7. HYPERBOLIC FUNCTIONS

Some formulae

1. $\sin hx = \frac{e^x - e^{-x}}{2}$

2. $\cos hx = \frac{e^x + e^{-x}}{2}$

3. $\tan hx = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

4. $\cos ec\, hx = \frac{2}{e^x - e^{-x}}$

5. $\sec hx = \frac{2}{e^x + e^{-x}}$

6. $\cot hx = \frac{e^x + e^{-x}}{e^x - e^{-x}}$

7. $\cos h^2 x - \sin h^2 x = 1$

8. $1 - \tan h^2 x = \sec h^2 x$

9. $\cot h^2 x - 1 = \cos ec\, h^2 x$

10. Prove that $\sin h^{-1} x = \log\left\{x + \sqrt{x^2+1}\right\}$

Let $\sin h^{-1} x = y \Rightarrow x = \sin \text{h}\, y$

$$x = \frac{e^y - e^{-y}}{2} = 2x = e^4 - \frac{1}{ey} \Rightarrow 2xe^y = \left(e^y\right)^2 - 1$$

$$\left(e^y\right)^2 - 2xe^4 - 1 = 0 \Rightarrow e^4 = \frac{2x \pm \sqrt{4x^2+4}}{2} = e^y = x + \sqrt{x^2+1} \Rightarrow y = \log_e\left(x + \sqrt{x^2+1}\right)$$

$$\boxed{\therefore \sin h^{-1} x = \log_e\left\{x + \sqrt{x^2+1}\right\}}$$

2. **Prove that** $\cos h^{-1} x = \log_e\left(x - \sqrt{x^2-1}\right)$

Solution:

Let $\cos h^{-1} x = y \Rightarrow x = \cos hy$

$$x = \frac{e^y + e^{-y}}{2} \Rightarrow 2x = e^y + \frac{1}{e^y}$$

$$2xe^y = \left(e^y\right)^2 + 1 \Rightarrow \left(e^y\right)^2 - 2xe^y + 1$$

$$e^y = \frac{2x \pm \sqrt{4x^2-4}}{2} \Rightarrow e^y = x \pm \sqrt{x^2-1}$$

$$e^y = x + \sqrt{x^2-1} \Rightarrow y = \log_e\left(x + \sqrt{x^2-1}\right)$$

$$\boxed{\cos h^{-1} x = \log_e\left(x + \sqrt{x^2-1}\right)}$$

3. **Prove that** $Tan\,h^{-1}x = \frac{1}{2}\ \log_e\left(\frac{4x}{1-x}\right)$

Let $Tan\,h^{-1}x = y \Rightarrow x = \tanh y$

$$x = \frac{e^y - e^{-y}}{e^y + e^{-y}} \Rightarrow \frac{1}{x} = \frac{e^y + e^{-y}}{e^y - e^{-y}}$$

Using component and dividend

$$\frac{1+x}{1-x} = \frac{e^y + e^{-y} + e^y - e^{-y}}{e^y + e^{-y} - e^y + e^{-y}}$$

$$\frac{1+x}{1-x} = \frac{2e^y}{2e^{-y}} \Rightarrow \frac{1+x}{1-x} = e^{2y}$$

$$2y = \left\{\frac{1+x}{1-x}\right\} \Rightarrow y = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$$

$$\boxed{\cot h^{-1}x = \frac{1}{2}\log\left(\frac{x+1}{x-1}\right)}$$

$$\rightarrow \sec h^{-1}x = \log\left\{\frac{1+\sqrt{1-x^2}}{x}\right\}$$

$$\rightarrow \cos ec\, h^{-1}x = \log\left(\frac{1-\sqrt{1+x^2}}{x}\right) x < 0$$

$$= \log\left\{\frac{1-\sqrt{1+x^2}}{x}\right\} x > 0$$

$$\rightarrow \sin h(x+y) = \sin hx \cos hy + \cosh x \sin hy$$

$$\rightarrow \sin h(x-y) = \sin hx \cos hy + \cos hx \sin hy$$

$$\rightarrow \cos h(x+y) = \cos hx \cos hy + \sin hx \sin hy$$

$$\rightarrow \cos h(x-y) = \cos hx \cos hy - \sin hx \sin hy$$

$$\rightarrow \sin h2x = 2\sin hx \cos hx = \frac{2Tan\,hx}{1-Tan\,h^2x}$$

$$\rightarrow \cosh 2x = \cosh^2 x + \sinh^2 x = 2\cosh^2 x - 1 = 1 + 2\sinh^2 x = \frac{1+Tanh^2x}{1-Tanh^2x}$$

$$\rightarrow Tanh(x+y)\frac{Tanhx + Tanhy}{1 - TanhxTanhy}$$

$$\rightarrow Tanh(x-y)\frac{Tanhx - Tanhy}{1 + TanhxTanhy}$$

$$\rightarrow \cot h(x+y)\frac{\cot hx \cot hy + 1}{\cot hy + \cot hx}$$

$$\rightarrow \cot h(x-y)=\frac{\cot hx \cot hy-1}{\cot hy-\cot hx}$$

$$\tan h2x=\frac{2\tan hx}{1+\tanh^2 x}$$

$$\rightarrow \cot h2x=\frac{\coth^2 x+1}{2\cot hx}$$

SOME PROBLEMS:

