

LIMITS AT INFINITY

Definition:

Let $f(x)$ be a function defined on $A = (K,)$.

- (i) A real number l is said to be the limit of $f(x)$ at $+\infty$ if to each $\epsilon > 0, \exists$, an $M > 0$ (however large M may be) such that $x \in A$ and $x > M \Rightarrow |f(x) - l| < \epsilon$ In this case we write $f(x) \rightarrow l$ as $x \rightarrow +\infty$ or $\lim_{x \rightarrow +\infty} f(x) = l$.
- (ii) A real number l is said to be the limit of $f(x)$ at $-\infty$ if to each $\epsilon > 0, \exists$ $M > 0$, an $M > 0$ (however large it may be) such that $x \in A$ and $x < -M \Rightarrow |f(x) - l| < \epsilon$ In this case, we write $f(x) \rightarrow l$ as $x \rightarrow -\infty$ or $\lim_{x \rightarrow -\infty} f(x) = l$.

INFINITE LIMITS

Definition:

- (i) Let f be a function defined in a deleted neighbourhood of D of a . (i) The limit of f at a is said to be $+\infty$ if to each $M > 0$ (however large it may be) a $\delta > 0$ such that $x \in D, 0 < |x - a| < \delta \Rightarrow f(x) > M$. In this case we write $f(x) \rightarrow +\infty$ as $x \rightarrow a$ or $\lim_{x \rightarrow a} f(x) = +\infty$.
- (ii) The limit of $f(x)$ at a is said to be $-\infty$ if to each $M > 0$ (however large it may be) a $\delta > 0$ such that $x \in D, 0 < |x - a| < \delta \Rightarrow f(x) < -M$. In this case we write $f(x) \rightarrow -\infty$ as $x \rightarrow a$ or $\lim_{x \rightarrow a} f(x) = -\infty$.

INDETERMINATE FORMS

While evaluating limits of functions, we often get forms of the type

$\frac{0}{0}, \frac{\infty}{\infty}, 0 \times \infty, \infty - \infty, 0^0, 1^\infty, \infty^0$ which are termed as indeterminate forms.

EXERCISE – 8 (d)

I. Compute the following limits.

1.
$$\lim_{x \rightarrow 3} \frac{x^2 + 3x + 2}{x^2 - 6x + 9}$$

Sol :
$$\lim_{x \rightarrow 3} \frac{x^2 + 3x + 2}{x^2 - 6x + 9} = \lim_{x \rightarrow 3} \frac{x^2 + 3x + 2}{(x - 3)^2} = \frac{9 + 9 + 2}{0} = \infty$$

2.
$$\lim_{x \rightarrow 1^-} \frac{1 + 5x^3}{1 - x^2}$$

Sol :
$$\lim_{x \rightarrow 1^-} \frac{1 + 5x^3}{1 - x^2} = \frac{1 + 5(1)^3}{1 - 1^2} = \frac{1 + 5}{1 - 1} = \frac{6}{0} = \infty$$

3.
$$\lim_{x \rightarrow \infty} \frac{3x^2 + 4x + 5}{2x^3 + 3x - 7}$$

Sol :
$$\lim_{x \rightarrow \infty} \frac{3x^2 + 4x + 5}{2x^3 + 3x - 7}$$

$$= \lim_{x \rightarrow \infty} \frac{\left(3 + \frac{4}{x} + \frac{5}{x^2}\right)x^2}{\left(2 + \frac{3}{x^2} - \frac{7}{x^3}\right)x^3} = \lim_{x \rightarrow \infty} \frac{\left(3 + \frac{4}{x} + \frac{5}{x^2}\right)}{2 + \frac{3}{x^2} - \frac{7}{x^3}} \cdot \frac{1}{x}$$

As $x \rightarrow \infty, \frac{1}{x}, \frac{1}{x^2}, \frac{1}{x^3} \rightarrow 0$

$$= \lim_{x \rightarrow \infty} \frac{3x^2 + 4x + 5}{2x^3 + 3x - 7} = \frac{(3 + 0 + 0)}{2 + 0 - 0} \cdot 0 = 0$$

