

CONTINUITY

CONTINUITY AT A POINT

Let f be a function defined in a neighbourhood of a point a . Then f is said to be continuous at the point a if and only if $\lim_{x \rightarrow a} f(x) = f(a)$.

In other words, f is continuous at a iff the limit of f at a is equal to the value of f at a .

NOTE:

1. If f is not continuous at a it is said to be discontinuous at a , and a is called a point of discontinuity of f .

2. Let f be a function defined in a nbd of a point a . Then f is said to be

(i) Left continuous at a iff $\lim_{x \rightarrow a^-} f(x) = f(a)$.

(ii) Right continuous at a iff $\lim_{x \rightarrow a^+} f(x) = f(a)$.

3. f is continuous at a iff f is both left continuous and right continuous at a

$$\text{i.e., } \lim_{x \rightarrow a} f(x) = f(a) \Leftrightarrow \lim_{x \rightarrow a^-} f(x) = f(a) = \lim_{x \rightarrow a^+} f(x)$$

CONTINUITY OF A FUNCTION OVER AN INTERVAL

I) A function f defined on (a, b) is said to be continuous (a, b) if

it is continuous at every point of (a, b) i.e., if $\lim_{x \rightarrow c} f(x) = f(c) \forall c \in (a, b)$

II) A function f defined on $[a, b]$ is said to be continuous on $[a, b]$ if

(i) f is continuous on (a, b) i.e., $\lim_{x \rightarrow c} f(x) = f(c) \forall c \in (a, b)$

(ii) f is right continuous at a i.e., $\lim_{x \rightarrow a^+} f(x) = f(a)$

(iii) f is left continuous at b i.e., $\lim_{x \rightarrow b^-} f(x) = f(b)$.

NOTE :

1. Let the functions f and g be continuous at a and $k \in \mathbb{R}$. Then $f + g$, $f - g$, kf , $kf + lg$, $f \cdot g$ are continuous at a and $\frac{f}{g}$ is continuous at a provided $g(a) \neq 0$.

2. All trigonometric functions, Inverse trigonometric functions, hyperbolic functions and inverse hyperbolic functions are continuous in their domains of definition.

3. A constant function is continuous on \mathbb{R}

4. The identity function is continuous on \mathbb{R} .

5. Every polynomial function is continuous on \mathbb{R} .

EXERCISE – 8 (e)

I.

1. **Is the function f defined by** $f(x) = \begin{cases} x^2 & \text{if } x \leq 1 \\ x & \text{if } x > 1 \end{cases}$ **Continuous on R ?**

Sol : $\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} x^2 = 1^2 = 1$

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} x = 1$$

$$\therefore \lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^+} f(x) = 1 \quad \text{and} \quad f(1) = 1^2 = 1$$

$$\lim_{x \rightarrow 1} f(x) = f(1)$$

f is continuous $x = 1$

Hence f is continuous on R.

2. **Is f defined by** $f(x) = \begin{cases} \frac{\sin 2x}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$ **Continuous at 0 ?**

Sol : $\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \frac{\sin 2x}{x} = 2$ But $f(0) = 1$

$$\therefore \lim_{x \rightarrow 0} f(x) \neq f(0)$$

Hence f is not continuous at 0

3. **Show that the function** $f(x) = [\cos(x^{10} + 1)]^{1/3}$, $x \in R$ **is a continuous functions.**

Sol : We know that $\cos x$ continuous for every $x \in R$

\therefore The given function $f(x)$ is continuous for every $x \in R$

II.

1. Check the continuity of the following function at 2 for the function

$$f(x) = \begin{cases} \frac{1}{2}(x^2 - 4) & \text{if } 0 < x < 2 \\ 2 - 8x^{-3} & \text{if } x > 2 \end{cases}$$

$$\text{Sol : } ll = \lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} \frac{1}{2}(x^2 - 4) = \frac{1}{2}(4 - 4) = 0$$

$$R.L = \lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} \left(2 - \frac{8}{x^3}\right) = 2 - \frac{8}{8} = 2 - 1 = 1$$

$$\lim_{x \rightarrow 2^-} f(x) \neq \lim_{x \rightarrow 2^+} f(x)$$

$\therefore \lim_{x \rightarrow 2} f(x)$ does not exist

Hence $f(x)$ is not continuous at 2.

2. Check the continuity of f given by $f(x) = \begin{cases} \frac{x^2 - 9}{x^2 - 2x - 3} & \text{if } 0 < x < 5 \text{ and } x \neq 3 \\ 1.5 & \text{if } x = 3 \end{cases}$ At

the point 3.

