Random Variables and Probability Distribution

Random Variable

Random variable: If S is the sample space P(S) is the power set of the sample space, P is the probability of the function then (S, P(S), P) is called the probability space,

In the probability space if $X : S \to \mathbb{R}$ is a function then x is called random variable

Frequency function of random variable (or) Probability density function: Function $f : X(S) \to \mathbb{R}$ is defined by

$f(r) = p[e / x(e) = r]$ is called the frequency function associated with random variable

Where i) $0 \le f(r) \le 1 \ \forall \ r \in x(S)$

ii) $\Sigma\, f(r) = 1 \ \forall\, r \in x(S)$

Arithmetic mean of the random variable: Arithmetic mean of the random variable x is denoted by $\bar{x}$ or all E(x) expected value of X and is defined as $\bar{x} = \Sigma\, r\ f(r)$

Variance of the random variable: If 'x' is a random variable then $E(x^2)$ is defined such that $E(x^2) = \Sigma r^2\ f(r)\ \forall\, x \in x(S)$. The variance of random variable (σ^2) and is defined as $F(x - \bar{x})^2$

Variance of the random variable

$$\sigma^2 = E(x - \bar{x})^2 = E(x^2 + \bar{x}^2 - 2x\ \bar{x})$$

$$\sigma^2 = E(x)^2 + \bar{x}^2 - 2\bar{x}\, E(x)$$

$$\sigma^2 = E(x)^2 + \bar{x}^2$$

Variable of random variable $\sigma^2 = E(x^2) - \mu^2$

$$\sigma^2 = \Sigma r^2\, f(r) - \mu^2$$

$$\sigma^2 + \mu^2 = \Sigma r^2\, f(r)$$

Standard deviation: It is the positive square root of the variance of the standard deviation of the random variable this is denoted by $\sigma = \sqrt{vairance}$

Note: Let X be a random variable on a sample space S. If $x \in R$ then we use the following symbols to denote some events in S.

i) $\{a \in S : X(a) = x\} = (X = x)$

ii) $\{a \in S : X(a) < x\} = (X < x)$

iii) $\{a \in S : X(a) \le x\} = (X \le x)$

iv) $\{a \in S : X(a) > x\} = (X > x)$

v) $\{a \in S : X(a) \ge x\} = (X \ge x)$

Def 2: Let S be a sample space and $X : S \to R$ be a random variable. The function $F : R \to R$ defined by $F(x) = P(X \le x)$, is called probability distribution function of the random variable X.

We now state some properties of probability distribution function for the random yond the scope of the book.

Theorem 2: Let F(x) be the probability distribution function for the random variable X. then

i) $0 \le F(x) \le 1, \forall x \in R$

ii) F (x) is an increasing function i.e. $x_1, x_2 \in R, x_1 < x_2 \Rightarrow F(x_1) F(x_2)$

iii) $\underset{x \to \infty}{Lt} F(x) = 1, \ \underset{x \to -\infty}{Lt} F(x) = 0$

Theorem 3: If $X : S \to R$ is a discrete random variable with range $\{x_1, x_2, x_3....\}$ then $\sum_{r=1}^{\infty} P(X = x_r) = 1$.

Mean and Variance

Def : Let $X : S \to R$ be a discrete random variable with range $\{x_1, x_2, x_3......\}$. If $\Sigma x_r \ P(X = x_r)$ exists, then $\Sigma x_r \ P(X = x_r)$ is called the mean of the random variable X. It is denoted by μ or $\bar{x}$. If $\Sigma (x_r - \mu)^2 P(X = x_r)$ exists, then $\Sigma (x_r - \mu)^2 P(X = x_r)$ is called variance of the random variable X. It is denoted by σ^2. The positive square root of the variance is called the standard deviation of the random variable X. It is denoted by σ.

Theorem 4: Let $X : S \to R$ be a discrete random variable with range $\{x_1, x_2, x_3,......\}$. If μ, σ^2 are the mean and variance of X then $\sigma^2 + \mu^2 = \Sigma x_r^2 P(X = x_r)$.

Def: Let n be a positive integer and p be a real number such that $0 \le p \le 1$. A random variable X with range {0, 1, 2, .n} is said to follows (or have) binomial distribution or Bernoulli distribution with parameters n and p if $P(X = r) = {}^n C_r \ p^r \ q^{n-r}$ for r = 0, 1, 2… n where q = 1 - p.

Theorem : If the random variable X follows a binomial distribution with parameters n and p then mean of X is np and the variance is npq where q = 1- p.

Def : Let $\lambda > 0$ be are real number. A random variable X with range {0, 1, 2….} is said to follows (have) Poisson distribution with parameter λ if $P(X = r) = \frac{e^{-\lambda} \lambda^r}{r!}$ for r = 0, 1, 2,…….

Theorem : If a random variable X follows Poisson distribution with parameter λ, then mean of X is λ and variance of X is λ.

