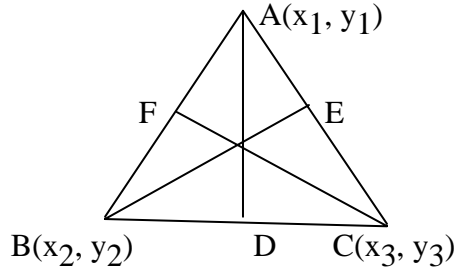


CONCURRENT LINES- PROPERTIES RELATED TO A TRIANGLE THEOREM

The medians of a triangle are concurrent.

Proof:

Let $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ be the vertices of the triangle



Let D, E, F be the mid points of $\overline{BC}, \overline{CA}, \overline{AB}$ respectively

$$\therefore D = \left(\frac{x_2 + x_3}{2}, \frac{y_2 + y_3}{2} \right), \quad E = \left(\frac{x_3 + x_1}{2}, \frac{y_3 + y_1}{2} \right)$$

$$F = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$

$$\text{Slope of } \overline{AD} \text{ is } \frac{\frac{y_2 + y_3}{2} - y_1}{\frac{x_2 + x_3}{2} - x_1} = \frac{y_2 + y_3 - 2y_1}{x_2 + x_3 - 2x_1}$$

Equation of \overline{AD} is

$$y - y_1 = \frac{y_2 + y_3 - 2y_1}{x_2 + x_3 - 2x_1} (x - x_1)$$

$$\Rightarrow (y - y_1)(x_2 + x_3 - 2x_1) = (x - x_1)(y_2 + y_3 - 2y_1)$$

$$\Rightarrow L_1 \equiv (x - x_1)(y_2 + y_3 - 2y_1)$$

$$- (y - y_1)(x_2 + x_3 - 2x_1) = 0.$$

Similarly, the equations to \overline{BE} and \overline{CF} respectively are $L_2 \equiv (x - x_2)(y_3 + y_1 - 2y_2)$

$$- (y - y_2)(x_3 + x_1 - 2x_2) = 0.$$

$$L_3 \equiv (x - x_3)(y_1 + y_2 - 2y_3)$$

$$- (y - y_3)(x_1 + x_2 - 2x_3) = 0.$$

$$\text{Now } 1. L_1 + 1.L_2 + 1. L_3 = 0$$

The medians $L_1 = 0, L_2 = 0, L_3 = 0$ are concurrent.

THEOREM

The altitudes of a triangle are concurrent.

Proof:

Let $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ be the vertices of the triangle ABC.

Let AD, BE, CF be the altitudes.

Slope of \overline{BC} is $\frac{y_3 - y_2}{x_3 - x_2}$ and $AD \perp BC$

Slope of the altitude through A is $-\frac{x_3 - x_2}{y_3 - y_2}$

Equation of the altitude through A is $y - y_1 = \frac{x_3 - x_2}{y_3 - y_2} (x - x_1)$

$$(y - y_1)(y_3 - y_2) = -(x - x_1)(x_3 - x_2)$$

$$L_1 = (x - x_1)(x_2 - x_3) + (y - y_1)(y_2 - y_3) = 0.$$

Similarly equations of the altitudes through B, C are

$$L_2 = (x - x_2)(x_3 - x_1) + (y - y_2)(y_2 - y_3) = 0,$$

$$L_3 = (x - x_3)(x_1 - x_2) + (y - y_3)(y_1 - y_2) = 0.$$

$$\text{Now } 1.L_1 + 1.L_2 + 1.L_3 = 0$$

The altitudes $L_1 = 0$, $L_2 = 0$, $L_3 = 0$ are concurrent.

THEOREM

The internal bisectors of the angles of a triangle are concurrent.

THEOREM

The perpendicular bisectors of the sides of a triangle are concurrent

EXERCISE – 3 (e)

I.

1. Find the in center of the triangle whose vertices are $(1, \sqrt{3})$, $(2, 0)$ and $(0, 0)$

Sol. let $A(0, 0)$, $B(1, \sqrt{3})$, $C(2, 0)$ be the vertices of ΔABC

$$a = BC = \sqrt{(1-2)^2 + (\sqrt{3}-0)^2} = \sqrt{1+3} = 2$$

$$b = CA = \sqrt{(2-0)^2 + (0-0)^2} = \sqrt{4} = 2$$

$$c = AB = \sqrt{(0-1)^2 + (0-\sqrt{3})^2} = \sqrt{4} = 2$$

$\therefore ABC$ is an equilateral triangle

co-ordinates of the in centre are

$$= \left(\frac{ax_1 + bx_2 + cx_3}{a+b+c}, \frac{ay_1 + by_2 + cy_3}{a+b+c} \right) = \left(\frac{2 \cdot 0 + 2 \cdot 1 + 2 \cdot 2}{2+2+2}, \frac{2 \cdot 0 + 2 \cdot \sqrt{3} + 2 \cdot 0}{2+2+2} \right)$$

$$= \left(\frac{6}{6}, \frac{2\sqrt{3}}{6} \right) = \left(1, \frac{1}{\sqrt{3}} \right)$$

