CHAPTER 4

PAIR OF STRAIGHT LINES

TOPICS:

1. Equation of a pair of lines passing through the origin

2.Angle between pair of lines

3.Bisectors of the angles between two lines.

4. pair of bisectors of angles between the pair of lines.

5.Equation of pair of lines passing through given point and parallel/perpendicular to the given pair of lines.

6.Condition for perpendicular and coincident lines

7. Area of the triangle formed by give pair of lines and a line.

8.pair of lines-second degree general equation

9. Conditions for parallel lines-distance between them.

10. Point of intersection of the pair of lines.

11.Homogenising a second degree equation w.r.t a 1st degree equation in x and y.

PAIR OF STRAIGHT LINES

Let $L_1=0$, $L_2=0$ be the equations of two straight lines. If $P(x_1,y_1)$ is a point on L_1 then it satisfies the equation $L_1=0$. Similarly, if $P(x_1,y_1)$ is a point on $L_2 = 0$ then it satisfies the equation.

If $P(x_1,y_1)$ lies on L_1 or L_2, then $P(x_1,y_1)$ satisfies the equation $L_1L_2= 0$.

$\therefore$ $L_1L_2= 0$ represents the pair of straight lines $L_1 = 0$ and $L_2 = 0$ and the joint equation of $L_1 = 0$ and $L_2 = 0$ is given by $L_1 . L_2 = 0$.-----(1)

On expanding equation (1) we get and equation of the form $ax^2+2hxy+by^2+2gx+2fy+c=0$ which is a second degree (non - homogeneous) equation in x and y.

Definition: If a, b, h are not all zero,then $ax^2+2hxy+by^2=0$ is the general form of a second degree homogeneous equation in x and y.

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THEOREM

If a, b, h are not all zero and $h^2 \geq ab$ then $ax^2+2hxy+by^2=0$ represents a pair of straight lines passing through the origin.

Proof:

Case (i) : Suppose a = 0.

Given equation $ax^2+2hxy+by^2=0$ reduces to $2hxy+by^2=0 \Rightarrow y(2hx+by)=0$.

$\therefore$ Given equation represents two straight lines

y = 0 -- (1) and $2hx+by=0$ -- (2) which pass through the origin.

Case (ii): Suppose $a \neq 0$.

Given equation $ax^2+2hxy+by^2=0$

$\Rightarrow a^2x^2+2ahxy+aby^2=0$

$\Rightarrow (ax)^2+2(ax)(hy)+(hy)^2-(h^2-ab)y^2=0$

$\Rightarrow (ax+hy)^2-(y\sqrt{h^2-ab})^2=0$

$\left[ax+y\left(h+\sqrt{h^2-ab}\right)\right]\left[ax+y\left(h-\sqrt{h^2-ab}\right)\right]=0$

$\therefore$Given equation represents the two lines

$ax+hy+y\sqrt{h^2-ab}=0$, $ax+hy-y\sqrt{h^2-ab}=0$ which pass through the origin.

Note 1: If $h^2 > ab$, the two lines are distinct.

Note 2: If $h^2 = ab$, the two lines are coincident.

Note 3: If $h^2 < ab$, the two lines are not real but intersect at a real point (the origin).

Note 4: If the two lines represented by $ax^2+2hxy+by^2=0$ are taken as $l_1x+m_1y=0$ and $l_2x+m_2y=0$ then

$ax^2+2hxy+by^2 \equiv (l_1x+m_1y)(l_2x+m_2y) \equiv \ell_1\ell_2x^2+(\ell_1m_2+\ell_2m_1)xy+m_1m_2y^2$

Equating the co – efficients of x^2, xy and y^2 on both sides, we get $l_1l_2 = a$, $l_1m_2 + l_2m_1 = 2h$, $m_1m_2 = b$.

THEOREM

If $ax^2 + 2hxy + by^2 = 0$ represent a pair of straight lines, then the sum of slopes of lines is $\frac{-2h}{b}$ and product of the slopes is $\frac{a}{b}$.

Proof:

Let $ax^2 + 2hxy + by^2 = 0$ represent the lines $l_1x + m_1y = 0$ -- (1) and $l_2x + m_2y = 0$ -- (2). Then $l_1l_2 = a$, $l_1m_2 + l_2m_1 = 2h$, $m_1m_2 = b$.

Slopes of the lines (1) and (2) are $-\frac{l_1}{m_1}$ and $-\frac{l_2}{m_2}$.

sum of the slopes = $\frac{-l_1}{m_1} + \frac{-l_2}{m_2} = -\frac{l_1m_2 + l_2m_1}{m_1m_2} = -\frac{2h}{b}$

Product of the slopes = $\left(\frac{-l_1}{m_1}\right)\left(-\frac{l_2}{m_2}\right) = \frac{l_1l_2}{m_1m_2} = \frac{a}{b}$

ANGLE BETWEEN A PAIR OF LINES

THEOREM :

If θ is the angle between the lines represented by $ax^2 + 2hxy + by^2 = 0$, then $\cos\theta = \pm\frac{a+b}{\sqrt{(a-b)^2 + 4h^2}}$

Proof:

Let $ax^2 + 2hxy + by^2 = 0$ represent the lines $l_1x + m_1y = 0$ -- (1) and $l_2x + m_2y = 0$ -- (2).

Then $l_1l_2 = a$, $l_1m_2 + l_2m_1 = 2h$, $m_1m_2 = b$.

Let θ be the angle between the lines (1) and (2). Then $\cos\theta = \pm\frac{l_1l_2 + m_1m_2}{\sqrt{\left(l_1^2 + m_1^2\right)\left(l_2^2 + m_2^2\right)}}$

$$= \pm\frac{l_1l_2 + m_1m_2}{\sqrt{l_1^2l_2^2 + m_1^2m_2^2 + l_1^2m_2^2 + l_2^2m_1^2}}$$

$$= \pm\frac{l_1l_2 + m_1m_2}{\sqrt{(l_1l_2 - m_1m_2)^2 + 2l_1l_2m_1m_2 + (l_1m_2 + l_2m_1)^2 - 2l_1m_2l_2m_1}}$$

$$= \pm\frac{a+b}{\sqrt{(a-b)^2 + 4h^2}}$$

Note 1: If θ is the accute angle between the lines $ax^2 + 2hxy + by^2 = 0$ then $\cos\theta = \frac{|a+b|}{\sqrt{(a-b)^2 + 4h^2}}$

Note 2: If θ is the accute angle between the lines $ax^2+2hxy+by^2=0$ then $\tan\theta=\pm\frac{2\sqrt{h^2-ab}}{a+b}$ and

$\sin\theta=\frac{2\sqrt{h^2-ab}}{\sqrt{(a-b)^2+4h^2}}$

CONDITIONS FOR PERPENDICULAR AND COINCIDENT LINES

1.If the lines $ax^2+2hxy+by^2=0$ are perpendicular to each other then $\boldsymbol{\theta=\pi/2}$ and $\cos\boldsymbol{\theta=0}\Rightarrow a+b=0$.

i.e., co-efficient of x^2 + coefficient of $y^2 = 0$.

