

PAIR OF LINES-SECOND DEGREE GENERAL EQUATION

THEOREM

If the equation $S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of straight lines then

$$\text{i) } \Delta \equiv abc + 2fgh - af^2 - bg^2 - ch^2 = 0 \quad \text{and (ii) } h^2 \geq ab, g^2 \geq ac, f^2 \geq bc$$

Proof:

Let the equation $S = 0$ represent the two lines $l_1x + m_1y + n_1 = 0$ and $l_2x + m_2y + n_2 = 0$.

Then

$$\begin{aligned} & ax^2 + 2hxy + by^2 + 2gx + 2fy + c \\ & \equiv (l_1x + m_1y + n_1)(l_2x + m_2y + n_2) = 0 \end{aligned}$$

Equating the co-efficients of like terms, we get

$$l_1l_2 = a, l_1m_2 + l_2m_1 = 2h, m_1m_2 = b, \text{ and } l_1n_2 + l_2n_1 = 2g, m_1n_2 + m_2n_1 = 2f, n_1n_2 = c$$

(i) Consider the product $(2h)(2g)(2f)$

$$\begin{aligned} & = (l_1m_2 + l_2m_1)(l_1n_2 + l_2n_1)(m_1n_2 + m_2n_1) \\ & = l_1l_2(m_1^2n_2^2 + m_2^2n_1^2) + m_1m_2(l_1^2n_2^2 + l_2^2n_1^2) + n_1n_2(l_1^2m_2^2 + l_2^2m_1^2) + 2l_1l_2m_1m_2n_1n_2 \\ & = l_1l_2[(m_1n_2 + m_2n_1)^2 - 2m_1m_2n_1n_2] + m_1m_2[(l_1n_2 + l_2n_1)^2 - 2l_1l_2n_1n_2] \\ & \quad + n_1n_2[(l_1m_2 + l_2m_1)^2 - 2l_1l_2m_1m_2] + 2l_1l_2m_1m_2n_1n_2 \\ & = a(4f^2 - 2bc) + b(4g^2 - 2ac) + c(4h^2 - 2ab) \\ & \quad 8fgh = 4[af^2 + bg^2 + ch^2 - abc] \\ & \therefore abc + 2fgh - af^2 - bg^2 - ch^2 = 0 \end{aligned}$$

$$\begin{aligned} \text{ii) } h^2 - ab & = \left(\frac{l_1m_2 + l_2m_1}{2} \right)^2 - l_1l_2m_1m_2 = \frac{(l_1m_2 + l_2m_1)^2 - 4l_1l_2m_1m_2}{4} \\ & = \frac{(l_1m_2 - l_2m_1)^2}{4} \geq 0 \end{aligned}$$

Similarly we can prove $g^2 \geq ac$ and $f^2 \geq bc$

NOTE :

If $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = 0$, $h^2 \geq ab$, $g^2 \geq ac$ and $f^2 \geq bc$, then the equation $S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of straight lines

CONDITIONS FOR PARALLEL LINES-DISTANCE BETWEEN THEM

THEOREM

If $S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of parallel lines then $h^2 = ab$ and $bg^2 = af^2$. Also the distance between the two parallel lines is

$$2\sqrt{\frac{g^2 - ac}{a(a+b)}} \quad (\text{or}) \quad 2\sqrt{\frac{f^2 - bc}{b(a+b)}}$$

Proof :

Let the parallel lines represented by $S = 0$ be
 $lx + my + n_1 = 0$ -- (1) $lx + my + n_2 = 0$ -- (2)

$$\therefore ax^2 + 2hxy + 2gx + 2fy + c$$

$$\equiv (lx + my + n_1)(lx + my + n_2)$$

Equating the like terms

$$l^2 = a \quad \text{-- (3)} \quad 2lm = 2h \quad \text{-- (4)}$$

$$m^2 = b \quad \text{-- (5)} \quad l(n_1 + n_2) = 2g \quad \text{-- (6)}$$

$$m(n_1 + n_2) = 2f \quad \text{-- (7)} \quad n_1n_2 = c \quad \text{-- (8)}$$

From (3) and (5), $l^2m^2 = ab$ and from (4) $h^2 = ab$.

$$\text{Dividing (6) and (7)} \quad \frac{l}{m} = \frac{g}{f} \Rightarrow \frac{l^2}{m^2} = \frac{g^2}{f^2},$$

$$\therefore \frac{a}{b} = \frac{g^2}{f^2} \Rightarrow bg^2 = af^2$$

Distance between the parallel lines (1) and (2) is

$$\begin{aligned} &= \left| \frac{n_1 - n_2}{\sqrt{l^2 + m^2}} \right| = \frac{\sqrt{(n_1 + n_2)^2 - 4n_1n_2}}{\sqrt{l^2 + m^2}} \\ &= \frac{\sqrt{(4g^2 / l^2) - 4c}}{\sqrt{a+b}} \quad \text{or} \quad \frac{\sqrt{(4f^2 / m^2) - 4c}}{\sqrt{a+b}} \\ &= 2\sqrt{\frac{g^2 - ac}{a(a+b)}} \quad (\text{or}) \quad 2\sqrt{\frac{f^2 - bc}{b(a+b)}} \end{aligned}$$

POINT OF INTERSECTION OF PAIR OF LINES THEOREM

The point of intersection of the pair of lines represented by

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \text{ when } h^2 > ab \text{ is } \left(\frac{hf - bg}{ab - h^2}, \frac{gh - af}{ab - h^2} \right)$$

Proof:

Let the point of intersection of the given pair of lines be (x_1, y_1) . Transfer the origin to (x_1, y_1) without changing the direction of the axes.

