

Reduction of the equation $ax + by + c = 0$ of a straight line into various forms.

1. SLOPE INTERCEPT FORM

Equation of the line is $ax + by + c = 0$

$$by = -ax - c \Rightarrow y = -\frac{a}{b}x - \frac{c}{b}$$

This equation is of the form $y = mx + K$. K is constant.

2. INTERCEPTS FORM

Equation of the line is $ax + by + c = 0$

$$ax + by = -c$$

$$\Rightarrow \frac{ax}{-c} + \frac{by}{-c} = 1 \Rightarrow \frac{x}{-c/a} + \frac{y}{-c/b} = 1 \text{ Which is of the form } \frac{x}{A} + \frac{y}{B} = 1$$

Here x - intercept = $-\frac{c}{a}$, y - intercept = $-\frac{c}{b}$.

3. NORMAL FORM

Case (i) Let $c \geq 0$

The equation of the line is $ax + by + c = 0$

$$ax + by = -c \quad (-c \geq 0) \Rightarrow \frac{a}{\sqrt{a^2 + b^2}}x + \frac{b}{\sqrt{a^2 + b^2}}y = \frac{-c}{\sqrt{a^2 + b^2}}$$

$x \cos \alpha + y \sin \alpha = p$, where $\cos \alpha = \frac{a}{\sqrt{a^2 + b^2}}$, $\sin \alpha = \frac{b}{\sqrt{a^2 + b^2}}$ and

$$p = \frac{-c}{\sqrt{a^2 + b^2}} > 0$$

Case (ii) Let $c < 0$.

Then $ax + by + c = 0$

$$ax + by = -c$$

$$(-a)x + (-b)y = c$$

$$\Rightarrow \left(\frac{-a}{\sqrt{a^2 + b^2}} \right)x + \left(\frac{-b}{\sqrt{a^2 + b^2}} \right)y = \frac{c}{\sqrt{a^2 + b^2}}$$

$x \cos \alpha + y \sin \alpha = p$, where $\cos \alpha = \frac{-a}{\sqrt{a^2 + b^2}}$, $\sin \alpha = \frac{-b}{\sqrt{a^2 + b^2}}$ and $p = \frac{c}{\sqrt{a^2 + b^2}}$

Note: The perpendicular distance from origin to the line $ax + by + c = 0$ is $\frac{|c|}{\sqrt{a^2 + b^2}}$.

The area of the triangle formed by the line $\frac{x}{a} + \frac{y}{b} = 1$ with the coordinate axes is $\frac{1}{2}|ab|$.

PARAMETRIC FORM

If $P(x, y)$ is any point on the line passing through $A(x_1, y_1)$ and having inclination θ , then

$x = x_1 + r \cos \theta$, $y = y_1 + r \sin \theta$ where $|r|$ is the distance from A to P . (r is a real parameter)

EXERCISE – 3(B)

I.

- Find the sum of the square of the intercepts of the line $4x - 3y = 12$ on the axes of co-ordinates.

Sol. Given line is $4x - 3y = 12$

$$\Rightarrow \frac{4x}{12} - \frac{3y}{12} = 1 \Rightarrow \frac{x}{3} + \frac{y}{-4} = 1$$

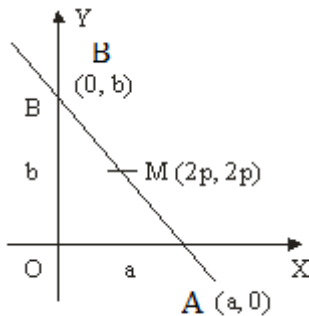
Intercepts are $a = 3$, $b = -4 \Rightarrow$ Sum of the squares = $a^2 + b^2 = 9 + 16 = 25$

- If the portion of a straight line intercepted between the axes of co-ordinates is bisected at $(2p, 2q)$, write the equation of the straight line.

Sol. Let a, b be the intercepts of the line and AB be the line segment between the axes.

Then points $A = (a, 0)$ and $B = (0, b)$

Equation of the line in the intercept form is $\frac{x}{a} + \frac{y}{b} = 1$ --- (1)



Mid-point of AB is $M = \left(\frac{a}{2}, \frac{b}{2}\right) = (2p, 2q)$ given

$$\Rightarrow \frac{a}{2} = 2p, \frac{b}{2} = 2q \Rightarrow a = 4p, b = 4q$$

Substituting in (1), $\frac{x}{4p} + \frac{y}{4q} = 1 \Rightarrow \frac{x}{p} + \frac{y}{q} = 4$

3. If the linear equation $ax + by + c = 0$

($a, b, c \neq 0$) and $lx + my + n = 0$ represent the same line and $r = \frac{l}{a} = \frac{m}{b} = \frac{n}{c}$, write the value of r in terms of m and b .