EXERCISE – 9(a)

1. A p.d.f of a discrete random variable is zero except at the points x = 0, 1, 2. At these points it has the value p (0) = $3c^3$, p(1) = 4c – $10c^2$, p(0) = 5c – 1 for some c > 0. Find the value of c.

Sol. P(x = 0) + p(x = 1) + p(x = 2) = 1

$3c^3 + 4c - 10c^2 + 5c - 1 = 1$

$3c^3 - 10^2 + 9c - 2 = 0$

Put c = 1, then 3 – 10 + 9 – 2 = 12 – 12 = 0

C = 1 satisfy the above equation

C = 1 $\Rightarrow$ p(x = 0) = 3 which is not possible dividing with c – 1, we get

$3c^2 - 7c + 2 = 0$

(c – 2) (3c – 1) = 0

c = 2 or c = 1/3

c = 2 $\Rightarrow$ p(x = 0) = 3.2^3 = 24 which is not possible

$\therefore$ c = 1/3

2. Find the constant C, so that $F(x) = C\left(\frac{2}{3}\right)^x, x = 1, 2, 3..........$ **is the p.d.f of a discrete random variable X.**

Sol. Given $F(x) = C\left(\frac{2}{3}\right)^x, x = 1, 2, 3$

We know that $p(x) = C\left(\frac{2}{3}\right)^x, x = 1, 2, 3 \ldots$

$$\because \sum_{x=1}^{\infty} p(x) = 1$$

$$\Rightarrow \sum_{x=1}^{\infty} c\left(\frac{2}{3}\right)^x = 1$$

$$\Rightarrow c\left[\left(\frac{2}{3}\right) + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + ...\infty\right] = 1$$

$$\Rightarrow C\frac{2}{3}\left[1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + ...\infty\right] = 1$$

$$\Rightarrow \frac{2c}{3}\left(\frac{1}{1-\frac{2}{3}}\right) = 1$$

$$\left[\because a + ar + ar^2 + ...\infty = \frac{a}{1-r}, \text{if } |r| <\right]$$

$$\Rightarrow \frac{2c}{3} \times 3 = 1 \Rightarrow c = \frac{1}{2}$$

3.

X=x	–2	–1	0	1	2	3
P(X=x)	0.1	k	0.2	k	0.3	k

is the probability distribution of a random variable x. find the value of K and the variance of x.

Sol. Sum of the probabilities = 1

$0.1 + k + 0.2 + 2k + 0.3 + k = 1$

$4k + 0.6 = 1$

$4k = 1 - 0.6 = 0.4$

$k = \frac{0.4}{4} = 0.1$

Mean = $(-2)(0.1) + (-1)k + 0(0.2) + 1(2k) + 2(0.3) + 3k$

$= -0.2 - k + 0 + 2k + 0.6 + 3k$

$= 4k + 0.4 = 4(0.1) + 0.4 = 0.4 + 0.4 = 0.8$

$\mu = 0.8$

Variance (σ^2) = $\sum_{1-1}^{0} x^2 p(x = x_1) - \mu^2$

$\therefore$ Variance = $4(0.1) + 1(k) + 0(0.2) + 1(2k) + 4(0.3) + 9k - \mu^2$

$= 0.4 + k + 0 + 2k + 4(0.3) + 9k - \mu^2$

$= 12k + 0.4 + 1.2 - (0.8)^2$

$= 12(0.1) + 1.6 - 0.64$

$= 1.2 + 1.6 - 0.64$

$\therefore \sigma^2 = 2.8 - 0.64 = 2.16$

4.

X=x	–3	–2	–1	0	1	2	3
P(X=x)	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$

is the probability distribution of a random variable x. find the variance of x.

Sol Mean (μ) = $-3\left(\frac{1}{9}\right) - 2\left(\frac{1}{9}\right) - 1\left(\frac{1}{9}\right) + 0\left(\frac{1}{9}\right) + 1\left(\frac{1}{9}\right) + 2\left(\frac{1}{9}\right) + 3\left(\frac{1}{9}\right)$

$= -\frac{3}{9} - \frac{2}{9} - \frac{1}{9} + 0 + \frac{1}{9} + \frac{2}{9} + \frac{3}{9} = (\mu) = 0$

Variance (σ^2) = $(-3)^2\frac{1}{9} + (-2)^2\frac{1}{9} + (-1)^2\frac{1}{9} + (0)^2\left(\frac{1}{3}\right) + (1)^2\frac{1}{9} + (2)^2\frac{1}{9} + (3)^2\frac{1}{9} - \mu^2$

$= \frac{9}{9} + \frac{4}{9} + \frac{1}{9} + 0 + \frac{1}{9} + \frac{4}{9} + \frac{9}{9} - 0^2 = \frac{28}{9} - 0$

$\sigma^2 = \frac{28}{9}$

5. A random variable x has the following probability distribution.

X=x	0	1	2	3	4	5	6	7
P(X=x)	0	k	2k	2k	3k	K^2	$2k^2$	$7k^2$+k

Find i) k ii) the mean and iii) p(0 < x < 5).

