CHAPTER 8

LIMITS

TOPICS:

1.INTERVALS AND NEIGHBOURHOODS
2.FUNCTIONS AND GRAPHS
3.CONCEPT OF LIMIT
4.ONE SIDED LIMITS
5. STANDARD LIMITS
6. INFINITE LIMITS AND LIMITS AT INFINITY
7. EVALUATION OF LIMITS BYDIRECT SUBSTITUTION METHOD
8. EVALUATION OF LIMITS BY FACTORISATION METHOD
9. EVALUATION OF LIMITS BYRATIONALISATION METHOD
10. EVALUATION OF LIMITS BY APPLICATION OF THE STANDARD LIMIT

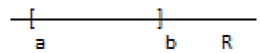
LIMITS

INTERVALS

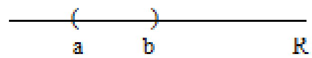
Definition:

Let a, b$\in$ R and a < b. Then the set {x$\in$ R: a$\leq$ x$\leq$ b} is called a closed interval. It is denoted by [a, b]. Thus

Closed interval [a, b] = {x$\in$ R: a $\leq$ x $\leq$ b}. It is geometrically represented by

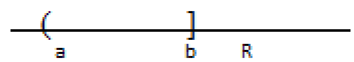


Open interval (a,b) = $\{x \in R : a < x < b\}$ It is geometrically represented by



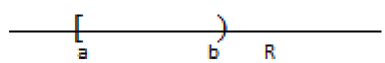
Left open interval

(a, b] = $\{x \in R : a < x \leq b\}$. It is geometrically represented by

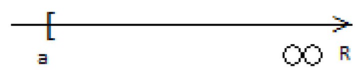


Right open interval

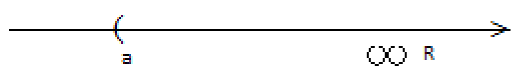
[a, b) = $\{x \in R : a \leq x < b\}$. It is geometrically represented by



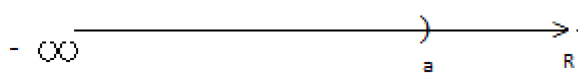
$[a, \infty) = \{x \in R : x \geq a\} = \{x \in R : a \leq x < \infty\}$ It is geometrically represented by



$(a, \infty) = \{x \in R : x > a\} = \{x \in R : a < x < \infty\}$



$(-\infty, a] = \{x \in R : x \leq a\} = \{x \in R : -\infty < x < a\}$



NEIGHBOURHOOD OF A POINT

Definition: Let $a \in R$. If $\delta > 0$ then the open interval $(a-\delta,\ a+\delta)$ is called the neighbourhood (δ - nbd) of the point a. It is denoted by $\boldsymbol{N_\delta(a)}$. a is called the centre and δ is called the radius of the neighbourhood .

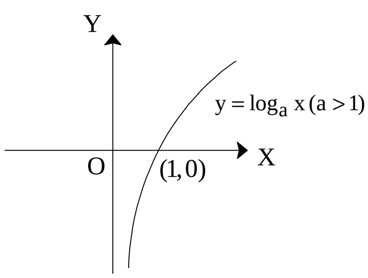
$\therefore N_\delta(a) = (a-\delta,\ a+\delta) = \{x \in R : a-\delta < x < a+\delta\} = \{x \in R : |x-a| < \delta\}$

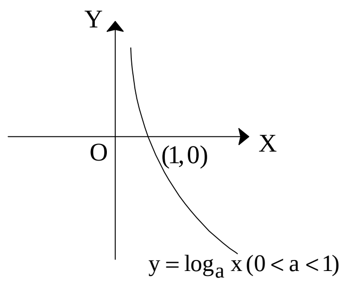
The set $N_\delta(a) - \{a\}$ is called a deleted δ - neighbourhood of the point a.