2. Find the orthocenter of the triangle are given by $x + y + 10 = 0$, $x - y - 2 = 0$ and

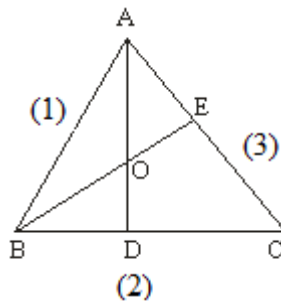
$$2x + y - 7 = 0$$

Sol. Let equation of

AB be $x + y + 10 = 0$ ---(1)

BC be $x - y - 2 = 0$ ---(2)

and AC be $2x + y - 7 = 0$ ---(3)



Solving (1) and (2) $B = (-4, -6)$

Solving (1) and (3) $A = (17, -27)$

Equation of BC is $x - y - 2 = 0$

Altitude AD is perpendicular to BC, therefore Equation of AD is $x + y + k = 0$

AD is passing through A $(17, -27)$

$$\Rightarrow 17 - 27 + k = 0 \Rightarrow k = 10$$

\therefore Equation of AD is $x + y + 10 = 0$ ----(4)

Altitude BE is perpendicular to AC.

\Rightarrow Let the equation of DE be $x - 2y = k$

BE is passing through D (-4, -6)

$\Rightarrow -4 + 12 = k \Rightarrow k = 8$

Equation of BE is $x - 2y = 8$ -----(5)

Solving (4) and (5), the point of intersection is (-4, -6).

Therefore the orthocenter of the triangle is (-4, -6).

- 3. Find the orthocentre of the triangle whose sides are given by $4x - 7y + 10 = 0$, $x + y = 5$ and $7x + 4y = 15$**

Sol. Ans: O (1, 2)

- 4. Find the circumcentre of the triangle whose sides are $x = 1$, $y = 1$ and $x + y = 1$**

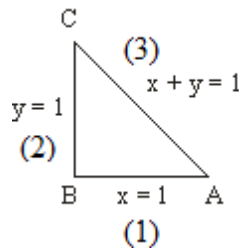
Sol. Let equation of AB be $x = 1$ -----(1)

BC be $y = 1$ -----(2)

and AC be $x + y = 1$ -----(3)

lines (1) and (2) are perpendicular to each other. Therefore, given triangle is a right triangle and $\angle B = 90^\circ$.

Therefore, circumcentre is the mid point of hypotenuse AC.



Solving (1) and (3), vertex A = (1, 0)

Solving (2) and (3), vertex c = (0, 1)

Circumcentre = mid point of AC = $\left(\frac{1}{2}, \frac{1}{2}\right)$

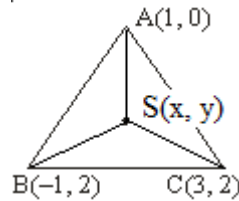
- 5. Find the incentre of the triangle formed by the lines $x = 1$, $y = 1$ and $x + y = 1$**

Sol. ANS: $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

6. Find the circumcentre of the triangle whose vertices are (1, 0), (-1, 2) and (3, 2)

Sol. vertices of the triangle are

A (1, 0), B (-1, 2), C (3, 2)



Let S (x, y) be the circumcentre of ΔABC .

Then $SA = SB = SC$

Let $SA = SB \Rightarrow SA^2 = SB^2$

$$(x-1)^2 + y^2 = (x+1)^2 + (y-2)^2$$

$$\Rightarrow x^2 - 2x + 1 + y^2 = x^2 + 2x + 1 + y^2 - 4y + 4$$

$$\Rightarrow 4x - 4y = -4 \Rightarrow x - y = -1 \quad \dots(1)$$

$SB = SC \Rightarrow SB^2 = SC^2$

$$(x+1)^2 + (y-2)^2 = (x-3)^2 + (y-2)^2$$

$$\Rightarrow x^2 + 2x + 1 = x^2 - 6x + 9$$

$$\Rightarrow 8x = 8 \Rightarrow x = 1$$

From (1), $1 - y = -1 \Rightarrow y = 2$

\therefore Circum centre is (1, 2)