Sol : Given $f(3) = 1.5$.

$$\begin{aligned} \lim_{x \rightarrow 3} f(x) &= \lim_{x \rightarrow 3} \frac{x^2 - 9}{x^2 - 2x - 3} \\ &= \lim_{x \rightarrow 3} \frac{(x - 3)(x + 3)}{(x - 3)(x + 1)} = \frac{3 + 3}{3 + 1} = \frac{6}{4} = 1.5 = f(3) \end{aligned}$$

$\therefore f(x)$ is continuous at $x = 3$.

3. Show that f , given by $f(x) = \frac{x - |x|}{x}$ ($x \neq 0$) is continuous on $R - \{0\}$.

Sol :

$$\text{Left limit at } x=0 \text{ is } \lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} 2 = 2$$

$$\text{Right limit at } x=0 \text{ is } \lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} 0 = 0$$

$$\lim_{x \rightarrow 0^-} f(x) \neq \lim_{x \rightarrow 0^+} f(x) \quad \therefore \lim_{x \rightarrow 0} f(x) \text{ does not exist.}$$

Hence the function is not continuous at $x=0$.

When $x < 0$, $f(x) = 2$, a constant. And it is continuous for all $x < 0$.

When $x > 0$, $f(x) = 0$, which is continuous for all $x > 0$.

Hence the function is continuous on $R - \{0\}$.

4. If f is a function defined by $f(x) = \begin{cases} \frac{x-1}{\sqrt{x}-1} & \text{if } x > 1 \\ 5-3x & \text{if } -2 \leq x \leq 1 \\ \frac{6}{x-10} & \text{if } x < -2 \end{cases}$ then discuss the

continuity of f

Sol: Case (i) continuity at $x = 1$

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} \frac{x-1}{\sqrt{x}-1} = \lim_{x \rightarrow 1^+} \frac{(\sqrt{x}-1)(\sqrt{x}+1)}{\sqrt{x}-1} = \lim_{x \rightarrow 1^+} \sqrt{x-1} = \sqrt{1-1} = 0$$

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} (5-3x) = 5-3 = 2$$

$$\therefore \lim_{x \rightarrow 1^-} f(x) \neq \lim_{x \rightarrow 1^+} f(x). \quad \text{Hence } f \text{ is not continuous at } x=1$$

Case (ii) continuity at $x = -2$

$$\lim_{x \rightarrow -2^-} f(x) = \lim_{x \rightarrow -2^-} \frac{6}{x-10} = \frac{6}{-2-10} = \frac{-6}{12} = \frac{-1}{2}$$

$$\lim_{x \rightarrow -2^+} f(x) = \lim_{x \rightarrow -2^+} (5-3x) = 5-3(-2) = 5+6 = 11$$

$$\therefore \lim_{x \rightarrow -2^-} f(x) \neq \lim_{x \rightarrow -2^+} f(x)$$

Hence $f(x)$ is not continuous at $x = -2$.

- 5. If f is given by $f(x) = \begin{cases} k^2x - k & \text{if } x \geq 1 \\ 2 & \text{if } x < 1 \end{cases}$ is a continuous function on \mathbf{R} , then find the values of k .**

Sol : $\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} 2 = 2$

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} (kx^2 - k) = k^2 - k \quad \text{Given } f(x) \text{ is continuous at } x = 1$$

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^+} f(x) \quad 2 = k^2 - k$$

GIVEN f is continuous on \mathbf{R} , hence it is continuous at $x=1$.

Therefore L.L = R.L

$$\Rightarrow k^2 - k - 2 = 0$$

$$\Rightarrow (k-2)(k+1) = 0 \Rightarrow k = 2 \text{ or } -1$$

- 6. Prove that the functions ' $\sin x$ ' and ' $\cos x$ ' are continuous on \mathbf{R} .**

Sol : i) Let $a \in \mathbf{R}$

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} \sin x = \sin a = f(a)$$

$\therefore f$ is continuous at a .

ii) Let $a \in R$

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} \cos x - \cos a = f(a)$$

$\therefore f$ is continuous at a .

III.

1 Check the continuity of 'f' given by $f(x) = \begin{cases} 4 - x^2 & \text{if } x \leq 0 \\ x - 5 & \text{if } 0 < x \leq 1 \\ 4x^2 - 9 & \text{if } 1 < x < 2 \\ 3x + 4 & \text{if } x \geq 2 \end{cases}$ at the points 0,

1 and 2.

Ans: f is continuous at $x = 0, 1, 2$

2. Find real constant a, b so that the function f given by

$$f(x) = \begin{cases} \sin x & \text{if } x \leq 0 \\ x^2 + a & \text{if } 0 < x < 1 \\ bx + 3 & \text{if } 1 \leq x \leq 3 \\ -3 & \text{if } x > 3 \end{cases} \text{ is continuous on } R.$$

Sol : Given $f(x)$ is continuous on R , hence it is continuous at $0, 1, 3$.