Let (X, Y) represent the new coordinates of (x, y) . Then $x = X + x_1$ and $y = Y + y_1$.

Now the given equation referred to new axes will be

$$\begin{aligned} a(X + x_1)^2 + 2h(X + x_1)(Y + y_1) + b(Y + y_1)^2 + 2g(X + x_1) + 2f(Y + y_1) + c &= 0 \\ \Rightarrow aX^2 + 2hXY + bY^2 + 2X(ax_1 + hy_1 + g) + 2Y(hx_1 + by_1 + f) \\ + (ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c) &= 0 \end{aligned}$$

Since this equation represents a pair of lines passing through the origin it should be a homogeneous second degree equation in X and Y . Hence the first degree terms and the constant term must be zero. Therefore,

$$ax_1 + hy_1 + g = 0 \quad \text{-- (1)}$$

$$hx_1 + by_1 + f = 0 \quad \text{-- (2)}$$

$$ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c = 0 \quad \text{-- (3)}$$

But (3) can be rearranged as

$$x_1(ax_1 + hy_1 + g) + y_1(hx_1 + by_1 + f) + (gx_1 + fy_1 + c) = 0$$

$$\Rightarrow gx_1 + fy_1 + c = 0 \quad \text{--(4)}$$

Solving (1) and (2) for x_1 and y_1

$$\frac{x_1}{hf - bg} = \frac{y_1}{gh - af} = \frac{1}{ab - h^2}$$

$$\therefore x_1 = \frac{hf - bg}{ab - h^2} \text{ and } y_1 = \frac{gh - af}{ab - h^2}$$

Hence the point of intersection of the given pair of lines is $\left(\frac{hf - bg}{ab - h^2}, \frac{gh - af}{ab - h^2} \right)$

THEOREM

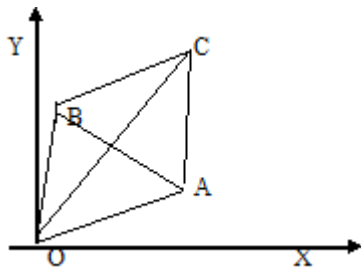
If the pair of lines $ax^2 + 2hxy + by^2 = 0$ and the pair of lines

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ form a rhombus then $(a-b)fg + h(f^2 - g^2) = 0$.

Proof:

The pair of lines $ax^2 + 2hxy + by^2 = 0$ -- (1) is parallel to the lines

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ -- (2)



Now the equation

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c + \lambda(ax^2 + 2hxy + by^2) = 0$$

Represents a curve passing through the points of intersection of (1) and (2).

Substituting $\lambda = -1$, in (3) we obtain $2gx + 2fy + c = 0$... (4) Equation (4) is a straight line passing through A and B and it is the diagonal \overline{AB}

The point of intersection of (2) is $C = \left(\frac{hf - bg}{ab - h^2}, \frac{gh - af}{ab - h^2} \right)$

$$\Rightarrow \text{Slope of } \overline{OC} = \frac{gh - af}{hf - bg}$$

In a rhombus the diagonals are perpendicular $\Rightarrow (\text{Slope of } \overline{OC})(\text{Slope of } \overline{AB}) = -1$

$$\Rightarrow \left(\frac{gh - af}{hf - bg} \right) \left(-\frac{g}{f} \right) = -1$$

$$\Rightarrow g^2h - afg = hf^2 - bfg$$

$$\Rightarrow (a-b)fg + h(f^2 - g^2) = 0$$

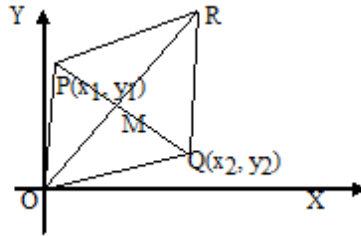
$$\frac{g^2 - f^2}{a-b} = \frac{fg}{h}$$

THEOREM

If $ax^2 + 2hxy + by^2 = 0$ be two sides of a parallelogram and $px + qy = 1$ is one diagonal, then the other diagonal is $y(bp - hq) = x(aq - hp)$

proof:

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be the points where the diagonal



$px + qy = 1$ meets the pair of lines.

\overline{OR} and \overline{PQ} bisect each other at $M(\alpha, \beta)$.

$$\therefore \alpha = \frac{x_1 + x_2}{2} \text{ and } \beta = \frac{y_1 + y_2}{2}$$

$$\text{Eliminating } y \text{ from } ax^2 + 2hxy + by^2 = 0 \quad \text{-- (1)}$$

$$\text{and } px + qy = 1 \quad \text{-- (2)}$$

$$ax^2 + 2hx\left(\frac{1-px}{q}\right) + b\left(\frac{1-px}{q}\right)^2 = 0$$

$$\Rightarrow x^2(aq^2 - 2hpq + bp^2) + 2x(hp - bp) + b = 0$$

The roots of this quadratic equation are x_1 and x_2 where

$$x_1 + x_2 = -\frac{2(hq - bp)}{aq^2 - 2hpq - bp^2}$$

$$\Rightarrow \alpha = \frac{(bp - hq)}{(aq^2 - 2hpq + bp^2)}$$

Similarly by eliminating x from (1) and (2) a quadratic equation in y is obtained and $y_1,$

y_2 are its roots where

$$y_1 + y_2 = -\frac{2(hp - aq)}{aq^2 - 2hpq - bp^2} \Rightarrow \beta = \frac{(aq - hp)}{(aq^2 - 2hpq + bp^2)}$$