7. Find the value of k, if the angle between the straight lines $kx + y + 9 = 0$ and

$3x - y + 4 = 0$ is $\pi/4$

Sol. Given lines are

$$kx + y + 9 = 0$$

$3x - y + 4 = 0$ and angle between the lines is $\pi/4$.

$$\therefore \cos \frac{\pi}{4} = \frac{|3k-1|}{\sqrt{k^2+1}\sqrt{9+1}} \Rightarrow \frac{1}{\sqrt{2}} = \frac{|3k-1|}{\sqrt{10}\sqrt{k^2+1}}$$

Squaring

$$\Rightarrow 5k^2 + 5 = (3k-1)^2 = 9k^2 - 6k + 1 \Rightarrow 4k^2 - 6k - 4 = 0 \Rightarrow 2k^2 - 3k - 2 = 0$$

$$\Rightarrow (k-2)(2k+1) = 0 \Rightarrow k = 2 \text{ or } -1/2$$

8. Find the equation of the straight line passing through the origin and also the point of intersection of the lines. $2x - y + 5 = 0$ and $x + y + 1 = 0$

Sol. Given lines are $L_1 = 2x - y + 5 = 0$

$$L_2 = x + y + 1 = 0$$

Equation of any line passing through the point of intersection of the lines $L_1=0$ and $L_2=0$ is $L_1 + KL_2 = 0$

$$\Rightarrow (2x - y + 5) + k(x + y + 1) = 0 \text{ -----(1)}$$

This line is passing through O (0, 0) $\Rightarrow 5 + k = 0 \Rightarrow k = -5$

Substituting in (1), equation of OA is $(x - y + 5) - 5(x + y + 1) = 0$

$$\Rightarrow 2x - y + 5 - 5x - 5y - 5 = 0$$

$$\Rightarrow -3x - 6y = 0 \Rightarrow x + 2y = 0$$

- 9. Find the equation of the straight line parallel to the lines $3x + 4y = 7$ and passing through the point of intersection of the lines $x - 2y - 3 = 0$ and $x + 3y - 6 = 0$**

Sol. Given lines are $L_1 = x - 2y - 3 = 0$ and

$$L_2 = x + 3y - 6 = 0$$

Equation of any line passing through the point of intersection of the lines $L_1=0$ and $L_2=0$ is $L_1 + KL_2 = 0$

$$\Rightarrow (x - 2y - 3) + k(x + 3y - 6) = 0$$

$$\Rightarrow (1 + k)x + (-2 + 3k)y + (-3 - 6k) = 0 \text{ -----(1)}$$

This line is parallel to $3x + 4y = 7$

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} \Rightarrow \frac{3}{1+k} = \frac{4}{-2+3k}$$

$$\Rightarrow 3(-2 + 3k) = (1 + k)4$$

$$\Rightarrow -6 + 9k = 4 + 4k \Rightarrow 5k = 10 \Rightarrow k = 2$$

Equation of the required line is

$$3x + 4y - 15 = 0$$

- 10. Find the equation of the straight line perpendicular to the line $2x + 3y = 0$ and passing through the point of intersection of the lines $x + 3y - 1 = 0$ and $x - 2y + 4 = 0$**

Sol. $L_1 = x + 3y - 1 = 0$

$$L_2 = x - 2y + 4 = 0$$

Equation of any line passing through the point of intersection of the lines $L_1=0$ and $L_2=0$ is $L_1 + KL_2 = 0$

$$\Rightarrow (x + 3y - 1) + k(x - 2y + 4) = 0$$

$$\Rightarrow (1 + k)x + (3 - 2k)y + (4k - 1) = 0 \text{ -----(1)}$$

This line is perpendicular to $2x + 3y = 0$,

$$a_1 a_2 + b_1 b_2 = 0 \Rightarrow 2(1 + k) + 3(3 - 2k) = 0$$

$$2 + 2k + 9 - 6k = 0 \Rightarrow 4k = 11 \Rightarrow k = \frac{11}{4}$$

Substituting in (1), equation of the required line is

$$\left(1 + \frac{11}{4}\right)x + \left(3 - \frac{11}{2}\right)y + (11 - 1) = 0$$

$$\frac{15}{4}x - \frac{5}{2}y + 10 = 0$$

$$\Rightarrow 15x - 10y = 40 = 0$$

$$\Rightarrow 3x - 2y + 8 = 0$$

- 11. Find the equation of the straight line making non – zero equal intercepts on the axes and passing through the point of intersection of the lines $2x - 5y + 1 = 0$ and $x - 3y - 4 = 0$**