Sol. Sum of the probabilities =

$0 + k + 2k + 2k + 3k + K^2 + 2k^2 + 7k^2 + k = 1$

$\Rightarrow 10k^2 + 9k = 1$
$\Rightarrow 10k^2 + 9k - 1 = 0$
$\Rightarrow 10k^2 + 10k - k - 1 = 0$
$\Rightarrow 10k\,(k + 1) - 1\,(k + 1) = 0$
$\Rightarrow (10k - 1)\,(k + 1) = 0$

$K = \frac{1}{10}, -1$

Since k > 0

$\therefore\ k = \frac{1}{10}$

i) $k = \frac{1}{10}$

ii) Mean = $\sum_{i=1}^{n} x_i p\left(x = x_i\right)$

Mean

$(\mu) = 0(0) + 1(k) + 2(2k) + 3(2k) + 4(3k) + 5(k^2) + 7(7k^2 + k)$

$= 0 + k + 4k + 6k + 12k + 5k^2 + 12k^2 + 49k^2 + 7k$

$= 66k^2 + 30k$

$= 66\left(\frac{1}{100}\right) + 30 \times \left(\frac{1}{10}\right)$

$= 0.66 + 3 = 3.66$

iii) $p(0 < x < 5)$

$p(0 < x < 5) = (x = 1) + p(x = 2) + p(x = 3) + p(x = 4)$

$= k + 2k + 2k + 3k = 8k$

$= 8\frac{1}{10} = 8\frac{1}{10} = \frac{4}{5}$

II.

1. The range of a random variable x is {0, 1, 2}. Given that $p(x = 0) = 3c^3$, $p(x = 1) = 4c - 10c^2$, $p(x = 2) = 5c - 1$

i) Find the value of c

ii) $p(x < 1)$, $p(1 < x \leq 3)$

Sol. $P(x = 0) + p(x = 1) + p(x = 2) = 1$

$3c^3 + 4c - 10c^2 + 5c - 1 = 1$

$3c^3 - 10c^2 + 9c - 2 = 0$

C = 1 satisfy this equation

$C = 1 \Rightarrow p(x = 0) = 3$ which is not possible dividing with c – 1, we get

$3c^2 - 7c + 2 = 0$

$(c - 2)\,(3c - 1) = 0$

d = 2 or c = 1/3

$c = 2 \Rightarrow p(x = 0) = 3.2^3 = 24$ which is not possible

$\therefore\ c = 1/3$

i) $P(x < 1) = p(x = 0)$

$= 3.c^3 = \left(\frac{1}{3}\right)^3 = 3.\frac{1}{27} = \frac{1}{9}$

ii) $p(1 < x \leq 2) = p(x = 2) = 5c - 1$

$= \frac{5}{3} - 1 = \frac{2}{3}$

iii) $p(0 < x \le 3) = p(x = 1) + p(x = 2)$

$= 4c - 10c^2 + 5c - 1$

$= 9c - 10c^2 - 1 = 9.\frac{1}{3} - 10.\frac{1}{9} - 1$

$= 3 - \frac{10}{9} - 1 = 2 - \frac{10}{9} = \frac{8}{9}$

2. The range of a random variable x is {1, 2, 3, ……..} and $p(x = k) = \frac{c^k}{k1}$ (k = 1, 2, 3, …………..)

Find the value of C and $p(0 < x < 3)$

Sol. Sum of the probabilities = 1

$\sum p(x = k) = 1$

$\sum_{k=1}^{n} \frac{c^k}{\lfloor k} = 1$

$c + \frac{c^2}{\lfloor 2} + \frac{c^3}{\lfloor 3} ....\infty = 1$

Add 1 on both sides

$1 + c + \frac{c^2}{\lfloor 2} + \frac{c^3}{\lfloor 3} ....\infty = 2$

$e^c = 2 \Rightarrow \log_e e^c = \log_e 2$

$\Rightarrow c = \log_e 2$

$P(0 < x < 3) = p(x = 1) = p(x = 2)$

$= c + \frac{c^2}{2} = \log_e{}^2 + \frac{(\log_e 2)^2}{2}$

EXERCISE – 9(b)

1. In the experiment of tossing a coin n times, if the variable x denotes the number of heads and P(X = 4), P(X = 5), P(X = 6) are in arithmetic progression then find n.