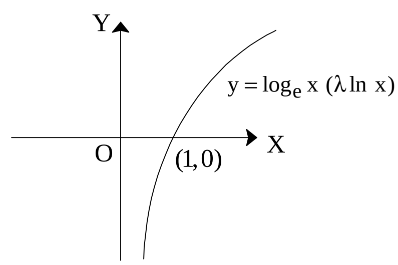
$\therefore N_\delta(a) - \{a\} = (a-\delta,\ a) \cup (a,\ a+\delta) = \{x \in R : 0 < |x-a| < \delta\}$

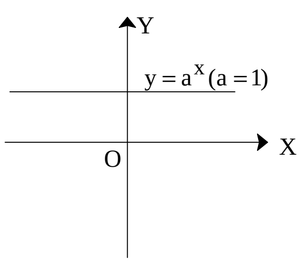
Note: $(a-\delta,\ a)$ is called left δ -neighbourhood, $(a,\ a+\delta)$ is called right δ - neighbourhood of a

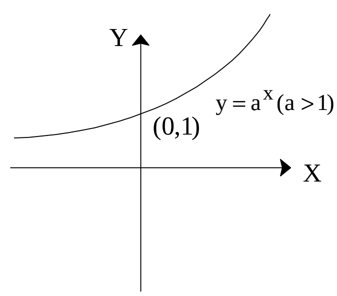
GRAPH OF A FUNCTION:







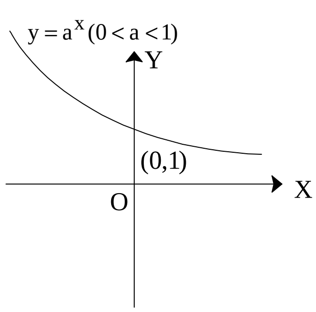


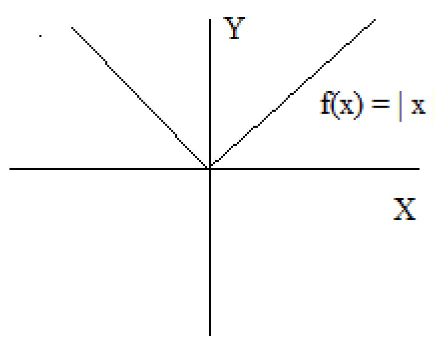


Mod function:

The function f : R-R defined by $f(x) = |x|$ is called the mod function or modulus function or absolute value function.

Dom f = R, Range f = [0,)



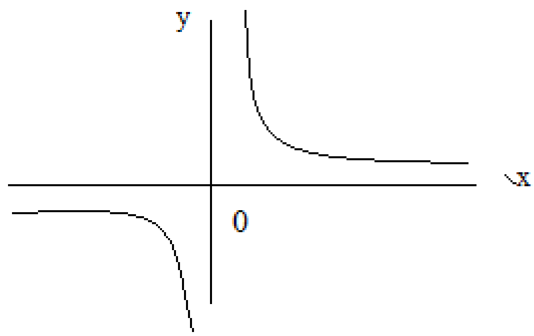


Reciprocal function :

The function $f : R - \{0\} - R$ defined by

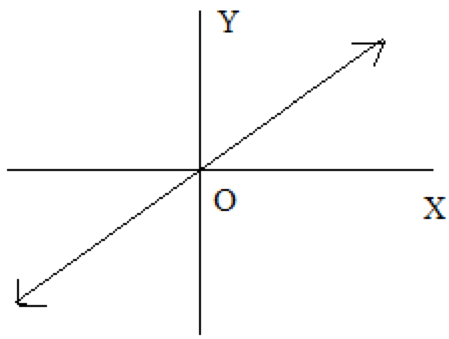
$f(x) = \frac{1}{x}$ is called the reciprocal function.

Dom $f = R - \{0\}$, Range $f = R$



Identity funciton:

The function f: R-R defined by f(x) = x is called the identity function on R. It is denoted by I(x) .



LIMIT OF A FUNCTION

Concept of limit:

Before giving the formal definition of limit consider the following example.

Let f be a function defined by $f(x) = \frac{x^2 - 4}{x - 2}$. clearly, f is not defined at x= 2.

