CHAPTER

3-D GEOMETRY

TOPICS:-

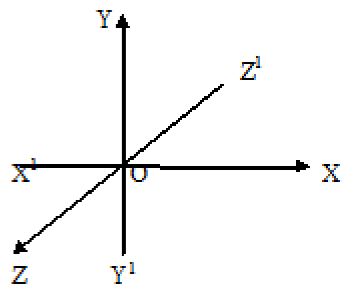
1. Introduction to 3-D system and coordinates Axes and coordinate planes, coordinate of a point in the space

2. Distance between two points, section formula, points of trisection and mid - point, Centroid of triangle and tetrahedron.

3. Translation of axes.

COORDINATES OF A POINT IN SPACE (3-D) (2 MARKS)

Let $\overrightarrow{X'OX}, \overrightarrow{Y'OY}$ and $\overrightarrow{Z'OZ}$ be three mutually perpendicular straight lines in space, intersecting at O. This point O is called origin.



Axes :

The three fixed straight lines $\overrightarrow{X'OX}, \overrightarrow{Y'OY}$ and $\overrightarrow{Z'OZ}$ are respectively called X-axis ,Y-axis and Z-axis. The three lines taken together are called rectangular coordinate axes.

COORDINATE PLANES

The plane containing the axes of Y and Z is called yz-plane. Thus yoz is the yz plane. Similarly the plane zox containing the axes of z and x is called ZX-plane and the plane xoy is called the xy-plane and contains x axis and y axis.

The above three planes are together called the rectangular co ordinate plane.

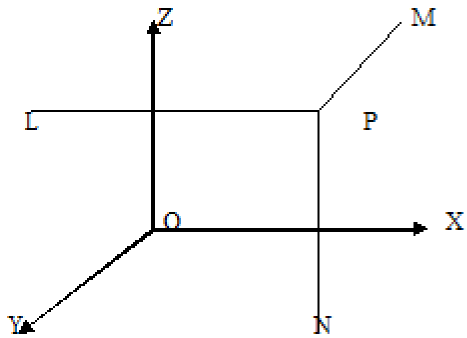
OCTANTS. The three co ordinate planes divide the whole space into 8 parts called octants.

COORDINATES OF A POINT.

Let P be any point in the space. Draw through P , three planes parallel to the three co ordinate planes meeting the axes of X,Y,Z in the points A,B and C respectively. Then if OA=x, OB=y and OC=z, the three numbers x,y,z taken in this order are called the co ordinates of the point P and we refer the point as (x,y,z) . Any one of these x,y,z will be positive of negative according as it is measured from O along the corresponding axis, in the positive or negative direction.

Another method of finding coordinates of a point.

The coordinates x,y,z of a point P are the perpendicular distances of P from the three co ordinate planes YZ,ZX and XY respectively.



From fig PN = z, PL = x and PM = y

Therefore point P =(x,y,z)

Note. On YZ- plane, a point has x coordinate as zero and similarly on zx-plane y coordinates and on xy-plane z coordinates are zero.

For any point on the

(i) X-axis, Y,Z coordinates are equal to zero,

(ii) Y-axis, X,Z coordinates are equal to zero,
(iii) Zaxis, X,Y coordinates are equal to zero.

DISTANCE BETWEEN THE POINTS

1.
The distance between the points and is given by $PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

Note:-
If is the origin and is a point in space, then
$OP = \sqrt{(x - 0)^2 + (y - 0)^2 + (z - 0)^2} = \sqrt{x^2 + y^2 + z^2}$

EXERCISE

I.

1. Find the distance of P(3, –2, 4) from the origin.

Sol. Origin 0=(0,0,0) and P(3, –2, 4)

$$OP = \sqrt{x^2 + y^2 + z^2} = \sqrt{9+4+16} = \sqrt{29} \text{ units}$$

2. Find the distance between the points (3, 4, –2) and (1, 0, 7).

Sol. Given points are P(3, 4, –2) and Q (1, 0, 7)

$$PQ = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2} = \sqrt{(3-1)^2 + (4-0)^2 + (-2-7)^2}$$

$$= \sqrt{4+16+81} = \sqrt{101} \text{ units}$$

II.

