Properties of Triangles

Key points:

1. **Sine rule : In ΔABC, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ = 2R Where R is the circum-radius.**

$\Rightarrow$ a = 2R sin A, b = 2R sin B, c = 2R sin C

a : b : c = sin A : sin B : sin C.

2. Cosine rule : In ΔABC,

$a^2 = b^2 + c^2 - 2bc \cos A$

$b^2 = c^2 + a^2 - 2ac \cos B$

$c^2 = a^2 + b^2 - 2ab \cos C$

or

$\cos A = \frac{b^2+c^2-a^2}{2bc}$; $\cos B = \frac{c^2+a^2-b^2}{2ca}$; $\cos C = \frac{a^2+b^2-c^2}{2ab}$ $\Rightarrow$ cos A : cos B : cos C.

$= a(b^2 + c^2 - a^2) : b(c^2 + a^2 - b^2) : c(a^2 + b^2 - c^2)$

3. Projection rule : In ΔABC

a = b cos C + c cos B, b = c cos A + a cos C, c = a Cos B + bcos A

4. Mollwiede's rule : In ΔABC

$\frac{a-b}{c} = \frac{\sin\frac{A-B}{2}}{\cos\frac{C}{2}}$, $\frac{a+b}{c} = \frac{\cos\frac{A-B}{2}}{\sin\frac{C}{2}}$

Similarly the other two can be written by symmetry.

5. Tangent rule (or) Napier's analogy : In ΔABC

$\tan\frac{A-B}{2} = \frac{a-b}{a+b}\cot\frac{C}{2}$;

$\tan\frac{B-C}{2} = \frac{b-c}{b+c}\cot\frac{A}{2}$;

$\tan\frac{C-A}{2} = \frac{c-a}{c+a}\cot\frac{B}{2}$

6. **Half angle formulae :**

i. $\sin\frac{A}{2} = \frac{\sqrt{(s-b)(s-c)}}{bc}$; $\sin\frac{B}{2} = \frac{\sqrt{(s-c)(s-a)}}{ca}$; $\sin\frac{C}{2} = \frac{\sqrt{(s-a)(s-b)}}{ab}$.

ii. $\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$; $\cos\frac{B}{2} = \sqrt{\frac{s(s-b)}{ac}}$; $\cos\frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}$

iii. $\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \sqrt{\frac{(s-b)(s-c)}{\Delta}}$; $\tan\frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}} = \sqrt{\frac{(s-a)(s-c)}{\Delta}}$;

$\tan\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}} = \sqrt{\frac{(s-a)(s-b)}{\Delta}}$

iv. $\cot\frac{A}{2} = \sqrt{\frac{s(s-a)}{(s-b)(s-c)}} = \sqrt{\frac{s(s-a)}{\Delta}}$;

$\cot\frac{B}{2} = \sqrt{\frac{s(s-b)}{(s-c)(s-a)}} = \sqrt{\frac{s(s-b)}{\Delta}}$; $\cot\frac{C}{2} = \sqrt{\frac{s(s-c)}{(s-a)(s-b)}} = \sqrt{\frac{s(s-c)}{\Delta}}$

7. $\cot A = \frac{b^2+c^2-a^2}{4\Delta}$; $\cot B = \frac{c^2+a^2-b^2}{4\Delta}$; $\cot C = \frac{a^2+b^2-c^2}{4\Delta}$.

8. Area of ΔABC is given by

i. $\Delta = \frac{1}{2}ab\sin C = \frac{1}{2}bc\sin A = \frac{1}{2}ca\sin B$

ii. $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$.

iii. $\Delta = \frac{abc}{4R}$

iv. $\Delta = 2R^2\sin A\sin B\sin C$

v. $\Delta = rs$

vi. $\Delta = \sqrt{r\,r_1\,r_2\,r_3}$

9. **If 'r' is radius of in circle and r_1, r_2, r_3 are the radii of ex-circles opposite to the vertices A, B, C of ΔABC respectively then**

i. $r = \frac{\Delta}{s}$, $r_1 = \frac{\Delta}{s-a}$, $r_2 = \frac{\Delta}{s-b}$, $r_3 = \frac{\Delta}{s-c}$

ii. $r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

$r_1 = 4R\sin\frac{A}{2}\cos\cos\frac{C}{2}$

$r_2 = 4R\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}$

$r_3 = 4R\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$

iii. $r = (s-a)\tan\frac{A}{2} = (s-b)\tan\frac{B}{2} = \tan(s-c)\frac{C}{2}$

$r_1 = s\tan\frac{A}{2} = (s-b)\cot\frac{C}{2} = (s-c)\cot\frac{B}{2}$

$r_2 = s\tan\frac{B}{2} = (s-c)\cot\frac{A}{2} = (s-a)\cot\frac{C}{2}$

$r_3 = s\tan\frac{C}{2} = (s-a)\cot\frac{B}{2} = (s-b)\cot\frac{A}{2}$

10. i) $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}$

ii) $rr_1r_2r_3 = \Delta^2$

11. i) $r_1r_2 + r_2r_3 + r_3r_1 = s^2$

ii) $r(r_1 + r_2 + r_3) = ab + bc + ca - s^2$.

i) $(r_1 - r)(r_2 + r_3) = a^2$

$(r_2 - r)(r_3 + r_1) = b^2$

$(r_3 - r)(r_1 + r_2) = c^2$

ii) $a = (r_2 + r_3)\sqrt{\frac{rr_1}{r_2r_3}}$, $b = (r_3 + r_1)\sqrt{\frac{rr_2}{r_3r_1}}$, $c =)r_1 + r_2)\sqrt{\frac{rr_3}{r_1r_2}}$

12. i) $r_1 - r = 4R\sin^2\frac{A}{2}$, $r_2 - r = 4R\sin^2\frac{B}{2}$, $r_3 - r = 4R\sin^2\frac{C}{2}$

ii) $r_1 + r_2 = 4R\cos^2\frac{C}{2}$, $r_2 + r_3 = 4R\cos^2\frac{A}{2}$, $r_3 + r_1 = 4R\cos^2\frac{B}{2}$

13. $r_1 + r_2 + r_3 = 4R$, $r + r_2 + r_3 - r_1 = 4R\cos A$, $r + r_3 + r_2 - r_2 = 4R\cos B$,

$r + r_1 + r_2 - r_3 = 4R\cos C$

14. In an equilateral triangle of side 'a'

i) area = $\frac{\sqrt{3}a^2}{4}$ ii) $R = a / \sqrt{3}$

iii) $r = R / 2$ iv) $r_1 = r_2 + r_3 = 3R / 2$

v) $r : R : r_1 = 1 : 2 : 3$

PROBLEMS

VSAQ'S

1. If the lengths of the sides of a triangle are 3,4,5 find the circum radius of the triangle.

Sol. sides of the triangle are 3,4,5

Therefore, the triangle is rt. Angled triangle and its hypotenuse is 5

Circum radius = $\frac{1}{2}.(hypotenuse) = \frac{5}{2}$

2. Show that $\sum a(\sin B - \sin C) = 0$

Solutoin: -

$$\sum 2R\sin A(\sin B - \sin C) \because a = 2R\sin A$$

$$2R\{\sin A(\sin B - \sin C) + \sin B(\sin C - \sin A) + \sin C(\sin A - \sin B)\} = 0$$

3. If $a=\sqrt{3}+1\,cms \;\; \angle B=30^0 \;\; \angle C=45^0$ **then find C**

Solution : -

$\angle B=30^0 \quad \angle C=45^0$ but $A+B+C=180^0 \Rightarrow A+75^0=180^0$

$A=105^0$

$$\frac{a}{\sin A}=\frac{c}{\sin C} \Rightarrow \frac{\sqrt{3}+1}{\sin 105^0}=\frac{C}{\sin 45^0} \Rightarrow \frac{\sqrt{3}+1}{\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)}=\frac{C}{\left(\frac{1}{\sqrt{2}}\right)}$$

4. If a = 2 , b = 3, c = 4 then find cos A

Solution : -

Cosine Rule we known that $\cos A=\frac{b^2+c^2-a^2}{2bc}$

$\cos A=\frac{9+16-4}{2\times 3\times 4} \Rightarrow \cos A=\frac{21}{24}=\frac{7}{8}$

5. If a = 26, b = 30 $\cos C=\frac{63}{65}$ **then find C**

Solution : -

From cosine rule $c^2=a^2+b^2-2ab\cos c$

$c^2=(26)^2+(30)^2-2(26)(30)\frac{63}{65} \Rightarrow c^2=676+900-1512$

5

$c^2=64 \Rightarrow c=8$

6. If the angles are in the ratio 1 : 5 : 6 then find the ratio of sides

Solution : -

Let A = x B = 5x C = 6x

We know that $A+B+C=180^0 \Rightarrow 12x=180^0 \Rightarrow x=15^0$

$\therefore A=15^0 \quad B=5\times 15^0 \quad C=6\times 15^0$

$A=15^0 \quad B=75^0 \quad C=90^0$

Ratio of sides $=a:b:c=2R\sin A:2R\sin B:2R\sin C$

$=\sin A:\sin B:\sin C$

$=\sin 15^0:\sin 75^0:\sin 90^0=\frac{\sqrt{3}-1}{2\sqrt{2}}:\frac{\sqrt{3}+1}{2\sqrt{2}}:1$

$=\left(\sqrt{3}-1\right):\left(\sqrt{3}+1\right):2\sqrt{2}$

7. **Prove tht** $2\left\{bc\cos A+ca\cos B+ab\cos C\right\}=a^2+b^2+c^2$

L.H.S $2bc\cos A+2ca\cos B+2ab\cos c$

From cosine rule $\cos A=\frac{b^2+c^2-a^2}{2bc} \quad \cos B=\frac{a^2+c^2-b^2}{2ac}$

$$\cos c = \frac{a^2 + b^2 - c^2}{2ab}$$

$$\therefore 2bc \frac{\{b^2 + c^2 - a^2\}}{2bc} + 2ac\left\{\frac{a^2 + c^2 - b^2}{2ac}\right\} + 2ab\frac{\{a^2 + b^2 - c^2\}}{2ab}$$

$$b^2 + c^2 - a^2 + a^2 + c^2 - b^2 + a^2 + b^2 - c^2 = a^2 + b^2 + c^2$$

8. . **Prove that** $\frac{a^2 + b^2 - c^2}{c^2 + a^2 - b^2} = \frac{\tan B}{\tan C}$

$$\tan B = \frac{\sin B}{\cos B} = \frac{\left(\frac{b}{2R}\right)}{\frac{a^2 + c^2 - b^2}{2ac}} = \frac{b}{2\text{R}} \times \frac{2ac}{a^2 + c^2 - b^2} = \frac{2\left(4R^2\ 4\right)}{2R\left(a^2 + c^2 - b^2\right)} \{\because abc = 4R\Delta\}$$

$$\tan B = \frac{4\Delta}{a^2 + c^2 - b^2} \qquad |||^{ly}\ \tan C = \frac{4\Delta}{a^2 + b^2 - c^2}$$

$$\text{R.H.S}\ \frac{\tan B}{\tan C} = \frac{\frac{4\Delta}{a^2 + c^2 - b^2}}{\frac{4\Delta}{a^2 + b^2 - c^2}} = \frac{a^2 + b^2 - c^2}{a^2 + c^2 - b^2}$$