When $x \neq 2, x - 2 \neq 0 and f(x) = \frac{(x-2)(x+2)}{x-2} = x + 2$

Now consider the values of f(x) when x≠2, but very very close to 2 and <2.

x	1.9	1.99	1.999	1.9999	1.99999
F(x)	3.9	3.99	3.999	3.9999	3.99999

It is clear from the above table that as x approaches 2 i.e.,$x\to 2$ through the values less than 2, the value of f(x) approaches 4 i.e., $f(x)\to 4$. We will express this fact by saying that left hand limit of f(x) as $x\to 2$ exists and is equal to 4 and in symbols we shall write $\underset{x\to 2^-}{\mathrm{lt}}\, f(x)=4$

Again we consider the values of f(x) when $x\neq 2$, but is very-very close to 2 and x>2.

x	2.1	2.01	2.001	2.0001	2.00001
F(x)	4.1	4.01	4.001	4.0001	4.00001

It is clear from the above table that as x approaches 2 i.e.,$x\to 2$ through the values greater than 2, the value of f(x) approaches 4 i.e., $f(x)\to 4$. We will express this fact by saying that right hand limit of f(x) as $x\to 2$ exists and is equal to 4 and in symbols we shall write $\underset{x\to 2^+}{\mathrm{lt}}\, f(x)=4$.

Thus we see that f(x) is not defined at x=2 but its left hand and right hand limits as $x\to 2$ exist and are equal.

When $\underset{x\to a^+}{\mathrm{lt}}\, f(x), \underset{x\to a^-}{\mathrm{lt}}\, f(x)$ are equal to the same number l, we say that $\underset{x\to a}{\mathrm{lt}}\, f(x)$ exist and equal to l.

Thus , in above example, $\underset{x\to 2^+}{\mathrm{lt}}\, f(x)=\underset{x\to 2^-}{\mathrm{lt}}\, f(x)=4$. $\therefore \underset{x\to 2}{\mathrm{lt}}\, f(x)=4$

ONE SIDED LIMITS

DEFINITION OF LEFT HAND LIMIT

Let f be a function defined on (a – h, a), h > 0. A number ℓ_1 is said to be the left hand limit (LHL) or left limit (LL) of f at a if to each $\varepsilon>0, \exists\ a\ \delta>0$ such that, $a-\delta<x<a \Rightarrow |f(x)-\ell_1|<\varepsilon$.

In this case we write $\underset{x\to a-}{Lt}\, f(x)=\ell_1$ (or) $\underset{x\to a-0}{Lt}\, f(x)=\ell_1$

DEFINITION OF RIGHT LIMIT:

Let f be a function defined on (a, a + h), h > 0. A number ℓ_2 is said to the right hand limit (RHL) or right limit (RL) of f at a if to each $\varepsilon>0, \exists\ a\ \delta>0$ such that $a<x<a+\delta \Rightarrow |f(x)-\ell_2|<\varepsilon$

In this case we write $\underset{x\to a+}{Lt}\, f(x)=\ell_2$ (or) $\underset{x\to a+0}{Lt}\, f(x)=\ell_2$.

DEFINITION OF LIMIT.

Let $A\subseteq R$, a be a limit point of A and $f : A\to R$. A real number l is said to be the limit of f at a if to each $\varepsilon>0, \exists\ a\ \delta>0$ such that $x\in A$, $0<|x-a|<\delta \Rightarrow |f(x)-l|<\in$.

In this case we write f(x) $\to$ l as x$\to$ a or $\underset{x\to a}{Lt} f(x) = \ell$

NOTE:

1.If a function *f* is defined on (*a–h, a*) for some *h* > 0 and is not defined on (*a, a+h*) and if $\underset{x\to a-}{Lt} f(x)$ **exists then** $\underset{x\to a}{Lt} f(x) = \underset{x\to a-}{Lt} f(x)$**.**

2. If a function *f* is defined on (*a, a + h*) for some *h* > 0 **and is not defined on (*a – h, a*) and if** $\underset{x\to a+}{Lt} f(x)$ **exists then** $\underset{x\to a}{Lt} f(x) = \underset{x\to a+}{Lt} f(x)$.

