LIMITS AT INFINITY

Definition:

Let f(x) be a function defined on A = (K,).

(i) A real number l is said to be the limit of f(x) at ∞ if to each $\varepsilon > 0, \exists$, an M > 0 (however large M may be) such that x$\in$ A and x > M $\Rightarrow |f(x) - \ell| < \varepsilon$ In this case we write $f(x) \to \ell$ as $x \to +\infty$ or $\underset{x\to+\infty}{Lt} f(x) = \ell$.

(ii) A real number l is said to be the limit of f(x) at $-\infty$ if to each $\varepsilon > 0, \exists$ > 0, an M > 0 (however large it may be) such that x$\in$ A and x < – M $\Rightarrow |f(x) - \ell| < \varepsilon$ In this case, we write $f(x) \to \ell$ as $x \to -\infty$ or $\underset{x\to-\infty}{Lt} f(x) = \ell$.

INFINITE LIMITS

Definition:

(i)Let f be a function defined is a deleted neighbourhood of D of a. (i) The limit of f at a is said to be ∞ if to each M > 0 (however large it may be) a δ > 0 such that x $\in$ D, 0 < |x–a| < δ $\Rightarrow$ f(x) > M. In this case we write f(x) $\to \infty$ as $x \to a$ or $\underset{x\to a}{Lt} f(x) = +\infty$

(ii) The limit of f(x) at a is said be $-\infty$ if to each M > 0 (however large it may be) a δ > 0 such that x$\in$ D, 0 < | x – a | < $\delta \Rightarrow$ f(x) < – M. In thise case we write f(x) $\to -\infty$ as $x \to a$ or $\underset{x\to a}{Lt} f(x) = -\infty$.

INDETERMINATE FORMS

While evaluation limits of functions, we often get forms of the type

$\frac{0}{0}, \frac{\infty}{\infty}, 0\times\infty, \infty-\infty, 0^0, 1^\infty, \infty^0$ which are termed as indeterminate forms.

EXERCISE – 8 (d)

I. Compute the following limits.

1. $\underset{x\to 3}{Lt}\dfrac{x^2+3x+2}{x^2-6x+9}$

Sol : $\underset{x\to 3}{Lt}\dfrac{x^2+3x+2}{x^2-6x+9}=\underset{x\to 3}{Lt}\dfrac{x^2+3x+2}{(x-3)^2}=\dfrac{9+9+2}{0}=\infty$

2. $\underset{x\to 1-}{Lt}\dfrac{1+5x^3}{1-x^2}$

Sol : $\underset{x\to 1-}{Lt}\dfrac{1+5x^3}{1-x^2}=\dfrac{1+5(1)^3}{1-1^2}=\dfrac{1+5}{1-1}=\dfrac{6}{0}=\infty$

3. $\underset{x\to\infty}{Lt}\dfrac{3x^2+4x+5}{2x^3+3x-7}$

Sol : $\underset{x\to\infty}{Lt}\dfrac{3x^2+4x+5}{2x^3+3x-7}$

$$=\underset{x\to\infty}{Lt}\dfrac{\left(3+\dfrac{4}{x}+\dfrac{5}{x^2}\right)x^2}{\left(2+\dfrac{3}{x^2}-\dfrac{7}{x^3}\right)x^3}=\underset{x\to\infty}{Lt}\dfrac{\left(3+\dfrac{4}{x}+\dfrac{5}{x^2}\right)}{2+\dfrac{3}{x^2}-\dfrac{7}{x^3}}.\dfrac{1}{x}$$

As $x\to\infty,\dfrac{1}{x},\dfrac{1}{x^2},\dfrac{1}{x^3}\to 0$

$$=\underset{x\to\infty}{Lt}\dfrac{3x^2+4x+5}{2x^3+3x-7}=\dfrac{(3+0+0)}{2+0-0}.0=0$$