4. $\lim_{x \rightarrow \infty} \frac{6x^2 - x + 7}{x + 3}$

Sol : $\lim_{x \rightarrow \infty} \frac{6x^2 - x + 7}{x + 3} = \lim_{x \rightarrow \infty} \frac{x^2 \left(6 - \frac{1}{x} + \frac{7}{x^2}\right)}{x \left(1 + \frac{3}{x}\right)}$

$$\lim_{x \rightarrow \infty} \frac{6 - \frac{1}{x} + \frac{7}{x^2}}{1 + \frac{3}{x}} \cdot \lim_{x \rightarrow \infty} x = \frac{6 - 0 + 0}{1 + 0} \cdot \infty = \infty$$

5. $\lim_{x \rightarrow \infty} e^{-x^2}$

Sol : $\lim_{x \rightarrow \infty} e^{-x^2} = \lim_{x \rightarrow \infty} \frac{1}{e^{x^2}} = \frac{1}{\infty} = 0$ (since $e > 1$)

6. $\lim_{x \rightarrow \infty} \frac{\sqrt{x^2 + 6}}{2x^2 - 1}$

Sol : $\lim_{x \rightarrow \infty} \frac{\sqrt{x^2 + 6}}{2x^2 - 1} = \lim_{x \rightarrow \infty} \frac{x \sqrt{1 + \frac{6}{x^2}}}{x^2 \left(2 - \frac{1}{x^2}\right)}$

$$= \lim_{x \rightarrow \infty} \frac{1}{x} \frac{\sqrt{1 + \frac{6}{x^2}}}{2 - \frac{1}{x^2}}$$

As $x \rightarrow \infty, \frac{1}{x} \rightarrow 0 \therefore \lim_{x \rightarrow \infty} \frac{\sqrt{x^2 + 6}}{2x^2 - 1} \cdot \lim_{x \rightarrow \infty} \frac{1}{x}$

$$\lim_{x \rightarrow \infty} \frac{\sqrt{1 + \frac{6}{x^2}}}{2 - \frac{1}{x^2}} = \frac{\sqrt{1 + 0}}{2 - 0} \cdot 0 = 0$$

II.

1.
$$\lim_{x \rightarrow 0} \frac{8|x| + 3x}{3|x| - 2x}$$

Sol : as $x \rightarrow \infty \Rightarrow |x| = x$ (\because here x is positive)

$$\lim_{x \rightarrow \infty} \frac{8|x| + 3x}{3|x| - 2x} = \lim_{x \rightarrow \infty} \frac{8x + 3x}{3x - 2x} = \lim_{x \rightarrow \infty} \frac{11x}{x} = 11$$

2.
$$\lim_{x \rightarrow \infty} \frac{x^2 + 5x + 2}{2x^2 - 5x + 1}$$

Sol :
$$= \lim_{x \rightarrow \infty} \frac{x^2 + 5x + 2}{2x^2 - 5x + 1} = \lim_{x \rightarrow \infty} \frac{x^2 \left[1 + \frac{5}{x} + \frac{2}{x^2} \right]}{x^2 \left(2 - \frac{5}{x} + \frac{1}{x^2} \right)}$$

As $x \rightarrow \infty, \frac{1}{x}$ and $\frac{1}{x^2} \rightarrow 0$

$$\lim_{x \rightarrow \infty} \frac{x^2 + 5x + 2}{2x^2 - 5x + 1} = \lim_{x \rightarrow \infty} \frac{1 + \frac{5}{x} + \frac{2}{x^2}}{2 - \frac{5}{x} + \frac{1}{x^2}} = \frac{1 + 0 + 0}{2 - 0 + 0} = \frac{1}{2}$$

