

ROTATION OF AXES (CHANGE OF DIRECTION)

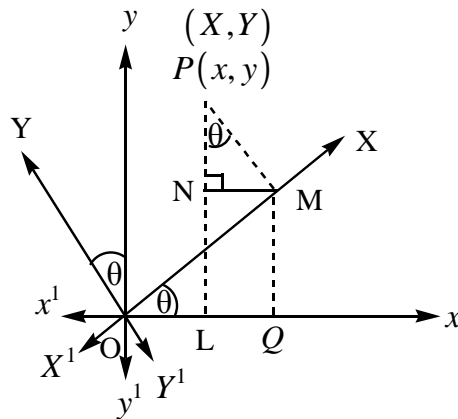
1. Definition: If the axes are rotated through an angle in the same plane by keeping the origin constant, then the transformation is called Rotation of axes.

2. Theorem: To find the co-ordinates of a point (x, y) are transformed to (X, Y) when the axes are rotated through an angle ' θ ' about the origin in the same plane.

Proof: Let x^1Ox , y^1OY^1 are the original axes

Let $P(x, y)$ be the co-ordinates of the point in the above axes.

After rotating the axes through an angle ' θ ', then the co-ordinates of P be (X, Y) w.r.t the new axes X^1OX and Y^1OY^1 as in figure.



Since θ is the angle of rotation, then $\angle xOX = \angle yOY = \theta$ as in the figure.

Since L, M is projections of P on Ox and OX respectively. We can see that $\angle LPM = \angle xOX = \theta$

Let N be the projection to PL from M

Now $x = OL = OQ - LQ = OQ - NM$

$$= OM \cos \theta - PM \sin \theta$$

$$= X \cos \theta - Y \sin \theta$$

$$y = PL = PN + NL = PN + MQ$$

$$PM \cos \theta + OM \sin \theta$$

$$= Y \cos \theta + X \sin \theta$$

$$\therefore x = X \cos \theta - Y \sin \theta \text{ and}$$

$$y = Y \cos \theta + X \sin \theta \text{ ----- (1)}$$

Solving the above equations to get X and Y, then $X = x \cos \theta + y \sin \theta$ and

$$Y = -x \sin \theta + y \cos \theta \text{ ---- (2)}$$

From (1) and (2) we can tabulate

	X	Y
x	$\cos \theta$	$-\sin \theta$
y	$\sin \theta$	$\cos \theta$

Note:

(i) If the axes are turned through an angle ' θ ', then the equation of a curve $f(x, y) = 0$ is transformed to $f(X \cos \theta - Y \sin \theta, X \sin \theta + Y \cos \theta) = 0$

(ii) If $f(X, Y) = 0$ is the transformed equation of a curve when the axes are rotated through an angle ' θ ' then the original equation of the curve is

$$f(x \cos \theta + y \sin \theta, -x \sin \theta + y \cos \theta) = 0$$

Theorem: To find the angle of rotation of the axes to eliminate xy term in the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

Proof: given equation is $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

Since the axes are rotated through an angle θ , then $x = X \cos \theta - Y \sin \theta$,
 $y = X \sin \theta + Y \cos \theta$

Now the transformed equation is

$$a(X \cos \theta - Y \sin \theta)^2 + 2h(X \cos \theta - Y \sin \theta)(X \sin \theta + Y \cos \theta)$$

$$+b(X \sin \theta + Y \cos \theta)^2 + 2g(X \cos \theta - Y \sin \theta) + 2f(X \sin \theta + Y \cos \theta) + c = 0$$

$$\Rightarrow a(X^2 \cos^2 \theta + Y^2 \sin^2 \theta - 2XY \cos \theta \sin \theta) + 2h[X^2 \cos \theta \sin \theta + XY(\cos^2 \theta - \sin^2 \theta) - Y^2 \sin \theta \cos \theta] + b(X^2 \sin^2 \theta + Y^2 \cos^2 \theta + 2XY \cos \theta \sin \theta)$$

$$+ 2g(X \cos \theta - Y \sin \theta) + 2f(X \sin \theta + Y \cos \theta) + c = 0$$

Since XY term is to be eliminated, coefficient of XY = 0.