Sol. X follows binomial distribution with

$p = \frac{1}{2,} q = \frac{1}{2}$ ($\because$ a coin is tossed)

Hint : a, b, c are in A.P,
2b = a + c (or) b – a = c – a

Given, P(X = 4), P(X = 5), P(X = 6) are in A.P

$\Rightarrow a\,{}^nC_4 \left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right)^{n-4}, {}^nC_5 \left(\frac{1}{2}\right)^5 \left(\frac{1}{2}\right)^{n-5},$

${}^nC_6 \left(\frac{1}{2}\right)^6 \left(\frac{1}{2}\right)^{n-6}$ are in A.P

$\Rightarrow {}^nC_4, {}^nC_5, {}^nC_6$ are in A.P

$\Rightarrow 2\,{}^nC_5 = {}^nC_4 + {}^nC_6$

$\Rightarrow \frac{2(n!)}{5!(n-5)!} = \frac{n!}{4!(n-4)!} + \frac{n!}{6!(n-6)!}$

$\Rightarrow \frac{2(n!)}{5\times 4!(n-5)(n-6)} =$

$$\frac{n!}{4!(n-4)(n-5)(n-6)!}+\frac{n!}{6\times 5\times 4!(n-6)!}$$

$$\Rightarrow \frac{2(n!)}{5\times 4!(n-5)(n-6)!}$$

$$=\frac{n!}{4!(n-6)!}\left[\frac{1}{(n-4)(n-5)}+\frac{1}{30}\right]$$

$$\Rightarrow \frac{2}{5(n-5)}=\frac{30+(n-4)(n-5)}{30(n-4)(n-5)}$$

$\Rightarrow 2 \times 30\ (n - 4) = 5[30 + n^2 - 9n + 20]$
$\Rightarrow 12n - 48 = n^2 - 9n - 50$
$\Rightarrow n^2 - 21n + 98 = 0$
$\Rightarrow n^2 - 14n - 7n + 98 = 0$
$n(n - 14) - 7(n - 14) = 0$
$(n - 7)\ (n - 14) = 0$
$\therefore\ n = 7$ or 14

2. Find the maximum number of times a fair coin must be tossed so that the probability of getting at least one head is at least 0.8.

Sol. Let n be number of times a fair coin tossed x denotes the number of heads getting x follows binomial distribution with parameters n and p =1/2 given $p(x \geq 1) \geq 0.8$
$\Rightarrow 1 - p(x = 0) \geq 0.8$
$\Rightarrow p(x = 0) \leq 0.2$

$$\Rightarrow {}^nC_0\left(\frac{1}{2}\right)^n \leq 0.2$$

$$\Rightarrow \left(\frac{1}{2}\right)^n \leq \frac{1}{5}$$

The maximum value of n is 3.

3. The probability of a bomb hitting a bridge is 1/2 and three direct hits (not necessarily consecutive) are needed to destroy it. Find the minimum number of bombs required so that the probability of the bridge being destroyed is greater than 0.9.

Sol. Let n be the minimum number of bombs required and x be the number of bombs that hit the bridge, then x follows binomial distribution with parameters n and p = 1/2.
Now $p(x \geq 3) > 0.9$
$\Rightarrow 1 - p(x < 3) > 0.9$
$\Rightarrow p(x < 3) < 0.1$
$\Rightarrow p(x = 0) + p(x = 1) + p(x = 2) < 0.1$

$$\Rightarrow {}^nC_0\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^n+{}^nC_1\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^{n-1}+{}^nC_2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^{n-2}<0.1$$

$$\Rightarrow 1.\frac{1}{2^n}+\frac{n}{2^n}+\frac{n(n-1)}{2}\frac{1}{2^2}<\frac{1}{10}$$

$$\Rightarrow 1.\frac{1}{2^n}+\frac{n}{2^n}+\frac{n^2-n}{2.2^n}<\frac{1}{10}$$

$$\Rightarrow \frac{1}{2^n}\left(1+n+\frac{n^2-n}{2}\right) < \frac{1}{10}$$

$$\Rightarrow \frac{1}{2^n}\left(\frac{2+2n+n^2-n}{2}\right) < \frac{1}{10}$$

$\Rightarrow 5(n^2 + n + 2) < 2^n$

By trial and error, we get $n \geq 9$

$\therefore$ The least value of n is 9

$\therefore$ n = 9

4. If the difference between the mean and the variance of a binomial variate is 5/9 then, find the probability for the event of 2 successes, when the experiment is conducted 5 times.

Sol. Given n = 5, let p be the parameters of the binomial distribution

Mean – Variance =5/9

np – npq = 5/9

np(1 – q) = 5/9 , $\because$ p + q = 1

$n.p^2 = 5/9 \Rightarrow 5p^2 = 5/9$

$p^2 = 1/9 \Rightarrow p = 1/3$

q = 1 – p = 1 – 1/3 = 2/3

$$p(x = 2) = {}^5C_2\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)^2 = 10\times\frac{8}{27}.\frac{1}{9} = \frac{80}{243}$$

$\therefore$ Prob. of the event of 2 success = $\frac{80}{243}$

5. One in 9 ships is likely to be wrecked, when they are set on sail, when 6 ships are on sail, find the probability for (a) Atleast one will arrive safely (b) Exactly, 3 will arrive safely.