3.
$$\lim_{x \rightarrow -\infty} \frac{2x^2 - x + 3}{x^2 - 2x + 5}$$

Sol ;
$$\lim_{x \rightarrow -\infty} \frac{2x^2 - x + 3}{x^2 - 2x + 5} = \lim_{x \rightarrow -\infty} \frac{x^2 \left(2 - \frac{1}{x} + \frac{3}{x^2} \right)}{x^2 \left(1 + \frac{2}{x} + \frac{5}{x^2} \right)}$$

$$\lim_{x \rightarrow -\infty} \frac{2x^2 - x + 3}{x^2 - 2x + 5} = \lim_{x \rightarrow -\infty} \frac{2 - \frac{1}{x} + \frac{3}{x^2}}{1 - \frac{2}{x} + \frac{5}{x^2}}$$

As $x \rightarrow -\infty, \frac{1}{x}$ and $\frac{1}{x^2} \rightarrow 0$

$$= \frac{2-0+0}{1-0+0} = \frac{2}{1} = 2$$

4. $\lim_{x \rightarrow -\infty} \frac{11x^3 - 3x + 4}{13x^3 - 5x^2 - 7}$

Sol: $\lim_{x \rightarrow -\infty} \frac{11x^3 - 3x + 4}{13x^3 - 5x^2 - 7}$

$$= \lim_{x \rightarrow \infty} \frac{x^3 \left(11 - \frac{3}{x^2} + \frac{4}{x^3} \right)}{x^3 \left(13 - \frac{5}{x} - \frac{7}{x^3} \right)}$$

As $x \rightarrow \infty$, $\frac{1}{x}$, $\frac{1}{x^2}$ and $\frac{1}{x^3} \rightarrow 0$

$$\lim_{x \rightarrow \infty} \frac{11x^3 - 3x + 4}{13x^3 - 5x^2 - 7}$$

$$= \lim_{x \rightarrow \infty} \frac{11 - \frac{3}{x^2} + \frac{4}{x^3}}{13 - \frac{5}{x} - \frac{7}{x^3}} = \frac{11-0+0}{13-0-0} = \frac{11}{13}$$

5. $\lim_{x \rightarrow 2} \left(\frac{1}{x-2} - \frac{4}{x^2-4} \right)$

Sol: $\lim_{x \rightarrow 2} \left(\frac{1}{x-2} - \frac{4}{x^2-4} \right) = \lim_{x \rightarrow 2} \frac{x+2-4}{x^2-4}$

$$= \lim_{x \rightarrow 2} \frac{x-2}{x^2-4} = \lim_{x \rightarrow 2} \frac{x-2}{(x-2)(x+2)} = \lim_{x \rightarrow 2} \frac{x+2-4}{x^2-4} = \frac{1}{4}$$

6. $\lim_{x \rightarrow -\infty} \frac{5x^3 + 4}{\sqrt{2x^4 + 1}}$

Sol: $\lim_{x \rightarrow -\infty} \frac{5x^3 + 4}{\sqrt{2x^4 + 1}} = \lim_{x \rightarrow -\infty} \frac{x^3 \left(5 + \frac{4}{x^3}\right)}{x^2 \sqrt{2 + \frac{1}{x^4}}}$

$$= \lim_{x \rightarrow -\infty} x \cdot \frac{5 + \frac{4}{x^3}}{\sqrt{2 + \frac{1}{x^4}}}$$

As $x \rightarrow -\infty, \frac{1}{x}, \frac{1}{x^2}, \frac{1}{x^3}, \frac{1}{x^4} \rightarrow 0$

$$\lim_{x \rightarrow -\infty} \frac{5x^3 + 4}{\sqrt{2x^4 + 1}} (-\infty) = (-\infty) \cdot \frac{5}{\sqrt{1}} = -\infty.$$