$$2(b-a) \cos \theta \sin \theta + 2h(\cos^2 \theta - \sin^2 \theta) = 0$$

$$\Rightarrow h \cos 2\theta + (b-a) \sin 2\theta = 0$$

$$\Rightarrow 2h \cos 2\theta = (a-b) \sin 2\theta$$

$$\Rightarrow \tan 2\theta = \frac{\sin 2\theta}{\cos 2\theta} = \frac{2h}{a-b}$$

$$\Rightarrow \text{Angle of rotation } (\theta) = \frac{1}{2} \text{Tan}^{-1} \left(\frac{2h}{a-b} \right)$$

Note: The angle of rotation of the axes to eliminate xy term in

$$ax^2 + 2hxy + ay^2 + 2gx + 2fy + c = 0 \text{ is } \frac{\pi}{4}$$

PROBLEMS

1. When the axes are rotated through an angle 30° , find the new co-ordinates of the following points.

- i) (0, 5) ii) (-2, 4) iii) (0, 0)**

Sol. i) Given $\theta = 30^\circ$

Old co-ordinates are (0,5)

i.e., x=0, y = 5

$$X = x \cos \theta + y \sin \theta$$

$$= 0 \cdot \cos 30^\circ + 5 \cdot \sin 30^\circ = \frac{5}{2}$$

$$Y = -x \sin \theta + y \cos \theta$$

$$-0 \cdot \sin 30^\circ + 5 \cdot \cos 30^\circ = \frac{5\sqrt{3}}{2}$$

New co-ordinates are $\left(\frac{5}{2}, \frac{5\sqrt{3}}{2}\right)$

ii) Old co-ordinates are (-2,4) ANS. $(\sqrt{3} + 2, 1 + 2\sqrt{3})$

iii) Given $(x, y) = (0, 0)$ and $\theta = 30^\circ$

$$\Rightarrow X = x \cdot \cos 30^\circ - y \sin 30^\circ$$

$$= 0 \cdot \frac{\sqrt{3}}{2} - 0 \cdot \frac{1}{2} = 0$$

$$Y = x \cdot \sin 30^\circ + y \cdot \cos 30^\circ = 0 \cdot \frac{1}{2} + 0 \cdot \frac{\sqrt{3}}{2} = 0$$

New co-ordinates of the point are (0, 0)

2. When the axes are rotated through an angle 60° , the new co-ordinates of three points are the following

i) (3, 4) ii) (-7, 2) iii) (2, 0) Find their original co-ordinates

Sol. i) Given $\theta = 60^\circ$

New co-ordinates are (3, 4)

$$X = 3, Y = 4$$

$$x = X \cos \theta - Y \sin \theta = 3 \cdot \cos 60^\circ - 4 \cdot \sin 60^\circ$$

$$= 3 \cdot \frac{1}{2} - \frac{4 \cdot \sqrt{3}}{2} = \frac{3 - 4\sqrt{3}}{2}$$

$$y = X \sin \theta + Y \cos \theta$$

$$= 3 \sin 60^\circ + 4 \cdot \cos 60^\circ = 3 \cdot \frac{\sqrt{3}}{2} + 4 \cdot \frac{1}{2} = \frac{4 + \sqrt{3}}{2}$$

Co-ordinates of P are $\left(\frac{3 - 4\sqrt{3}}{2}, \frac{4 + 3\sqrt{3}}{2} \right)$

ii) New coordinates are (-7,2) ANS. $\left(\frac{-7 - 2\sqrt{3}}{2}, \frac{2 - 7\sqrt{3}}{2} \right)$

iii) New co-ordinates are (2, 0)

ans. $(1, \sqrt{3})$

3. **Find the angle through which the axes are to be rotated so as to remove the xy term in the equation.** $x^2 + 4xy + y^2 - 2x + 2y - 6 = 0$

Sol. Comparing the equation

$$x^2 + 4xy + y^2 - 2x + 2y - 6 = 0 \text{ with } ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

$$a = 1, \quad h = 2, \quad b = 1, \quad g = -1, \quad f = 1, \quad c = -6$$

Let ' θ ' be the angle of rotation of axes, then $\theta = \frac{1}{2} \tan^{-1} \left(\frac{2h}{a-b} \right)$

$$= \frac{1}{2} \tan^{-1} \left(\frac{4}{1-1} \right) = \frac{1}{2} \tan^{-1} \left(\frac{4}{0} \right)$$

$$= \frac{1}{2} \tan^{-1}(\infty) = \frac{1}{2} \times \frac{\pi}{2}$$

$$\therefore \theta = \frac{\pi}{4}$$

SHORT ANSWERS QUESTIONS

1. When the axes are rotated through an angle 45° , the transformed equation of a curve is $17x^2 - 16xy + 17y^2 = 225$. Find the original equation of the curve.?