Sol. P = probability of ship to be wrecked = 1/9

$q = 1 – p = 1 – \frac{1}{9} = \frac{8}{9}$

Number of ships = n = 6

$$p(x=0) = {}^6C_0\left(\frac{8}{9}\right)^{6-6}\left(\frac{1}{9}\right)^6 = \left(\frac{1}{9}\right)^6$$

a) Probability of atleast one will arrive safely = p(x > 0) = 1 – p(x = 0)

$$= 1-\left(\frac{1}{9}\right)^6 = 1-\frac{1}{9^6}$$

b) $p(x = 3) = {}^6C_3\left(\frac{8}{9}\right)^3\left(\frac{1}{9}\right)^3$

$$= \frac{1}{9^6}.{}^6C_3.8^3 = 20\left(\frac{8^3}{9^6}\right)$$

6. If the mean and variance of a binomial variable x are 2.4 and 1.44 respectively, find $p(1 < x \le 4)$.

Sol. Mean = np = 2.4 …(1)

Variance = npq = 1.44 …(2)

Dividing (2) by (1)

$$\frac{npq}{np} = \frac{1.44}{2.4}$$

q = 0.6 = 3/5

p = 1 – q = 1 – 0.6 = 0.4 = 2/5

Substituting in (1)

n(0.4) = 2.4

$$n = \frac{2.4}{0.4} = 6$$

$P(1 < x \le 4) = p(x = 2) + p(x = 3) + p(x = 4)$

$= {}^6C_2 q^4.p^2 + {}^6C_3 q^3.p^3 + {}^6C_4 q^2.p^4$

$$= {}^6C_2\left(\frac{3}{5}\right)^4\left(\frac{2}{5}\right)^2 + {}^6C_3\left(\frac{3}{5}\right)^3\left(\frac{2}{5}\right)^3 + {}^6C_4\left(\frac{3}{5}\right)^2\left(\frac{2}{5}\right)^4$$

$$= \frac{6^2}{5^6}(15.9 + 20.6 + 15.4)$$

$$= \frac{36}{15625}(135 + 120 + 60) = \frac{315 \times 36}{15625}$$

$$= \frac{36 \times 63}{3125} = \frac{2268}{3125}$$

7. It is given that 10% of the electric bulbs manufactured by a company are defective. In a sample of 20 bulbs, find the probability that more than 2 are defective.

Sol. p = probability of defective bulb = 1/10

$$q = 1 - p = 1 - \frac{1}{10} = \frac{9}{10}$$

n = number of bulbs in the sample = 20

$p(x > 2) = 1 - p(x \le 2)$

$= 1 - [p(x = 0) + p(x = 1) + p(x = 2)]$

$$p(x = 0) = {}^{20}C_0\left(\frac{9}{10}\right)^{20} = \left(\frac{9}{10}\right)^{20}$$

$$p(x = 1) = {}^{20}C_1\left(\frac{9}{10}\right)^{19} = \left(\frac{1}{10}\right) = \frac{20.9^{19}}{10^{20}}$$

$$p(x = 2) = {}^{20}C_2\left(\frac{9}{10}\right)^{18} = \left(\frac{1}{10}\right)^2 = \frac{190.9^{18}}{10^{20}}$$

$$p(x > 2) = 1 - \left(\frac{9^{20}}{10^{20}} + \frac{20.9^{10}}{10^{20}} + \frac{190.9^{18}}{10^{20}}\right)$$

$$= 1 - \sum_{k=0}^{2} {}^{20}C_k\left(\frac{9}{10}\right)^{20-k}\left(\frac{1}{10}\right)^k$$

$$=1-\sum_{k=0}^{2} {}^{20}C_k \frac{9^{20-k}}{10^{20}} = \sum_{k=3}^{20} {}^{20}C_k \left(\frac{9^{20-k}}{10^{20}}\right)$$

8. On an average, rain falls on 12 days in every 30 days, find the probability that, rain will fall on just 3 days of a given week.

Sol. Given p = $\frac{12}{30}=\frac{2}{5}$

q = 1– p = 1 – $\frac{2}{5}=\frac{3}{5}$

n = 7, r = 3

p(x = 3)

$$= {}^{n}C_{r}.q^{n-r}.p^{r} = {}^{7}C_3\left(\frac{3}{5}\right)^4\left(\frac{2}{5}\right)^3$$

$$=35.\left(\frac{3}{5}\right)^4\left(\frac{2}{5}\right)^3 = \frac{35\times 2^3\times 3^4}{5^7}$$

9. For a binomial distribution with mean 6 and variance 2, find the first two terms of the distribution.

Sol. Let n, p be the parameters of a binomial distribution

Mean (np) = 6 … (1)

and variance (n pq) = 2 …(2)

then $\frac{npq}{np}=\frac{2}{6}\Rightarrow q=\frac{1}{3}$

$\because p=1-q=1-\frac{1}{3}=\frac{2}{3}$

From (1) n p = 6

$$n\left(\frac{2}{3}\right)=6 \Rightarrow n=\frac{18}{2}=9$$

First two terms of the distribution are

$p(x=0)={}^{9}C_0\left(\frac{1}{3}\right)^9=\frac{1}{3^9}$ and

$p(x=1)={}^{9}C_1\left(\frac{1}{3}\right)^8\left(\frac{2}{3}\right)=\frac{2}{3^7}$

10. In a city 10 accidents take place in a span of 50 days. Assuming that the number of accidents follows the poisson distribution, find the probability that there will be 3 or more accidents in a day.