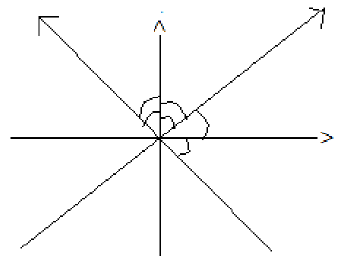
2.If the two lines are parallel to each other then $\theta=0$.

$\Rightarrow$ The two lines are coincident $\Rightarrow h^2=ab$.

BISECTORS OF ANGLES.

THEOREM

The equations of bisectors of angles between the lines $\boldsymbol{a_1x+b_1y+c_1=0, a_2x+b_2y+c_2=0}$



are $\boldsymbol{\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}}=\pm\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}}$.

PAIR OF BISECTORS OF ANGLES.

The equation to the pair bisectors of the angle between the pair of lines $\boldsymbol{ax^2+2hxy+by^2=0}$ **is**

$\boldsymbol{h(x^2-y^2)=(a-b)xy}$ (or) $\boldsymbol{\frac{x^2-y^2}{a-b}=\frac{xy}{h}}$.

Proof:

Let $ax^2+2hxy+by^2=0$ represent the lines $l_1x+m_1y=0$ -- (1) and $l_2x+m_2y=0$ -- (2).

Then $l_1l_2=a$, $l_1m_2+l_2m_1=2h$, $m_1m_2 = b$.

The equations of bisectors of angles between (1) and (2) are $\frac{l_1x+m_1y}{\sqrt{l_1^2+m_1^2}}-\frac{l_2x+m_2y}{\sqrt{l_2^2+m_2^2}}=0$ AND

$$\frac{l_1x+m_1y}{\sqrt{l_1^2+m_1^2}}+\frac{l_2x+m_2y}{\sqrt{l_2^2+m_2^2}}=0$$

The combined equation of the bisectors is

$$\left(\frac{l_1x+m_1y}{\sqrt{l_1^2+m_1^2}}-\frac{l_2x+m_2y}{\sqrt{l_2^2+m_2^2}}\right)\left(\frac{\ell_1x+m_1y}{\sqrt{\ell_1^2+m_1^2}}+\frac{\ell_2x+m_2y}{\sqrt{\ell_2^2+m_2^2}}\right)=0$$

$$\Rightarrow\left(\frac{l_1x+m_1y}{\sqrt{l_1^2+m_1^2}}\right)^2-\left(\frac{l_2x+m_2y}{\sqrt{l_2^2+m_2^2}}\right)^2=0$$

$$\Rightarrow\left(l_2^2+m_2^2\right)(l_1x+m_1y)^2-\left(l_1^2+m_1^2\right)(l_2x+m_2y)^2=0$$

$$\Rightarrow x^2\left[l_1^2\left(l_2^2+m_2^2\right)-l_2^2\left(l_1^2+m_1^2\right)\right]y^2\left[m_2^2\left(l_1^2+m_1^2\right)-m_1^2\left(l_2^2+m_2^2\right)\right]-2xy\left[l_2m_2\left(l_1^2+m_1^2\right)-l_1m_1\left(l_2^2+m_2^2\right)\right]=0$$

$$\Rightarrow x^2\left(l_1^2l_2^2+l_1^2m_2^2-l_1^2l_2^2-l_2^2m_1^2\right)-y^2\left(l_1^2m_2^2+m_1^2m_2^2-m_1^2l_2^2-m_1^2m_2^2\right)-2xy\left(l_2m_2l_1^2+l_2m_2m_1^2-l_1m_1l_2^2-l_1m_1m_2^2\right)=0$$

$$\Rightarrow x^2\left(l_1^2m_2^2-l_2^2m_1^2\right)-y^2\left(l_1^2m_2^2-l_2^2m_2^2\right)\qquad=2xy[l_1l_2(l_1m_2-l_2m_1)-m_1m_2(l_1m_2-l_2m_1)]$$

$$\Rightarrow(x^2-y^2)\left(l_1^2m_2^2-l_2^2m_1^2\right)=2xy\left(l_1l_2-m_1m_2\right)\left(l_1m_2-l_2m_1\right)$$

$$\Rightarrow(x^2-y^2)(l_1m_2+l_2m_1)=2xy(l_1l_2-m_1m_2)$$

$$\Rightarrow 2h(x^2-y^2)=2xy(a-b)$$

$$\therefore h(x^2-y^2)=(a-b)xy\qquad\text{OR }\frac{x^2-y^2}{a-b}=\frac{xy}{h}$$

THEOREM

The equation to the pair of lines passing through (x_0,y_0) and parallel $ax^2 + 2hxy + by^2 = 0$ is $\boldsymbol{a(x-x_0)^2+2h(x-x_0)(y-y_0)+b(y-y_0)^2=0}$

Proof :

Let $ax^2+2hxy+by^2=0$ represent the lines $l_1x+m_1y=0$ -- (1) and $l_2x+m_2y=0$ -- (2).
Then $l_1l_2=a$, $l_1m_2+l_2m_1=2h$, $m_1m_2 = b$.

The equation of line parallel to (1) and passing through (x_0,y_0) is

$l_1(x-x_0)+m_1(y-y_0)=0$ -- (3)

The equation of line parallel to (2) and passing through (x_0,y_0) is $l_2(x-x_0)+m_2(y-y_0)=0$ -- (4)

The combined equation of (3), (4) is

$$[l_1(x-x_0)+m_1(y-y_0)][l_2(x-x_0)+m_2(y-y_0)]=0$$

$$\Rightarrow l_1l_2(x-x_0)^2+(l_1m_2+l_2m_1)(x-x_0)(y-y_0)+m_1m_2(y-y_0)^2=0$$

$$\Rightarrow a(x-x_0)^2+2h(x-x_0)(y-y_0)+b(y-y_0)^2=0$$

THEOREM

The equation to the pair of lines passing through the origin and perpendicular to $ax^2+2hxy+by^2=0$ **is** $bx^2-2hxy+ay^2=0$ **.**

Let $ax^2+2hxy+by^2=0$ represent the lines $l_1x+m_1y=0$ -- (1) and $l_2x+m_2y=0$ -- (2).
Then $l_1l_2=a$, $l_1m_2+l_2m_1=2h$, m_1m_2 = b.