Sol. Angle of rotation = $\theta = 45$

$$X = x \cos \theta + y \sin \theta = x \cos 45 + y \sin 45 = \frac{x+y}{\sqrt{2}}$$

$$Y = -x \sin \theta + y \cos \theta = -x \sin 45 + y \cos 45 = \frac{-x+y}{\sqrt{2}}$$

The original equation of

$$17X^2 - 16XY + 17Y^2 = 225 \text{ is}$$

$$\Rightarrow 17\left(\frac{x+y}{\sqrt{2}}\right)^2 - 16\left(\frac{x+y}{\sqrt{2}}\right)\left(\frac{-x+y}{\sqrt{2}}\right) + 17\left(\frac{-x+y}{\sqrt{2}}\right)^2 = 225$$

$$\Rightarrow 17\frac{(x^2 + y^2 + 2xy)}{2} - 16\frac{(y^2 - x^2)}{2} + 17\frac{(x^2 + y^2 - 2xy)}{2} = 225$$

$$\Rightarrow 17[(x+y)^2 + (x-y)^2] - 16(x^2 - y^2) = 450$$

$$\Rightarrow 17[2(x^2 + y^2)] - 16(x^2 - y^2) = 450$$

$$\Rightarrow 17(x^2 + y^2) - 8(x^2 - y^2) = 225$$

$$\Rightarrow 9x^2 + 25y^2 = 225 \text{ is the original equation}$$

2. when the axes are rotated through an angle α , find the transformed equation of $x \cos \alpha + y \sin \alpha = p$?

Sol. The given equation is $x \cos \alpha + y \sin \alpha = p$

\therefore The axes are rotated through an angle α

$$x = X \cos \alpha - Y \sin \alpha$$

$$y = X \sin \alpha + Y \cos \alpha$$

The given equation transformed to

$$(X \cos \theta - Y \sin \theta) \cos \theta + (X \sin \theta + Y \cos \theta) \sin \theta = p$$

$$\Rightarrow X (\cos^2 \theta + \sin^2 \theta) = p \Rightarrow X = p$$

The equation transformed to $x = p$

- 3. When the axes are rotated through an angle $\pi/6$. Find the transformed equation of $x^2 + 2\sqrt{3}xy - y^2 = 2a^2$**

Sol. Since $\theta = \frac{\pi}{6}$, $x = X \cos \theta - Y \sin \theta$

$$x = X \cos \frac{\pi}{6} - Y \sin \frac{\pi}{6}$$

$$X \cdot \frac{\sqrt{3}}{2} - Y \cdot \frac{1}{2} = \frac{\sqrt{3}X - Y}{2}$$

$$y = X \sin \theta + Y \cos \theta = X \cdot \sin \frac{\pi}{6} + Y \cos \frac{\pi}{6} = X \cdot \frac{1}{2} + Y \cdot \frac{\sqrt{3}}{2} = \frac{X + \sqrt{3}Y}{2}$$

Transformed equation is

$$\left(\frac{\sqrt{3}X - Y}{2} \right)^2 + 2\sqrt{3} \left(\frac{\sqrt{3}X - Y}{2} \right) \left(\frac{X + \sqrt{3}Y}{2} \right) - \left(\frac{X + \sqrt{3}Y}{2} \right)^2 = 2a^2$$

$$\Rightarrow \frac{3x^2 - 2\sqrt{3} + Y^2}{4} + \frac{2\sqrt{3}[\sqrt{3}X^2 - XY + 3XY - \sqrt{3}Y^2]}{4} = \frac{X^2 + 3Y^2 + 2\sqrt{3}XY}{4} = 2a^2$$