9. **Prove that** $(b + c)\cos A + (c + a)\cos B + (a + b)\cos C = a + b + c$

Solution : -

$(b + c)\cos A + (c + a)\cos B + (a + b)\cos C$

$b\cos A + \text{c}\cos A + \text{c}\cos B + a\cos B + a\cos c + b\cos c$

$(b\cos A + a\cos B) + (b\cos c + c\cos B) + (c\cos A + a\cos c)$

c +a +b {from projection rule}
$= a + b + c$

9. **Prove that** $(b - a\cos c)\sin A = a\cos A\sin c$

Solution : -

From projection rule $b = a\cos c + c\cos A$

L.H.S $= (a\cos C + c\cos A - a\cos C)\sin A = c\cos A\sin A$

$= 2R\sin C\cos A\sin A$

$= (2R\sin A)\cos A\sin C$

$= a\cos A\sin C$

10. **If 4, 5 are two sides of a triangle and the include angle is** 60^0 **find the area**

Solution: -

Given b = 4 C = 5 A = 60^0

Area of triangle $\Delta = \frac{1}{2}bc\sin A = \frac{1}{2} \times 4 \times 5\sin 60^0 = 5\sqrt{3}$

11. **Show that** $b\cos^2\frac{C}{2}+\cos^2\frac{B}{2}=S$

Solution : -

We known that $\cos\frac{C}{2}=\sqrt{\frac{S(S-C)}{ab}}\ \frac{\cos B}{2}=\sqrt{\frac{S(S-b)}{ac}}$

$$b\cos^2\frac{C}{2}+c\cos^2\frac{B}{2}=b\frac{S(S-C)}{ab}+c\frac{S(S-b)}{ac}$$
$$=\frac{S}{a}\{s-c+s\}=\frac{s}{a}[2s-b-c]=\frac{s}{a}\{a+\not{b}+\not{c}-\not{b}-\not{c}\}$$
$$=S$$

12. **If** $\frac{a}{\cos A}=\frac{b}{\cos B}=\frac{c}{\cos C}$ **then show that triangle ABC is equilateral**

Solution: -

$$\frac{a}{\cos A}=\frac{b}{\cos B}=\frac{c}{\cos A}\Rightarrow\frac{2R\sin A}{\cos A}=\frac{2R\sin B}{\cos B}=\frac{2R\sin C}{\cos C}$$
$$\tan A=\tan B=\tan C\Rightarrow A=B=C$$

$\therefore$ Triangle ABC is equilateral

13. Show that Σ a cot A = 2(R + r).

Sol. $\Sigma a\cot A=\Sigma 2R\sin A\frac{\cos A}{\sin A}$

$$=2R\Sigma\cos A$$

$$=2R\left(1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)\quad(\because\ \text{from transformations})$$

$$=2\left(R+4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)=2(R+r)$$

14. In ΔABC, prove that $r_1 + r_2 + r_3 - r = 4R$.

Sol. $r_1+r_2+r_3-r=4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}+4R\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}+4R\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$

$$-4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

$$=4R\sin\frac{A}{2}\left[\cos\frac{B}{2}\cos\frac{C}{2}-\sin\frac{B}{2}\sin\frac{C}{2}\right]+4R\cos\frac{A}{2}\left[\sin\frac{B}{2}\cos\frac{C}{2}+\cos\frac{B}{2}\sin\frac{C}{2}\right]$$

$$=4R\sin\frac{A}{2}\cos\left(\frac{B+C}{2}\right)+4R\cos\frac{A}{2}\sin\left(\frac{B+C}{2}\right)$$

$$=4R\sin\frac{A}{2}\sin\frac{A}{2}+4R\cos\frac{A}{2}\cos\frac{A}{2}$$

$$=4R\left[\sin^2\frac{A}{2}+\cos^2\frac{A}{2}\right]$$

$$=4R\left(\because\sin^2\frac{A}{2}+\cos^2\frac{A}{2}=1\right)$$

15. Show that $\dfrac{c-b\cos A}{b-\cos A}=\dfrac{\cos B}{\cos C}$

Hint: Apply projection formula i.e., C = b coA + a cosB

$b=a\cos C+C\cos A$

16. Prove that $a\{b\cos C-\text{c}\cos B\}=b^2-c^2$

Solution : -

$a\{b\cos c-\text{c}\cos B\}=ab\cos c-ac\cos B$

Write $\cos C=\dfrac{a^2+b^2-c^2}{ab}$ $\cos B=\dfrac{a^2+c^2-b^2}{ac}$ and simplify to get R.H.S

17. In ΔABC, show that Σ(b + c) cos A = 2s.

Sol. L.H.S.

= (b + c)cos A + (c + a)cos B + (a + b)cos C

= (b cos A + a cos B) + (c cos B + b cos C) + (a cos C + c cos A)

= c + a + b = 2s = R.H.S.

18. In ΔABC, if (a + b + c), (b + c – a) = 3bc, find A.

Sol. 2s(2s – 2a) = 3bc

$\Rightarrow\dfrac{s(s-a)}{bc}=\dfrac{3}{4}\Rightarrow\cos^2\dfrac{A}{2}=\dfrac{3}{4}$

$\Rightarrow\cos\dfrac{A}{2}=\dfrac{\sqrt{3}}{2}=\cos 30°$

$\therefore\dfrac{A}{2}=30°\Rightarrow A=60°$

19. In ΔABC, prove that $\dfrac{1}{r_1}+\dfrac{1}{r_2}+\dfrac{1}{r_3}=\dfrac{1}{r}$**.**

Sol. L.H.S. = $\dfrac{1}{r_1}+\dfrac{1}{r_2}+\dfrac{1}{r_3}=\dfrac{s-a}{\Delta}+\dfrac{s-b}{\Delta}+\dfrac{s-c}{\Delta}$

$=\dfrac{3s-(a+b+c)}{\Delta}=\dfrac{3s-2s}{\Delta}=\dfrac{s}{\Delta}=\dfrac{1}{r}=\text{R.H.S.}$

20. Show that $rr_1r_2r_3=\Delta^2$**.**

Sol. L.H.S. = $rr_1r_2r_3=\dfrac{\Delta}{s}\cdot\dfrac{\Delta}{s-a}\cdot\dfrac{\Delta}{s-b}\cdot\dfrac{\Delta}{s-c}$

$=\dfrac{\Delta^4}{\Delta^2}=\Delta^2=\text{R.H.S.}$

21. In an equilateral triangle, find the value of $\frac{r}{R}$.

Sol. $\frac{r}{R} = \frac{4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}{R} = 4\sin^2 30°$

$= 4\left(\frac{1}{2}\right)^3 = \frac{1}{2}$ $(\because A = B = C = 60°)$

22. If $rr_2 = r_1r_3$, then find B.

Sol. $rr_2 = r_1r_3 \Rightarrow \frac{\Delta}{s}\cdot\frac{\Delta}{s-b} = \frac{\Delta}{s-a}\cdot\frac{\Delta}{s-c}$

$\Rightarrow (s-a)(s-c) = s(s-b)$

$\Rightarrow \frac{(s-c)(s-a)}{s(s-b)} = 1 \Rightarrow \tan^2\frac{B}{2} = 1$

$\Rightarrow \tan\frac{B}{2} = 1 \Rightarrow \frac{B}{2} = 45° \Rightarrow B = 90°$

23. If A = 90°, show that 2(r + R) = b + c.

Sol. L.H.S. = 2r + 2R = 2(s – a) $\tan\frac{A}{2} + 2R\cdot 1$

= 2(s – a) tan 45° + 2R sin A

= (2s – 2a)1 + a $(\because A = 90°)$

= b + c

= R.H.S.

24. In a triangle ABC express $\sum r_1 \cot\frac{A}{2}$ in terms of S

Solution : -

$\sum r_1 \cot A/2 = r_1 \cot A/2 + r_2 \cot B/2 + r_3 \cot\frac{C}{2}$

$r_1 = s\tan A/2 \quad r_2 = s\tan B/2 \quad r_3 = s\tan C/2$

$= s\tan\frac{A}{2}\cot A/2 + s\tan B/2\cot B/2 + s\tan\frac{C}{2}\cot\frac{C}{2}$

$= s + s + s = 3s$

25. Show that $\sum \frac{r_1}{(s-b)(s-c)} = \frac{3}{r}$

Solution : -

$$\sum \frac{r_1}{(s-b)(s-c)} = \frac{r_1}{(s-b)(s-c)} + \frac{r_2}{(s-c)(s-a)} + \frac{r_3}{(s-a)(s-b)}$$

But $r_1 = \frac{\Delta}{s-a}\ r_2 = \frac{\Delta}{s-b}\ r_3 = \frac{\Delta}{s-c}$

$$= \frac{\Delta}{(s-a)(s-b)(s-c)} + \frac{\Delta}{(s-a)(s-b)(s-c)} + \frac{\Delta}{(s-a)(s-b)(s-c)}$$

$$= \frac{3\Delta s}{s(s-a)(s-b)(s-c)} = \frac{3\cancel{\Delta}s}{\Delta^2} = \frac{3}{\left(\frac{\Delta}{s}\right)} = \frac{3}{r}$$

SAQ'S

26. Prove that in a triangle ABC $a\cos A + b\cos B + C\cos c = 4R\sin A\sin B\sin C$

Solution : -

L.H.S $= a\cos A + b\cos B + C\cos C = 2R\sin A\cos A + 2\sin B\cos B + 2R\sin C\cos C$

$= R\{\sin 2A + \sin^2 B + \sin^2 C\}$

Given that $A + B + C = 180^0$

$\therefore L.H.S\ = R\{\sin(A+B)\cos(A-B) + \sin 2C\}$

$= R\{2\sin C\cos(A-B) + 2\sin C\cos C\} = 2R\sin C\{\cos(A-B) + \cos C\}$

$= 2R\sin C\left\{\cos(A-B) + \cos\left(180^0 - \overline{A+B}\right)\right\}$

$= 2R\sin C\{\cos(A-B) - \cos(A+B)\} = 4R\sin A\sin B\sin C$

27. Prove that $\sum a^3\sin(B-C) = 0$

Solution : -

$\sum a^3\sin(B-C) = \sum (2R\sin A)^3\sin(B-C) = 8R^3\sum \sin^3 A\sin(B-C)$

$8R^3\sum \sin^2 A\sin A\sin(B-C) = 8R^3\sum \sin^2 A\sin\left(180^0 - B + \overline{C}\right)\sin(B-C)$

$\{\because A = 180^0 - \overline{B+C}\}$

$= 8R^3\sum \sin^2 A\sin(B+C).\sin(B-C) = 8R^3\sum \sin^2 A\left(\sin^2 B - \sin^2 C\right)$

$= 8R^3\left\{\sin^2 A\left(\sin^2 B - \sin^2 C\right) + \sin^2 B\left(\sin^2 C - \sin^2 A\right) + \sin^2 C\left(\sin^2 A - \sin^2 B\right)\right\}$

$= 0$

28. In a triangle ABC prove that $\frac{a\sin(B-C)}{b^2-c^2}=\frac{b\sin(C-A)}{C^2-a^2}=\frac{C\sin(A-B)}{a^2-b^2}$

Solution : -

$$\frac{a\sin(B-C)}{b^2-c^2}=\frac{2R\sin A\sin(B-C)}{4R^2\{\sin^2 B-\sin^2 C\}}\begin{Bmatrix}\because a=2R\sin A : b=2R\sin B\\ C=2R\sin C \ \ from\ \sin e\ Rule\end{Bmatrix}$$

$$\frac{1}{2R}\,\frac{\sin(B+C).\sin(B-C)}{\sin^2 B-\sin^2 C}\left\{\because \sin A\ =\sin(B+C)\ In\ a\ triangle\ ABC\right\}$$

$$\frac{1}{2R}\,\frac{\sin^2 B-\sin^2 C}{\sin^2 B-\sin^2 C}=\frac{1}{2R}$$

|||ly we have prove that $\frac{b\sin(C-A)}{c^2-a^2}=\frac{1}{2R}$ and $\frac{C\sin(A-B)}{a^2-b^2}=\frac{1}{2R}$