4. $\underset{x\to\infty}{Lt}\dfrac{6x^2 - x + 7}{x+3}$

Sol : $\underset{x\to\infty}{Lt}\dfrac{6x^2 - x + 7}{x+3} \quad \underset{x\to\infty}{Lt}\dfrac{x^2\left(6-\frac{1}{x}+\frac{7}{x^2}\right)}{x\left(1+\frac{3}{x}\right)}$

$$\underset{x\to\infty}{Lt}\frac{6-\frac{1}{x}+\frac{7}{x^2}}{1+\frac{3}{x}}.\underset{x\to\infty}{Lt}\, x = \frac{6-0+0}{1+0}\infty = \infty$$

5. $\underset{x\to\infty}{Lt}\; e^{-x^2}$

Sol : $\underset{x\to\infty}{Lt}\; e^{-x^2} = \underset{x\to\infty}{Lt}\dfrac{1}{e^{x^2}} = \dfrac{1}{\infty} = 0$ (since $e > 1$)

6. $\underset{x\to\infty}{Lt}\dfrac{\sqrt{x^2+6}}{2x^2-1}$

Sol : $\underset{x\to\infty}{Lt}\dfrac{\sqrt{x^2+6}}{2x^2-1} = \underset{x\to\infty}{Lt}\dfrac{x\sqrt{1+\frac{6}{x^2}}}{x^2\left(2-\frac{1}{x^2}\right)}$

$$= \underset{x\to\infty}{Lt}\;\frac{1}{x}\frac{\sqrt{1+\frac{6}{x^2}}}{2-\frac{1}{x^2}}$$

As $x \to \infty, \dfrac{1}{x} \to 0 \quad \therefore \quad \underset{x\to\infty}{Lt}\dfrac{\sqrt{x^2+6}}{2x^2-1}.\underset{x\to\infty}{Lt}\;\dfrac{1}{x}$

$$\underset{x\to 0}{Lt}\frac{\sqrt{1+\frac{1}{x^2}}}{2-\frac{1}{x^2}} = \frac{\sqrt{1+0}}{2-0}0 = 0$$

II.

1. $\underset{x \to 0}{Lt} \dfrac{8|x| + 3x}{3|x| - 2x}$

Sol : $as \;\; x \to \infty \Rightarrow |x| = x$ $(\because$ *here* x is positive$)$

$$\underset{x \to \infty}{Lt} \frac{8|x| + 3x}{3|x| - 2x} = \underset{x \to \infty}{Lt} \frac{8x + 3x}{3x - 2x} = \underset{x \to \infty}{Lt} \frac{11x}{x} = 11$$

2. $\underset{x \to \infty}{Lt} \dfrac{x^2 + 5x + 2}{2x^2 - 5x + 1}$

Sol : $= \underset{x \to \infty}{Lt} \dfrac{x^2 + 5x + 2}{2x^2 - 5x + 1} = \underset{x \to \infty}{Lt} \dfrac{x^2\left[1 + \frac{5}{x} + \frac{2}{x^2}\right]}{x^2\left(2 \frac{-5}{x} + \frac{1}{x^2}\right)}$

As $x \to \infty . \dfrac{1}{x} and \dfrac{1}{x^2} \to 0$

$$\underset{x \to \infty}{Lt} \frac{x^2 + 5x + 2}{2x^2 - 5x + 1} = \underset{x \to \infty}{Lt} \frac{1 + \frac{5}{x} + \frac{2}{x^2}}{2 - \frac{5}{x} + \frac{1}{x^2}} = \frac{1 + 0 + 0}{2 - 0 + 0} = \frac{1}{2}$$