Sol. The equations $ax + by + c = 0$ and

$$lx + my + n = 0 \text{ are representing the same line } \Rightarrow \frac{l}{a} = \frac{m}{b} = \frac{n}{c} = r \Rightarrow \frac{m}{b} = r$$

4. Find the angle made by the straight line $y = -\sqrt{3}x + 3$ with the positive X-axis measured in the counter clock-wise direction.

Sol. Equation of the given line is $y = -\sqrt{3}x + 3$

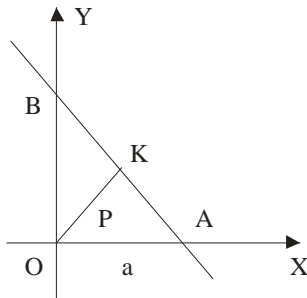
let α be the inclination of the line.

$$\text{Then } \tan \alpha = -\sqrt{3} = \tan \frac{2\pi}{3} \Rightarrow \alpha = \frac{2\pi}{3}$$

5. The intercepts of a straight line on the axes of co-ordinates are a and b . If P is the length of the perpendicular drawn from the origin to this line. Write the value of P in terms of a and b .

Sol. Equation of the line in the intercept form is

$$\frac{x}{a} + \frac{y}{b} - 1 = 0$$



P = length of the perpendicular from origin

$$p = \frac{|0+0-1|}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}}$$

$$\Rightarrow \frac{1}{p} = \sqrt{\frac{1}{a^2} + \frac{1}{b^2}}$$

Square on both sides

$$\Rightarrow \frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2} = \frac{a^2 + b^2}{a^2 b^2} \Rightarrow p^2 = \frac{a^2 b^2}{a^2 + b^2} \Rightarrow p = \frac{|ab|}{\sqrt{a^2 + b^2}}$$

II.

1. In what follows, P denotes the distance of the straight line from the origin and α is the angle made by the normal ray drawn from the origin to the straight line with \overline{OX} measured in the anti-clock wise sense. Find the equations of the straight lines with the following values of P and α .

i) $p = 5, \alpha = 60^\circ$ ii) $p = 6, \alpha = 150^\circ$ iii) $p = 1, \alpha = \frac{7\pi}{4}$

iv) $p = 4, \alpha = 90^\circ$ iv) $p = 0, \alpha = 0$ v) $p = 2\sqrt{2}, \alpha = \frac{5\pi}{4}$

Sol. Equation of the line in the normal form is $x \cos \alpha + y \sin \alpha = p$

i) given $p = 5, \alpha = 60^\circ$

$$\cos \alpha = \cos 60^\circ = \frac{1}{2} \text{ and } \sin \alpha = \sin 60^\circ = \frac{\sqrt{3}}{2}$$

Equation of the line is $x \cos \alpha + y \sin \alpha = p$

$$\Rightarrow x \frac{1}{2} + y \frac{\sqrt{3}}{2} = 5 \Rightarrow x + \sqrt{3}y = 10$$

ii) ans : $\sqrt{3}x + y + 12 = 0$

iii) $p = 4, \alpha = \frac{7\pi}{4}$

$$\cos \alpha = \cos 315^\circ = \frac{1}{\sqrt{2}} \text{ and } \sin \alpha = \sin 315^\circ = -\frac{1}{\sqrt{2}}$$

Equation of the line is $x \cdot \frac{1}{\sqrt{2}} - y \cdot \frac{1}{\sqrt{2}} = 4$

$$x - y = \sqrt{2} \Rightarrow x - y - \sqrt{2} = 0$$

iv) $P = 4, \alpha = 90^\circ$

$$\cos \alpha = \cos 90^\circ = 0 \text{ and } \sin \alpha = \sin 90^\circ = 1$$

$$\text{Equation of the line is } x \cos \alpha + y \sin \alpha = p \Rightarrow x \cdot 0 + y \cdot 1 = 4 \Rightarrow x = 4$$

- v) ans: $x = 0$
vi) ans: $x + y + 4 = 0$

2. Find the equation of the straight line in the symmetric form, given the slope and a point on the line in each part of the question.