The equation of the line perpendicular to (1) and passing through the origin is $m_1x-l_1y=0$ -- (3)

The equation of the line perpendicular to (2) and passing through the origin is $m_2x-l_2y=0$ -- (4)

The combined equation of (3) and (4) is

$$(m_1x-l_1y)(m_2x-l_2y)=0$$

$$\Rightarrow m_1m_2x^2-(l_1m_2+l_2m_1)ny+l_1l_2y^2=0$$

$$\Rightarrow bx^2-2hxy+ay^2=0$$

THEOREM

The equation to the lines passing through (x_0,y_0) **and perpendiculars to ax^2 + 2hxy + by^2 = 0 is** $b(x-x_0)^2-2h(x-x_0)(y-y_0)+a(y-y_0)^2=0$**.**

Try yourself.

AREA OF THE TRIANGLE.

THEOREM

The area of triangle formed by the lines ax^2 + 2hxy + by^2 = 0and $\boldsymbol{lx+my+n=0}$ is $\dfrac{n^2\sqrt{h^2-ab}}{\left|am^2-2h\ell m+b\ell^2\right|}$

Let $ax^2+2hxy+by^2=0$ represent the lines $l_1x+m_1y=0$ -- (1) and $l_2x+m_2y=0$ -- (2).
Then $l_1l_2=a$, $l_1m_2+l_2m_1=2h$, m_1m_2 = b.

The given straight line is $\boldsymbol{lx+my+n=0}$ -- (3) Clearly (1) and (2) intersect at the origin.

Let A be the point of intersection of (1) and (3). Then

x	y	1	
m_1	0		m_1
m	n	l	m

$$\Rightarrow \frac{x}{m_1n-0}=\frac{y}{0-nl_1}=\frac{1}{l_1m-lm_1}$$

$$\Rightarrow x=\frac{m_1n}{l_1m-lm_1} \text{ AND } y=\frac{-nl_1}{l_1m-lm_1}$$

$$\therefore A=\left(\frac{m_1 n}{l_1 m-l m_1},\frac{-l_1 n}{l_1 m-l m_1}\right)=(x_1,y_1)$$

$$B=\left(\frac{m_2 n}{l_2 m-l m_2},\frac{-l_2 n}{l_2 m-l m_2}\right)=(x_2,y_2)$$

$$\therefore \text{The area of } \Delta OAB=\frac{1}{2}\left|x_1 y_2-x_2 y_1\right|$$

$$=\frac{1}{2}\left|\left(\frac{m_1 n}{l_1 m-l m_1}\right)\left(\frac{-l_2 n}{l_2 m-l m_2}\right)-\left(\frac{m_2 n}{l_2 m-l m_2}\right)\left(\frac{-n l_1}{l_1 m-l m_1}\right)\right|$$

$$\frac{1}{2}\left|\frac{l_1 m_2 n^2-l_2 m_1 n^2}{(l_1 m-l m_1)(l_2 m-l m_2)}\right|$$

$$\frac{n^2}{2}\left|\frac{(l_1 m_2-l_2 m_1)}{l_1 l_2 m^2-(l_1 m_2+l_2 m_1)lm+m_1 m_2 l^2}\right|$$

$$\frac{n^2}{2}\left|\frac{\sqrt{(l_1 m_2+l_2 m_1)^2-4 l_1 m_2 l_2 m_1}}{am^2-2hlm+bl^2}\right|$$

$$=\frac{n^2}{2}\frac{\sqrt{4h^2-4ab}}{\left|am^2-2hlm+bl^2\right|}$$

$$=\frac{n^2\sqrt{h^2-ab}}{\left|am^2-2hlm+bl^2\right|}$$

THEOREM

The product of the perpendiculars from (α, β) to the pair of lines $ax^2+2hxy+by^2=0$ **is**

$$\frac{\left|a\alpha^2+2h\alpha\beta+b\beta^2\right|}{\sqrt{(a-b)^2+4h^2}}$$

Proof:

Let $ax^2+2hxy+by^2=0$ represent the lines $l_1 x+m_1 y=0$ -- (1) and $l_2 x+m_2 y=0$ -- (2).

Then $l_1 l_2=a$, $l_1 m_2+l_2 m_1=2h$, $m_1 m_2 = b$.

The lengths of perpendiculars from (α, β) to

the line (1) is p= $=\frac{\left|l_1\alpha+m_1\beta\right|}{\sqrt{l_1^2+m_1^2}}$

and to the line (2) is q= $=\frac{\left|l_2\alpha+m_2\beta\right|}{\sqrt{l_2^2+m_2^2}}$

$\therefore$ The product of perpendiculars is

$$pq = \left|\frac{l_1\alpha + m_1\beta}{\sqrt{l_1^2 + m_1^2}}\right| \cdot \left|\frac{l_2\alpha + m_2\beta}{\sqrt{l_2^2 + m_2^2}}\right|$$

$$= \left|\frac{l_1 l_2\alpha^2 + (l_1 m_2 + l_2 m_1)\alpha\beta \quad + m_1 m_2\beta^2}{\sqrt{l_1^2 l_2^2 + l_1^2 m_2^2 + l_2^2 \; m_1^2 + m_1^2 m_2^2}}\right|$$

$$= \frac{\left|a\alpha^2 + 2h\alpha\beta + b\beta^2\right|}{\sqrt{(l_1 l_2 - m_1 m_2)^2 + 2l_1 l_2 m_1 m_2 + (l_1 m_2 + l_2 m_1)^2 - 2l_1 m_2 l_2 m_1}} = \frac{\left|a\alpha^2 + 2h\alpha\beta + b\beta^2\right|}{\sqrt{(a-b)^2 + 4h^2}}$$

EXERCISE 4A

I

1. Find the acute angle between the pair of line represented by the following equations.

i) $x^2 - 7xy + 12y^2 = 0$ **ii)** $y^2 - xy - 6x^2 = 0$ **iii)** $(x\cos\alpha - y\sin\alpha)^2 = (x^2 + y^2)\sin^2\alpha$

iv) $x^2 + 2xy\cot\alpha - y^2 = 0$

Sol. i) Given eq is $x^2 - 7xy + 12y^2 = 0$**.** Comparing with $ax^2 + 2hxy + by^2 = 0$

$$a = 1,\ b = 12,\ h = -\frac{7}{2}$$

Let θ be the angle between the lines, then $\tan\theta = \frac{2\sqrt{h^2 - ab}}{a+b} = \frac{2\sqrt{\frac{49}{4} - 12}}{1+12} = \frac{2\sqrt{\frac{1}{4}}}{13} = \frac{\sqrt{1}}{13}$

