

$k = \dfrac{0.4}{4} = 0.1$

Mean $= (-2)(0.1) + (-1)k + 0(0.2) + 1(2k) + 2(0.3) + 3k$

$= -0.2 - k + 0 + 2k + 0.6 + 3k$

$= 4k + 0.4 = 4(0.1) + 0.4 = 0.4 + 0.4 = 0.8$

$\mu = 0.8$

Variance $(\sigma^2) = \displaystyle\sum_{1-1}^{0} x^2 p(x = x_1) - \mu^2$

$\therefore$ Variance $= 4(0.1) + 1(k) + 0(0.2) + 1(2k) + 4(0.3) + 9k - \mu^2$

$= 0.4 + k + 0 + 2k + 4(0.3) + 9k - \mu^2$

$= 12k + 0.4 + 1.2 - (0.8)^2$

$= 12(0.1) + 1.6 - 0.64$

$= 1.2 + 1.6 - 0.64$

$\therefore \sigma^2 = 2.8 - 0.64 = 2.16$

4.

X=x	−3	−2	−1	0	1	2	3
P(X=x)	$\dfrac{1}{9}$	$\dfrac{1}{9}$	$\dfrac{1}{9}$	$\dfrac{1}{9}$	$\dfrac{1}{9}$	$\dfrac{1}{9}$	$\dfrac{1}{9}$

is the probability distribution of a random variable x. find the variance of x.

Sol Mean $(\mu) = -3\left(\dfrac{1}{9}\right) - 2\left(\dfrac{1}{9}\right) - 1\left(\dfrac{1}{9}\right) + 0\left(\dfrac{1}{9}\right) + 1\left(\dfrac{1}{9}\right) + 2\left(\dfrac{1}{9}\right) + 3\left(\dfrac{1}{9}\right)$

$= -\dfrac{3}{9} - \dfrac{2}{9} - \dfrac{1}{9} + 0 + \dfrac{1}{9} + \dfrac{2}{9} + \dfrac{3}{9} = (\mu) = 0$

Variance $(\sigma^2) = (-3)^2\dfrac{1}{9} + (-2)^2\dfrac{1}{9} + (-1)^2\dfrac{1}{9} + (0)^2\left(\dfrac{1}{3}\right) + (1)^2\dfrac{1}{9} + (2)^2\dfrac{1}{9} + (3)^2\dfrac{1}{9} - \mu^2$

$= \dfrac{9}{9} + \dfrac{4}{9} + \dfrac{1}{9} + 0 + \dfrac{1}{9} + \dfrac{4}{9} + \dfrac{9}{9} - 0^2 = \dfrac{28}{9} - 0$

$\sigma^2 = \dfrac{28}{9}$

5. A random variable x has the following probability distribution.

X=x	0	1	2	3	4	5	6	7
P(X=x)	0	k	2k	2k	3k	K^2	$2k^2$	$7k^2+k$

Find i) k ii) the mean and iii) p(0 < x < 5).

Sol. Sum of the probabilities =

$0 + k + 2k + 2k + 3k + K^2 + 2k^2 + 7k^2 + k = 1$