THEOREM

If $\underset{x\to a}{Lt} f(x)$ **exists then** $\underset{x\to a}{Lt} f(x) = \underset{x\to 0}{Lt} f(x+a) = \underset{x\to 0}{Lt} f(a-x)$

THEOREMS ON LIMITS WITH OUT PROOFS

1. If f : R$\to$R defined by f(x) = c, a constant then $\underset{x\to a}{Lt} f(x) = c$ **for any a $\in$ R.**

2. If f: R$\to$R defined by f(x) = x, then $\underset{x\to a}{Lt} f(x) = a$ *i.e.,* $\underset{x\to a}{Lt} x = a \quad (a \in R)$

3. Algebra of limits

Let $\underset{x\to a}{Lt} f(x) = \ell, \underset{x\to a}{Lt} g(x) = m.$ **then**

i) $\underset{x\to a}{Lt} (f+g)(x) = \underset{x\to a}{Lt} (f(x)+g(x)) = \ell + m$

ii) $\underset{x\to a}{Lt} (f-g)(x) = \underset{x\to a}{Lt} (f(x)-g(x)) = \ell - m$

iii) $\underset{x\to a}{Lt} (cf)(x) = \underset{x\to a}{Lt} c.f(x) = c \underset{x\to a}{Lt} f(x) = c\ell$

iv) $\underset{x\to a}{Lt} (fg)(x) = \underset{x\to a}{Lt} (f(x).g(x)) = \ell.m$

v) $\underset{x\to a}{Lt} \left(\frac{f}{g}\right)(x) = \underset{x\to a}{Lt} \left(\frac{f(x)}{g(x)}\right) = \frac{\ell}{m}$ **(m≠0)**

vi) $\underset{x\to a}{Lt} (f(x)-\ell) = 0$ **and vii)** $\underset{x\to a}{Lt} |f(x)| = |\ell|$

vii) If *f*(*x*) $\le$ *g*(*x*) in some deleted neighbourhood of *a*, then $\underset{x\to a}{Lt} f(x) \le \underset{x\to a}{Lt} g(x)$

viii) If *f*(*x*) $\le$ *h*(*x*) $\le$ *g*(*x*) in a deleted nbd of *a* and $\underset{x\to a}{Lt} f(x) = \ell = \underset{x\to a}{Lt} g(x)$ **then** $\underset{x\to a}{Lt} h(x) = \ell$

ix) If $\underset{x\to a}{Lt} f(x) = 0$ and $g(x)$ is a bounded function in a deleted nbd of a then $\underset{x\to a}{Lt} f(x)g(x) = 0$**.**

THEOREM

If n is a positive integer then $\underset{x\to a}{Lt}\ x^n = a^n, a \in R$

THEOREM

If f(x) is a polynomial function, then $\underset{x\to a}{Lt}\ f(x) = f(a)$

EVALUATION OF LIMITS

A) Evaluation of limits involving algebraic functions.

To evaluate the limits involving algebraic functions we use the following methods:

1) **Direct substitution method**
2) **Factorisation method**
3) **Rationalisation method**
4) **Application of the standard limits.**

1) **Direct substitution method:**

This method can be used in the following cases:

(i) If $f(x)$ is a polynomial function, then $\underset{x\to a}{Lt}\ f(x) = f(a)$.

(ii) If $f(x) = \dfrac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomial functions then $\underset{x\to a}{Lt}\ f(x) = \dfrac{P(a)}{Q(a)}$, provided $Q(a) \neq 0$.

2) **Factiorisation Method:**

This method is used when $\underset{x\to a}{Lt}\ \dfrac{f(x)}{g(x)}$ is taking the indeterminate form of the type $\dfrac{0}{0}$ by the substitution of $x = a$.