$\tan\theta = \frac{1}{13} \Rightarrow \theta = \tan^{-1}\left(\frac{1}{13}\right)$

ii) $y^2 - xy - x^2 = 0$ **ans** $\therefore \theta = \frac{\pi}{4}$

iii) $(x\cos\alpha - y\sin\alpha)^2 = (x^2 + y^2)\sin^2\alpha$

$x^2\cos^2\alpha + y^2\sin^2\alpha - 2xy\cos\alpha\sin\alpha = x^2\sin^2\alpha + y^2\sin^2\alpha$

$\therefore x^2(\cos^2\alpha - \sin^2\alpha) - 2xy\cos\alpha\sin\alpha = 0$

$x^2.\cos 2\alpha - xy\sin 2\alpha = 0 \Rightarrow a = \cos 2\alpha, b = 0, 2h = -\sin 2\alpha$

Let θ be the angle between the lines, then

$$\cos\theta = \frac{|\cos 2\alpha + 0|}{\sqrt{(\cos 2\alpha - 0)^2 + \sin^2 2\alpha}} = \cos 2\alpha \qquad \therefore \theta = 2\alpha$$

iv) $x^2 + 2xy\cot\alpha - y^2 = 0$

Coefficient of x^2 + coefficient of y^2 = $a + b = 1 - 1 = 0$ $\therefore \theta = \frac{\pi}{2}$

II

1. Show that the following pairs of straight lines have the same set of angular bisector (that, is they are equally inclined to each other).

i) $2x^2 + 6xy + y^2 = 0, 4x^2 + 18xy + y^2 = 0$

ii) $a^2x^2 + 2h(a+b)xy + b^2y^2 = 0$, $ax^2 + 2hxy + by^2 = 0, a + b \neq 0$

iii) $ax^2 + 2hxy + by^2 + \lambda(x^2 + y^2) = 0; (\lambda \in R)$,

$ax^2 + 2hxy + by^2 = 0$

Sol. i) equation of 1st pair of lines is $2x^2 + 6xy + y^2 = 0$

Equation of the pair of bisectors is $3(x^2 - y^2) = (2-1)xy$

$3(x^2 - y^2) = xy$ ………(1)

Equation of 2nd pair of lines is is $4x^2 + 18xy + y^2 = 0$

Equation of the pair of bisector is $9(x^2 - y^2) = (4-1)xy$

$9(x^2 - y^2) = 3xy \Rightarrow \quad 3(x^2 - y^2) = xy$ ……….(2)

(1), (2) are some

Given pairs have same angular bisectors. Hence they are inclined to each other.

ii) and iii) same as above .

2. Find the value of h, if the slopes of the lines represented by $6x^2 + 2hxy + y^2 = 0$ are in the ratio 1 : 2.

Sol. Combined equation of the pair of lines is $6x^2 + 2hxy + y^2 = 0$

Let $y = m_1x$ and $y = m_2x$ be the lines represented by $6x^2 + 2hxy + y^2 = 0$

$\therefore m_1 + m_2 = \frac{-2h}{6} = -\frac{h}{3}, \quad m_1m_2 = \frac{1}{6}$

Given $\frac{m_1}{m_2} = \frac{1}{2} \Rightarrow m_2 = 2m_1$

$\therefore\ m_1 + 2m_1 = -\frac{h}{3} \Rightarrow 3m_1 = -\frac{h}{3}; \quad 2m_1^2 = \frac{1}{6}$

$m_1 = -\frac{h}{9}; m_1^2 = \frac{1}{12} \quad \Rightarrow \quad \left(-\frac{h}{9}\right)^2 = \frac{1}{12} \Rightarrow \frac{h^2}{81} = \frac{1}{12}$

$h^2 = \frac{81}{12} = \frac{27}{4} \quad \Rightarrow h = \pm\sqrt{\frac{27}{4}} = \pm\frac{3\sqrt{3}}{2}$

3. **If** $ax^2 + 2hxy + by^2 = 0$ **represents two straight lines such that the slope of one line is twice the slope of the other, prove that** $\mathbf{8h^2 = 9ab}$**.**

Sol. Equation of the pair of lines is $ax^2 + 2hxy + by^2 = 0$

Let $y = m_1 x$ and $y = m_2 x$ be the lines represented by $ax^2 + 2hxy + by^2 = 0$

$\therefore m_1 + m_2 = -\frac{2h}{b}, m_1 m_2 = \frac{a}{b}$

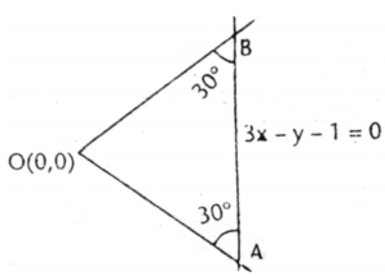
Given $m_2 = 2m_1 \quad \therefore m_1 + 2m_1 = -\frac{2h}{b}, \quad m_1.2m_1 = \frac{a}{b}$

$\therefore 3m_1 = -\frac{2h}{b}; \qquad 2m_1^2 = \frac{a}{b}$

$m_1 = -\frac{2h}{3};$ and $m_1^2 = \frac{a}{2b}$

$\therefore \left(-\frac{2h}{3b}\right)^2 = \frac{a}{2b} \Rightarrow \frac{4h^2}{9b^2} = \frac{a}{2b} \Rightarrow 8h^2 = 9ab$

4. **Show that the equation of the pair of straight lines passing through the origin and making an angle of 30º with the line 3x – y – 1 = 0 is** $13x^2 + 12xy - 3y^2 = 0$.



Sol. **let** the Equation of AB be 3x – y – 1 = 0.

Let OA, OB be the lines which are making angles of 30º with AB and passing through the origin.

let slope of OA be m ,then equation of OA is y – 0 = m(x – 0) or mx – y =0

$$\cos\underline{|OAB} = \frac{|3m+1|}{\sqrt{9+1}\sqrt{m^2+1}} \Rightarrow \cos\underline{|OAB} = \cos 30° = \frac{\sqrt{3}}{2}$$

$$\therefore \frac{\sqrt{3}}{2} = \frac{|3m+1|}{\sqrt{10}\sqrt{m^2+1}}$$

Squaring and cross multiplying

$$\frac{3(m^2+1)}{4} = \frac{(3m+1)^2}{10} \Rightarrow 15(m^2+1) = 2(3m+1)^2$$

$$\Rightarrow 15m^2 + 15 = 2(9x^2 + 6m + 1) = 18m^2 + 12m + 2$$

$$\Rightarrow 3m^2 + 12m - 13 = 0$$

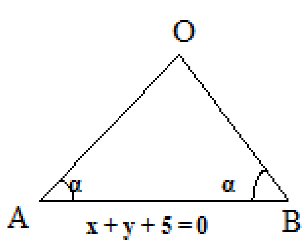
If m_1, m_2 are two roots of the equation $m_1 + m_2 = -4$ and $m_1 m_2 = \frac{-13}{3}$

Combined equation of OA and OB is

$$(m_1x - y)(m_2x - y) = 0 \Rightarrow m_1m_2x^2 - (m_1 + m_2)xy + y^2 = 0$$

$$\Rightarrow \frac{-13}{3}x^2 + 4xy + y^2 = 0 \qquad \Rightarrow -13x^2 + 12xy + 3y^2 = 0 \Rightarrow \text{ or } 13x^2 - 12xy - 3y^2 = 0$$

5. Find the equation to the pair of straight lines passing through the origin and making an acute angle α with the straight line x + y + 5 = 0.