Sol. Let $L_1 = 2x + 5y + 1 = 0$, $L_2 = x - 3y - 4 = 0$

Equation of any line passing through the point of intersection of the lines $L_1 = 0$ and $L_2 = 0$

is $L_1 + KL_2 = 0$

$$\Rightarrow (2x - 5y + 1) + k(x - 3y - 4) = 0$$

$$\Rightarrow (2 + k)x - (5 + 3k)y + (1 - 4k) = 0 \quad (1)$$

Intercepts on co-ordinates axes are equal, coefficient of $x =$ coefficient of y

$$\Rightarrow 2 + k = -5 - 3k$$

$$\Rightarrow 4k = -7 \Rightarrow k = -7/4$$

Substituting in (1)

Equation of the required line is

$$\Rightarrow \left(-2\frac{7}{4}\right)x - \left(5 - \frac{21}{4}\right)y + (1 + 7) = 0$$

$$\Rightarrow \frac{1}{4}x + \frac{1}{4}y + 8 = 0 \Rightarrow x + y + 32 = 0$$

- 12. Find the length of the perpendicular drawn from the point of intersection of the lines $3x + 2y + 4 = 0$ and $2x + 5y - 1 = 0$ to the straight line $7x + 24y - 15 = 0$**

Sol. Given lines are

$$3x + 2y + 4 = 0 \quad \text{-----(1)}$$

$$2x + 5y - 1 = 0 \quad \text{-----(2)}$$

Solving (1) and (2), point of intersection is $P (-2, 1)$.

Length of the perpendicular from $P (-2, 1)$ to the line $7x + 24y - 15 = 0$ is

$$= \frac{|-14 + 24 - 15|}{\sqrt{49 + 576}} = \frac{5}{25} = \frac{1}{5}$$

- 13. Find the value of ‘a’ if the distance of the points (2, 3) and (-4, a) from the straight line $3x + 4y - 8 = 0$ are equal.**

Sol. Equation of the line is $3x + 4y - 8 = 0$ ---(1)

Given points P (2, 3), (-4, a)

Perpendicular from P(2,3) to (1) = perpendicular from Q(-4,a) to (1)

$$\Rightarrow \frac{|3 \cdot 2 + 4 \cdot 3 - 8|}{\sqrt{9+16}} = \frac{|3 \cdot (-4) + 4a - 8|}{\sqrt{9+16}}$$

$$\Rightarrow 10 = |4a - 20|$$

$$\Rightarrow 4a - 20 = \pm 10 \Rightarrow 4a = 20 \pm 10 = 30 \text{ or } 10$$

$$\Rightarrow a = \frac{30}{4} \text{ or } \frac{10}{4}$$

$$\therefore a = \frac{15}{2} \text{ or } 5/2$$

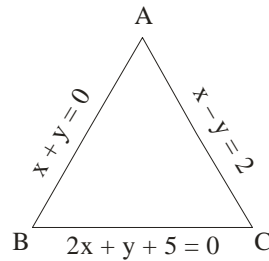
14. Find the circumcentre of the triangle formed by the straight lines $x + y = 0$, $2x + y + 5 = 0$ and $x - y = 2$

Sol. let the equation of

AB be $x + y = 0$ ---(1)

BC be $2x + y + 5 = 0$ ---(2)

And AC be $x - y = 2$ ---(3)



Solving (1) and (2), vertex B = (-5, 5)

Solving (2) and (3), vertex C = (-1, -3)

Solving (1) and (3), vertex A = (1, -1)

Let S (x, y) be the circumcentre of ΔABC .

Then SA = SB = SC

$$SA = SB \Rightarrow SA^2 = SB^2$$

$$(x + 5)^2 + (y - 5)^2 = (x + 1)^2 + (y + 3)^2$$

$$x^2 + 10x + 25 + y^2 - 10y + 25 = x^2 + 2x + 1 + y^2 + 6y + 9$$

$$\Rightarrow 8x - 16y = -40$$

$$\Rightarrow x - 2y = -5 \quad \text{---(4)}$$

$$SB = SC \Rightarrow SB^2 = SC^2$$

$$\Rightarrow (x + 1)^2 + (y + 3)^2 = (x - 1)^2 + (y + 1)^2$$

$$\Rightarrow x^2 + 2x + 1 + y^2 + 6y + 9 = x^2 - 2x + 1 + y^2 + 2y + 1$$

$$\Rightarrow 4x + 4y = -8$$

$$\Rightarrow x + y = -2 \quad \text{---(5)}$$

Solving (4) & (5), point of intersection is (-3, 1)
 circumcentre is S(-3, 1)

15. If θ is the angle between the lines $\frac{x}{a} + \frac{y}{b} = 1$ and $\frac{x}{b} + \frac{y}{a} = 1$, find the value of $\sin \theta$,
 when $a > b$.