- i) $\sqrt{3}, (2, 3)$ ii) $-\frac{1}{\sqrt{3}}, (-2, 0)$ iii) $-1, (1, 1)$

Sol. i) point $(x_1, y_1) = (2, 3)$

$$\text{slope } m = \tan \alpha = \sqrt{3} \Rightarrow \alpha = 60^\circ \Rightarrow \cos \alpha = \cos 60^\circ \Rightarrow \sin \alpha = \sin 60^\circ$$

$$\text{Equation of the line in the symmetric form is } \frac{x - x_1}{\cos \alpha} = \frac{y - y_1}{\sin \alpha} = r \Rightarrow \frac{x - 2}{\cos \frac{\pi}{3}} = \frac{y - 3}{\sin \frac{\pi}{3}}$$

ii) $(x_1, y_1) = (-2, 0)$

$$\tan \alpha = -\frac{1}{\sqrt{3}} \Rightarrow \alpha = 180^\circ - 30^\circ = 150^\circ$$

Equation of the line in the symmetric form is

$$\frac{x - x_1}{\cos \alpha} = \frac{y - y_1}{\sin \alpha} = r \Rightarrow \frac{x - 2}{\cos \frac{\pi}{3}} = \frac{y - 3}{\sin \frac{\pi}{3}} \Rightarrow \frac{x + 2}{\cos 150^\circ} = \frac{y}{\sin 150^\circ}$$

iii) ans: $\frac{x-1}{\cos\left(\frac{3\pi}{4}\right)} = \frac{y-1}{\sin\left(\frac{3\pi}{4}\right)}$

3. Transform the following equation into a) Slope-intercept form

b) Intercept form and c) Normal form

i) $3x + 4y = 5$ ii) $4x - 3y + 12 = 0$ iii) $\sqrt{3}x + y = 4$ iv) $x + y + 2 = 0$

v) $x + y - 2 = 0$ vi) $\sqrt{3}x + y + 10 = 0$

Sol. i) equation of the line is $3x + 4y = 5$

1. Slope-intercept form

$$4y = -3x + 5 \Rightarrow y = \left(-\frac{3}{4}\right)x + \left(\frac{5}{4}\right) \text{ which is of the form } y = mx + c.$$

2. Intercept form : $3x + 4y = 5$

$$\frac{3x}{5} + \frac{4y}{5} = 5 \Rightarrow \frac{x}{\left(\frac{5}{3}\right)} + \frac{y}{\left(\frac{5}{4}\right)} = 1 \text{ which is of the form } \frac{x}{a} + \frac{y}{b} = 1$$

3. Normal form : $3x + 4y = 5$

$$\text{Dividing with } \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = 5$$

$$\frac{3}{5}x + \frac{4}{5}y = 1 \Rightarrow \text{let } \cos \alpha = \frac{3}{5}, \sin \alpha = \frac{4}{5} \text{ and } p=1$$

Then the above eq. reduces to $x \cos \alpha + y \sin \alpha = p$

$$\text{ii) ans: } y = \left(\frac{4}{3}\right)x + 4, \frac{x}{(-3)} + \frac{y}{4} = 1, \left(\frac{-4}{5}\right)x + \left(\frac{3}{5}\right)y = \frac{12}{5}$$

$$\text{iii) } y = -\sqrt{3}x + 4, \frac{x}{\left(\frac{4}{\sqrt{3}}\right)} + \frac{y}{4} = 1, x \cos\left(\frac{\pi}{6}\right) + y \sin\left(\frac{\pi}{6}\right) = 32$$

$$\text{iv) } x + y + 2 = 0$$

1. Slope-intercept form :

$$x + y + 2 = 0$$

$$\Rightarrow y = -x - 2 = (-1)x + (-2) \text{ which is of the form } y = mx + c.$$

Intercept form :

$$x + y + 2 = 0 \Rightarrow -x - y = 2$$

$$\Rightarrow -\frac{x}{2} - \frac{y}{2} = 1 \Rightarrow \frac{x}{-2} + \frac{y}{-2} = 1 \text{ which is of the form } \frac{x}{a} + \frac{y}{b} = 1$$

$$\text{3. Normal form: } x + y + 2 = 0 \Rightarrow -x - y = 2$$

Dividing with $\sqrt{1+1} = \sqrt{2}$

$$\Rightarrow \left(-\frac{1}{\sqrt{2}}\right)x + \left(-\frac{1}{\sqrt{2}}\right)y = \sqrt{2} \Rightarrow x \cos\left(\frac{5\pi}{4}\right) + y \sin\left(\frac{5\pi}{4}\right) = \sqrt{2}$$