29. Prove that $\frac{a}{bc}+\frac{\cos A}{a}=\frac{b}{ca}+\frac{\cos B}{b}=\frac{c}{ab}+\frac{\cos C}{c}$

Solution : -

$$\frac{a}{bc}+\frac{\cos A}{a}=\frac{a^2+bc\cos A}{abc}$$

We knonw that $\cos A=\frac{b^2+c^2-a^2}{2bc}$

$$\frac{a^2+bc\cos A}{abc}=\frac{a^2+\frac{\cancel{bc}\left[b^2+c^2-a^2\right]}{2bc}}{abc}=\frac{2a^2+b^2+c^2-a^2}{2abc}$$

$$\frac{a^2+b^2+c^2}{2abc}$$

$$|||^{ly}\quad \frac{b}{ca}+\frac{\cos B}{b}=\frac{b^2+ac\cos B}{abc}=\frac{b^2+\frac{\cancel{ac}\left(a^2+c^2-b^2\right)}{2\cancel{ac}}}{abc}=\frac{a^2+b^2+c^2}{2abc}$$

$$|||^{ly}\quad \frac{c}{ab}+\frac{\cos c}{c}=\frac{a^2+b^2+c^2}{2abc}$$

30 **Prove that** $\dfrac{1+\cos(A-B)\cos C}{1+\cos(A-C)\cos B}=\dfrac{a^2+b^2}{a^2+c^2}$

Solution : -

In a triangle ABC $A+B+C=180^0 \Rightarrow C=180^0-\overline{A+B}$

$B=\left(180^0-\overline{A+C}\right)$

$$\frac{1+\cos(A-B)\cos C}{1+\cos(A-C)\cos B}=\frac{1+\cos(A-B)\cos\left(180^0-\overline{A+D}\right)}{1+\cos(A-C).\cos\left(180^0-A+C\right)}$$

$$=\frac{1-\cos(A-B)\cos(A+B)}{1-\cos(A-C)\cos(A+C)}=\frac{1-\left\{\cos^2 A-\sin^2 B\right\}}{1-\left\{\cos^2 A-\sin^2 C\right\}}$$

$$=\frac{1-\cos^2 A+\sin^2 B}{1-\cos^2 A+\sin^2 C}=\frac{\sin^2 A+\sin^2 B}{\sin^2 A+\sin^2 C}=\frac{\frac{a^2}{4R^2}+\frac{b^2}{4R^2}}{\frac{a^2}{4R^2}+\frac{c^2}{4R^2}}=\frac{a^2+b^2}{a^2+c^2}$$

31. If $C=60^0$ **then show that (i)** $\dfrac{a}{b+c}+\dfrac{b}{c+a}=1$ **(ii)** $\dfrac{b}{c^2-a^2}+\dfrac{a}{c^2-b^2}=0$

Solution : -

Given $C=60^0$

$C^2=a^2+b^2-2ab\cos C$

$C^2=a^2+b^2-\not{2}ab\left(\frac{1}{\not{2}}\right)$

$C^2+ab=a^2+b^2$

$\dfrac{a}{b+c}+\dfrac{b}{c+a}=$

$$\frac{ac+a^2+bc+b^2}{(b+c)(c+a)}=\frac{a^2+b^2+ac+bc}{bc+ab+c^2+ac}$$

But $a^2+b^2=c^2+ab$

$$\frac{c^2+ab+ac+bc}{bc+ab+c^2+ac}=1$$

(ii) $\dfrac{b}{c^2-a^2}+\dfrac{a}{c^2-b^2}=0 \Rightarrow \dfrac{b\left\{c^2-b^2\right\}+a\left\{c^2-a^2\right\}}{\left(c^2-a^2\right)\left(c^2-b^2\right)}$

$$=\frac{bc^2-b^3+ac^2-a^3}{\left(c^2-a^2\right)\left(c^2-b^2\right)}=\frac{c^2\left(a+b\right)-\left(a+b\right)\left\{a^2+b^2-ab\right\}}{\left(c^2-a^2\right)\left(c^2-b^2\right)}$$

$$=\frac{\left(a+b\right)\left[c^2-\left\{a^2+b^2-ab\right\}\right]}{\left(c^2-a^2\right)\left(c^2-b^2\right)}=\frac{\left(a+b\right)\left[c^2-\left(c^2+\cancel{ab}-\cancel{ab}\right)\right]}{\left(c^2-a^2\right)\left(c^2-b^2\right)}$$

$= 0$

(iii) **In a triangle ABC** $\dfrac{1}{a+c}+\dfrac{1}{b+c}=\dfrac{3}{a+b+c}$ **show that** $c=60^0$

Solution : -

Given $\dfrac{1}{a+c}+\dfrac{1}{b+c}=\dfrac{3}{a+b+c}$ $\qquad \dfrac{a+b+c}{a+c}+\dfrac{a+b+c}{b+c}=3$

$$\frac{\left(a+c\right)+b}{a+c}+\frac{a+\left(b+c\right)}{b+c}=3\Rightarrow\frac{\cancel{a}+c}{\cancel{a}-c}+\frac{b}{a+c}+\frac{a}{b+c}+\frac{\cancel{b+c}}{\cancel{b+c}}=3$$

$$\frac{b}{a+c}+\frac{a}{b+c}=1\Rightarrow\frac{b^2+bc+a^2+ac}{ab+ac+bc+c^2}=1$$

$$a^2+b^2-c^2=ab\Rightarrow 2ab\cos C=ab\left\{\because a^2+b^2-c^2=2abc\cos C\right\}$$

$$\cos C=\frac{ab}{2ab}=\frac{1}{2}\Rightarrow C=60^0\ \cos C=\frac{\cancel{ab}}{\cancel{2ab}}=\frac{1}{2}\Rightarrow C=60^0$$

32. If a : b : c = 7 : 8 : 9 then find $\cos A=\cos B=\cos C$

Solution : - $\cos A=\dfrac{b^2+c^2-a^2}{2bc}=\dfrac{64k^2+81k^2-49k^2}{2\left(8k\right)\left(9k\right)}=\dfrac{\cancel{96}^{12}k^2}{2\times\cancel{8}k\times 9\cancel{k}}$

$$\cos B=\frac{a^2+c^2-b^2}{2ac}=\frac{49k^2+81k^2-64k^2}{2\times 7k\times 9k}=\frac{66^{33}}{\cancel{2}\times 63}=\frac{11}{21}$$

$$\cos C=\frac{a^2+b^2-c^2}{2ab}=\frac{49k^2+64k^2-81k^2}{2\times 7k\times 8k}=\frac{32k^2}{20k\times 8k}=\frac{2}{7}$$

$$\therefore \cos A=\cos B=\text{co}\,C=\frac{2}{3}=\frac{11}{21}=\frac{2}{7}=\frac{2}{3}\times 21=\frac{11}{21}\times 21=\frac{2}{7}\times 21$$

$=14:11:6$

33. Show that $\frac{\cos A}{a}+\frac{\cos B}{b}+\frac{\cos C}{c}=\frac{a^2+b^2+c^2}{2abc}$

LHS $\frac{\cos A}{a}+\frac{\cos B}{b}+\frac{\cos C}{c}=\frac{bc\cos A+ca\cos B+ab\cos C}{abc}$

$=\frac{2bc\cos A+2ca\cos B+2ab\cos C}{2bac}$

$=\frac{b^2+c^2-a^2+a^2+c^2-b^2+a^2+b^2-c^2}{2abc}$

$\left\{\because 2bc\cos A=b^2+c^2-a^2 : 2ac\cos B=a^2+c^2-b^2 \quad 2ab\cos c=a^2+b^2+c^2\right\}$

$=\frac{a^2+b^2+c^2}{2abc}$

34. Prove that $(b-a)\cos c+c(\cos B-\cos A)=c\sin\left(\frac{A-B}{2}\right)\cos ec\left(\frac{A+B}{2}\right)$

Solution : - $(b-a)\cos c+c\{\cos B-\cos A\}$

$b\cos c-a\cos c+\cos B-\cos A=(b\cos c+\cos B)-ca\cos C+\cos A$

$a-b\{\textit{from projection Rule}\}$

$=2R\{\sin A-\sin B\}$

$=2R\left\{2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)\right\}$

$\frac{2R\left\{2\sin\frac{C}{2}\sin\left(\frac{A-R}{2}\right)\right\}\cos\frac{C}{2}}{\cos\frac{C}{2}}=\frac{2R\left\{2\sin\frac{C}{2}\cos\frac{C}{2}\right\}\sin\left(\frac{A-B}{2}\right)}{\cos\frac{C}{2}}$

$=\frac{2R\sin C\sin\left(\frac{A-B}{2}\right)}{\sin\left(\frac{A+B}{2}\right)}=c\sin C\sin\left(\frac{A-B}{2}\right)\cos ec\left(\frac{A+B}{2}\right)$

$\left\{\because \textit{In a triangle ABC} \quad \frac{C}{2}=\left(90^0-\overline{\frac{A+B}{2}}\right);\frac{A+B}{2}=90^0-\frac{C}{2}\right\}$

35. Express $a\sin^2\frac{C}{2}+c\sin^2\frac{A}{2}=a\frac{(s-a)(s-b)}{ab}+c\frac{(s-b)(s-c)}{bc}$

$$\frac{(s-b)\{s-a+s-c\}}{b}$$

$$\frac{(s-b)\{2s-ac\}}{b}=\frac{(s-b)\{a+b+c-a-c\}}{b}=(s-b)$$

36. If $b+c=3a$ **then find the value of** $\cot\frac{B}{2}\cot\frac{c}{2}$

Solution : -

$$\cot\frac{B}{2}\cot\frac{c}{2}=\sqrt{\frac{s(s-b)}{(s-a)(s-c)}}\sqrt{\frac{s(s-c)}{(s-a)(s-b)}}$$

$$=\sqrt{\frac{s(s-b)}{(s-a)(s-c)}\cdot\frac{s(s-c)}{(s-a)(s-b)}}=\frac{s}{s-a}=\frac{2s}{2(s-a)}$$

$$=\frac{b+c+a}{b+c-a}=\frac{3a+a}{3a-a}=\frac{4a}{2a}=2$$

37. Show that $(b-c)^2\cos^2\frac{A}{2}+(b+c)^2\sin^2\frac{A}{2}=a^2$

Solution : -

$$(b-c)^2\cos^2\frac{A}{2}+(b+c)^2\sin^2\frac{A}{2}$$

$$\{b^2+c^2-2bc\}\cos^2\frac{A}{2}+\{b^2+c^2+2bc\}\sin^2\frac{A}{2}$$

$$b^2\left\{\cos^2\frac{A}{2}+\sin^2\frac{A}{2}\right\}+c^2\left\{\cos^2\frac{A}{2}+\sin^2\frac{A}{2}\right\}-2bc\left\{\cos^2\frac{A}{2}-\sin^2\frac{A}{2}\right\}$$

$b^2+c^2-2bc\cos A=a^2$ (from cosine rule)

$|||^{ly}$ prove that (i) $(c-a)^2\cos^2\frac{C}{2}+(c+a)^2\sin^2\frac{B}{2}=b^2$