Sol. Let **the e**quation of line AB be x + y + 5 = 0 ……..(1)

let OA and OB be the required lines making angles α with OA

let the equation of OA be y=mx $\Rightarrow$ mx-y=0

$$\Rightarrow \cos\alpha = \frac{|a_1a_2 + b_1b_2|}{\sqrt{a_1^2+b_1^2}\sqrt{a_2^2+b_2^2}} = \frac{|m-1|}{\sqrt{2}\sqrt{m^2+1}}$$

$$\Rightarrow 2\left(m^2+1\right)\cos^2\alpha = (m-1)^2$$

$$\Rightarrow 2\left(m^2+1\right) = \frac{(m-1)^2}{\cos^2\alpha} = (m-1)^2\sec^2\alpha$$

$$\Rightarrow 2m^2+2 = m^2\sec^2\alpha - 2m\sec^2\alpha + \sec^2\alpha$$

$$\Rightarrow m^2\left(\sec^2\alpha - 2\right) - 2m\sec^2\alpha + \left(\sec^2\alpha - 2\right) = 0$$

$$\Rightarrow m_1 + m_2 = \frac{2\sec^2\alpha}{\sec^2\alpha - 2}, m_1m_2 = 1$$

Combined equation of OA and OB is $(y - m_1x)(y - m_2x) = 0$

$$y^2 - (m_1+m_2)xy + m_1m_2x^2 = 0$$

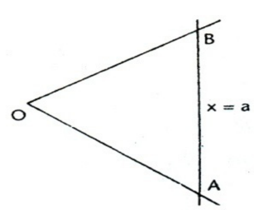
$$\Rightarrow y^2 + \frac{2\sec^2\alpha}{\sec^2\alpha - 2}.xy + x^2 = 0$$

$$\Rightarrow m_1 + m_2 = \frac{2\sec^2\alpha}{\sec^2\alpha - 2} = \frac{2}{1-2\cos^2\alpha} \quad = \frac{-2}{2\cos^2\alpha - 1} = \frac{-2}{\cos 2\alpha} = -2\sec 2\alpha$$

Therefore required pair of lines is $x^2 + 2xy\sec 2\alpha + y^2 = 0$

6. Show that the straight lines represented by $(x+2a)^2 - 3y^2 = 0$ and x = a form an equilateral triangle.

Sol.



Given equation is $(x+2a)^2 - 3y^2 = 0$

$\Rightarrow (x+2a)^2 - (\sqrt{3}y)^2 = 0 \Rightarrow \quad (x+2a+\sqrt{3}y)(x+2a-\sqrt{3}y) = 0$

Equations of the lines are $x+\sqrt{3}y+2a=0$ ……(1) and $x-\sqrt{3}y+2a=0$ …..(2)

Equation of 3rd line is x – a = 0 ------(3)

Angle between I and iii is $\cos\alpha = \dfrac{|1+0|}{\sqrt{1+3}\sqrt{1+0}} = \dfrac{1}{2} = \cos 60°$

$\therefore \alpha = 60°$

Angle between ii and iii is

$\cos\beta = \dfrac{|1-0|}{\sqrt{1+3}\sqrt{1+0}} = \dfrac{1}{2} = \cos 60°$

$\therefore \beta = 60°$

$\therefore$ angle between i and ii $= 180° - (\alpha+\beta)$

$= 180° - (60° + 60°) = 180° - 120° = 60°$

$\therefore \Delta OAB$ is an equilateral traingle.

7. Show that the pair of bisectors of the angle between the straight lines $(ax+by)^2=c(bx-ay)^2$, c > 0 are parallel and perpendicular to the line ax+by+k=0.

Sol. Equation of pair of lines is $(ax+by)^2 = c(bx-ay)^2$

$\Rightarrow a^2x^2 + b^2y^2 + 2abxy = c(b^2x^2 + a^2y^2 - 2abxy) = cb^2x^2 + ca^2y^2 - 2cabxy$

$\Rightarrow (a^2 - cb^2)x^2 + 2ab(1+c^2)xy + (b^2 - ca^2)y^2 = 0$

Equation of the pair of bisector is $h(x^2 - y^2) = (a-b)xy$

$ab(1+c)(x^2-y^2)=(a^2-cb^2-b^2+ca^2)(x^2-y^2)==(a^2-b^2)(1+c)xy$

i.e. $ab(x^2-y^2)-(a^2-b^2)xy=0$

but $(ax+by)(bx-ay)=abx^2-a^2xy+b^2xy-aby^2$

$=ab(x^2-y^2)-(a^2-b^2)xy$

$\therefore$ The equation of the pair of bisectors are (ax+by) (bx – ay) = 0

The bisectors are ax + by = 0 and bx – ay =0.

The line ax + by = 0 is parallel to ax+by+k = 0 and the line bx – ay = 0 is perpendicular to ax + by + k = 0

8. The adjacent sides of a parallelogram are $2x^2-5xy+3y^2=0$ **and one diagonal is x + y + 2= 0. Find the vertices and the other diagonal.**

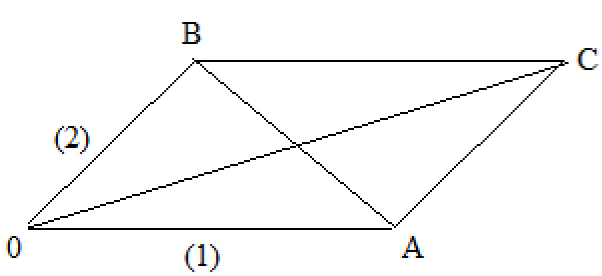
Sol. Let OACB be the given parallelogram.