1. Find x if the distance between (5, –1, 7) and (x, 5, 1) is 9 units.

Sol. Given Points are P(5, –1, 7), Q(x, 5, 1) and given that PQ = 9

$$PQ = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2} = 9$$

$$\Rightarrow \sqrt{(5-x)^2 + (-1-5)^2 + (7-1)^2} = 9$$

Squaring on both sides

$\Rightarrow (5 - x)^2 + 36 + 36 = 81$

$\Rightarrow (5 - x)^2 = 81 - 72 = 9$

$\Rightarrow 5 - x = \pm 3$

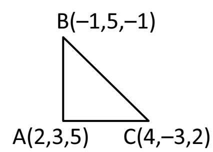
$\Rightarrow 5 - x = 3$ or $5 - x = -3$

$\Rightarrow x = 5 - 3$ or $x = 5 + 3$

$\Rightarrow x = 2$ or 8

2. Show that the points (2, 3, 5), (–1, 5, –1) and (4, –3, 2) form a right angled isosceles triangle.

Sol. Given points are A(2, 3, 5), B(–1, 5, –1), C(4, –3, 2)



$$AB = \sqrt{(2+1)^2 + (3-5)^2 + (5+1)^2}$$

$\Rightarrow AB^2 = (2 + 1)^2 + (3 - 5)^2 + (5 + 1)^2$

$= 9 + 4 + 36 = 49$

Similarly, $BC^2 = (-1 - 4)^2 + (5 + 3)^2 + (-1-2)^2 = 25 + 64 + 9 = 98$

And $CA^2 = (4 - 2)^2 + (-3 - 3)^2 + (2 - 5)^2 = 4 + 36 + 9 = 49$

from above values $AB^2 + AC^2 = BC^2$

Therefore, ABC is a right angled isosceles triangle.

3. Show that the points (1, 2, 3), (2, 3, 1) and (3, 1, 2) form an equilateral triangle.

Sol. Given points are A(1, 2, 3), B(2, 3, 1) and C(3, 1, 2)

$$AB = \sqrt{(1-2)^2 + (2-3)^2 + (3-1)^2} = \sqrt{1+1+4} = \sqrt{6}$$

$$BC = \sqrt{(2-3)^2 + (3-1)^2 + (1-2)^2} = \sqrt{6}$$

$$CA = \sqrt{(3-1)^2 + (1-2)^2 + (2-3)^2} = \sqrt{6}$$

$\Rightarrow$ AB = BC = CA

$\Rightarrow$ABC is an equilateral triangle.

4. P is a variable point which moves such that 3PA = 2PB. If A = (–2, 2, 3) and B = (13, –3, 13). prove that P satisfies the equation $x^2 + y^2 + z^2 + 28x - 12y + 10z - 247 = 0$.

Sol. Given points are A(–2, 2, 3) and B = (13, –3, 13)

Let P(x, y, z) be any point on the locus.

Given that 3PA = 2PB $\Rightarrow$ 9 PA^2 = 4 PB^2

$9[(x+2)^2 + (y-2)^2 + (z-3)^2] = 4[(x-13)^2 + (y+3)^2 + (z-13)^2]$

$\Rightarrow 9(x^2 + 4x + 4 + y^2 - 4y + 4 + z^2 - 6z + 9) = x^2 - 26x + 169 + y^2 + 6y + 9 + z^2 - 26z + 169)$

$\Rightarrow 9x^2 + 9y^2 + 9z^2 + 36x - 36y - 54z + 153 = 4x^2 + 4y^2 + 4z^2 - 104x + 24y - 104z + 1388$

$\Rightarrow 5x^2 + 5y^2 + 5z^2 + 140x - 60y + 50z - 1235 = 0$

$\Rightarrow x^2 + y^2 + z^2 + 28x - 12y + 10z - 247 = 0.$

Locus of P is $x^2 + y^2 + z^2 + 28x - 12y + 10z - 247 = 0.$

5. Show that the points (1, 2, 3) (7, 0, 1) and (–2, 3, 4) are collinear.

Sol. Given points are A(1, 2, 3), B(7, 0, 1) and C(–2, 3, 4)

$$AB = \sqrt{(1-7)^2 + (2-0)^2 + (3-1)^2} = \sqrt{36+4+4} = \sqrt{44} = 2\sqrt{11}$$

$$BC = \sqrt{(7+2)^2 + (0-3)^2 + (1-4)^2} = \sqrt{81+9+9} = \sqrt{99} = 3\sqrt{11}$$

$$CA = \sqrt{(-2-1)^2 + (3-2)^2 + (4-3)^2} = \sqrt{9+1+1} = \sqrt{11}$$

From above values $AB + AC = 2\sqrt{11} + \sqrt{11} = 3\sqrt{11} = BC$

Hence, the points A, B, C are collinear.