In such a case the numerator (Nr.) and the denominator (Dr.) are factorized and the common factor $(x - a)$ is cancelled. After eliminating the common factor the substitution $x = a$ gives the limit, if it exists.

3) **Rationalisation Method :** This method is used when $\underset{x\to a}{Lt}\ \dfrac{f(x)}{g(x)}$ is a $\frac{0}{0}$ form and either the Nr. or Dr. consists of expressions involving radical signs.

4) **Application of the standard limits.**

In order to evaluate the given limits , we reduce the given limits into standard limits form and then we apply the standard limits.

EXERCISE – 8(a)

I. Compute the following limits.

1. $\underset{x \to a}{Lt} \dfrac{x^2 - a^2}{x - a}$

Sol : Given limit $= \underset{x \to a}{Lt} \dfrac{x^2 - a^2}{x - a} = \underset{x \to a}{Lt} \dfrac{(x+a)(x-a)}{x - a} = \underset{x \to a}{Lt}(x+a) \quad = a + a$

$= 2a$

2. $\underset{x \to a}{Lt}\left(x^2 + 2x + 3\right)$

Sol : given function $f(x) = x^2 + 2x + 3$ is a polynomial.

$\therefore \underset{x \to 1}{Lt}\left(x^2 + 2x + 3\right) = 1^2 + 2.1 + 3 \quad = 1 + 2 + 3 = 6$

3. $\underset{x \to 0}{Lt} \dfrac{1}{x^2 - 3x + 2}$

Sol : $\underset{x \to 0}{Lt} \dfrac{1}{x^2 - 3x + 2} = \dfrac{1}{0 - 0 + 2} = \dfrac{1}{2}$

4. $\underset{x \to 3}{Lt} \dfrac{1}{x + 1}$

Sol : $\underset{x \to 3}{Lt} \dfrac{1}{x + 1} = \dfrac{1}{3 + 1} = \dfrac{1}{4}$

5. $\underset{x \to 1}{Lt} \dfrac{2x + 1}{3x^2 - 4x + 5}$

Sol : $\underset{x \to 1}{Lt} \dfrac{2x + 1}{3x^2 - 4x + 5} = \dfrac{2.1 + 1}{3.1^2 - 4.1 + 5} = \dfrac{3}{4}$

6. $\underset{x \to 1}{Lt} \dfrac{x^2 + 2}{x^2 - 2}$

Sol : $\underset{x \to 1}{Lt} \dfrac{x^2 + 2}{x^2 - 2} = \dfrac{1^2 + 2}{1^2 - 2} = \dfrac{1+2}{1-2} = \dfrac{3}{-1} = -3$

7. $\underset{x \to 1}{Lt} \left(\dfrac{2}{x+1} - \dfrac{3}{x}\right)$

Sol : G.L.= $\underset{x \to 2}{Lt} \left(\dfrac{2}{x+1} - \dfrac{3}{x}\right) = \dfrac{2}{2+1} - \dfrac{3}{2} = \dfrac{2}{3} - \dfrac{3}{2} = \dfrac{4-9}{6} = \dfrac{-5}{6}$

8. $\underset{x \to 0}{Lt} \left[\dfrac{x-1}{x^2 + 4}\right]$

Sol : $\underset{x \to 0}{Lt} \left[\dfrac{x-1}{x^2 + 4}\right] = \dfrac{0-1}{0+4} = -\dfrac{1}{4}$

9. $\underset{x \to 0}{Lt} x^{3/2} \, (x > 0)$

Sol : $\underset{x \to 0}{Lt} x^{3/2} \, (x > 0) = 0^{3/2} = 0$

10. $\underset{x \to 0}{Lt} \left(\sqrt{x} + x^{5/2}\right)(x > 0)$

Sol : $\underset{x \to 0}{Lt} \left(\sqrt{x} + x^{5/2}\right) = \sqrt{0} + 0^{5/2} = 0 + 0 = 0$