Let Combined equation of OA and OB be $2x^2-5xy+3y^2=0$

$2x^2-5xy + 3y^2=0 \Rightarrow 2x^2-2xy\text{-}3xy + 3y^2=0$
$\Rightarrow (x-y)(2x-3y)=0$

$\Rightarrow x-y=0---(1)$ and $2x-3y=0--(2)$

The point of intersection of above lines is O(0,0)



Equation of diagonal AB is x + y + 2 =0 ($\because$ it is not passing through O)

Solving (1) and (3), we get vertex A =(-1,-1)

Soving (2) and (3), we get vertex B = (-6/5,-4/5)

4^{th} vertex C = A+B-0 = (-1-6/5, -1-4/5) = (-11/5, -9/5)

Equation of diagonal OC is

$$2x^2 - 5xy + 3y^2 = 0 \Rightarrow 2x^2 - 2xy\text{-}3xy + 3y^2 = 0$$
$$\Rightarrow (x-y)(2x-3y) = 0$$
$$\Rightarrow x - y = 0 \text{ and } 2x - 3y = 0$$

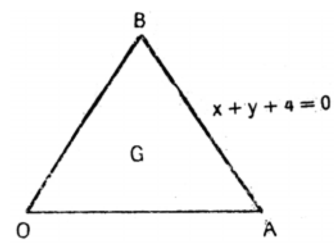
$$y - 0 = \frac{\frac{-9}{5}}{\frac{-11}{5}}(x-0)$$

i.e., 11y = 9x

9. Find the centroid and the area of the triangle formed by the following lines.

i) $2y^2 - xy - 6x^2 = 0, x + y + 4 = 0$

ii) $3x^2 - 4xy + y^2 = 0, 2x - y = 6$



Sol. i) The point of intersection of $2y^2 - xy - 6x^2 = 0$ is 0(0,0)

Let ΔOAB be the triangle formed by given lines.

Let the combined equation of OA,OB be $2y^2 - xy - 6x^2 = 0$ …….(1)

Equation of AB is x + y + 4 = 0 $\Rightarrow y = -(x+4)$ …….(2)

Substituting in (1)

$$2(x+4)^2 + x(x+4) - 6x^2 = 0 \Rightarrow 2(x^2 + 8x + 16) + x^2 + 4x - 6x^2 = 0$$

$$\Rightarrow 2x^2 + 16x + 32 + x^2 + 4x - 6x^2 = 0 \Rightarrow -3x^2 + 20x + 32 = 0$$

$$\Rightarrow (3x+4)(x-8) = 0 \Rightarrow 3x + 4 = 0 \text{ or } x - 8 = 0$$

$\Rightarrow 3x = -4$ or x = 8 $\Rightarrow x = -\frac{4}{3}$ or 8

Case (i) : $x = -\frac{4}{3} \Rightarrow y = -(x+4)$

$= -\left(\frac{-4}{3} + 4\right) = -\frac{8}{3}$

Co-ordinates of A are $\left(-\frac{4}{3}, -\frac{8}{3}\right)$

Case (ii): x = 8 $\Rightarrow y = -(x+4) = -(8+4) = -12$

Co-ordinates of B are (8, - 12)

Let G be the centriod of ΔAOB

Co-ordinates of G are $\left(\frac{0-\frac{4}{3}+8}{3}, \frac{0-\frac{8}{3}-12}{3}\right) = \left(\frac{20}{9}, -\frac{44}{9}\right)$

Area of $\Delta OAB = \frac{1}{2}\left|x_1y_2 - x_2y_1\right| = \frac{1}{2}\left|\left(-\frac{4}{3}\right)(-12) - \left(-\frac{8}{3}\right)(8)\right|$

$= \frac{1}{2}\left|\frac{48}{3} + \frac{64}{3}\right| = \frac{1}{2} . \frac{112}{3} \quad = \frac{56}{3}$ sq.units.

ii) ans 36 sq.units

10. Find the equation of the pair of lines intersecting at (2, - 1) and

(i) Perpendicular to the pair $6x^2 - 13xy - 5y^2 = 0$ **and (ii) parallel to the pair** $6x^2 - 13xy - 5y^2 = 0$.

Sol. Given point is **(2, - 1)=** (x_1, y_1)

Equation of pair of lines is $6x^2 - 13xy - 5y^2 = 0$

i) Equation of the pair of lines through (x_1,y_1) and perpendicular to $ax^2+2hxy+by^2=0$ is

$b(x-x_1)^2-2h(x-x_1)(y-y_1)+a(y-y_1)^2=0$

$\Rightarrow$ Require pair of lines is $-5(x-2)^2+13(x-2)(y+1)+6(y+1)^2=0$

$\Rightarrow -5(x^2-4x+4)+13(xy+x-2y-2)+6(y^2+2y+1)=0$ $-5x^2+20x-20+13xy+13x-26y-26+6y^2+12y-6=0$

$\Rightarrow -5x^2+13xy+6y^2+33x-14y-40=0$

or $5x^2-13xy-6y^2-33x+14y+40=0$

ii) Equation of the pair of lines through (x_1,y_1) and parallel to $ax^2+2hxy+by^2=0$ **is**

$a(x-x_1)^2+2h(x-x_1)(y-y_1)+b(y-y_1)^2=0$

$\Rightarrow$ Required Equation of the pair of parallel lines is $6(x-2)^2-13(x-2)(y+1)-5(y+1)^2=0$

$\Rightarrow 6(x^2-4x+4)-13(xy+x-2y-2)\ -5(y^2+2y+1)=0$

$\Rightarrow 6x^2-24x+24-13xy-13x+26y+26\ -5y^2-10y-5=0$

$\Rightarrow 6x^2-13xy-5y^2-37x+16y+45=0$

11. Find the equation of the bisector of the acute angle between the lines

$3x-4y+7=0$ **and** $12x+5y-2=0$

Sol. Given lines 3x – 4y + 7 = 0 ………(1)

12x + 5y – 2 = 0 ……….(2)

The equations of bisector's angles between (1) & (2) is

$$\frac{3x-4y+7}{\sqrt{3^2+4^2}}\pm\frac{12x+5y-2}{\sqrt{12^2+5^2}}=0$$

$$\Rightarrow \frac{3x-4y+7}{5} \pm \frac{12x+5y-2}{13} = 0$$

$$\Rightarrow 13(3x-4y+7) \pm 5(12x+5y-2) = 0$$

$$\Rightarrow (39x-52y-91) \pm (60x+25y-10) = 0$$

$$\Rightarrow 39x-52y+91+60x+25y-10=0 \text{ And } (39x-52y+91)-(60x+25y-10)=0$$

$$\Rightarrow 99x-27y+81=0 \text{ and } \quad 21x+77y-81=0$$

$\Rightarrow 11x-3y+9=0$ ----3 and $21x+77y-81=0$ ------4

Let θ be the angle between (1), (4)

$$\tan\theta = +\left|\frac{a_1b_2-a_2b_1}{a_1a_2+b_1b_2}\right| = \left|\frac{231+84}{63-308}\right| = \frac{315}{225} > 1$$

$\therefore$ (4) is the bisector of obtuse angle , then other one (3) is the bisectorof acute angle.