Sol. Average number of accidents per day

$$\lambda=\frac{10}{50}=\frac{1}{5}=0.2$$

The prob. That there win be 3 or more accidents in a day p(x ≥ 3)

$$\sum_{k=3}^{\infty} e^{-\lambda}\frac{\lambda^k}{k!},\ \lambda=0.2$$

II.

1. Five coins are tossed 320 times. Find the frequencies of the distribution of number of heads and tabulate the result.

Sol. 5 coins are tossed 320 times

Prob. of getting a head on a coin

$p = \frac{1}{2}, \; n = 5$

Prob. of having x heads

$p(x = x) = {}^5C_x \,(q)^5 \left(\frac{1}{2}\right)^x$

$= {}^5C_x \left(\frac{1}{2}\right)^{5-x} \left(\frac{1}{2}\right)^x = {}^5C_x \left(\frac{1}{2}\right)^5 \; x = 0, 1, 2, 3, 4, 5$

Frequencies of the distribution of number of heads = N.P(X = x)

$= 320\left[{}^5C_x \left(\frac{1}{2}\right)^5\right]; \; x = 0, 1, 2, 3, 4, 5$

Frequency of

Having 0 head = $320 \times {}^5C_0 \times \left(\frac{1}{2}\right)^5 = 10$

Having 1 head = $320 \times {}^5C_1 \times \left(\frac{1}{2}\right)^5 = 50$

Having 2 head = $320 \times {}^5C_2 \times \left(\frac{1}{2}\right)^5 = 100$

Having 3 head = $320 \times {}^5C_3 \times \left(\frac{1}{2}\right)^5 = 100$

Having 4 head = $320 \times {}^5C_4 \times \left(\frac{1}{2}\right)^5 = 50$

Having 5 head = $320 \times {}^5C_5 \times \left(\frac{1}{2}\right)^5 = 10$

N(H)	0	1	2	3	4	5
f	10	50	100	100	50	10

2. Find the probability of guessing at least 6 out of 10 of answers in (i) true or false type examination ii) multiple choice with 4 possible answers.

Sol. i) Since the answers are in true or false type.

Prob. of success p = $\frac{1}{2}$, q = $\frac{1}{2}$

Prob. of guessing at least 6 out of 10

$$p(x \geq 6) = \sum_{6}^{10} {}^{10}C_6 \left(\frac{1}{2}\right)^{10-6} \left(\frac{1}{2}\right)^6$$

$$= \sum_{6}^{10} {}^{10}C_k \left(\frac{1}{2}\right)^{10}$$

ii) Since the answers are in multiple choice with 4 possible answers

Prob. of success p = 1/4, q = 3/4

Prob. of guessing at least 6 out of 10

$$p(x \geq 6) = {}^{10}C_6 \left(\frac{3}{4}\right)^{10-6} \left(\frac{1}{4}\right)^{6}$$

$$= \sum_{6}^{10} {}^{10}C_k \left(\frac{1}{4}\right)^{k} \left(\frac{3}{4}\right)^{10-k}$$

13. The number of persons joining a cinema ticket counter in a minute has poisson distribution with parameter 6. Find the probability that i) no one joins the queue in a particular minute ii) two or more persons join the queue in a minute.

Sol. Here $\lambda = 6$

i) prob. That no one joins the queune in a particular minute

$$p(x = 0) = \frac{e^{-\lambda}\lambda^{0}}{0!} = e^{-6}$$

ii) prob. that two or more persons join the queue in a minute

$p(x \geq 2) = 1 - p(x \leq 1)$

$= 1 - [p(x = 0) + p(x = 1)]$

$$= 1 - \left[e^{-\lambda}\frac{\lambda^{0}}{0!} + \frac{e^{-\lambda}\lambda^{1}}{1!}\right]$$

$$= 1 - \left[e^{6} + \frac{e^{6}(6)}{1!}\right] = 1 - 7.e^{-6}$$

<u>EXAMPLE PROBLEMS</u>

1. A cubical die is thrown. Find the mean and variance of x, giving the number on the face that shows up.

Sol. Let S be the sample space and x be the random variable associated with S, where p(x) is given by the following table

$X=x_i$	1	2	3	4	5	6
$P(X=x_i)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