6. Show that ABCD is a square where A, B, C, D are the points (0, 4, 1), (2, 3, –1), (4, 5, 0) and (2, 6, 2) respectively.

Sol. Given points A = (0, 4, 1), B = (2, 3, –1), C = (4, 5, 0) and D = (2, 6, 2)

$$AB = \sqrt{(0-2)^2 + (4-3)^2 + (1+1)^2} = 3$$

$$BC = \sqrt{(2-4)^2 + (3-5)^2 + (-1-0)^2} = 3$$

$$CD = \sqrt{(4-2)^2 + (5-6)^2 + (0-2)^2} = 3$$

$$DA = \sqrt{(2-0)^2 + (6-4)^2 + (2-1)^2} = 3$$

$\therefore$ AB = BC = CD = DA

$$AC = \sqrt{(0-4)^2 + (4-5)^2 + (1-0)^2} = \sqrt{18}$$

$$BD = \sqrt{(2-2)^2 + (3-6)^2 + (-1-2)^2} = \sqrt{18}$$

$\Rightarrow$ AC = BC and $AB^2 + BC^2 = 9 + 9 = 19 = AC^2$

$\Rightarrow \angle ABC = 90°$

A, B, C, D are the vertices of a square.

SECTION FORMULA

(i) **$P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be two points in space and let R be a point on the line segment joining P and Q such that it divides $\overline{PQ}$ internally in the ratio $m:n$. Then the coordinates of are** $\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}, \frac{mz_2 + nz_1}{m+n}\right)$

(ii) **$P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be two points in space and let R be a point on the line segment joining P and Q such that it divides $\overline{PQ}$ externally in the ratio $m:n$. Then the coordinates of are** $\left(\frac{mx_2 - nx_1}{m-n}, \frac{my_2 - ny_1}{m-n}, \frac{mz_2 - nz_1}{m-n}\right), m \neq n$

MID POINT

The mid point of the linesegment joining the points $P(x_1, y_1, z_1)$ **and** $Q(x_2, y_2, z_2)$ is

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$$

Centroid of a triangle.

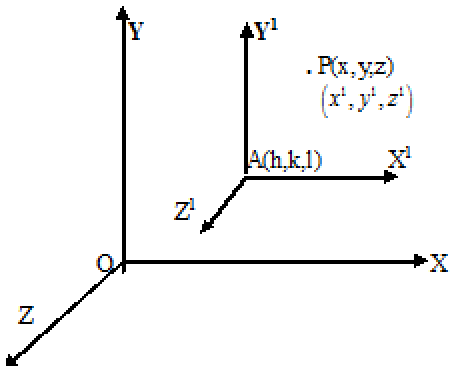
The coordinates of the centriod of the triangle with vertices $A(x_1, y_1, z_1), B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$ is $\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3} + \frac{z_1 + z_2 + z_3}{3}\right)$

Centroid of a tetrahydron.

The coordinates of the centriod of the tetrahedron with vertices $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2), C(x_3, y_3, z_3)$ and $D(x_4, y_4, z_4)$

is $\left(\frac{x_1 + x_2 + x_3 + x_4}{4}, \frac{y_1 + y_2 + y_3 + y_4}{4}, \frac{z_1 + z_2 + z_3 + z_4}{4}\right)$

TRANSLATION OF AXES



Let P(x,y,z) and A(h,k,l) be two points is space w.r.t the frame of reference OXYZ. Now treating A as the origin, let $\overrightarrow{AX^1}, \overrightarrow{AY^1}, \overrightarrow{AZ^1}$ be the new axes parallel to

$\overrightarrow{OX}$,$\overrightarrow{OY}, \overrightarrow{OZ}$ respectively. If $(x^1, y^1, z^1)$are the coordinates of P w.r.t

$AX^1Y^1Z^1$,then $x^1 = x - h, y^1 = y - k, z^1 = z - l$.