7. $\lim_{x \rightarrow \infty} (\sqrt{x+1} - \sqrt{x})$

Sol: $\lim_{x \rightarrow \infty} (\sqrt{x+1} - \sqrt{x})$

$$= \lim_{x \rightarrow \infty} \frac{(\sqrt{x+1} - \sqrt{x})(\sqrt{x+1} + \sqrt{x})}{\sqrt{x+1} + \sqrt{x}}$$

$$= \lim_{x \rightarrow \infty} \frac{x+1-x}{\sqrt{x+1} + \sqrt{x}}$$

$$= \lim_{x \rightarrow \infty} \frac{1}{\sqrt{x} \left(\sqrt{1 + \frac{1}{x}} + 1 \right)}$$

$$= \lim_{x \rightarrow \infty} \frac{1}{\sqrt{x} \left(\sqrt{1 + \frac{1}{x}} + 1 \right)} = \frac{0}{1+1} = 0$$

$$8. = \lim_{x \rightarrow \infty} (\sqrt{x^2 + x} - x)$$

$$\text{Sol: } = \lim_{x \rightarrow \infty} \sqrt{x^2 + x} - x$$

$$= \lim_{x \rightarrow \infty} \frac{(\sqrt{x^2 + x} - x)(\sqrt{x^2 + x} + \sqrt{x})}{\sqrt{x^2 + x} + \sqrt{x}} = \lim_{x \rightarrow \infty} \frac{x^2 + x - x^2}{x \left(\sqrt{1 + \frac{1}{x}} + 1 \right)}$$

$$= \lim_{x \rightarrow \infty} \frac{x}{x \left(\sqrt{1 + \frac{1}{x}} + 1 \right)}$$

$$= \frac{1}{\sqrt{1+0}+1} = \frac{1}{1+1} = \frac{1}{2}$$

III.

$$1. \lim_{x \rightarrow -\infty} \left(\frac{2x+3}{\sqrt{x^2-1}} \right)$$

$$\lim_{x \rightarrow -\infty} \frac{2x+3}{\sqrt{x^2-1}}$$

$$\text{Sol: } = \lim_{x \rightarrow -\infty} \frac{x \left(2 + \frac{3}{x} \right)}{-x \sqrt{1 - \frac{1}{x^2}}} \left(\because \text{here } x \rightarrow -\infty \text{ i.e., } x \text{ is negative } \Rightarrow \sqrt{x^2} = -x \right)$$

$$\text{As } x \rightarrow -\infty, \frac{1}{x}, \frac{1}{x^2} \rightarrow 0$$

$$\lim_{x \rightarrow -\infty} \frac{2x+3}{\sqrt{x^2-1}} = -\frac{2+0}{\sqrt{1-0}} = -\frac{2}{1} = -2$$

2. $\lim_{x \rightarrow \infty} \frac{2 + \sin x}{x^2 + 3}$

Sol : $\lim_{x \rightarrow \infty} \frac{2 + \sin x}{x^2 + 3} = \lim_{x \rightarrow \infty} \frac{2 + \sin x}{x^2 \left(1 + \frac{3}{x^2}\right)}$

$$= \lim_{x \rightarrow \infty} \frac{2 + \sin x}{x^2 \left(1 + \frac{3}{x^2}\right)} = \lim_{x \rightarrow \infty} \frac{\frac{2}{x^2} + \frac{\sin x}{x^2}}{\left(1 + \frac{3}{x^2}\right)}$$

as $x \rightarrow \infty$, $\frac{1}{x^2}$ and $\frac{\sin x}{x^2} \rightarrow 0$. ($\because -1 \leq \sin x \leq 1$) $= \frac{0 + 0}{1 + 0} = 0$

3. $\lim_{x \rightarrow \infty} \frac{2 + \cos^2 x}{x + 2007}$ (ans 0)

4. $\lim_{x \rightarrow \infty} \frac{6x^2 - \cos 3x}{x^2 + 5}$ ans. 6

5. $\lim_{x \rightarrow \infty} \frac{\cos x + \sin^2 x}{x + 1}$ try yourself. ans = $\lim_{x \rightarrow \infty} \frac{\cos x + \sin^2 x}{x + 1} = 0$