Mean of $x = \mu = \Sigma(X = x_i)\ P(X = x_i)$

$$= 1\left(\frac{1}{6}\right) + 2\left(\frac{1}{6}\right) + 3\left(\frac{1}{6}\right) + 4\left(\frac{1}{6}\right) + 5\left(\frac{1}{6}\right) + 6\left(\frac{1}{6}\right)$$

$$= \frac{1}{6}(1+2+3+4+5+6)$$

$$= \frac{1}{6}\frac{(6)(6+1)}{2} = \frac{7}{2} = 3.5$$

Variance of $x = \sigma^2 = \sum x_i^2\ p(X = x_i) - \mu^2$

$$(\text{i.e.,})\sigma^2 = 1^2\left(\frac{1}{6}\right) + 2^2\left(\frac{1}{6}\right) + 3^2\left(\frac{1}{6}\right) + 4^2\left(\frac{1}{6}\right) + 5^2\left(\frac{1}{6}\right) + 6^2\left(\frac{1}{6}\right) - \left(\frac{7}{2}\right)^2$$

$$= \frac{1}{6}\left(1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2\right) - \frac{49}{4}$$

$$= \frac{1}{6}\frac{(6)(6+1)(2\times 6+1)}{6} - \frac{49}{4}$$

$= \frac{91}{6} - \frac{49}{4} = \frac{182-147}{12} = \frac{35}{12}$

2. The probability distribution of a random variable x is given below. Find the value of k, and the mean and variance of x

$X=x_i$	1	2	3	4	5
$P(X=x_i)$	K	2k	3k	4k	5k

Sol. we have $\sum_{r=1}^{5} p(X = x_1) = 1$

$\Rightarrow k + 2k + 3k + 4k + 5k = 1 \;\Rightarrow k = \frac{1}{15}$

Mean μ of x = $\sum_{r-1}^{5} r.p(x = x_1)$ = $\sum_{r-1}^{5} r(rk)$

= 1.(k) + 2.(2K) + 3.(3k) + 4.(4k) + 5.(5k)

= 55k

$= 55 \times \frac{1}{15} = \frac{11}{3}$

Variance (σ^2) = $(1)^2$. k + $(2)^2$ 2k + $(3)^2$ 3k + $(4)^2$ 4k + $(5)^2$ (5k) – μ^2

= k + 8k + 27k + 64K +125k – $\left(\frac{11}{3}\right)^2$

$= 225k - \frac{121}{9} = 225 \times \frac{1}{15} - \frac{121}{9}$

$= \frac{135-121}{9} = \frac{14}{9}$

3. If X is a random variable with the probability distribution. P(X = k) = $\frac{(k+1)c}{2^k}$, (k = 0, 1, 2, 3, …) then find C.

Sol. given p(x = k) = $\frac{(k+1)c}{2^k}$ (k = o, 1, 2, 3, …)

$\sum_{k=0}^{0} p(x = k) = 1$

$\sum_{k=0}^{0} \frac{(k+1)c}{2^k} = c\left(1 + 2\frac{1}{2} + 3\left(\frac{1}{2}\right)^2 + k\alpha\right) = 1$

$c\left[\frac{1}{1-\frac{1}{2}} + \frac{1.\frac{1}{2}}{\left(1-\frac{1}{2}\right)^2}\right] = 1$

Hint: in A.G.P. $S_\infty = \frac{a}{1-r} + \frac{dr}{(1-r)^2}$

Here a = 1, d = 1, r = 1/2

$$c\left[\frac{1}{\frac{1}{2}}+\frac{\frac{1}{2}}{\left(\frac{1}{2}\right)^2}\right]=1$$

c [2 + 2] = 1

$$\therefore c=\frac{1}{4}$$

4. **Let x be a random variable such that p(x = –2) = p(x = –1) = p(x = 2) = p(x =3) =1/6 and p(x = 0) = 1/3. Find the mean and variance of x.**

Sol. Mean

$$=(-2)\frac{1}{6}+(-1)\frac{1}{6}+2\left(\frac{1}{6}\right)+(1)\left(\frac{1}{6}\right)+0.\left(\frac{1}{3}\right)$$

$$=-\frac{2}{6}-\frac{1}{6}+\frac{2}{6}+\frac{1}{6}+0$$

$\mu = 0$

Variance (σ^2) = $(-2)^2\left(\frac{1}{6}\right)+(-1)^2\left(\frac{1}{6}\right)-0^2\left(\frac{1}{3}\right)+2^2\left(\frac{1}{6}\right)+1^2\left(\frac{1}{6}\right)$

$$=\frac{4}{6}+\frac{1}{6}+0+\frac{4}{6}+\frac{1}{6}$$

$$=\frac{10}{6}-\frac{5}{3}$$

5. **Two dice are rolled at random. Find the probability distribution of the sum of the numbers on them. Find the mean of the random variable.**

Sol. When two dice are rolled, the sample space S contains $6 \times 6 = 36$ sample points.

S = {(1, 1), (1, 2)…(1, 6), (2, 1), (2, 2)…(6,6)}

Let x denote the sum of the numbers on the two dice

Then the range x = {2, 3, 4, …12}

Probability Distribution of x is given by the following table.