Ex. Origin is shifted to the point (1,2 -3). Find the new coordinates of (1,0,-1)

Sol. (x, y, z) = (1,0,-1) and(h, k, l) = (1,2 -3).

Now new coordinates are X = x-h = 1 -1 = 0

Y = y –k = 0-2 = -2

Z = z-l = -1 +3 =2

herefore new coordinates are (0,-2,2)

EXERCISE 5

1. Find the ratio in which the xz-plane divides the line joining A(–2, 3, 4) and B(1, 2, 3).

Sol. The Ratio in which xz plane divides the line segment joining the points A(–2, 3, 4) and (1, 2, 3) is $-y_1 : y_2 = -3 : 2$

2. Find the coordinates of the vertex C of ΔABC if its centroid is the origin and the vertices A, B are (1, 1, 1) and (–2, 4, 1) respectively.

Sol. A(1, 1, 1), B(–2, 4, 1) are the vertices of ΔABC.

Let C= (x, y, z)

Given O is the centroid of ΔABC

$$\Rightarrow \left(\frac{1-2+x}{3}, \frac{1+4+y}{3}, \frac{1+1+z}{3}\right) = (0,0,0)$$

$\Rightarrow \frac{x-1}{3} = 0, \frac{y+5}{3} = 0, \frac{z+2}{3} = 0$

$x - 1 = 0,\ y + 5 = 0,\ z + 2 = 0 \Rightarrow x = 1,\ y = -5,\ z = -2$

$\therefore$ Coordinates of c are (1, –5, –2).

3. If (3, 2, –1), (4, 1, 1) and (6, 2, 5) are three vertices and (4, 2, 2) is the centroid of a tetrahedron, find the fourth vertex.

Sol. A(3, 2, –1), B(4, 1, 1), C(6, 2, 5), let D=(x, y, z) be the 4th vertex of the tetrahedron.

Given centroid G = (4, 2, 2)

But G = $\left(\frac{3+4+6+x}{4}, \frac{2+1+2+y}{4}, \frac{-1+1+5+z}{4}\right)$

Therefore, $\left(\frac{13+x}{4}, \frac{5+y}{4}, \frac{5+z}{4}\right) = (4,2,2)$

$\Rightarrow \frac{13+x}{4} = 4, \frac{5+y}{4} = 2, \frac{5+z}{4} = 2$

$\Rightarrow$ 13 + x = 16, 5 + y = 8, 5+ z = 8

$\Rightarrow$ x = 3, y = 3, z = 3

Coordinates of D are (3, 3, 3)

4. Find the distance between the midpoint of the line segment $\overrightarrow{AB}$ and the point (3,–1, 2) where A = (6, 3, –4) and B = (–2, –1, 2).

Sol. Given points are A= (6, 3, –4), B = (–2, –1, 2)

Mid point of AB is Q= $\left(\frac{6-2}{2}, \frac{3-1}{2}, \frac{-4+2}{2}\right) = (2,1,-1)$

Given point P = (3, –1, 2)

$\Rightarrow PQ = \sqrt{(3-2)^2 + (-1-1)^2 + (2+1)^2} = \sqrt{1+4+9} = \sqrt{14}$ units.

II.

1. Show that the points (5, 4, 2) (6, 2, –1) and (8, –2, –7) are collinear.

Sol. Given points are A(5, 4, 2), B(6, 2, –1), C(8, –2, –7)

Show that AB + BC = AC.

2. Show that the points A(3, 2, –4), B(5, 4, –6) and C(9, 8, –10) are collinear and find the ratio in which B divides $\overline{AC}$.