(ii) $(a-b)^2\cos^2\frac{c}{2}+(a+b)^2\sin^2\frac{C}{2}=c^2$

38. In ΔABC, prove that r + r_1 + r_2 – r_3 = 4R cos C.

Sol.

$$r+r_1+r_2-r_3=4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}+4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}+4R\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}-4R\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$$

$$=4R\sin\frac{A}{2}\left[\cos\frac{B}{2}\cos\frac{C}{2}+\sin\frac{B}{2}\sin\frac{C}{2}\right]+4R\cos\frac{A}{2}\left[\sin\frac{B}{2}\cos\frac{C}{2}-\cos\frac{B}{2}\cos\frac{C}{2}\right]$$

$$=4R\sin\frac{A}{2}\cos\left(\frac{B-C}{2}\right)+4R\cos\frac{A}{2}\sin\left(\frac{B-C}{2}\right)$$

$$=4R\cos\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right)+4R\sin\left(\frac{B+C}{2}\right)\sin\left(\frac{B-C}{2}\right)$$

$$=4R\left[\cos\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right)+\sin\left(\frac{B+C}{2}\right)\sin\left(\frac{B-C}{2}\right)\right]$$

$$=4R\cos\left(\frac{B+C}{2}-\frac{B-C}{2}\right)$$

$$=4R\cos\left(\frac{B+C-B+C}{2}\right)$$

$$=4R\cos\frac{2C}{2}=4R\cos C$$

39. If r_1 + r_2 + r_3 = r, then show that ∠C = 90°.

Sol. $r_1+r_2=r-r_3\Rightarrow\frac{r_1+r_2}{r-r_3}=1$ …(1)

$$r_1+r_2=4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}+4R\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}$$

$$=4R\cos\frac{C}{2}\left[\sin\frac{A}{2}\cos\frac{B}{2}+\cos\frac{A}{2}\sin\frac{B}{2}\right]$$

$$=4R\cos\frac{C}{2}\left[\sin\frac{A+B}{2}\right]$$

$$=4R\cos\frac{C}{2}\cdot\cos\frac{C}{2}$$

$$=4R\cos^2\frac{C}{2}$$

$$r - r_3 = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} - 4R\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$$

$$= 4R\sin\frac{C}{2}\left[\sin\frac{A}{2}\sin\frac{B}{2} - \cos\frac{A}{2}\cos\frac{B}{2}\right]$$

$$= 4R\sin\frac{C}{2}\left[-\cos\left(\frac{A+B}{2}\right)\right]$$

$$= 4R\sin\frac{C}{2}\left[-\sin\frac{C}{2}\right]$$

$$= -4R\sin^2\frac{C}{2}$$

$$\frac{r - r_3}{r_1 + r_2} = \frac{4R\sin^2\frac{C}{2}}{4R\cos^2\frac{C}{2}} = \tan^2\frac{C}{2}$$

$$\therefore \tan^2\frac{C}{2} = \tan 45° \qquad \text{From(1)}$$

$$\frac{C}{2} = 45° \qquad \therefore \angle C = 90°$$

40. Prove that $4(r_1r_2 + r_2r_3 + r_3r_1) = (a + b + c)^2$.

Sol. $4(r_1r_2 + r_2r_3 + r_3r_1) = 4\left[\frac{\Delta}{S-a}\cdot\frac{\Delta}{S-b} + \frac{\Delta}{S-b}\cdot\frac{\Delta}{S-c} + \frac{\Delta}{S-c}\cdot\frac{\Delta}{S-a}\right]$

$$= 4\Delta^2\left[\frac{S-c+S-a+S-b}{(S-a)(S-b)(S-c)}\right]$$

$$= 4\Delta^2\left[\frac{3S-(a+b+c)}{(S-a)(S-b)(S-c)}\right]$$

$$= 4\Delta^2\left[\frac{S^2}{S(S-a)(S-b)(S-c)}\right]$$

$$= 4\Delta^2\frac{S^2}{\Delta^2} = 4S^2$$

$$= (2S)^2 = (a+b+c)^2$$

41. Prove that $\left(\frac{1}{r} - \frac{1}{r_1}\right)\left(\frac{1}{r} - \frac{1}{r_2}\right)\left(\frac{1}{r} - \frac{1}{r_3}\right) = \frac{abc}{\Delta^3} = \frac{4R}{r^2S^2}$.

Sol. $\frac{1}{r} - \frac{1}{r_1} = \frac{S}{\Delta} - \frac{S-a}{\Delta} = \frac{S-S+a}{\Delta} = \frac{a}{\Delta}$

Similarly we get

$$\frac{1}{r} - \frac{1}{r_2} = \frac{b}{\Delta} \text{ and } \frac{1}{r} - \frac{1}{r_3} = \frac{c}{\Delta}$$

$$\text{L.H.S.} = \left(\frac{1}{r} - \frac{1}{r_1}\right)\left(\frac{1}{r} - \frac{1}{r_2}\right)\left(\frac{1}{r} - \frac{1}{r_3}\right)$$

$$= \frac{a}{\Delta}\frac{b}{\Delta}\frac{c}{\Delta} = \frac{abc}{\Delta^3}$$

$$= \frac{4R \cdot \Delta}{\Delta^3} = \frac{4R}{\Delta^2} = \frac{4R}{(rS)^2} = \text{R.H.S.}$$

42. Prove that $\dfrac{r_1(r_2 + r_3)}{\sqrt{r_1 r_2 + r_2 r_3 + r_3 r_3}} = a$

Solution : - LHS

$$\frac{\dfrac{\Delta}{s-a}\left\{\dfrac{\Delta}{s+b} + \dfrac{\Delta}{s-c}\right\}}{\sqrt{\dfrac{\Delta^2}{(s-a)(s-b)} + \dfrac{\Delta^2}{(s-b)(s-c)} + \dfrac{\Delta^2}{(s-c)(s-a)}}} = \frac{\dfrac{\Delta^2}{(s-a)(s-b)(s-c)}\{s-b+s-c\}}{\sqrt{\dfrac{\Delta^2}{(s-a)(s-b)(s-c)}}\,(s-a)(s-b)+(s-c)}$$

$$\frac{\Delta^{\not 2}}{(s-a)(s-b)(s-c)} \times \frac{\sqrt{\cancel{(s-a)}\,\cancel{(s-b)}\,\cancel{(s-c)}}}{\cancel{\Delta}} \times \frac{a}{\sqrt{3s-2s}}$$

$$\frac{a\Delta}{\sqrt{s(s-a)(s-b)(s-c)}} = \frac{a\Delta}{\Delta} = a$$

43. Prove that $r(r_1 + r_2 + r_3) = ab + bc + ca - S^2$.

Sol. L.H.S. = $r(r_1 + r_2 + r_3)$

$$= \frac{\Delta}{S}\left(\frac{\Delta}{S-a} + \frac{\Delta}{S-b} + \frac{\Delta}{S-c}\right)$$

$$= \frac{\Delta^2}{S}\left[\frac{(S-b)(S-c)+(S-a)(S-c) + (S-a)(S-b)}{(S-a)(S-b)(S-c)}\right]$$

$$= \frac{\Delta^2}{\Delta^2}\left[S^2 - Sc - Sb + bc + S^2 - Sc - Sa + ac + S^2 - Sb - Sa + ab\right]$$

$$= 3S^2 - 2S(a+b+c) + bc + ca + ab$$

$$= 3S^2 - 4S^2 + bc + ca + ab$$

$$= ab + bc + ca - S^2$$

$$= \text{R.H.S.}$$

44. Show that $(r_1 + r_2)\tan\frac{C}{2} = (r_3 - r)\cot\frac{C}{2} = c$ **.**

Sol. $(r_1 + r_2)\tan\frac{C}{2}$

$$= 4R\cos^2\frac{C}{2}\tan\frac{C}{2}$$

$$= 4R\cos^2\frac{C}{2}\frac{\sin\frac{C}{2}}{\cos\frac{C}{2}}$$

$$= 4R\sin\frac{C}{2}\cos\frac{C}{2}$$

$$= 2R\sin C = c \qquad ...(1)$$

$$(r_3 - r)\cot\frac{C}{2} = 4R\sin\frac{C}{2}\cot\frac{C}{2}$$

$$= 4R\sin^2\frac{C}{2}\frac{\cos\frac{C}{2}}{\sin\frac{C}{2}}$$

$$= 4R\sin\frac{C}{2}\cos\frac{C}{2}$$

$$= 2R\sin C = c \qquad ...(2)$$

From (1) and (2)

$$(r_1 + r_2)\tan\frac{C}{2} = (r_3 - r)\cot\frac{C}{2} = c$$

45. In a ΔABC, show that the sides a, b, c are in A.P. if and only if r_1, r_2, r_3 are in H.P.

Sol. r_1, r_2, r_3 are in H.P. $\Leftrightarrow \frac{1}{r_1}, \frac{1}{r_2}, \frac{1}{r_3}$ are in A.P.

$\Leftrightarrow \frac{s-a}{\Delta}, \frac{s-b}{\Delta}, \frac{s-c}{\Delta}$ are in A.P.

$\Leftrightarrow s-a, s-b, s-c$ are in A.P.

$\Leftrightarrow$ –a, –a, –c are in A.P.

$\Leftrightarrow$ a, b, c are in A.P.

46. If A, A_1, A_2, A_3 are the areas of incircle and excircle of a triangle respectively then prove that $\frac{1}{\sqrt{A_1}}+\frac{1}{\sqrt{A_2}}+\frac{1}{\sqrt{A_3}}=\frac{1}{\sqrt{A}}$

Solution : -

Here $A=\pi r^2 \quad A_1=\pi r_1^2 \ ; A_2=\pi r_2^2 \ ; A_3=\pi r_3^2$

LHS $\frac{1}{\sqrt{A_1}}+\frac{1}{\sqrt{A_2}}+\frac{1}{\sqrt{A_3}}=\frac{1}{\sqrt{\pi r_1^2}}+\frac{1}{\sqrt{\pi r_2^2}}+\frac{1}{\sqrt{\pi r_3^2}}$

$$\frac{1}{\sqrt{\pi}}\left\{\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}\right\}=\frac{1}{\sqrt{\pi}}\left\{\frac{s-a}{\Delta}+\frac{s-b}{\Delta}+\frac{s-c}{\Delta}\right\}$$

$$\frac{1}{\sqrt{\Delta}}\left\{\frac{3s-(a+b+c)}{\Delta}\right\}=\frac{1}{\sqrt{\pi}}\frac{s}{\Delta}=\frac{1}{\sqrt{\pi}}\times\frac{1}{\left(\frac{\Delta}{s}\right)}$$

$$\frac{1}{\sqrt{\pi}}\cdot\frac{1}{r}=\frac{1}{\sqrt{\pi r^2}}=\frac{1}{\sqrt{A}}$$

LAQ'S

47. In a triangle ABC prove that $\cot A+\cot B+\cot C=\frac{a^2+b^2+c^2}{4\Delta}$

Solution : -

$$\cot A=\frac{\cos A}{\sin A}$$

$$\cos A=\frac{b^2+c^2-a^2}{2bc}$$

$$\therefore \cot=\frac{\frac{b^2+c^2-a^2}{2bc}}{\sin A}=\frac{b^2+c^2-a^2}{2bc\sin A}=\frac{b^2+c^2-a^2}{4\left\{\frac{1}{2}bc\sin A\right\}}$$

$$\cot A=\frac{b^2+c^2-a^2}{4\Delta}\left\{\because \Delta=\frac{1}{2}bc\sin A\right\}$$