$X=x_i$	2	3	4	5	6	7	8	9	10	11	12
$P(X=x_i)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Mean of x = $\mu=\sum_{1-2}^{12} x_1 p(X=x_1)$

$$=2.\frac{1}{36}+3.\frac{2}{36}+4.\frac{3}{36}+5.\frac{4}{36}+6.\frac{5}{36}+7.\frac{6}{36}+8.\frac{5}{36}+9.\frac{4}{36}+10.\frac{3}{36}+11.\frac{2}{36}+12.\frac{1}{36}$$

$$=\frac{1}{36}(2+6+12+20+30+42+40$$

$$+36+30+22+12)$$

$$=\frac{252}{36}=7$$

6. 8 coins are tossed simultaneously. Find the probability of getting at least 6 heads.

Sol. p = probability of getting head $=\frac{1}{2}$

$q = 1 - p = 1 - \frac{1}{2} = \frac{1}{2} : n = 8$

$p(x \geq 6) = p(x = 6) + p(x = 7) + p(x = 8)$

$= {}^8C_6\left(\frac{1}{2}\right)^6\left(\frac{1}{2}\right)^2 + {}^8C_7\left(\frac{1}{2}\right)^7\left(\frac{1}{2}\right)^1 + {}^8C_7\left(\frac{1}{2}\right)^8\left(\frac{1}{2}\right)^0$

$= \left(\frac{1}{2}\right)^8\left[{}^8C_6 + {}^8C_7 + {}^8C_8\right]$

$= \frac{1}{256}[28+8+1] = \frac{37}{256}$

7. The mean and variance of a binomial distribution are 4 and 3 respectively. Fix the distribution and find p(x ≥ 1)

Sol. Given distribution is binomial distribution with mean = np = 4

Variance = npq = 3

$\therefore \frac{npq}{np} = \frac{3}{4} \Rightarrow q = \frac{3}{4}$

So that $p = 1 - q = 1 - \frac{3}{4} = \frac{1}{4}$

$\therefore$ np = 4

$n\frac{1}{4} = 4 \Rightarrow n = 16$

$P(x \geq 1) = 1 - p(x = 0)$

$= 1 - {}^{16}C_0\left(\frac{1}{4}\right)^0\left(\frac{3}{4}\right)^{16-0} = 1 - \left(\frac{3}{4}\right)^{16}$

$\therefore p(x \geq 1)$

$= 1 - \left(\frac{3}{4}\right)^{16}$

8. The probability that a person chosen at Random is left handed (in hand writing) is 0.1. What is the probability that in a group of ten people there is one, who is left handed ?

Sol. here n = 10

P = 0.1

q = 1 – p = 1 – 0.1 = 0.9

$p(x = 1) = {}^{10}C_1 (0.1)^1(0.9)^{10-1}$

$= 10 \times 0.1 \times (0.9)^9$

$= 1 \times (0.9)^9 = (0.9)^9$

9. In a book of 450 pages, there are 400 typographical errors. Assuming that following the passion law, the number of errors per page, find the probability that a random sample of 5 pages will contain no typographical error.

Sol. The average number of errors per page in the book is

$\lambda = \frac{400}{450} = \frac{8}{9}$

Here r = 0

$$p(x=r)=\frac{e^{-\lambda}\lambda^{r}}{r!}$$

$$p(x=0)=\frac{e^{-8/9}\left(\frac{8}{9}\right)^{0}}{0!}=e^{-8/9}$$

$\therefore$ The required probability that a random sample of 5 pages will contain no error is

$[p(x = 0)]^5 = \left(e^{-8/9}\right)^5$

10. Deficiency of red cells in the blood cells is determined by examining a specimen of blood under a microscope. Suppose a small fixed volume contains on an average 20 red cells for normal persons. Using the poisson distribution, find the probability that a specimen of blood taken from a normal person will contain less than 15 red cells.

Sol. Here $\lambda = 20$

$$p(x<15)=\sum_{r=0}^{14}p(x=r)$$

$$=\sum_{r=0}^{14}\frac{e^{-\lambda}\lambda^{r}}{r!}=\sum_{r=0}^{14}e^{-20}\frac{20^{r}}{r!}$$

11. A poisson variable satisfies p(x = 1) = p(x = 2). Find p(x = 5)

Sol. Given p(x = 1) = p(x = 2)

$$p(x=r)=\frac{\lambda^{r}e^{-\lambda}}{r!},\lambda>0$$

$$\frac{\lambda^{r}e^{-\lambda}}{1!}=\frac{\lambda^{2}e^{-\lambda}}{2!}$$

$\lambda = 2$, $(\therefore \lambda > 0)$

$$\therefore p(x=5)=\frac{2^{5}e^{-2}}{5!}$$

$$=\frac{32}{120e^{2}}=\frac{4}{15e^{2}}$$