At $x=0$.

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0} (x^2 + a) = 0 + a = a$$

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0} \sin x = 0$$

Since $f(x)$ is continuous at $x=0$,

$$\therefore L.L = R.L \rightarrow a=0.$$

At $x=3$

$$\text{R.L.} = \lim_{x \rightarrow 3^+} f(x) = \lim_{x \rightarrow 3} -3 = -3$$

$$\text{L.L.} = \lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3} (bx + 3) = 3b + 3$$

Since $f(x)$ is continuous AT $X=3$, L.L=R.L

$$\Rightarrow 3b + 3 = -3 \Rightarrow 3b = -6 \Rightarrow b = -2$$

3. Show that $f(x) = \begin{cases} \frac{\cos ax - \cos bx}{x^2} & \text{if } x \neq 0 \\ \frac{1}{2}(b^2 - a^2) & \text{if } x = 0 \end{cases}$

Where a and b are real constant, is continuous at 0.

Sol : $\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \frac{\cos ax - \cos bx}{x^2}$

$$= \lim_{x \rightarrow 0} \frac{2 \sin \frac{(a+b)x}{2} \sin \frac{(b-a)x}{2}}{x^2}$$

$$= 2 \lim_{x \rightarrow 0} \frac{\sin(a+b) \frac{x}{2}}{x} \lim_{x \rightarrow 0} \frac{\sin(b-a) \frac{x}{2}}{x}$$

$$= \frac{2(b+a)}{2} \frac{(b-a)}{2} = \frac{b^2 - a^2}{2}$$

$$\text{Given } f(0) = \frac{b^2 - a^2}{2}. \quad \therefore \lim_{x \rightarrow 0} f(x) = f(0)$$

$\therefore f(x)$ is continuous at $x = 0$

PROBLEMS FOR PRACTICE

1. If f is given by $f(x) = \begin{cases} ax - b & \text{if } x \leq -1 \\ 3x^2 - 4ax + 2b & \text{if } -1 < x < 1 \\ 10 & \text{if } x \geq 1 \end{cases}$ a continuous function is on \mathbf{R} , then find the values of a, b .

2. Check the continuity of the function f given below at 1 and at 2.

$$f(x) = \begin{cases} x + 1 & \text{if } x \leq 1 \\ 2x & \text{if } 1 < x < 2 \\ 1 + x^2 & \text{if } x \geq 2 \end{cases}$$

3. Show that $f(x) = (x)$ ($x \in \mathbf{R}$) is continuous at only those real numbers that are not integers.

Sol : Case i) if $a \in \mathbf{z}$, $f(a) = (a) = a$

$$\lim_{x \rightarrow a^-} f(x) = \lim_{h \rightarrow 0} (a - h) = a +$$

$$\lim_{x \rightarrow a^+} f(x) = \lim_{h \rightarrow 0} (a - h) = a$$

$$\therefore \lim_{x \rightarrow a^-} f(x) \neq \lim_{x \rightarrow a^+} f(x)$$

$\lim_{x \rightarrow a} f(x)$ does not exist

$\therefore f$ is not continuous at $x = a \in \mathbf{z}$.

Case ii) : if $a \notin \mathbf{z}$, then $\exists n \in \mathbf{z}$ such that $n < a < n + 1$ then $f(a) = (a) = n$.

$$\lim_{x \rightarrow a^-} f(x) = \lim_{h \rightarrow 0} (a - h) = n,$$

$$\lim_{x \rightarrow a^+} f(x) = \lim_{h \rightarrow 0} (a + h) = n,$$

$$\therefore \lim_{x \rightarrow a} f(x) = n = f(a)$$

$\therefore f(x)$ is continuous at $x = a \notin z$.

4. If $f : R \rightarrow R$ is such that $f(x + y) = f(x) + f(y)$ for all $x, y \in R$ then f is continuous on R if it is continuous at a single point in R .

Sol : Let f be continuous at $x_0 \in R$

$$\lim_{t \rightarrow x_0} f(t) = f(x_0) \quad \lim_{h \rightarrow 0} f(x_0 + h) = f(x_0) \quad x \in R, f(x_0)$$

$$\Rightarrow f(x + h) - f(x) = f(h) = f(x_0 + h) - f(x_0) \quad \lim_{h \rightarrow 0} \{f(x + h) - f(x)\}$$

$$= \lim_{h \rightarrow 0} \{f(x_0 + h) - f(x_0)\} = 0$$

$\therefore f$ is continuous at x .

Since $x \in R$ is arbitrary, f is continuous on R .