$\therefore$ $99x-27y+41=0$ is the acute angle bisector.

12. Find the equation of the bisector of the obtuse angle between the lines x + y – 5 =0 and x – 7y + 7 =0.

Ans: 3x – y – 9 = 0 is the obtuse angle bisector.

III

1. Show that the lines represented by $(lx+my)^2-3(mx-ly)^2=0$ and $lx+my+n=0$ from an equilateral triangle with area $\frac{n^2}{\sqrt{3}(l^2+m^2)}$.

Equation of the pair of lines is $(lx+my)^2-3(mx-ly)^2=0$------(1)

$$\Rightarrow l^2x^2+m^2y^2+2lmxy-3m^2x^2-3l^2y^2+6lmxy=0$$

$$\Rightarrow (l^2-3m^2)x^2+8lmxy+(m^2-3l^2)y^2=0.$$

The point of intersection of above lines is 0(0,0).

Let θ be the angle between the lines, then $\cos\theta = \frac{|a+b|}{\sqrt{(a-b)^2+4n^2}}$

$$= \frac{|l^2-3m^2+m^2-3l^2|}{\sqrt{(l^2-3m^2-m^2+3l^2)^2+64l^2m^2}} \qquad = \frac{2|l^2+m^2|}{4\sqrt{(l^2-m^2)^2+4l^2m^2}} = \frac{2|l^2+m^2|}{4(l^2+m^2)} = \frac{1}{2}$$

$$\Rightarrow \theta = 60^\circ.$$

$$(lx+my)^2 - 3(mx-ly)^2 = 0$$

$$\Rightarrow (lx+my)^2 - \left(\sqrt{3}(mx-ly)\right)^2 = 0$$
$$\Rightarrow \left((lx+my)-\sqrt{3}(mx-ly)\right)\left((lx+my)+\sqrt{3}(mx-ly)\right) = 0$$

$$(lx+my)-\sqrt{3}(mx-ly) = 0 \text{ and } (lx+my)+\sqrt{3}(mx-ly) = 0$$

$$\Rightarrow \left(l-m\sqrt{3}\right)x+\left(m+\sqrt{3}l\right)y = 0 -----(2)$$
$$and \ \left(l+m\sqrt{3}\right)x+\left(m-\sqrt{3}l\right)y = 0 ----(3)$$

Equation of given line is lx + my + n = 0 ------(4)

Let the Angle between (2) and (4) be α, then $\cos\alpha = \frac{l\left(l+\sqrt{3}m\right)+m\left(m-\sqrt{3}l\right)}{\sqrt{\left(l+\sqrt{3}m\right)^2+\left(m-\sqrt{3}l\right)^2}\sqrt{l^2+m^2}}$

$$= \cos\alpha = \frac{l^2+m^2}{\sqrt{4\left(l^2+m^2\right)}\sqrt{l^2+m^2}} = \frac{1}{2} \Rightarrow \alpha = 60^0$$

Now Angle between (3) and (4) = $180^0 - (60+60) = 60^0$

Therefore the angles of the triangle are 60^0, 60^0, 60^0

Hence the triangle is an equilateral triangle

Let p = Length of the perpendicular from p to line lx + my + n = 0 is $= \frac{|n|}{\sqrt{l^2+m^2}}$

$\therefore$ Area of $\Delta OAB = \frac{p^2}{\sqrt{3}} = \frac{n^2}{\sqrt{3}\left(l^2 + m^2\right)}$ sq units.

2. Show that the straight lines represented by $3x^2 + 48xy + 23y^2 = 0$ and $3x - 2y + 13 = 0$ form an equilateral triangle of area $\frac{13}{\sqrt{3}}$ sq. units.

Sol. Equation of pair of lines is $3x^2 + 48xy + 23y^2 = 0$ …………(1)

Equation of given line is $3x - 2y + 13 = 0$ ……(2)

$\Rightarrow$ slope $= 3/2$

$\therefore$ the line (2) is making an angle of $\tan^{-1}\frac{3}{2}$ with the positive direction of x-axis. Therefore no straight line which makes an angle of 60^0 with (2) is vertical.

Let m be the slope of the line passing through origin and making an angle of 60^0 with line (2).

$$\therefore \tan 60^0 = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| \Rightarrow \sqrt{3} = \left|\frac{\frac{3}{2} - m}{1 + \frac{3}{2}m}\right| \Rightarrow \sqrt{3} = \left|\frac{3 - 2m}{2 + 3m}\right|$$

Squaring on both sides , $3 = \frac{(3-2m)^2}{(2+3m)^2} \Rightarrow 23m^2 + 48m + 3 = 0$,which is a quadratic equation in m.

Let the roots of this quadratic equation be m_1, m_2, which are the slopes of the lines.

Now , $m_1 + m_2 = \frac{-48}{23}$ and $m_1.m_2 = \frac{3}{23}$.

The equation of the lines passing through origin and having slopes m_1, m_2 are $m_1x - y = 0$ and $m_2x - y = 0$.

Their combined equation is $(m_1x - y)(m_2x - y) = 0$

$$\Rightarrow m_1m_2x^2-(m_1+m_2)xyy^2=0$$
$$\Rightarrow \frac{3}{23}x^2-\left(-\frac{48}{23}\right)xy+y^2=0$$
$$\Rightarrow 3x^2+48xy+23y^2=0$$

Which is the given pair of lines.

Therefore, given lines form an equilateral triangle.

$$\therefore \text{ Area of } \Delta=\left|\frac{n^2\sqrt{h^2-ab}}{am^2-2hlm+bl^2}\right|=\frac{169\sqrt{576-69}}{\left|3(-2)^2-48.3(-2)+23(3)^2\right|}$$

$$=\frac{169\sqrt{507}}{|12+288+207|}=\frac{169.13\sqrt{3}}{507}=\frac{13\sqrt{3}}{3}=\frac{13}{\sqrt{3}}\text{sq.units.}$$

3. Show that the equation of the pair of lines bisecting the angles between the pair of bisectors of the angles between the pair of lines ax^2 + 2hxy +by^2 = 0 is (a – b) (x^2 – y^2) + 4hxy = 0.

Sol. Equation of the given lines is $ax^2+2hxy+by^2=0$

Equation of the pair of bisector is $h\left(x^2-y^2\right)=(a-b)xy$

$$hx^2-(a-b)xy-hy^2=0-----(1)$$

$$\therefore A=h, B=-h, 2H=-(a-b)$$

Equation of the pair of bisector of (1) is

$$H\left(x^2-y^2\right)=(A-B)xy$$

$$\Rightarrow -\frac{(a-b)}{2}\left(x^2-y^2\right)=2hxy$$

$$\Rightarrow -(a-b)\left(x^2-y^2\right)=4hxy$$

$\Rightarrow$ $(a - b)(x^2 - y^2) + 4hxy = 0$

$\therefore$ Equation of the pair of bisector of the pair of bisectors of $ax^2 + 2hyx + by^2 = 0$ is

$(a-b)(x^2 - y^2) + 4hxy = 0$

4. If one line of the pair of lines $ax^2 + 2hyx + by^2 = 0$ bisects the angle between the co-ordinate axes, prove that (a+b) = 4h^2.