Sol. Given points are A(3, 2, –4), B(5, 4, –6) and C(9, 8, –10)

AB = $\sqrt{(3-5)^2 + (2-4)^2 + (-4+6)^2} = \sqrt{4+4+4} = \sqrt{12} = 2\sqrt{3}$

BC = $\sqrt{(5-9)^2 + (4-8)^2 + (-6+10)^2} = \sqrt{16+16+16} = \sqrt{48} = 4\sqrt{3}$

CA = $\sqrt{(9-3)^2 + (8-2)^2 + (-10+4)^2} = \sqrt{36+36+36} = \sqrt{108} = 6\sqrt{3}$

From above values AB + BC = $2\sqrt{3}+4\sqrt{3}=6\sqrt{3}=CA$

$\therefore$ A, B, C are collinear.

The Ratio in which B divides AC is AB: BCi.e., $2\sqrt{3}:4\sqrt{3}=1:2$

III.

1. If A(4, 8, 12), B(2, 4,6), C(3, 5, 4) and D(5, 8, 5) are four points, show that the line $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersect. Also find their point of intersection.

Sol. Given points are A(4, 8, 12), B(2, 4,6), C(3, 5, 4), D(5, 8, 5)

Equation of $\overrightarrow{AB}$ is $\frac{x-2}{2}=\frac{y-4}{4}=\frac{z-6}{6}=t$ i.e.,

$\overrightarrow{AB}$ is $\frac{x-2}{1}=\frac{y-4}{2}=\frac{z-6}{3}=t$

Any point on this line is P (2+t, 4+2t,6+3t)

Equation of $\overrightarrow{CD}$ is $\frac{x-3}{2}=\frac{y-5}{3}=\frac{z-4}{1}=s$

Any of on this line is Q (3+2s,5+3s,4+s)

Lines are intersecting , P =Q

(2+t, 4+2t,6+3t) = (3+2s,5+3s,4+s)

t+2 =3+2s , 4+2t = 5+3s and 6+3t =4+s

$\Rightarrow$ t-2s =1---------(1) ,

2t -3s =1 ---------- -(2) and

3t-s = -2--------------(3)

Solving (1) and (2), t= -1 , s=-1

Substituting these values in (3), 3(-1) - (-1) = -2

-2 = -2

Therefore eq.(3) is satisfied by the values of t and s.

Hence the lines are intersecting lines.

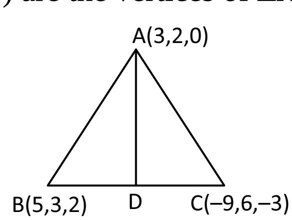
Sub t = -1 in point P then P=(2-1, 4-2, 6-3) = (1,2,3)

2. Find the point of intersection of the lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ where A = (7, –6, 1), B = (17, –18, –3), C = (1, 4, –5) and D = (3, –4, 11).

Same as above problem.

3. A(3, 2, 0), B(5, 3, 2), C(–9, 6, –3) are vertices of a triangle. $\overline{AD}$, the bisector of $\angle$BAC meets $\overline{BC}$ at D. Find the coordinates of D.

Sol. A(3, 2, 0), B(5, 3, 2), C(–9, 6, –3) are the vertices of ΔABC.



AB $= \sqrt{(3-5)^2+(2-3)^2+(0-2)^2} \quad = \sqrt{4+1+4} = 3$

AC $= \sqrt{(3+9)^2+(2-6)^2+(0+3)^2} \quad = \sqrt{144+16+9} = \sqrt{169} = 13$

Let AD be the bisector of $\angle$BAC. Then D divides BC in the ratio AB : AC i.e., 3 : 13

$\therefore$ Coordinates of D are $\left(\frac{3(-9)+13\cdot 5}{3+13}, \frac{3\cdot 6+13\cdot 3}{3+13}, \frac{3(-3)+13\cdot 2}{3+13}\right) \quad = \left(\frac{38}{16}, \frac{57}{16}, \frac{17}{16}\right)$

4. Show that the points O(0, 0, 0), A(2, –3, 3), B(–2, 3, –3) are collinear. Find the ratio in which each point divides the segment joining the other two.