3. $\underset{x \to -\infty}{Lt} \dfrac{2x^2 - x + 3}{x^2 - 2x + 5}$

Sol ; $\underset{x \to -\infty}{Lt} \dfrac{2x^2 - x + 3}{x^2 - 2x + 5} = \underset{x \to -\infty}{Lt} \dfrac{x^2\left(2 - \frac{1}{x} + \frac{3}{x^2}\right)}{x^2\left(1 + \frac{2}{x} + \frac{5}{x^2}\right)}$

$$\underset{x \to -\infty}{Lt} \frac{2x^2 - x + 3}{x^2 - 2x + 5} = \underset{x \to 0 - \infty}{Lt} \frac{2 - \frac{1}{x} + \frac{3}{x^2}}{1 - \frac{2}{x} + \frac{5}{x^2}}$$ As $x \to -\infty, \dfrac{1}{x} and \dfrac{1}{x^2} \to 0$

$$= \frac{2-0+0}{1-0+0} = \frac{2}{1} = 2$$

4. $$\underset{x\to-\infty}{Lt} \frac{11x^3 - 3x + 4}{13x^3 - 5x^2 - 7}$$

Sol : $$\underset{x\to-\infty}{Lt} \frac{11x^3 - 3x + 4}{13x^3 - 5x^2 - 7}$$

$$= \underset{x\to\infty}{Lt} \frac{x^3\left(11 - \frac{3}{x^2} + \frac{4}{x^3}\right)}{x^3\left(13 - \frac{5}{x} - \frac{7}{x^3}\right)}$$

As $x \to \infty, \frac{1}{x}, \frac{1}{x^2}$ and $\frac{1}{x^3} \to 0$

$$\underset{x\to\infty}{Lt} \frac{11x^3 - 3x + 4}{13x^3 - 5x^2 - 7}$$

$$= \underset{x\to\infty}{Lt} \frac{11 - \frac{3}{x^2} + \frac{4}{x^3}}{13 - \frac{5}{x} - \frac{7}{x^3}} = \frac{11-0+0}{13-0-0} = \frac{11}{13}$$

5. $$\underset{x\to 2}{Lt}\left(\frac{1}{x-2} - \frac{4}{x^2-4}\right)$$

Sol : $$\underset{x\to 2}{Lt}\left(\frac{1}{x-2} - \frac{4}{x^2-4}\right) = \underset{x\to 2}{Lt} \frac{x+2-4}{x^2-4}$$

$$= \underset{x\to 2}{Lt} \frac{x-2}{x^2-4} = \underset{x\to 2}{Lt} \frac{x-2}{(x-2)(x+2)} = \underset{x\to 2}{Lt} \frac{x+2-4}{x^2-4} \quad = \frac{1}{4}$$

6. $\underset{x\to-\infty}{Lt}\frac{5x^3+4}{\sqrt{2x^4+1}}$

Sol : $\underset{x\to-\infty}{Lt}\frac{5x^3+4}{\sqrt{2x^4+1}}=\underset{x\to-\alpha}{Lt}\frac{x^3\left(5+\frac{4}{x^3}\right)}{x^2\sqrt{2+\frac{1}{x^4}}}$

$$=\underset{x\to-\infty}{Lt}\ x.\frac{5+\frac{4}{x^3}}{\sqrt{2}+\frac{1}{x^4}}$$

As $x\to-\infty,\frac{1}{x},\frac{1}{x^2}.\frac{1}{x^3},\frac{1}{x^4}\to 0$

$$\underset{x\to-\infty}{Lt}\frac{5x^3+4}{\sqrt{2x^4}+1}(-\infty)=(-\infty).\frac{5}{\sqrt{1}}=-\infty.$$

7. $\underset{x\to\infty}{Lt}\left(\sqrt{x+1}-\sqrt{x}\right)$

Sol : $\underset{x\to\infty}{Lt}\left(\sqrt{x+1}-\sqrt{x}\right)$

$$=\underset{x\to\infty}{Lt}\frac{\left(\sqrt{x+1}-\sqrt{x}\right)\left(\sqrt{x+1}+\sqrt{x}\right)}{\sqrt{x+1}+\sqrt{x}}$$

$$=\underset{x\to\infty}{Lt}\frac{x+1-x}{\sqrt{x+1}+\sqrt{x}}$$

$$=\underset{x\to\infty}{Lt}\frac{1}{\sqrt{x}\left(\sqrt{1}+\frac{1}{x}+1\right)}$$

$$=\underset{x\to\infty}{Lt}\frac{\frac{1}{\sqrt{x}}}{\sqrt{1}+\frac{1}{x}+1}=\frac{0}{1+1}=0$$