$|||^{ly}$ $\cot B=\frac{a^2+c^2-b^2}{4\Delta} \quad \cot c=\frac{a^2+b^2-c^2}{4\Delta}$

$$\cot A+\cot B+\cot C=\frac{b^2+c^2-a^2}{4\Delta}+\frac{a^2+c^2-b^2}{4\Delta}+\frac{a^2+b^2-c^2}{4\Delta}$$

$$=\frac{b^2+c^2-\cancel{a^2}+\cancel{a^2}+\cancel{c^2}-\cancel{b^2}+a^2+\cancel{b^2}-\cancel{c^2}}{4\Delta}$$

$$=\frac{a^2+b^2+c^2}{4\Delta}$$

48. If $\cot\frac{A}{2} : \cot\frac{B}{2}\ \ \cot\frac{C}{2} = 3:5:7$ **show that a : b : c = 6 : 5 : 4**

Solution : -

$$\cot\frac{A}{2} : \cot\frac{B}{2} : \cot\frac{C}{2} = \frac{s(s-a)}{\Delta} : \frac{s(s-c)}{\Delta} = 6:5:4$$

$$\therefore s-a : s-b : s-c = 6:5=4$$

Let $s-a=3k \quad s-b=5k \quad s-c=7k$

$$(s-a)+(s-b)+(s-c)=3k+5k+7k$$

$$3s-2s\ =15k \Rightarrow s=15k$$

$$s-a=3k \Rightarrow 15k-a=3k \Rightarrow a=12k$$

$$s-b=5k \Rightarrow 15k-5k=b \Rightarrow b=10k$$

$$s-c=7k \Rightarrow 15k-7k=c \Rightarrow c=8k$$

$$a:b:c=12k:10k:8k \Rightarrow a:b:c=6:5:4$$

49. In a triangle ABC show that $(a+b+c)\left\{\tan\frac{A}{2}+\tan\frac{B}{2}\right\}=2c\cot\frac{C}{2}$

Solution : - $(a+b+c)\left[\tan\frac{A}{2}+\tan\frac{B}{2}\right]=2s\left\{\frac{(s-b)(s-c)}{\Delta}+\frac{(s-c)(s-a)}{\Delta}\right\}$

$$=\frac{2s(s-c)}{\Delta}\{s-b+s-a\}$$

$$=2\cot\frac{C}{2}\{2s-b-a\} \Rightarrow 2\cot\frac{C}{2}\{\cancel{a}+\cancel{b}+c-\cancel{b}-\cancel{a}\}$$

$$=2c\cot\frac{C}{2}$$

50. Show that cos A + cos B + cos C = $1+\frac{\mathrm{r}}{\mathrm{R}}$**.**

Sol. L.H.S. = cos A + cos B + cos C

$$=1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

$$=1+\frac{4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}{R}$$ ($\because$ From transformations Prove This In Exam)

$$=1+\frac{r}{R}=\text{R.H.S.}$$

51. Show that $\cos^2\frac{A}{2}+\cos^2\frac{B}{2}+\cos^2\frac{C}{2}=2+\frac{r}{2R}$**.**

Sol. L.H.S. = $\cos^2\frac{A}{2}+\cos^2\frac{B}{2}+\cos^2\frac{C}{2}$

$$=2+2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

(from transformationsPROVE THIS IN EXAM)

$$=2+\frac{4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}{2R}$$

$$=2+\frac{r}{2R}=\text{R.H.S.}$$

52. Prove that (i) $\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\frac{s^2}{\Delta}$

Solution : -

$$\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\frac{s(s-a)}{\Delta}+\frac{s(s-b)}{\Delta}+\frac{s(s-c)}{\Delta}$$

$$=\frac{S}{\Delta}\{s-a+s-b+s-c\}=\frac{s}{\Delta}\{3s-(a+b+c)\}=\frac{s}{\Delta}(3s-2s)$$

$$=\frac{s}{\Delta}(s)=\frac{s^2}{\Delta}$$

(i) **Prove that** $\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}=\frac{bc+ca+ab-s^2}{\Delta}$

Solution : -

$$\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}$$

$$\frac{(s-b)(s-c)}{\Delta}+\frac{(s-c)(s-a)}{\Delta}+\frac{(s-a)(s-b)}{\Delta}$$

$$\frac{s^2-cs-bs+bc+s^2-as-cs+ac+s^2-bs-as+ab}{\Delta}$$

$$\frac{3s^2-2as-2bs-2cs+bc+ca+ab}{\Delta}$$

$$\frac{bc+ca+ab+3s^2-2s(a+b+c)}{\Delta}=\frac{bc+ca+ab+3s^2-2s(2s)}{\Delta}$$

$$\frac{bc+ca+ab+3s^2-4s^2}{\Delta}=\frac{bc+ca+ab-s^2}{\Delta}$$

(iii) $\dfrac{\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}}{\cot A+\cot B+\cot C}$

Solution : -

$$\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\frac{s(s-a)}{\Delta}+\frac{s(s-b)}{\Delta}+\frac{s(s-c)}{\Delta}$$

$$\frac{s}{\Delta}\{s-a+s-b+s-c\}=\frac{s}{\Delta}\{3s-a-b-c\}=\frac{s}{\Delta}\times(3s-2s)=\frac{s^2}{\Delta}$$

$$\cot A=\frac{\cos A}{\sin A}=\frac{\frac{b^2+c^2-a^2}{2bc}}{\sin A}=\frac{b^2+c^2-a^2}{bc\sin A}=\frac{b^2+c^2-a^2}{\Delta\left\{\frac{1}{2}bc\sin A\right\}}$$

$$=\frac{b^2+c^2-a^2}{4\Delta}\quad\left\{\because\frac{1}{2}bc\sin A=\Delta\right\}$$

$$\cot A+\cot B+\cot C=\frac{b^2+c^2-a^2}{4\Delta}+\frac{c^2+a^2-b^2}{4\Delta}+\frac{a^2+b^2-c^2}{4\Delta}$$

$$=\frac{b^2+c^2-a^2+c^2+a^2-b^2+a^2+b^2-c^2}{4\Delta}=\frac{a^2+b^2+c^2}{4\Delta}$$

$$\frac{\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}}{\cot A+\cot B+\cot C}=\frac{\frac{s^2}{\Delta}}{\frac{a^2+b^2+c^2}{4\Delta}}=\frac{s^2}{\cancel{\Delta}}\times\frac{4\cancel{\Delta}}{a^2+b^2+c^2}$$

$$=\frac{4s^2}{a^2+b^2+c^2}=\frac{(2s)^2}{a^2+b^2+c^2}=\frac{(a+b+c)^2}{a^2+b^2+c^2}$$

53. Show that (i) $\sum (a+b)\tan\left(\frac{A-B}{2}\right)=0$

Solution : - $\sum (a+b)\tan\left(\frac{A-B}{2}\right)$ from Napiries Rules

$$\tan\left(\frac{A-B}{2}\right)=\frac{a-b}{a+b}\cot\frac{C}{2}$$

$$\therefore\sum (a+b)\tan\left(\frac{A-B}{2}\right)=\sum \cancel{(a+b)}\left(\frac{a-b}{\cancel{a+b}}\right)\cot\frac{c}{2}$$

$$\sum\frac{(a-b)s(s-c)}{\Delta}=\frac{s}{\Delta}\sum (a-b)(s-c)$$

$$\frac{s}{\Delta}\sum s(a-b)-c(a-b)=0$$

$$\frac{s}{\Delta}\left[s(a-b)+s(b-c)+x(c-a)\right]-\left\{c(a-b)+b(c-a)+a(b-c)\right\}=0$$

(ii) $\frac{b-c}{b+c}\cot\frac{A}{2}+\frac{b+c}{b-c}\tan\frac{A}{2}=2\cos ec(B-C)$

Solution : - $\frac{b-c}{b+c}\cot A/2+\frac{b+c}{b-c}\tan A/2$

From Napiers Rule $\frac{b-c}{b+c}\cot\frac{A}{2}=\tan\left(\frac{B-C}{2}\right)$

$$\frac{b+c}{b-c}\tan\frac{A}{2}=\cot\left(\frac{B-C}{2}\right)$$

$$\therefore\ LHS=\tan\left(\frac{B-C}{2}\right)+\cot\left(\frac{B-C}{2}\right)$$

$$=\frac{\sin\left(\frac{B-C}{2}\right)}{\cos\left(\frac{B-C}{2}\right)}+\frac{\cos\left(\frac{B-C}{2}\right)}{\sin\left(\frac{B-C}{2}\right)}=\frac{\sin^2\left(\frac{B-C}{2}\right)+\cos^2\left(\frac{B-C}{2}\right)}{\cos\left(\frac{B-C}{2}\right)\sin\left(\frac{B-C}{2}\right)}$$

$$\frac{2}{2\sin\left(\frac{B-C}{2}\right)\cos\left(\frac{B-C}{2}\right)}=\frac{2}{\sin(B-C)}=2\cos ec(B-C)$$

54. . (i) It $\sin\theta=\frac{a}{b+c}$ **then show that** $\cos\theta=\frac{2\sqrt{bc}}{b+c}\cos\frac{A}{2}$

Solution : - $\sin\theta=\frac{a}{b+c}\Rightarrow\cos^2\theta=1-\sin^2\theta$

$$\therefore\cos^2\theta=1-\frac{a^2}{(b+c)^2}\Rightarrow\cos^2\theta=\frac{(b+c)^2-a^2}{(b+c)^2}$$

$$\cos^2\theta=\frac{b^2+c^2+2bc-a^2}{(b+c)^2}=\frac{\left(b^2+c^2-a^2\right)+2bc}{(b+c)^2}$$

$$=\frac{2bc\cos A+2bc}{(b+c)^2}=\frac{2bc\{1+\cos A\}}{(b+c)^2}$$

$$\cos^2\theta=\frac{2bc\times2\cos^2\frac{A}{2}}{(b+c)^2}$$

$$\cos\theta=\frac{2\sqrt{bc}}{b+c}\cos\frac{A}{2}$$

(ii) **If** $a=(b+c)\cos\theta$ **then prove that** $\sin\theta=\frac{2\sqrt{bc}}{b+c}\cos\frac{A}{2}$

Solution: - $a=(b+c)\cos\theta\Rightarrow\frac{a}{b+c}=\cos\theta$

$$\sin^2\theta=1-\cos^2\theta=1-\frac{a^2}{(b+c)^2}\Rightarrow\sin^2\theta=\frac{(b+c)^2-a^2}{(b+c)^2}$$

$$\sin^2\theta=\frac{b^2+c^2+2bc-a^2}{(b+c)^2}\Rightarrow\sin^2\theta=\frac{(b^2+c^2-a^2)+2bc}{(b+c)^2}$$

$$\sin^2\theta=\frac{2bc\cos A+2bc}{(b+c)^2}\Rightarrow\sin^2\theta=\frac{2bc\{1+\cos A\}}{(b+c)^2}$$

$$\sin^2\theta=\frac{2bc\left(2\cos^2\frac{A}{2}\right)}{(b+c)^2}\Rightarrow\sin\theta=\frac{2\sqrt{bc}}{b-c}\cos\frac{A}{2}$$

(iii) **If** $a=(b-c)\sec\theta$ **prove that** $\tan\theta=\frac{2\sqrt{bc}}{b-c}\sin\frac{A}{2}$