Which is of the form $x \cos \alpha + y \sin \alpha = p$

$$\text{v) } y = -x + 2, \frac{x}{2} + \frac{y}{2} = 1, x \cos\left(\frac{\pi}{4}\right) + y \sin\left(\frac{\pi}{4}\right) = \sqrt{2}$$

$$\text{vi) } y = -\sqrt{3}x - 10, \frac{x}{\left(\frac{10}{\sqrt{3}}\right)} + \frac{y}{-10} = 1, x \cos 30^\circ + y \sin 30^\circ = 5$$

4. If the product of the intercepts made by the straight line

$$x \tan \alpha + y \sec \alpha = 1 \left(0 \leq \alpha < \frac{\pi}{2}\right), \text{ on the co-ordinate axes is equal to } \sin \alpha, \text{ find } \alpha.$$

Sol. Equation of the line is $x \tan \alpha + y \sec \alpha = 1$

$$\Rightarrow \frac{x}{\cot \alpha} + \frac{y}{\cos \alpha} = 1 \Rightarrow a = \cot \alpha, b = \cos \alpha$$

Given product of intercepts = $\sin \alpha$

$$\begin{aligned} \Rightarrow \cot \alpha \cdot \cos \alpha &= \sin \alpha \\ \Rightarrow \frac{\cos^2 \alpha}{\sin \alpha} &= \sin \alpha \Rightarrow \cos^2 \alpha = \sin^2 \alpha \\ \Rightarrow \tan^2 \alpha &= 1 \Rightarrow \tan \alpha = \pm 1 \Rightarrow \alpha = 45^\circ \end{aligned}$$

5. **If the sum of the reciprocals of the intercept made by a variable straight line on the axes of co-ordinates is a constant then prove that the line always passes through a fixed point.**

Sol. Equation of the line in the intercept form is $\frac{x}{a} + \frac{y}{b} = 1$ --- (1)

Sum of the reciprocals of the intercepts

$$\frac{1}{a} + \frac{1}{b} = k \Rightarrow \frac{1}{ak} + \frac{1}{bk} = k \Rightarrow \left(\frac{1}{k}\right) + \left(\frac{1}{k}\right) = 1$$

Comparing this equation with (1),

The line (1) passes through the fixed point $\left(\frac{1}{k}, \frac{1}{k}\right)$

6. **Line L has intercepts a and b on the axes of co-ordinates. When the axes are rotated through a given angle, keeping the origin fixed, the same line L has intercepts p and q on the transformed axes. Prove that $\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{p^2} + \frac{1}{q^2}$.**

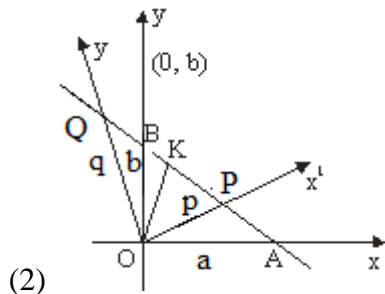
Sol. Equation of the line in the old system in intercept form is $\frac{x}{a} + \frac{y}{b} = 1 \Rightarrow \frac{x}{a} + \frac{y}{b} - 1 = 0$

$$\text{Length of the perpendicular from origin} = \frac{|0+0-1|}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}} \text{ ---(1)}$$

Equation of the line in the new system in intercept form

$$\text{is } \frac{X}{p} + \frac{Y}{q} = 1 \Rightarrow \frac{X}{p} + \frac{Y}{q} - 1 = 0$$

$$\text{Length of the perpendicular from origin} = \frac{|0+0-1|}{\sqrt{\frac{1}{p^2} + \frac{1}{q^2}}} \text{ ---}$$



Since the position of the origin and the given line remain unchanged, perpendicular distances in both the systems are same.

$$\frac{|-1|}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}} = \frac{|-1|}{\sqrt{\frac{1}{p^2} + \frac{1}{q^2}}} \Rightarrow \frac{1}{\left(\frac{1}{a^2} + \frac{1}{b^2}\right)} = \frac{1}{\left(\frac{1}{p^2} + \frac{1}{q^2}\right)} \Rightarrow \frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{p^2} + \frac{1}{q^2}$$

7. Transform the equation $\frac{x}{a} + \frac{y}{b} = 1$ into the normal form when $a > 0$ and $b > 0$.

If the perpendicular distance of the straight line from the origin is p , deduce that

$$\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2}.$$

Sol. Equation of the line is $\frac{x}{a} + \frac{y}{b} = 1$

$$\Rightarrow bx + ay = ab$$

Dividing with $\sqrt{a^2 + b^2}$

$$\frac{b}{\sqrt{a^2 + b^2}} \cdot x + \frac{a}{\sqrt{a^2 + b^2}} \cdot y = \frac{ab}{\sqrt{a^2 + b^2}} \text{ ----(1)}$$

$$\text{Let } \cos \alpha = \frac{b}{\sqrt{a^2 + b^2}}, \sin \alpha = \frac{a}{\sqrt{a^2 + b^2}} \text{ and } p = \frac{ab}{\sqrt{a^2 + b^2}}$$

Now eq. (1) reduces to $x \cos \alpha + y \sin \alpha = p$ which is normal form.