11. $\underset{x \to 0}{Lt} x^2 \cos \dfrac{2}{x}$

Sol : $\underset{x \to 0}{Lt} x^2 . \underset{x \to 0}{Lt} \cos \dfrac{2}{x} = 0.k$ Where $|k| \leq 1 = 0$

EXERCISE – 8(b)

I. **Find the right and left hand limits of the functions in 1,2,3 of I and 1,2,3 of II at the point a mentioned against them. Hence, check whether the functions have limits at those a' s.**

1. $f(x) = \begin{cases} 1-x & \text{if} \quad x \le 1 \\ 1+x & \text{if} \quad x > 1 \end{cases}$; $a = 1$.

Sol : *Left* lim*it at* x=1 is $\underset{x \to 1-}{Lt} (1-x) = 1-1 = 0$

Right limit at x=1 is $\underset{x \to 1+}{Lt} (1+x) = 1+1 = 2$

$\underset{x \to 1-}{Lt} f(x) \neq \underset{x \to 1+}{Lt} f(x)$

$\therefore \underset{x \to 1}{Lt} f(x)$ does not exist.

2. $f(x) = \begin{cases} x+2 & \text{if} \quad -1 < x \le 3 \\ x^2 & \text{if} \quad 3 < x < 5 \end{cases}$; $a = 3$.

Sol : $L.L = \underset{x \to 3-}{Lt} (x+2) = 3+2 = 5$

$R.L = \underset{x \to 3+}{Lt} x^2 = 3^2 = 9$

$\underset{x \to 3-}{Lt} f(x) \neq \underset{x \to 3+}{Lt} f(x)$

$\underset{x \to 3}{Lt} f(x)$ does not exists.

3. $f(x) = \begin{cases} \dfrac{x}{2} & if \quad x < 2 \\ \dfrac{x^2}{3} & if \quad x \geq 2 \end{cases}$; $x = 2.$

Sol: At x =21

$$LL = \underset{x \to 2-}{Lt} f(x) = \underset{x \to 2-}{Lt} \frac{x}{2} = \frac{2}{2} = 1$$

$$R.L = \underset{x \to 2+}{Lt} f(x) = \underset{x \to 2+}{Lt} \frac{x^2}{3} = \frac{4}{3}$$

$$\underset{x \to 2-}{Lt} f(x) \neq \underset{x \to 2+}{Lt} f(x)$$

$\underset{x \to 2}{Lt} f(x)$ does not exist.

II.

1. $f(x) = \begin{cases} 1 & \text{if} \quad x < 0 \\ 2x + 1 & \text{if} \quad 0 \leq x < 1 \\ 3x & \text{if} \quad x > 1 \end{cases}$;a $= 1.$

Sol : At x =1

$$L.L = \underset{x \to 1-}{Lt} f(x) = \underset{x \to 1-}{Lt} 2x + 1 = 2(1) + 1 = 3$$

$$R.L = \underset{x \to 1+}{Lt} f(x) = \underset{x \to 1+}{Lt} 3x = 3(1) = 3$$

$$\underset{x \to 1-}{Lt} f(x) = \underset{x \to 1+}{Lt} f(x) = 3 \quad \therefore \underset{x \to 1}{Lt} f(x) = 3$$

2. $f(x) = \begin{cases} x^2 & \text{if} \quad x \le 1 \\ x & \text{if} \quad 1 < x \le 2 \\ x-3 & \text{if} \quad x > 2 \end{cases}$; $a = 2$.

At x =2

$$L.L = \underset{x \to 2-}{Lt} f(x) = \underset{x \to 2-}{Lt} x = 20$$

$$R.L = \underset{x \to 2+}{Lt} f(x) = \underset{x \to 2+}{Lt} (x-3) \qquad = 2 - 3 = -1$$

$\underset{x \to 2-}{Lt} f(x) \neq \underset{x \to 2+}{Lt} f(x)$. hence $\underset{x \to 2}{Lt} f(x)$ does not exist.