Sol. Given equations are $\frac{x}{a} + \frac{y}{b} = 1 \Rightarrow bx + ay = ab$

And $\frac{x}{b} + \frac{y}{a} = 1 \Rightarrow ax + by = ab$

Let θ be angle between the lines, then

$$\begin{aligned} \cos \theta &= \frac{|a_1 a_2 + b_1 b_2|}{\sqrt{a_1^2 + b_1^2} \sqrt{a_2^2 + b_2^2}} \\ &= \frac{|ab + ab|}{\sqrt{b^2 + a^2} \sqrt{b^2 + a^2}} = \frac{2ab}{a^2 + b^2} \\ \sin^2 \theta &= 1 - \cos^2 \theta = 1 - \frac{4a^2 b^2}{(a^2 + b^2)^2} \Rightarrow \sin \theta = \frac{a^2 - b^2}{a^2 + b^2} \end{aligned}$$

II.

1. Find the equation of the straight lines passing through the point (-10, 4) and making an angle θ with the line $x - 2y = 10$ such that $\tan \theta = 2$.

Sol: Given line is $x - 2y = 10$ ---- (1) and point (-10, 4).

$$\tan \theta = 2 \Rightarrow \cos \theta = \frac{1}{\sqrt{5}}$$

Let m be the slope of the require line. This line is passing through (-10, 4), therefore equation of the line is

$$\begin{aligned} y - 4 &= m(x + 10) = mx + 10m \\ \Rightarrow mx - y + (10m + 4) &= 0 \text{ -----(2)} \end{aligned}$$

Given θ is the angle between (1) and (2), therefore, $\cos \theta = \frac{|a_1 a_2 + b_1 b_2|}{\sqrt{a_1^2 + b_1^2} \sqrt{a_2^2 + b_2^2}}$

$$\frac{1}{\sqrt{5}} = \frac{|m + 2|}{\sqrt{1 + 4} \sqrt{m^2 + 1}}$$

Squaring

$$m^2 + 1 = (m + 2)^2 = m^2 + 4m + 4$$

$$\Rightarrow 4m + 3 = 0 \Rightarrow m = -\frac{3}{4}$$

Case (i): Co-efficient of $m^2 = 0$

\Rightarrow One of the root is ∞

Hence the line is vertical.

\therefore Equation of the vertical line passing through $(-10, 4)$ is $x + 10 = 0$

Case (ii): $m = -\frac{3}{4}$

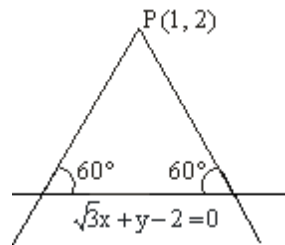
Substituting in (1)

$$\text{Equation of the line is } -\frac{3}{4}x - y + \left(-\frac{30}{4} + 4\right) = 0$$

$$\frac{-3x - 4y - 14}{4} = 0 \Rightarrow 3x + 4y + 14 = 0$$

2. Find the equation of the straight lines passing through the point $(1, 2)$ and making an angle of 60° with the line $\sqrt{3}x + y - 2 = 0$.

Sol: equation of the given line is $\sqrt{3}x + y - 2 = 0$.-----(1)



Let $P(1, 2)$. let m be the slope of the required line.

Equation of the line passing through $P(1, 2)$ and having slope m is

$$y - 2 = m(x - 1) = mx - m$$

$$mx - y + (2 - m) = 0 \quad \text{---(2)}$$

This line is making an angle of 60° with (1), therefore,

$$\cos \theta = \frac{|a_1 a_2 + b_1 b_2|}{\sqrt{a_1^2 + b_1^2} \sqrt{a_2^2 + b_2^2}} \Rightarrow \cos 60^\circ = \frac{|\sqrt{3}m - 1|}{\sqrt{3+1} \sqrt{m^2 + 1}}$$

$$\Rightarrow \frac{1}{2} = \frac{|\sqrt{3}m - 1|}{2\sqrt{m^2 + 1}}$$

$$\text{Squaring on both sides, } \Rightarrow m^2 + 1 = (\sqrt{3}m - 1)^2 = 3m^2 + 1 - 2\sqrt{3}m$$

$$\Rightarrow 2m^2 - 2\sqrt{3}m = 0 \Rightarrow 2m(m - \sqrt{3}) = 0$$

$$\Rightarrow m = 0 \text{ or } \sqrt{3}$$

Case (i): $m = 0, P(1, 2)$