Sol. The angular bisectors of the co-ordinate axes are $y = \pm x$

Case (i) let y = x be one of the lines of $ax^2 + 2hxy + by^2 = 0$

$\Rightarrow x^2(a + 2h + b) = 0$

$\Rightarrow a + 2h + b = 0$ ………(1)

Case (ii) let y = - x is one of the lines of $ax^2 + 2hxy + by^2 = 0$

$\Rightarrow x^2(a - 2h + b) = 0$

$\Rightarrow a - 2h + b = 0$ ……..(2)

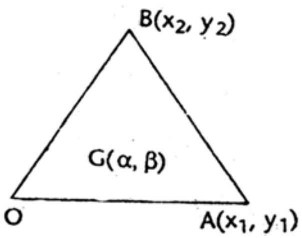
Multiplying (1) and (2), we get

$(a + b + 2h).(a + b - 2h) = 0 \Rightarrow (a + b)^2 - 4h^2 = 0$

$\Rightarrow (a + b)^2 = 4h^2$

5. **If** (α,β) **is the centroid of the triangle formed by the lines** $ax^2+2hxy+by^2=0$ **and** $lx+my=1$**,**

prove that $\dfrac{\alpha}{bl-hm}=\dfrac{+\beta}{am-hl}=\dfrac{2}{3\left(bl^2-2hlm+am^2\right)}$



Sol. Given equation of pair of lines is $ax^2+2hxy+by^2=0$ ....(1)

Point of intersection of the lines is O(0,0)

Let O,A,B the vertices of the triangle and Let A (x_1,y_1) and B (x_2,y_2).

Let equation of AB be $lx+my=1 \Rightarrow my=1-lx$

$\Rightarrow y=\dfrac{1-lx}{m}$ ....(2)

from (1) and (2) $ax^2+2hx\dfrac{(1-lx)}{m}+b\dfrac{(1-lx)^2}{m^2}=0$

$\Rightarrow am^2x^2+2hx(1-lx)+b\left(1+l^2x^2-2lx\right)=0$

$\Rightarrow am^2x^2+2hmx-2hlmx^2+b+bl^2x^2-2blx=0$

$\Rightarrow \left(am^2-2hlm+bl^2\right)x^2-2(bl-hm)x+b=0$

Let A (x_1,y_1) and B (x_2,y_2), then $x_1+x_2=\dfrac{2(bl-hm)}{am^2-2hlm+bl^2}$ ....(3)

A and B are points on $lx+my=1 \Rightarrow lx_1+my_1=1$ and $lx_2+my_2=1$

$\Rightarrow l(x_1+x_2)+m(y_1+y_2)=2$

$$\Rightarrow m(y_1+y_2)=2-l(x_1-x_2)=2-\frac{l.2(bl-hm)}{am^2-2hlm+bl^2}$$

$$=\frac{2(am^2-2hlm+bl^2-bl^2+hlm)}{am^2-2hlm+bl^2}=\frac{2(am^2-hlm)}{am^2-2hlm+bl^2}=\frac{2m(am-hl)}{am^2-2hlm+bl^2}$$

$$\Rightarrow y_1+y_2=\frac{2(am-hl)}{am^2-2hlm+bl^2} \quad \ldots\ldots\ldots\ldots(4)$$

Now centroid G = $\left(\frac{x_1+x_2}{3},\frac{y_1+y_2}{3}\right)=(\alpha,\beta)\Rightarrow\frac{x_1+x_2}{3}=\alpha$

$$\Rightarrow \alpha=\frac{2(bl-hm)}{3(am^2-2hlm+bl^2)}$$

$$\frac{\alpha}{bl-hm}=\frac{2}{3(bl^2-2hlm+am^2)} \quad \ldots.(5)$$

$$\frac{y_1+y_2}{3}=\beta \qquad \Rightarrow \beta=\frac{2(am-hl)}{3(bl^2-2hlm+am^2)}$$

$$\therefore \frac{\beta}{am-hl}=\frac{2}{3(bl^2-2hlm+am^2)} \quad \ldots..(6)$$

From (5) and (6), we get

$$\frac{\alpha}{bl-hm}=\frac{\beta}{am-hl}=\frac{2}{3(bl^2-2hlm+am^2)}$$

6. Prove that the distance from the origin to the orthocenter of the triangle formed by the line $\frac{x}{\alpha}+\frac{y}{\beta}=1$ and $ax^2+2hxy+by^2=0$ is $(\alpha^2+\beta^2)^{1/2}\left|\frac{(a+b)\alpha\beta}{a\alpha^2-2h\alpha\beta+b\beta^2}\right|$

7. **The straight line $lx + my + n = 0$ bisects an angle between the pair of lines of which one is px+qy+r=0. Show that the other line is (px+qy+r) (l^2 +m^2) - 2(lp+mq) (lx+my+n) = 0**

Sol. Given line is $L_{1=}$ px+qy+r=0

Equation of the bisector is L_2= lx + my + n = 0

Equation of the line is passing through the point of intersection of L_1=0 and L_2=0 is $L_1 + \lambda L_2 = 0$

$\Rightarrow (px + qy + r) + \lambda(lx + my + n) = 0$ …..(1)

let (α, β) be any point on L_2 =0 so that $l\alpha + m\beta + n = 0$----(2)

If (α, β) be a point on the bisector then its perpendicular distance from the lines L_2=0 and (2) are equal.

$$\Rightarrow \frac{(p\alpha + q\beta + r) + \lambda(l\alpha + m\beta + n)}{\sqrt{\left[(q + l\lambda)^2 + (q + m\lambda)^2\right]}} = \pm \frac{p\alpha + q\beta + r}{\sqrt{p^2 + q^2}}$$

$$\Rightarrow (p + l\lambda)^2 + (q + m\lambda)^2 = p^2 + q^2 \text{ (From (2), } l\alpha + m\beta + n = 0)$$

$$\Rightarrow 2\lambda(pl + qm) + \lambda^2(l^2 + m^2) = 0$$

$$\therefore \lambda = -2\frac{pl + qm}{l^2 + m^2}$$

Substitute λ value in (1), $(px + qy + r)(l^2 + m^2) - 2(pl + qm)(lx + my + n) = 0$