$$\Rightarrow 3X^2 - 2\sqrt{3}XY + Y^2 + 2\sqrt{3}[\sqrt{3}X^2 + 2XY + \sqrt{3}Y^2] - (X^2 + 3Y^2 + \sqrt{3}XY) = 8a^2$$

$$\Rightarrow 3X^2 - 2\sqrt{3} + Y^2 + 6X^2 + 4\sqrt{3}XY - 6Y^2 - X^2 - 3Y^2 - 2\sqrt{3}XY = 8a^2$$

$$\Rightarrow 8X^2 - 8Y^2 = 8a^2 \Rightarrow X^2 - Y^2 = a^2$$

4. When the axes are rotated through an angle $\frac{\pi}{4}$, find the transformed equation of $3x^2 + 10xy + 3y^2 = 9$

Sol. Given equation is $3x^2 + 10xy + 3y - 9 = 0$ (1)

$$\text{Angle of rotation of axes} = \theta = \frac{\pi}{4}$$

Let (X, Y) be the new co-ordinates of (x, y)

$$x = X \cos \theta - Y \sin \theta$$

$$= X \cos \frac{\pi}{4} - Y \sin \frac{\pi}{4} = \frac{X - Y}{\sqrt{2}}$$

$$y = X \sin \theta + Y \cos \theta = X \sin \frac{\pi}{4} + Y \cos \frac{\pi}{4} = \frac{X + Y}{\sqrt{2}}$$

Transformed equation of (1) is

$$3\left(\frac{X - Y}{\sqrt{2}}\right)^2 + 10\left(\frac{X - Y}{\sqrt{2}}\right)\left(\frac{X + Y}{\sqrt{2}}\right) + 3\left(\frac{X + Y}{\sqrt{2}}\right)^2 - 9 = 0$$

$$3\frac{(X^2 - 2XY + Y^2)}{2} + 10\frac{(X^2 - Y^2)}{2} + 3\frac{(X^2 + 2XY + Y^2)}{2} - 9 = 0$$

$$\Rightarrow 3X^2 - 6XY + 3Y^2 + 10X^2 - 10Y^2 + 3X^2 + 6XY + 3Y^2 - 18 = 0$$

$$\Rightarrow 16X^2 - 4Y^2 - 18 = 0$$

$\therefore 8X^2 - 2Y^2 = 9$ is the transformed equation.

5. Find the transformed equation of $17x^2 - 16xy + 17y^2 = 225$ when the axes are rotated through an angle 45°

Sol. Let (x, y) the original equation of (X, Y)

$$\text{Angle of rotation } \theta = 45^\circ$$

$$\text{Now } X = x \cos \theta - y \sin \theta$$

$$= x \cos 45^\circ - y \sin 45^\circ = \frac{x-y}{\sqrt{2}}$$

$$Y = x \sin \theta + y \cos \theta$$

$$= x \sin 45^\circ + y \cos 45^\circ = \frac{x+y}{\sqrt{2}}$$

The transformed equation is 45°

$$f\left(\frac{x-y}{\sqrt{2}}, \frac{x+y}{\sqrt{2}}\right) = 0$$

$$\Rightarrow 17\left(\frac{x-y}{\sqrt{2}}\right)^2 - 16\left(\frac{x-y}{\sqrt{2}}\right)\left(\frac{x+y}{\sqrt{2}}\right) + 17\left(\frac{x+y}{\sqrt{2}}\right)^2 = 225$$

$$\Rightarrow 17\left(\frac{x^2 + y^2 - 2xy}{2}\right) - 16\left(\frac{x^2 - y^2}{2}\right) + 17\left(\frac{x^2 + y^2 + 2xy}{2}\right) = 225$$

$$\Rightarrow 17X^2 + 17Y^2 - 34XY - 16X^2 + 16Y^2 + 17X^2 + 17Y^2 + 34XY = 450$$