8. $= \underset{x\to\infty}{Lt}\left(\sqrt{x^2+x}-x\right)$

Sol : $= \underset{x\to\infty}{Lt}\ \sqrt{x^2+x}-x$

$$= \underset{x\to\infty}{Lt}\frac{\left(\sqrt{x^2+x}-x\right)\left(\sqrt{x^2+x+}\sqrt{x}\right)}{\sqrt{x^2+x}+x} = \underset{x\to\infty}{Lt}\frac{x^2+x-x^2}{x\left(\sqrt{1+\frac{1}{x}}+1\right)}$$

$$= \underset{x\to\infty}{Lt}\frac{x}{x\left(\sqrt{1+\frac{1}{x}}+1\right)}$$

$$= \frac{1}{\sqrt{1+0}+1} = \frac{1}{1+1} = \frac{1}{2}$$

III.

1. $\underset{x\to-\infty}{Lt}\left(\frac{2x+3}{\sqrt{x^2-1}}\right)$

$$\underset{x\to-\infty}{Lt}\frac{2x+3}{\sqrt{x^2-1}}$$

Sol :

$$= \underset{x\to-\infty}{Lt}\frac{x\left(2+\frac{3}{x}\right)}{-x\sqrt{1-\frac{1}{x^2}}}\left(\because \textit{here}\ \ \text{x} \to \text{-}\infty\ \ \text{i.e., x is negative} \Rightarrow \sqrt{x^2} = -x\right)$$

As $x \to -\infty,\ \frac{1}{x},\ \frac{1}{x^2} \to 0$

$$\underset{x\to-\alpha}{Lt}\frac{2x+3}{\sqrt{x^2-1}} = -\frac{2+0}{\sqrt{1-0}} = -\frac{2}{1} = -2$$

2. $\underset{x\to\infty}{Lt}\ \dfrac{2+\sin x}{x^2+3}$

Sol : $\underset{x\to\infty}{Lt}\ \dfrac{2+\sin x}{x^2+3} = \underset{x\to\infty}{Lt}\ \dfrac{2+\sin x}{x^2\left(1+\dfrac{3}{x^2}\right)}$

$$= \underset{x\to\infty}{Lt}\ \frac{2+\sin x}{x^2\left(1+\dfrac{3}{x^2}\right)} = \underset{x\to\infty}{Lt}\ \frac{\dfrac{2}{x^2}+\dfrac{\sin x}{x^2}}{\left(1+\dfrac{3}{x^2}\right)}$$

$as\ \ \text{x} \to \infty,\ \dfrac{1}{x^2}\, and\ \ \dfrac{\text{sinx}}{x^2} \to 0.(\because -1 \le \sin x \le 1) = \dfrac{0+0}{1+0} = 0$

3. $\underset{x\to\infty}{Lt}\ \dfrac{2+\cos^2 x}{x+2007}$ **(ans 0)**

4. $\underset{x\to\infty}{Lt}\ \dfrac{6x^2-\cos\ 3x}{x^2+5}$ **ans. 6**

5. $\underset{x\to\infty}{Lt}\ \dfrac{\cos x+\sin^2 x}{x+1}$ try yourself. $ans = \underset{x\to\infty}{Lt}\ \dfrac{\cos x+\sin^2 x}{x+1} = 0$

PROBLEMS FOR PRACTICE

1. **Evaluate** $\underset{x\to -3}{Lt}\ \dfrac{1}{x+2}$.

2. **Compute** $\underset{x\to 2}{Lt}\ \dfrac{x-2}{x^3-8}$.