3. **Show that** $\underset{x \to 2-}{Lt} \dfrac{|x-2|}{x-2} = -1$

Sol : $x \to 2- \Rightarrow x < 2 \Rightarrow \quad x - 2 < 0$

Then, $|x-2| = -(x-2)$

$$\underset{x \to 2-}{Lt} \frac{|x-2|}{x-2} = \underset{x \to 2-}{Lt} \frac{-(x-2)}{(x-2)} = -1$$

4. **Show that** $\underset{x \to 0+}{Lt} \left(\dfrac{2|x|}{x} + x + 1 \right) = 3.$

Sol : $x \to 0+ \Rightarrow x > 0 \Rightarrow \quad |x| = x$

$$\therefore \underset{x \to 0+}{Lt} \left(\frac{2|x|}{x} + x + 1 \right)$$

$$= \underset{x \to 0+}{Lt} \left(\frac{2x}{x} + x + 1 \right) = \underset{x \to 0+}{Lt} (2 + x + 1) = \underset{x \to 0+}{Lt} (2 + 0 + 1) = 3$$

5. **Compute** $\underset{x \to 2+}{Lt}\ ([x]+x)$ **and** $\underset{x \to 2-}{Lt}\ ([x]+x)$.

Sol : $\underset{x \to 2^+}{Lt}\ \{[x]+x\}. = \underset{h \to 0+}{Lt}\ \{[2+h]+(2+h)\} \ = [2+0]+2+0 \quad \left(\because \left[2^+\right] = 2\right)$

$= 2 + 2 = 4$

$\underset{x \to 2-}{Lt}\ \{[x]+x\} = \left[2^-\right] + 2 \quad = 1 + 2 = 3$

6. **Show that** $\underset{x \to 0-}{Lt}\ x^3 \cos \dfrac{3}{x} = 0$

Sol : For any $x, -1 \le \cos \dfrac{3}{x} \le 1$

$\underset{x \to 0-}{Lt}\ -x^3 . \cos \dfrac{3}{x} = -\underset{x \to 0-}{Lt}\ x^3 . \underset{x \to 0-}{Lt}\ \cos \dfrac{3}{x} = 0.k \ \ =0.$ where $-1 \le k \le 1$

III. **1.** **Compute** $\underset{x \to 2-}{Lt}\ \sqrt{2-x}\ (x<2)$. **What is** $\underset{x \to 2}{Lt}\ \sqrt{2-x}$?

Sol : $\underset{x \to 2-}{Lt}\ \sqrt{2-x} = \underset{h \to 0^+}{Lt}\ \sqrt{2-(2-h)} \ \left(\because \underset{x \to a-}{Lt}\ f(x) = \underset{h \to o^+}{Lt}\ f(x)\right)$

$= \underset{h \to 0+}{Lt}\ \sqrt{h} = 0$ The function is not defined when x>2. Therefore we consider only the left limit.. Hence we will not consider the right limit of the function.

So we consider $\underset{x \to 2}{Lt}\ \sqrt{2-x} = \underset{x \to 2-}{Lt}\ \sqrt{2-x} \qquad \therefore \underset{x \to 2}{Lt}\ \sqrt{2-x} = 0$

2. **Compute** $\underset{x\to\left(-\frac{1}{2}\right)+}{Lt}\sqrt{1+2x}$. **Hence find** $\underset{x\to-\frac{1}{2}}{Lt}\sqrt{1+2x}$.

Sol : $\underset{x\to-\frac{1}{2}+}{Lt}\sqrt{1+2x} = \underset{h\to 0+}{Lt}\sqrt{1+2\left\{\left(-\frac{1}{2}\right)+h\right\}} = \underset{h\to 0+}{Lt}\sqrt{1-1+2h} = 0$

The function is not defined When $x < \frac{-1}{2}$ is not defined.

Hence $\underset{x\to-\frac{1}{2}}{Lt}\sqrt{1+2x} = \underset{x\to-\frac{1}{2}+}{Lt}\sqrt{1+2x} = 0$