The perpendicular distance from O to the line is $p = \frac{ab}{\sqrt{a^2 + b^2}} \Rightarrow p^2 = \frac{a^2 b^2}{a^2 + b^2}$

$$\Rightarrow \frac{1}{p^2} = \frac{a^2 + b^2}{a^2 b^2} = \frac{a^2}{a^2 b^2} + \frac{b^2}{a^2 b^2} = \frac{1}{b^2} + \frac{1}{a^2} \quad \therefore \frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2}$$

III. 1. A straight line passing through A(-2, 1) makes an angle 30° with \overline{OX} in the positive direction. Find the points on the straight line whose distance from A is 4 units.

Sol. Let $(x_1, y_1) = (-2, 1)$, $r = 4$ and $\alpha = 30^\circ$

Co-ordinates of any point on the given line are

$$(x_1 \pm r \cos \alpha, y_1 \pm r \sin \alpha) \quad \cos \alpha = \cos 30^\circ = \frac{\sqrt{3}}{2}, \sin \alpha = \sin 30^\circ = \frac{1}{2}$$

Require points are

$$\left(-2+4\cdot\frac{\sqrt{3}}{2}, 1+4\cdot\frac{1}{2}\right) = (-2+2\sqrt{3}, 3) \quad \text{And}$$

$$\left(-2+4\cdot\frac{\sqrt{3}}{2}, 1-4\cdot\frac{1}{2}\right) = (-2-2\sqrt{3}, -1)$$

- 2. Find the points on the line $3x - 4y - 1 = 0$ which are at a distance of 5 units from the point $(3, 2)$.**

Sol. Equation of the line is $3x - 4y - 1 = 0 \Rightarrow$ slope of the line is $\tan\theta = \frac{3}{4}$
 $\Rightarrow \sin\theta = \frac{3}{5}$ and $\cos\theta = \frac{4}{5}$

Given Point is $(3, 2) = (x_1, y_1)$ and $r = 5$.

Co-ordinates of any point on the given line are $(x_1 \pm r \cos \theta, y_1 \pm r \sin \theta)$

Co-ordinates of the points are $\left(3+5\cdot\frac{4}{5}, 2+5\cdot\frac{3}{5}\right) = (7, 5)$ And

$$\left(3-5\cdot\frac{4}{5}, 2-5\cdot\frac{3}{5}\right) = (-1, -1)$$

- 3. A straight line whose inclination with the positive direction of the X – axis measured in the anti clock wise sense is $\pi/3$ makes positive intercept on the Y – axis. If the straight line is at a distance of 4 from the origin, find its equation.**

Sol. Let $\alpha = \pi/3, P = 4$

$$\text{Slope } m = \tan \alpha = \tan 60^\circ = \sqrt{3}$$

Equation of the line in the slope – intercept

$$\text{form is } y = \sqrt{3}x + c \Rightarrow \sqrt{3}x - y + c = 0$$

Distance from the origin = 4

$$\frac{|0-0+c|}{\sqrt{3+1}} = 4 \Rightarrow |c| = 4 \times 2 = 8 \Rightarrow c = \pm 8$$

$$\text{Given } c > 0 \quad \therefore c = 8$$

$$\text{Equation of the line } \sqrt{3}x - y + 8 = 0$$

- 4. A straight line L is drawn through the point $A(2, 1)$ such that its point of intersection with the straight line $x + y = 9$ is at a distance of $3\sqrt{2}$ from A. Find the angle which the line L makes with the positive direction of the X – axis.**

Sol. Suppose α is the angle made by L with the positive X – axis

$$\text{Given point } (2, 1) = (x_1, y_1) \quad \text{and } r = 3\sqrt{2}$$

$$\text{Any point on the line is } (x_1 + r \cos \theta, y_1 + r \sin \theta) = (2 + 3\sqrt{2} \cos \theta + 3\sqrt{2} \sin \theta)$$

This is a point on the line $x + y = 9$