PROBLEMS FOR PRACTICE

1. Evaluate $\lim_{x \rightarrow -3} \frac{1}{x + 2}$.

2. Compute $\lim_{x \rightarrow 2} \frac{x - 2}{x^3 - 8}$.

3. Find $\lim_{x \rightarrow 1} (x + 2)(2x + 1)$

4. Compute $\lim_{x \rightarrow 0} x^2 \cdot \sin \frac{1}{x}$.

5. Show that $\lim_{x \rightarrow 0^+} \frac{|x|}{x} = 1$ and $\lim_{x \rightarrow 0^-} \frac{|x|}{x} = -1 (x \neq 0)$

6. Let $f: R \rightarrow R$ be defined by $f(x) = \begin{cases} 2x-1 & \text{if } x < 3 \\ 5 & \text{if } x \geq 3 \end{cases}$ show that

$$\lim_{x \rightarrow 3} f(x) = 5.$$

7. Show that $\lim_{x \rightarrow -2} \sqrt{x^2 - 4} = 0, = \lim_{x \rightarrow 2} \sqrt{x^2 - 4}$

8. If $f(x) = \begin{cases} x^2, & x \leq 1 \\ 2x-1 & x > 1 \end{cases}$, then find $\lim_{x \rightarrow 1^+} f(x)$ and $\lim_{x \rightarrow 1^-} f(x)$.

9. Show that $\lim_{x \rightarrow 0} \frac{\tan x}{x} = 1$.

10. Find $\lim_{x \rightarrow 0} \left\{ \frac{\sqrt{1+x-1}}{x} \right\}$

11. Compute $\lim_{x \rightarrow 0} \left[\frac{e^x - 1}{\sqrt{1+x-1}} \right]$.

Sol : For $0 < |x| < 1$.

$$\frac{e^x - 1}{\sqrt{1+x-1}} = \frac{e^x - 1}{\sqrt{1+x-1}} \times \frac{\sqrt{1+x+1}}{\sqrt{1+x+1}}$$

$$= \frac{e^x - 1(\sqrt{1+x+1})}{14x - y} = \frac{e^x - 1}{x}(\sqrt{1+x+1})$$

$$\lim_{x \rightarrow 0} \frac{e^x - 1}{\sqrt{1+x-1}} = \lim_{x \rightarrow 0} \frac{e^x - 1}{x}$$

$$\lim_{x \rightarrow 0} \sqrt{1+x+1} = 1.(\sqrt{1+0+1}) = (1+1) = 2$$

12. Show that $\lim_{x \rightarrow 0} \frac{x-3}{\sqrt{|x^2-9|}} = 0$

Sol : For $x^2 \neq 9$, $\left| \frac{x-3}{\sqrt{|x^2-9|}} \right| = \sqrt{\frac{|x-3|}{|x+3|}}$ (1)

$$\lim_{x \rightarrow 3} \sqrt{|x-3|} = 0, \quad \lim_{x \rightarrow 3} \sqrt{|x+3|} = \sqrt{6}$$

$$\therefore \lim_{x \rightarrow 3} \frac{x-3}{\sqrt{|x^2-9|}} = 0$$

13. Compute $\lim_{x \rightarrow 0} \frac{a^x - 1}{b^x - 1}$ ($a > 0, b > 0, b \neq 1$).

Sol : For $x \neq 0$, $\frac{a^x - 1}{b^x - 1} = \frac{\left[\frac{a^x - 1}{x} \right]}{\left[\frac{b^x - 1}{x} \right]}$

$$\lim_{x \rightarrow 0} \frac{a^x - 1}{b^x - 1} = \frac{\lim_{x \rightarrow 0} \frac{a^x - 1}{x}}{\lim_{x \rightarrow 0} \frac{b^x - 1}{x}} = \frac{\log_e^a}{\log_e^b}$$

14. Show that $\lim_{x \rightarrow \infty} \frac{1}{x^2} = 0$.

15. Show that $\lim_{x \rightarrow \infty} e^x = \infty$.

16. Compute $\lim_{x \rightarrow 2} \frac{x^2 + 2x - 1}{x^2 - 4x + 4}$

17. Evaluate $\lim_{x \rightarrow \infty} \frac{3x^5 - 1}{4x^2 + 1}$