Solution : -

$$\sec\theta=\frac{a}{b-c}\Rightarrow\tan^2\theta=\sec^2\theta-1=\frac{a^2}{(b-c)^2}=1$$

$$\tan^2\theta=\frac{a^2-(b-c)^2}{(b-c)^2}\Rightarrow\tan^2\theta=\frac{a^2-b^2-c^2+2bc}{(b-c)^2}$$

$$\tan^2\theta=\frac{2bc-\{b^2+c^2-a^2\}}{(b-c)^2}$$

$$\tan^2\theta=\frac{2bc-2bc\cos A}{(b-c)^2}=\frac{2bc[1-\cos A]}{(b-c)^2}$$

$$\tan^2\theta=\frac{2bc\left(2\sin^2\frac{A}{2}\right)}{(b-c)^2}\Rightarrow\tan\theta=\frac{2\sqrt{bc}}{b-c}\sin\frac{A}{2}$$

(iv) **For any angle** θ **show that** $a\cos\theta=b\cos(C+\theta)+c\cos(B-\theta)$

Soluton : -

$$RHS=b\cos(c+\theta)+c\cos(B-\theta)=b\{\cos c\cos\theta-\sin c\sin\theta\}$$

$$+c\{\cos B\cos\theta+\sin B\sin\theta\}$$

$$=b\cos c\cos\theta-b\sin\sin\theta+c\cos B\cos\theta+c\sin B\sin\theta$$

$$=\cos\theta\{b\cos c+c\cos B\}-\frac{bc}{2R}/\sin\theta+\frac{cb}{2R}\sin\theta=a\cos\theta$$

55. If the angles of a triangle ABC are in AP and $b:c=\sqrt{3}:\sqrt{2}$ then show that $A=75^0$

Solution : -

Angles of a triangle ABC are in AP

$\therefore 2B=A+C \Rightarrow 3B=A+B+C$ but $A+B+C=180^0$

$\therefore 3B=180^0 \Rightarrow B=60^0$

Given that $b:c=\sqrt{3}:\sqrt{2}$

$2R\sin B=2R\sin C=\sqrt{3}:\sqrt{2} \Rightarrow \sin B=\sin C=\sqrt{3}\ =\sqrt{2}$

$\sin 60^0=\sin C=\sqrt{3}:\sqrt{2}$

$\frac{\sqrt{3}}{2}:\sin C=\sqrt{3}:\sqrt{2}=\frac{\sqrt{3}}{2}\times\sqrt{2}=\sqrt{3}\ \sin C$

$\Rightarrow \sin c=\frac{1}{\sqrt{2}} \Rightarrow C=45^0$

$A+B+C=180^0 \Rightarrow A+60^0+45^0=180^0 \Rightarrow A=75^0$ c (proved)

56. If $\frac{a^2+b^2}{a^2-b^2}=\frac{\sin C}{\sin(A-B)}$ then prove that triangle ABC is either isosceles or right angled

Solution : -

Given $\frac{a^2+b^2}{(a^2-b^2)}=\frac{\sin C}{\sin(A-B)}$

$\Rightarrow (a^2+b^2)\sin(A-B)=(a^2-b^2)\sin C$

Using sine rule we have

$4R^2\{\sin^2 A+\sin^2 B\}\sin(A-B)=4R^2\{\sin^2 A-\sin^2 B\}\sin C$

$\{\sin^2 A+\sin^2 B\}\sin(A-B)-\sin(A-B)\sin(A+B)\sin C=0$

But in triangle ABC $\sin(A+B)=\sin C$

$\therefore (\sin^2 A+\sin^2 B)\sin(A-B)-\sin(A-B)(\sin C)(\sin C)=0$

$\sin(A-B)\{\sin^2 A+\sin^2 B-\sin^2 C\}=0$

$\sin(A-B)=0\ or\ \sin^2 A+\sin^2 B=\sin^2 C$

$A=B\ or\ a^2+b^2=c^2$

$\therefore$ triangle either isosceles or right angled

57. If $\cos A+\cos B+\cos C=\frac{3}{2}$ **then show that the triangle is equilateral**

Solution: - $\cos A+\cos B+\cos C=\frac{3}{2}\Rightarrow 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)+\cos C=\frac{3}{2}$

$$2\cos\left(90^0-\frac{C}{2}\right)\cos\left(\frac{A-B}{2}\right)+\cos C=\frac{3}{2}$$

$$2\sin\frac{C}{2}\cos\left(\frac{A-B}{2}\right)+1-2\sin^2\frac{C}{2}=3/2$$

$$2\sin\frac{C}{2}\cos\left(\frac{A-B}{2}\right)-2\sin^2\frac{C}{2}=\frac{1}{2}$$

$$4\sin\frac{C}{2}\cos\left(\frac{A-B}{2}\right)-4\sin^2\frac{C}{2}=1\Rightarrow 1+4\sin^2\frac{C}{2}-4\sin\frac{C}{2}\cos\left(\frac{A-B}{2}\right)=0$$

$$\left(2\sin\frac{C}{2}\right)^2-2\left(2\sin\frac{C}{2}\cos\frac{A-B}{2}\right)+\cos^2\left(\frac{A-B}{2}\right)-\cos^2\left(\frac{A-B}{2}\right)+1=0$$

$$\left\{2\sin\frac{C}{2}-\cos\left(\frac{A-B}{2}\right)\right\}^2+\sin^2\left(\frac{A-B}{2}\right)=0$$

$$\therefore 2\sin\frac{C}{2}-\cos\left(\frac{A-B}{2}\right)=0 \text{ and } \sin\frac{A-B}{2}=0$$

$$\therefore 2\sin\frac{C}{2}=\cos\left(\frac{A-B}{2}\right) \text{ and } A-B=0$$

$$\therefore 2\sin\frac{C}{2}=1\Rightarrow\frac{C}{2}=30^0\Rightarrow c=60^0$$

$$A=B\therefore A=B=60^0$$

Hence trangile is equilateral

58. It $\cos^2 A+\cos^2 B+\cos^2 C=1$ **then show that triangle ABC is right angled**

Solution: -

$$\cos^2 A+\cos^2 B+\cos^2 C=1\Rightarrow\cos^2 A+\cos^2 B-1+\cos^2 C=0$$

$$\cos^2 A-\sin^2 B+\cos^2 C=0\Rightarrow\cos(A-B)\cos(A-B)+\cos^2 C=0$$

$$\cos\left(180^0-c\right).\cos(A-B)+\cos^2 C=0$$

$$-\cos C\cos(A-B)+\cos C=0$$

$$-\cos C\left\{\cos(A-B)-\cos C\right\}=0$$

$$-\cos C\left\{\cos(A-B)-\cos C\right\}=0$$

$$-\cos c\left\{\cos(A-B)-\cos\left(180^0-\overline{A+B}\right)\right\}=0$$

$$-\cos c\left\{\cos(A-B)+\cos(A+B)\right\}=0$$

$$-\cos c\left\{\cos(A-B)+\cos(A+B)\right\}=0$$

$$2\cos A\cos B\cos C=0$$

$$\Rightarrow \cos A=0 \ or\ \cos B=0\ (or)\ \cos C=0$$

$$A=90^0\ (or)\ B=90^0\ or\ C=90^0$$

$\therefore$ Triangle is right angled triangle

59. If $a^2+b^2+c^2=8R^2$ then prove that the triangle is right angled

Solutin :

- Given $a^2+b^2+c^2=8R^2 \Rightarrow 4R^2\left\{\sin^2 A+\sin^2 B+\sin^2 C\right\}=8R^2$

$$\sin^2 A+\sin^2 B+\sin^2 C=2 \Rightarrow 1-\cos^2 A+\sin^2 B+\sin^2 C=2$$

$$1-\left\{\cos^2 A-\sin^2 B\right\}+1-\cos^2 C=2$$

$$-\cos(A-B)\cos(A-B)-\cos^2 C=0$$

$$\cos C\cos(A-B)-\cos^2=0 \Rightarrow \cos C\left\{\cos(A-B)-\cos C\right\}=0$$

$$\cos C\left\{\cos(A-B)+\cos(A+B)\right\}=0 \Rightarrow 2\cos A\cos B\cos C=0$$

$$\cos A=0\ or\ \cos B=0\ (or)\ \cos C=0 \Rightarrow A=90^0\ (or)\ B=90^0\ (or)\ C=90^0$$

60. If $\cot\frac{A}{2}, \cot\frac{B}{2}, \cot\frac{C}{2}$ are in AP then prove that a, b, c are in AP

Solution ; -

$\cot\frac{A}{2}, \cot\frac{B}{2} \cot\frac{C}{2}$ are in AP

$$2\cot\frac{B}{2}=\cot\frac{A}{2}+\cot\frac{C}{2} \Rightarrow \frac{2s(s-b)}{\Delta}=\frac{s(s-a)}{\Delta}+\frac{s(s-c)}{\Delta}$$

$$\Rightarrow 2(s-b)=(s-a)+(s-c) \Rightarrow a-b+c=2s-a-c$$

$$a+c-b=\cancel{a}+\cancel{b}+\cancel{c}-\cancel{a}-\cancel{c} \Rightarrow a+c=2b$$

61. If $\sin^2\frac{A}{2}, \sin^2\frac{B}{2}, \sin^2\frac{C}{2}$ **are in HP then show that a, b, c are in HP**

Solution : -

$\sin^2\frac{A}{2}, \sin^2\frac{B}{2}, \sin^2\frac{C}{2}$ are in HP

$\frac{(s-b)(s-c)}{bc}, \frac{(s-c)(s-a)}{ac}, \frac{(s-a)(s-b)}{ab}$ are in HP

$\frac{bc}{(s-b)(s-c)}, \frac{ac}{(s-a)(s-c)}, \frac{ab}{(s-a)(s-b)}$ are in HP

Multiplying with $\frac{(s-a)(s-b)(s-c)}{abc}$ we have

$\frac{bc\,(s-a)(s-b)(s-c)}{abc\,(s-b)(s-c)}, \frac{ac\,(s-a)(s-b)(s-c)}{abc\,(s-a)(s-c)}, \frac{ab\,(s-a)(s-b)(s-c)}{abc\,(s-a)(s-b)}$ and P

$\frac{s-a}{a}, \frac{s-b}{b}, \frac{s-c}{c}$ are in AP

Adding '1' to every term we here

$\frac{s-a}{a}+1, \frac{s-b}{b}+1, \frac{s-c}{c}+1$ are in AP

$\frac{s}{a}, \frac{s}{b}, \frac{s}{c}$ are in AP $\Rightarrow \frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in AP

a, b, c are in HP

62. Prove that $a^2\cot A + b^2\cot B + c^2\cot C = \frac{abc}{R}$

Solution: -

$a^2\cot A + b^2\cot B + c^2\cot C$

$4R^2\sin^2 A\times\frac{\cos A}{\sin A} + 4R^2\sin^2 B\frac{\cos B}{\sin B} + 4R^2\sin^2 C\frac{\cos C}{\sin C}$