3. **Find** $\underset{x\to 1}{Lt}(x+2)(2x+1)$

4. **Compute** $\underset{x\to 0}{Lt}\ x^2.\sin\dfrac{1}{x}$.

5. **Show that** $\underset{x\to 0+}{Lt}\frac{|x|}{x}=1$ **and** $\underset{x\to 0-}{Lt}\frac{|x|}{x}=-1(x\neq 0)$

6. **Let f :** $R\to R$ **be defined by** $f(x)=\begin{cases}2x-1 & \text{if} \quad x<3\\ 5 & \text{if} \quad x\geq 3\end{cases}$ **show that**

$\underset{x\to 3}{Lt}f(x)=5.$

7. **Show that** $\underset{x\to -2}{Lt}\sqrt{x^2-4}=0,=\underset{x\to 2}{Lt}\sqrt{x^2-4}$

8. **If** $f(x)=\begin{cases}x^2, & x\leq 1\\ 2x-1 & x>1\end{cases}$, **then find** $\underset{x\to 1+}{Lt}f(x)$ **and** $\underset{x\to 1-}{Lt}f(x).$

9. **Show that** $\underset{x\to 0}{Lt}\frac{\tan x}{x}=1.$

10. **Find** $\underset{x\to 0}{Lt}\left\{\frac{\sqrt{1+x-1}}{x}\right\}$

11. **Compute** $\underset{x\to 0}{Lt}\left[\frac{e^x-1}{\sqrt{1+x}-1}\right].$

Sol : For $0<|x|<1.$

$$\frac{e^x-1}{\sqrt{1+x}-1}=\frac{e^x-1}{\sqrt{1+x}-1}\times\frac{\sqrt{1+x}+1}{\sqrt{1+x}+1}$$

$$=\frac{e^x-1\left(\sqrt{1+x}+1\right)}{14x-y}=\frac{e^x-1}{x}\left(\sqrt{1+x}+1\right)$$

$$\underset{x\to 0}{Lt}\frac{e^x-1}{\sqrt{1+x}-1}=\underset{x\to 0}{Lt}\frac{e^x-1}{x}$$

$$\underset{x\to 0}{Lt}\sqrt{1+x}+1=1.\left(\sqrt{1+0}+1\right)=(1+1)=2$$

12. **Show that** $\underset{x\to 0}{Lt}\dfrac{x-3}{\sqrt{|x^2-9|}}=0$

Sol : For $x^2 \neq 9, \left|\dfrac{x-3}{\sqrt{x^2-9}}\right| = \sqrt{\left|\dfrac{x-3}{x+3}\right|}$ .....(1)

$\underset{x\to 3}{Lt}\sqrt{|x-3|}=0,\ \underset{x\to 3}{Lt}\sqrt{|x+3|}=\sqrt{6}$

$\therefore \underset{x\to 3}{Lt}\dfrac{x-3}{\sqrt{x^2-9}}=0$

13. **Compute** $\underset{x\to 0}{Lt}\dfrac{a^x-1}{b^x-1}(a>0, b>0, b\neq 1).$

Sol : For $x\neq 0, \dfrac{a^x-1}{b^x-1}=\dfrac{\left[\dfrac{a^x-1}{x}\right]}{\left[\dfrac{b^x-1}{x}\right]}$

$$\underset{x\to 0}{Lt}\frac{a^x-1}{b^x-1}=\frac{\underset{x\to 0}{Lt}\dfrac{a^x-1}{x}}{\underset{x\to 0}{Lt}\dfrac{b^x-1}{x}}=\frac{\log_e^a}{\log_e^b}$$

14. **Show that** $\underset{x\to\infty}{Lt}\dfrac{1}{x^2}=0.$

15. **Show that** $\underset{x\to\infty}{Lt}\ e^x=\infty.$

16. **Compute** $\underset{x\to 2}{Lt}\dfrac{x^2+2x-1}{x^2-4x+4}$

17. **Evaluate** $\underset{x\to\infty}{Lt}\dfrac{3x^5-1}{4x^2+1}$