1. **If** $\sin hx=\frac{3}{4}$ **find** $\cos h2x \quad ad \ \sin h2x$

Solution:

$$\sin hx=\frac{3}{4}$$

$$\cos h^2 x=1+\sin h^2 x$$

$$\cos h^2 x=1+\frac{9}{16}\Rightarrow \cos h^2 x=\frac{25}{16}$$

$$\cos hx=\frac{5}{4}$$

$$\sin^2 x=2\sin hx\cos hx=2\times\frac{3}{4}\times\frac{5}{4}=\frac{15}{8}$$

$$\cos^2 x=\cos\text{h}^2 x+\sin h^2 x=\frac{25}{16}+\frac{9}{16}=\frac{34}{16}=\frac{17}{8}$$

2. **If** $\sin hx=3 \quad x=\log\left(3-1\sqrt{10}\right)$

Solution:

$$\sin hx=3 \ \Rightarrow \ x=\sin h^{-1}3$$

$$\sin h^{-1}y=\cos\left(y+\sqrt{y^2+1}\right)$$

$$\sin h^{-1}3=\log\left\{3+\sqrt{9+1}\right\}$$

$$=\log\left\{3+\sqrt{10}\right\}$$

3. **Prove that (i) Tan h(x-y) =** $\frac{\tan hx - \tan hy}{1 - \tan hx \tan hy}$

$$\text{R.H.S} = \frac{\tan hx - \tan hy}{1 - \tan hx \tan hy} = \frac{\frac{\sin hx}{\cos hx} - \frac{\sin hy}{\cos hy}}{1 - \frac{\sin hx}{\cos hx} \cdot \frac{\sin hy}{\cos hy}}$$

$$\frac{\sin hx \cos hy - \cos hx \sin hy}{\cos hx \cos hy - \sin hx \sin hy} = \frac{\sin h(x-y)}{\cos h(x-y)} = \tan h(x-y)$$

(ii) $\cot h(x-y) = \frac{\cot hx \ \cot hy - 1}{\cot hy - \cot hx}$

$$\text{RHS} \quad \frac{\cot hx \ \cot hy - 1}{\cot hy - \cot hx} = \frac{\frac{\cos hx}{\sin hx} \cdot \frac{\cos hy}{\sin hy} - 1}{\frac{\cos hy}{\sin hy} - \frac{\cos hx}{\sin hx}}$$

$$\frac{\frac{\cos hx \cos hy - \sin hx \sin hy}{\sin hx \sin hy}}{\frac{\sin hx \cos hy - \cos hx \sin hy}{\sin hx \sin hy}} = \frac{\cos h(x-y)}{\sin h(x-y)} = \cot h(x-y)$$

4. **Prove that** $(\cos hx - \sin hx)^n = \cos h\, nx - \sin hnx$

Solution:

$$(\cos hx - \sin hx)^n = \left\{\frac{e^x + e^{-x}}{2} - \frac{e^x - e^{-x}}{2}\right\}^n$$

$$= \left\{\frac{e^x + e^{-x} - e^x + e^{-x}}{2}\right\} = \left(e^{-x}\right) = e^{-nx}$$

$$\text{RHS= coshnx-sinhnx} = \frac{e^{nx} + e^{-nx}}{2} - \left(\frac{e^{nx} - e^{-nx}}{2}\right)$$

$$= \frac{e^{nx} + e^{-nx} - e^{nx} + e^{-nx}}{2} = e^{-nx}$$

(ii) $(\cos hx + \sin hx)^n = \cos h\, nx + \sin h\, nx$

$$\text{LHS} = (\cos hx + \sin hx)^n = \left(\frac{e^x + e^{-x} + e^x - e^{-x}}{2}\right)^n = \left(e^x\right)^n = e^{nx}$$

$$\text{RHS} \ \cos h\, nx + \sin h\, nx \frac{e^{nx} + e^{-nx} + e^{nx} - e^{-nx}}{2} = e^{nx}$$

LHS = RHS

5. **Prove that** $\frac{\tan hx}{\sec hx - 1} + \frac{\tan hx}{\sec hx + 1} == -2\cos ec\, hx$

$$\frac{\tan hx}{\sec hx - 1} + \frac{\tan hx}{\sec hx + 1} = \frac{\tan hx(\sec hx + 1) + \tan hx(\sec hx - 1)}{(\sec hx - 1)(\sec hx + 1)}$$

$$\frac{\tan hx\{\sec hx + y + \sec hx - y\}}{\sec h^2 x - 1}$$

$$\frac{2\tan hx.\sec hx}{-\tanh^2 x} = \frac{\frac{-2}{\cos hx}}{\frac{\sin hx}{\cos hx}} = -\cos ec\, hx$$