$2R^2\{\sin 2A + \sin 2B + \sin 2C\}$

$2R^2\{2\sin(A+B)\cos(A-B) + \sin 2C\}$

$2R^2\{2\sin C\cos(A-B) + 2\sin C\cos C\}$

$2R^2\left[2\sin C\{\cos(A-B)+\cos C\}\right] = 4R\sin C\{\cos(A-B)-\cos(A+B)\}$

$4R^2\sin C\sin A\sin B = \frac{2\{2R^2\sin A\sin B\sin C\}}{R} =$

$$\frac{(2R\sin A)(2R\sin B)(2R\sin C)}{R}=\frac{abc}{R}$$

63. Show that $a\cos^2\frac{A}{2}+b\cos^2\frac{B}{2}+c\cos^2\frac{C}{2}=s+\frac{\Delta}{R}$

Solution: -

$$a\cos^2\frac{A}{2}+b\cos^2\frac{B}{2}+c\cos^2\frac{C}{2}$$

$$\frac{a(1+\cos A)}{2}+\frac{b(1+\cos B)}{2}+\frac{c(1+\cos C)}{2}$$

$$\frac{a+a\cos A+b+b\cos B+C+c\cos C}{2}=\frac{(a+b+c)+}{2}$$

$$\frac{(a+b+c)+\{a\cos A+b\cos B+\text{c}\cos C\}}{2}$$

$$\frac{2s+2R\sin A\cos A+2R\sin B\cos B+2R\sin C\cos C}{2}$$

$$\frac{2S+R(\sin 2A+\sin 2B+\sin 2C)}{2}$$

$$\frac{2S+R\{2\sin(A+B)\cos(A-B)+\sin 2C\}}{2}$$

$$\frac{2S+R\{2\sin C\cos(A-B)+2\sin C\cos C\}}{2}$$

$$\frac{2S+2R\sin C\{\cos(A-B)+\cos C\}}{2}$$

$$\frac{2S+2R\sin C\{\cos(A-B)-\cos(A+B)\}}{2}$$

$$\frac{2S+4R\sin A\sin B\sin C}{2}=S+2\text{R}\sin A\sin B\sin C$$

$$\frac{S+2R^2\sin A\sin B\sin C}{R}=S+\frac{\Delta}{R}$$

64. Show that $\sin^2\frac{A}{2}+\sin^2\frac{B}{2}+\sin^2\frac{C}{2}=1-\frac{r}{2R}$ **.**

Sol. L.H.S. = $\sin^2\frac{A}{2}+\sin^2\frac{B}{2}+\sin^2\frac{C}{2}$

$$=\frac{1-\cos A}{2}+\frac{1-\cos B}{2}+\frac{1-\cos C}{2}$$

$$=\frac{3}{2}-\frac{1}{2}(\cos A+\cos B+\cos C)$$

$$=\frac{3}{2}-\frac{1}{2}\left[1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right]$$

($\because$ from transformations)

$$=\frac{3}{2}-\frac{1}{2}\left[1+\frac{4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}{R}\right]$$

$$=\frac{3}{2}-\frac{1}{2}\left[1+\frac{r}{R}\right]$$

$$=\frac{3}{2}-\frac{1}{2}-\frac{r}{2R}=1-\frac{r}{2R}=\text{R.H.S.}$$

65. Show that i. $\Delta=r_1r_2\sqrt{\frac{4R-r_1-r_2}{r_1+r_2}}$ **. ii.** $a=(r_2+r_3)\sqrt{\frac{rr_1}{r_2r_3}}$

Sol. i)R.H.S. $r_1r_2\sqrt{\frac{4R-r_1-r_2}{r_1+r_2}}$

$$=r_1r_2\sqrt{\frac{4R-(r_1+r_2)}{r_1+r_2}}$$

$$\left(\because r_1+r_2=4R\cos^2\frac{C}{2}\right)$$

$$=r_1r_2\sqrt{\frac{4R\left(1-\cos^2\frac{C}{2}\right)}{4R\cos^2\frac{C}{2}}}$$

$$= r_1 r_2 \sqrt{\frac{\sin^2 \frac{C}{2}}{\cos^2 \frac{C}{2}}} = r_1 r_2 \tan\frac{C}{2}$$

$$= \frac{\Delta}{S-a} \cdot \frac{\Delta}{S-b} \tan\frac{C}{2}$$

$$= \frac{\Delta^2}{(S-a)(S-b)} \sqrt{\frac{(S-b)(S-a)}{S(S-c)}}$$

$$= \frac{\Delta^2}{\sqrt{S(S-a)(S-b)(S-c)}} = \frac{\Delta^2}{\Delta} = \Delta = \text{R.H.S.}$$

ii) RHS $= (r_2 + r_3)\sqrt{\frac{rr_1}{r_2 r_3}} = \left(\frac{\Delta}{s-b} + \frac{\Delta}{s-c}\right)\sqrt{\frac{\frac{\Delta^2}{s(s-a)}}{\frac{\Delta^2}{(s-b)(s-c)}}}$

$$= \frac{\Delta\{s-c+s-b\}}{(s-b)(s-c)} \sqrt{\frac{\Delta^2}{s(s-a)} \cdot \frac{(s-b)(s-c)}{\Delta^2}}$$

$$= \frac{\Delta.(2s-b-c)}{\left\{\sqrt{(s-b)(s-c)}\right\}^2} \frac{\sqrt{(s-b)(s-c)}}{\sqrt{s(s-a)}} = \frac{\Delta\{a+b+c-b-c\}}{\sqrt{s(s-a)(s-b)(s-c)}} = a$$

66. Prove that $r_1^2 + r_2^2 + r_3^2 + r^2 = 16R^2 - (a^2 + b^2 + c^2)$.

Sol.

$(r_1 + r_2 + r_3 - r)^2 = r_1^2 + r_2^2 + r_3^2 + r^2 - 2r(r_1 + r_2 + r_3) + 2(r_1 r_2 + r_2 r_3 + r_3 r_1)$...(1)

But $[r_1 + r_2 + r_3 - r] = 4R$ and $r_1 r_2 + r_2 r_3 + r_3 r_1 = S^2$

$16R^2 = [r_1 + r_2 + r_3 - r]^2$

$16R^2 = r_1^2 + r_2^2 + r_3^2 + r^2 - 2r(r_1 + r_2 + r_3) + 2(r_1 r_2 + r_2 r_3 + r_3 r_1)$

$= r_1^2 + r_2^2 + r_3^2 + r^2 - 2r(r_1 + r_2 + r_3) + 2S^2$

$16R^2 = r_1^2 + r_2^2 + r_3^2 + r^2 - 2(rr_1 + rr_2 + rr_3) + 2S^2$ Consider $2(rr_1 + rr_2 + rr_3) =$

$$= 2\left[\frac{\Delta^2}{S(S-a)} + \frac{\Delta^2}{S(S-b)} + \frac{\Delta^2}{S(S-c)}\right]$$

$$= 2\Delta^2 \frac{[(S-b)(S-c)+(S-a)(S-c)+(S-a)(S-b)]}{S(S-a)(S-b)(S-c)}$$

$$= \frac{2\Delta^2}{\Delta^2}\left[S^2 - Sc - Sb + bc + S^2 - Sc - Sa + ac + S^2 - Sb - Sa + ab\right]$$

$$= 2\left[3S^2 - 2S(a+b+c) + ab + bc + ca\right]$$

$$= 2\left[3S^2 - 4S^2 + ab + bc + ca\right]$$

$$= 2\left[ab + bc + ca - S^2\right]$$

$$= 2\left[ab + bc + ca\right] - 2S^2$$

From (2)

$$\Rightarrow r_1^2 + r_2^2 + r_3^2 + r^2$$

$$= 16R^2 + 2r(r_1 + r_2 + r_3) - 2S^2$$

$$= 16R^2 - 2S^2 + 2(ab + bc + ca) - 2S^2$$

$$= 16R^2 - 4S^2 + 2(ab + bc + ca)$$

$$= 16R^2 - 4\left(\frac{a+b+c}{2}\right)^2 + 2(ab + bc + ca)$$

$$= 16R^2 - \left[(a+b+c)^2 - 2(ab + bc + ca)\right]$$

$$= 16R^2 - (a^2 + b^2 + c^2)$$

67. If P_1, P_2, P_3 are the altitudes from the vertices A, B, C to the opposite side of a triangle respectively, then show that

(i) $\frac{1}{P_1} + \frac{1}{P_2} + \frac{1}{P_3} = \frac{1}{r}$ **(ii)** $\frac{1}{P_1} + \frac{1}{P_2} - \frac{1}{P_3} = \frac{1}{r_3}$ **(iii)** $P_1 P_2 P_3 = \frac{(abc)^2}{8R^3} = \frac{8\Delta^3}{abc}$

Sol. We know that

$$\Delta = \frac{1}{2}aP_1, \Delta = \frac{1}{2}bP_2, \Delta = \frac{1}{2}cP_3$$

$$\Rightarrow P_1 = \frac{2\Delta}{a}, P_2 = \frac{2\Delta}{b} \text{ and } P_3 = \frac{2\Delta}{c}$$

i) $\frac{1}{P_1} + \frac{1}{P_2} + \frac{1}{P_3} = \frac{a}{2\Delta} + \frac{b}{2\Delta} + \frac{c}{2\Delta}$

$$= \frac{a+b+c}{2\Delta} = \frac{2S}{2\Delta} = \frac{1}{r}$$

ii) $\frac{1}{P_1} + \frac{1}{P_2} - \frac{1}{P_3} = \frac{a}{2\Delta} + \frac{b}{2\Delta} - \frac{c}{2\Delta}$

$$= \frac{a+b-c}{2\Delta} = \frac{2S-2c}{2\Delta} = \frac{S-c}{\Delta} = \frac{1}{r_3}$$

iii) $P_1P_2P_3 = \frac{2\Delta}{a}\cdot\frac{2\Delta}{b}\cdot\frac{2\Delta}{c} = \frac{8\Delta^3}{abc}$

68. If a = 13, b = 14, c = 15, show that $R = \frac{65}{8}$**, r = 4,** $r_1 = \frac{21}{2}$**,** r_2 **= 12 and** r_3 **= 14.**

Sol. Given that a = 13, b = 14, c = 15

$$S = \frac{a+b+c}{2} = \frac{13+14+15}{2} = \frac{42}{2} = 21$$

$$S-a = 21-13 = 8, S-b = 21-14 = 7$$

$$S-c = 21-15 = 6$$

$$\Delta = \sqrt{S(S-a)(S-b)(S-c)}$$

$$= \sqrt{21(8)(7)(6)} = \sqrt{21\times16\times21}$$

$$= \sqrt{21\times21\times4\times4} = 21\times4 = 84$$

$$R = \frac{abc}{4\Delta} = \frac{13\times14\times15}{4\times84} = \frac{65}{8}$$

$$r = \frac{\Delta}{S} = \frac{84}{21} = 4$$

$$r_1 = \frac{\Delta}{S-a} = \frac{84}{8} = \frac{21}{2}$$

$$r_2 = \frac{\Delta}{S-b} = \frac{84}{7} = 12$$

$$r_3 = \frac{\Delta}{S-c} = \frac{84}{6} = 14$$

69. If r_1 **= 2,** r_2 **= 3,** r_3 **= 6 and r = 1, prove that a = 3, b = 4 and c = 5.**

Sol. $\Delta^2 = rr_1r_2r_3 = 1\cdot2\cdot3\cdot6 = 36$

$$\Delta^2 = 36 \Rightarrow \Delta = 6$$

$$r = \frac{\Delta}{S} = \frac{6}{S} \Rightarrow S = 6 \qquad (\because r = 1)$$

$$r_1 = \frac{\Delta}{S-a} \Rightarrow S-a = \frac{\Delta}{r_1}$$

$$\therefore a = S - \frac{\Delta}{r_1} = 6 - \frac{6}{2} = 6-3 = 3$$

$$r_2 = \frac{\Delta}{S-b} \Rightarrow S-b = \frac{\Delta}{r_2}$$

$$\therefore b = S - \frac{\Delta}{r_2} = 6-3 = 3$$

$$r_3 = \frac{\Delta}{S-c} \Rightarrow S-c = \frac{\Delta}{r_3}$$

$$\therefore c = S - \frac{\Delta}{r_3} = 6-1 = 5$$

70. Show that $a^2 \cot A + b^2 \cot B + c^2 \cos C = \frac{abc}{R}$**.**

Sol. L.H.S. $a^2 \cot A + b^2 \cot B + c^2 \cos C$

$$= 4R^2 \sin^2 A \frac{\cos A}{\sin A} + 4R^2 \sin^2 B \frac{\cos B}{\sin B} \quad +4R^2 \sin^2 C \frac{\cos C}{\sin C} \text{(by sine rule)}$$