Sol. Given points are O(0, 0, 0), A(2, –3, 3), B(–2, 3, –3)

OA $= \sqrt{(0-2)^2+(0+3)^2+(0-3)^2} \quad = \sqrt{4+9+9} = \sqrt{22}$

OB $= \sqrt{(0+2)^2+(0-3)^2+(0+3)^2} \quad = \sqrt{4+9+9} = \sqrt{22}$

AB $= \sqrt{(2+2)^2+(-3-3)^2+(3+3)^2} \quad = \sqrt{16+36+36} = \sqrt{88} = 2\sqrt{22}$

OA + OB = $\sqrt{22}+\sqrt{22} = 2\sqrt{22}$ = AB

$\therefore$ O, A, B are collinear.

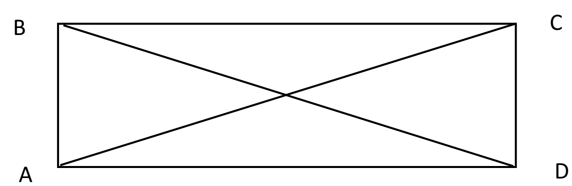
The Ratio in which O divides AB is OA : OB = $\sqrt{22}:\sqrt{22} = 1:1$

The Ratio in which A divides OB is OA : AB = $\sqrt{22}:2\sqrt{22} = 1:2$

The Ratio in which B divides OA is AB : BO = $2\sqrt{22}:\sqrt{22} = 2:1$

5. Find the fourth vertex of the parallelogram whose consecutive vertices are (2, 4, –1), (3, 6, –1) are (4, 5, 1).

Sol.



ABCD is a parallelogram

where A = (2, 4, –1), B = (3, 6, –1), C = (4, 5, 1)

let D(x, y, z) be the fourth vertex

since A B C D is a parallelogram, therefore Mid point of AC = Mid point of BD

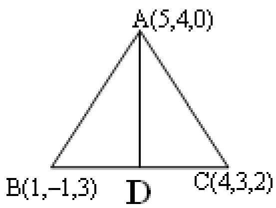
$$\left(\frac{2+4}{2}, \frac{4+5}{7}, \frac{-1+1}{2}\right) = \left(\frac{3+x}{2}, \frac{6+y}{2}, \frac{-1+z}{2}\right)$$

$$\frac{3+x}{2} = \frac{6}{2} \Rightarrow x = 3, \frac{6+y}{2} = \frac{9}{2} \Rightarrow y = 3, \frac{z-1}{2} = \frac{0}{2} \Rightarrow z = 1$$

$\therefore$ Coordinates of the fourth vertex are D (3, 3, 1)

6. A(5, 4, 6), B(1, –1, 3), C(4, 3, 2) are three points. Find the coordinates of the point in which the bisector of $\angle$BAC meets the side $\overline{BC}$.

Sol. We know that if AB is the bisector of $\angle$BAC divides BC in the ratio AB : AC



$$AB = \sqrt{(5-1)^2+(4+1)^2+(6-3)^2} = \sqrt{16+25+9} = \sqrt{50} = 5\sqrt{2}$$

$$AC = \sqrt{(5-4)^2+(4-3)^2+(6-2)^2} = \sqrt{1+1+16} = \sqrt{18} = 3\sqrt{2}$$

D divides BC in the ratio AB :AC i.e., $5\sqrt{2} : 3\sqrt{2}$ i.e., 5 : 3.

Coordinates of D are $\left(\frac{5\cdot4+3\cdot1}{5+3}, \frac{5\cdot3+3\cdot(-1)}{5+3}, \frac{5\cdot2+3\cdot3}{5+3}\right) = \left(\frac{23}{8}, \frac{3}{2}, \frac{19}{8}\right)$

Problems for practice

1. **Show that the points A(–4, 9, 6), B(–1, 6, 6) and C(0, 7, 10) from a right angled isosceles triangle.**
2. **Show that the point whose distance from Y-axis is thrice its distance from (1, 2, –1) satisfies the equation $8x^2 + 9y^2 + 8z^2 - 18x - 36y + 18z + 54 = 0$.**
3. **Show that the points A(3, –2, 4), B(1, 1, 1) and C(–1, 4, 2) are collinear.**
4. **Find the ratio in which YZ-plane divides the line joining A(2, 4, 5) and B(3, 5, –4). Also find the point of intersection.**

   Ans. (0, 2, 23)
5. **Show that the points A(3, –2, 4), B(1, 1, 1) and C(–1, 4, –2) are collinear.**