6. **Prove that** $\frac{\cos hx}{1 - \tan hx} + \frac{\sin hx}{1 - \cot hx} = \sin hx + \cos hx$

Solution:

$$\frac{\cos hx}{1 - \tan hx} + \frac{\sin hx}{1 - \cot hx}$$

$$\frac{\cos hx}{1 - \frac{\sin hx}{\cos hx}} + \frac{\sin hx}{1 - \frac{\cos hx}{\sin hx}} = \frac{\cosh^2 x}{\cos hx - \sin hx} + \frac{\sin h^2 x}{\sin hx - \cos hx}$$

$$= \frac{\cosh^2 x}{\cos hx - \sin hx} - \frac{\sin h^2 x}{\cos hx - \sin hx}$$

$$\frac{\cosh^2 x - \sin h^2 x}{\cos hx - \sin hx} = \frac{(\cos hx + \sin x)(\cos hx - \sin hx)}{(\cos hx - \sin hx)} = \cos hx + \sin hx$$

7. **Prove that** $\cos h^4 x - \sin h^4 x = \cos h^2 x$

Solution:

$$\cos h^4 x - \sin h^4 x = \left(\cos h^2 x\right)^2 - \left(\sin h^2 x\right)^2$$

$$\cos h^4 x - \sin h^4 x = \left(\cos h^2 x\right)^2 - \left(\sin h^2 x\right)^2$$

$$= \left(\cos h^2 x + \sin h^2 x\right)\left(\cos h^2 x - \sin h^2 x\right)$$

$$= \left(\cosh^2 x\right)(1) = \cos h2x$$

8. **If** $\mu = \log_e \left\{ \tan\left(\frac{\pi}{4} + \frac{\pi}{2} \right) \right\}$ **the prove that** $\cos h\mu = \sec\theta$

Solution:

$$\mu = \log_e \left\{ \tan\left(\frac{\pi}{4} + \frac{\pi}{2} \right) \right\} \Rightarrow e^{\mu} = \tan\left(\frac{\pi}{4} + \frac{\theta}{2} \right)$$

$$e^{-\mu} = \frac{1}{e^{\mu}} = \cot\left(\frac{\pi}{4} + \frac{\pi}{2} \right)$$

$$\cosh \mu = \frac{e^{\mu} + e^{-\mu}}{2} = \frac{1}{2}\left(\tan\left(\frac{\pi}{4} + \frac{\theta}{2} \right) + \cot\left(\frac{\pi}{4} + \frac{\theta}{2} \right) \right)$$

$$= \frac{1}{2}\left(\frac{\sin\left(\frac{\pi}{4} + \frac{\theta}{2} \right)}{\cos\left(\frac{\pi}{4} + \frac{\theta}{2} \right)} + \frac{\cos\left(\frac{\pi}{4} + \frac{\theta}{2} \right)}{+\sin\left(\frac{\pi}{4} + \frac{\theta}{2} \right)} \right)$$

$$= \frac{1}{2}\left(\frac{\sin^2\left(\frac{\pi}{4} + \frac{\theta}{2} \right)\cos^2\left(\frac{\pi}{4} + \frac{\theta}{2} \right)}{\cos\left(\frac{\pi}{4} + \frac{\theta}{2} \right)\sin\left(\frac{\pi}{4} + \frac{\theta}{2} \right)} \right)$$

$$= \frac{1}{2\sin\left(\frac{\pi}{4} + \frac{\theta}{2} \right)\cos\left(\frac{\pi}{4} + \frac{\theta}{2} \right)}$$

$$= \frac{1}{\sin\left\{ 2\left(\frac{\pi}{4} + \frac{\theta}{2} \right) \right\}} = \frac{1}{\sin\left(\frac{\pi}{4} + \theta \right)} = \frac{1}{\cos\theta} = \sec\theta$$

$$\therefore \cos h4 = \sec\theta$$

PROBLEMS FOR PRACTICE

1. **If** $\sin hx = 5$ **show that** $x = \log\left(5+\sqrt{20}\right)$

2. **Show that** $\tan \mathrm{h}^{-1}\left(\frac{1}{2}\right) = \frac{1}{2}\log_{e}^{3}$

3. **If** $x = \log\left\{\cot\left(\frac{\pi}{4}+\theta\right)\right\}$ **then prove that** $\cos hx = \sec 2\theta$ **and** $\sin hx = -\tan 2\theta$

4. **If** $\cos hx = \frac{5}{2}$ **find the values of** $\cos h2x$ **and** $\sin h2x$

5. **Prove that** $\sin h3x = 3\sin hx + 4\sinh^{3} x$

6. **Prove that** $\tan h3x = \dfrac{3\tan hx + \tanh^{3} x}{1+3\tan h^{2}x}$