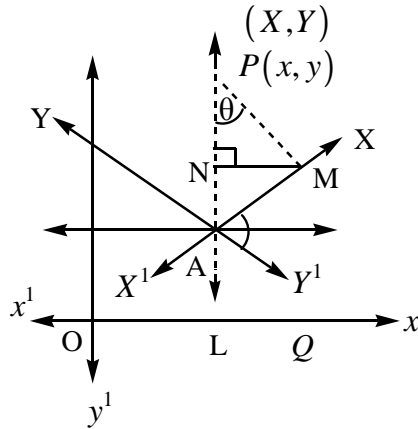
$$\Rightarrow 18X^2 + 50Y^2 = 450$$

$$9X^2 + 25Y^2 = 225$$

GENERAL TRANSFORMATIONS

1. Definition: If the axes are rotated through an angle θ after shifting the origin in the same plane, then the transformation is called “General Transformation”

New origin $A = (x_1, y_1)$, angle of rotation = θ as in figure



We get the transformed equations as

$$x = x_1 + X \cos \theta - Y \sin \theta$$

$$y = y_1 + X \sin \theta + Y \cos \theta$$

$$X = (x - x_1) \cos \theta + (y - y_1) \sin \theta$$

$$Y = (x - x_1) \sin \theta + (y - y_1) \cos \theta$$

We can easily understand the translation and rotation satisfy commutative property.

PROBLEMS.

1. When the origin is shifted to $(-2, -3)$ and the axes are rotated through an angle 45° find the transformed of $2x^2 + 4xy - 5y^2 + 20x - 22y - 14 = 0$?

Sol. Here $(h, k) = (-2, -3), h = -2, k = -3$

$$\theta = 45^\circ$$

Let (x^1, y^1) be the new co-ordinates of any point (x, y) is the plane after transformation

$$x = x^1 \cos \theta - y^1 \sin \theta + h = -2x + x^1 \cos 45^\circ - y^1 \sin 45^\circ$$

$$= -2 + \frac{x^1 - y^1}{\sqrt{2}}$$

$$y = x^1 \sin \theta + y^1 \cos \theta + k = x^1 \sin 45^\circ + y^1 \cos 45^\circ - 3$$

$$-3 + \frac{x^1 + y^1}{\sqrt{2}}$$

The transformed equation is

$$\Rightarrow 2 \left(\frac{x^1 - y^1}{\sqrt{2}} - 2 \right)^2 + 4 \left(\frac{x^1 - y^1}{\sqrt{2}} - 2 \right) \left(\frac{x^1 + y^1}{\sqrt{2}} - 3 \right)$$

$$-5 \left(\frac{x^1 + y^1}{\sqrt{2}} - 3 \right)^2 + 20 \left(\frac{x^1 - y^1}{\sqrt{2}} - 2 \right) - 22 \left(\frac{x^1 + y^1}{\sqrt{2}} - 3 \right) - 14 = 0$$

$$\Rightarrow 2 \left(\frac{(x^1 + y^1)^2}{2} + 42\sqrt{2}(x^1 - y^1) \right) + 4$$

$$\left(\frac{x^{1^2} - y^{1^2}}{2} - 3 \frac{(x^1 - y^1)}{\sqrt{2}} - 2 \frac{(x^1 + y^1)}{\sqrt{2}} + 6 \right)$$

$$-5 \left(\frac{(x^1 + y^1)^2}{2} + 9 - 3\sqrt{2}(x^1 + y^1) \right) + 10\sqrt{2} \left[(x^1 - y^1) - 2\sqrt{2} \right] - 11\sqrt{2}$$

$$\left[(x^1 + y^1) - 3\sqrt{2} \right] - 14 = 0$$

$$(x^1 + y^1)^2 + 8 - 4\sqrt{2}(x^1 - y^1) + 2(x^{1^2} - y^{1^2}) - 6\sqrt{2}(x^1 - y^1)$$

$$-4\sqrt{2}(x^1 + y^1) + 24 = 0 - \frac{5}{2}(x^1 + y^1)^2 - 45 + 15\sqrt{2}(x^1 + y^1) + 10\sqrt{2}(x^1 - y^1)$$

$$-40 - 11\sqrt{2}(x^1 + y^1) + 66 - 14 = 0$$

$$x^{1^2} + y^{1^2} - 2x^1 y^1 + 2x^{1^2} - 2y^{1^2} - \frac{5}{2}$$

$$(x^{1^2} + y^{1^2} + 2x^1 y^1) - 1 = 0$$