$$= 2R^2(2\sin A\cos A + 2\sin B\cos B + 2\sin C\cos C)$$

$$= 2R^2(\sin 2A + \sin 2B + \sin 2C)$$

$$= 2R^2(4\sin A\sin B\sin C)$$

$$= \frac{1}{R}(2R\sin A)(2R\sin B)(2R\sin C)$$

$$= \frac{abc}{R} = \text{R.H.S.}$$

71. In ΔABC, if $\frac{1}{a+c} + \frac{1}{b+c} = \frac{3}{a+b+c}$**, show that C = 60°.**

Sol. $\frac{1}{a+c} + \frac{1}{b+c} = \frac{3}{a+b+c}$

$$\Rightarrow \frac{b+c+a+c}{(a+c)(b+c)} = \frac{3}{a+b+c}$$

$$\Rightarrow 3(a+c)(b+c) = (a+b+2c)(a+b+c)$$

$$\Rightarrow 3(ab+ac+bc+c^2)$$

$$= (a^2+b^2+2ab) + 3c(a+b) + 2c^2$$

$$\Rightarrow ab = a^2+b^2-c^2 = 2ab\cos C$$

(from cosine rule)

$$\Rightarrow \cos C = \frac{1}{2} \Rightarrow C = 60°$$

72. Prove that $\cot A + \cot B + \cot C = \dfrac{a^2 + b^2 + c^2}{4\Delta}$**.**

Sol. L.H.S. = $\Sigma \cot A = \Sigma \dfrac{\cos A}{\sin A}$

$$= \Sigma\left(\frac{b^2 + c^2 - a^2}{2bc \sin A}\right) \text{(by cosine rule)}$$

$$= \Sigma \frac{b^2 + c^2 - a^2}{4\Delta}\left[\because \Delta = \frac{1}{2} bc \sin A\right]$$

$$= \frac{1}{4\Delta}[b^2 + c^2 - a^2 + c^2 + a^2 - b^2 + a^2 + b^2 - c^2]$$

$$= \frac{a^2 + b^2 + c^2}{4\Delta} = \text{R.H.S.}$$

73. In ∆ABC, if a cos A = b cos B, prove that the triangle is either isosceles or right angled.

Sol. a cos A = b cos B

$\Rightarrow$ 2R sin A cos A = 2R sin B cos B

$\Rightarrow$ sin 2A = sin 2B = sin(180° – 2B)

Hence 2A = 2B or 2A = 180° – 2B

$\Rightarrow$ A = B or A = (90° – B)

$\Rightarrow$ a = b or (A + B) = 90°

$\Rightarrow$ a = b or C = 90°

$\therefore$ The triangle is isosceles or right angled.

74. If $\cot \dfrac{A}{2} : \cot \dfrac{B}{2} : \cot \dfrac{C}{2} = 3 : 5 : 7$**, show that** $a : b : c = 6 : 5 : 4$**.**

Sol. $\cot \dfrac{A}{2} : \cot \dfrac{B}{2} : \cot \dfrac{C}{2} = 3 : 5 : 7$

$$\Rightarrow \frac{s(s-a)}{\Delta} \cdot \frac{s(s-b)}{\Delta} \cdot \frac{s(s-c)}{\Delta} = 3 : 5 : 7$$

$$\Rightarrow (s-a) : (s-b) : (s-c) = 3 : 5 : 7$$

$$\Rightarrow \frac{s-a}{3} = \frac{s-b}{5} = \frac{s-c}{7} = k \text{ (say)}$$

Then s – a = 3k, s – b = 5k, s – c = 4k

Adding these equations,

3s – (a + b + c) = 3k + 5k + 7k = 15k

$\Rightarrow$ 3s – 2s = 15k $\Rightarrow$ s = 15k

Hence a = 12k, b = 10k, c = 8k

$\therefore$ a : b : c = 12k : 10k : 8k = 6 : 5 : 4

75. Prove that $a^3 \cos(B - C) + b^3 \cos(C - A) + c^3 \cos(A - B) = 3abc$.

Sol. L.H.S. $= \Sigma a^3 \cos(B - C)$

$$= \Sigma a^2 (2R \sin A) \cos(B - C)$$
$$= R\Sigma a^2 \cdot [2\sin(B + C)\cos(B - C)]$$
$$= R\Sigma a^2 (\sin 2B + \sin 2C)$$
$$= R\Sigma a^2 (2\sin B \cos B + 2\sin C \cos C)$$
$$= \Sigma[a^2 (2R \sin B)\cos B + a^2 (2R \sin C)\cos C]$$
$$= \Sigma(a^2 b \cos B + a^2 c \cos C)$$
$$= (a^2 b \cos B + a^2 c \cos C) + (b^2 c \cos C + b^2 a \cos A) + (c^2 a \cos A + c^2 b \cos B)$$
$$= ab(a\cos B + b\cos A) + bc(b\cos C + c\cos B)$$
$$+ ca(c\cos A + a\cos C)$$
$$= ab(c) + bc(a) + ca(b)$$
$$= 3abc = \text{R.H.S.}$$

76. If p_1, p_2, p_3 are the altitudes of the vertices A, B, C of a triangle respectively, show that $\frac{1}{p_1^2} + \frac{1}{p_2^2} + \frac{1}{p_3^2} = \frac{\cot A + \cot B + \cot C}{\Delta}$.

Sol. Since p_1, p_2, p_3 are the altitudes of ΔABC, we have

$$\Delta = \frac{1}{2} a p_1 = \frac{1}{2} b p_2 = \frac{1}{2} c p_3$$
$$\Rightarrow p_1 = \frac{2\Delta}{a}, p_2 = \frac{2\Delta}{b}, p_3 = \frac{2\Delta}{c}$$

Now $\frac{1}{p_1^2} + \frac{1}{p_2^2} + \frac{1}{p_3^2} = \frac{a^2 + b^2 + c^2}{4\Delta^2}$

$$= \frac{1}{\Delta}(\cot A + \cot B + \cot C) = \text{R.H.S.}$$

$[\because \cot A + \cot B + \cot C = \frac{a^2 + b^2 + c^2}{4\Delta}]$

77. If $(r_2 - r_1)(r_3 - r_1) = 2r_2r_3$. Show that A = 90°.

Sol. $(r_2 - r_1)(r_3 - r_1) = 2r_2r_3$

$$\Rightarrow \left[\frac{\Delta}{(s-b)} - \frac{\Delta}{(s-a)}\right]\left[\frac{\Delta}{(s-c)} - \frac{\Delta}{(s-a)}\right]$$

$$= 2\frac{\Delta}{(s-b)}\frac{\Delta}{(s-c)}$$

$$\Rightarrow \Delta\left[\frac{s-a-s+b}{(s-b)(s-a)}\right]\cdot\Delta\left[\frac{s-a-s+c}{(s-c)(s-a)}\right]$$

$$= \frac{2\Delta^2}{(s-b)(s-c)}$$

$$\Rightarrow (b-a)(c-a) = 2(s-a)^2$$

$$\Rightarrow (b-a)(c-a) = 2\left(\frac{b+c-a}{2}\right)^2$$

$$\Rightarrow 2(bc - ca - ab + a^2)$$

$$= b^2 + c^2 + a^2 + 2bc - 2ca - 2ab$$

$$\Rightarrow 2a^2 = b^2 + c^2 + a^2$$

$$\Rightarrow b^2 + c^2 = a^2$$

Hence ΔABC is right angled and A = 90°.

78. In a triangle ABC prove that $\sum (r + r_1)\tan\left(\frac{B-C}{2}\right) = 0$

Solution: -

$$r_1r_2 = 4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} + 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

$$= 4R\sin\frac{A}{2}\cos\left(\frac{B-C}{2}\right)$$

$$\sum (r_1 + r)\tan\left(\frac{B-C}{2}\right) = \sum 4R\sin\frac{A}{2}\cos\left(\frac{B-C}{2}\right)\frac{\sin\left(\frac{B-C}{2}\right)}{\cos\left(\frac{B-C}{2}\right)}$$

$$\sum 3R\,2\sin\left(90^0 - \frac{B+C}{2}\right).\sin\left(\frac{B-C}{2}\right)$$

$$\sum 2R\left\{2\cos\left(\frac{B+C}{2}\right)\sin\left(\frac{B-C}{2}\right)\right\}$$

$$\sum 2R\{\sin B - \sin C\} = \sum 2R\sin B - 2R\sin C$$

$$\sum b - c = b - c + c - a + a - b = 0$$

79. Show that $\frac{1}{r^2}+\frac{1}{r_1^2}+\frac{1}{r_2^2}+\frac{1}{r_3^2}=\frac{a^2+b^2+c^2}{\Delta^2}$**.**

Sol. L.H.S. = $\frac{1}{r^2}+\frac{1}{r_1^2}+\frac{1}{r_2^2}+\frac{1}{r_3^2}$

$$=\frac{s^2}{\Delta^2}+\frac{(s-a)^2}{\Delta^2}+\frac{(s-b)^2}{\Delta^2}+\frac{(s-c)^2}{\Delta^2}$$

$$=\frac{1}{\Delta^2}\left[s^2+(s-a)^2+(s-b)^2+(s-c)^2\right]$$

$$=\frac{1}{\Delta^2}\left[s^2+s^2-2as+a^2+s^2-2bs+b^2+s^2-2cs+c^2\right]$$

$$=\frac{1}{\Delta^2}\left[4s^2-2s(a+b+c)+a^2+b^2+c^2\right]$$

$$=\frac{1}{\Delta^2}\left[4s^2-2s(2s)\right]+\frac{a^2+b^2+c^2}{\Delta^2}$$

$$=\frac{a^2+b^2+c^2}{\Delta^2}=\text{R.H.S.}$$

80. Show that $\frac{r_1}{bc}+\frac{r_2}{ca}+\frac{r_3}{ab}=\frac{1}{r}-\frac{1}{2R}$**.**

Sol. L.H.S. = $\frac{r_1}{bc}+\frac{r_2}{ca}+\frac{r_3}{ab}=\frac{1}{abc}[ar_1+br_2+cr_3]$

$$=\frac{1}{abc}\left[\Sigma a\cdot s\tan\frac{A}{2}\right]=\frac{s}{abc}\Sigma 2R\sin A\tan\frac{A}{2}\qquad\left(\because \Delta=\frac{abc}{4R}\right)$$

$$=\frac{s}{abc}\Sigma\left[2R\cdot 2\sin\frac{A}{2}\cos\frac{A}{2}\cdot\left(\frac{\sin\frac{A}{2}}{\cos\frac{A}{2}}\right)\right](\because r=\Delta/s)$$

$$=4\frac{Rs}{abc}\Sigma\left(\sin^2\frac{A}{2}\right)=\frac{s}{\Delta}\Sigma\left(\frac{1-\cos A}{2}\right)$$

$$=\frac{1}{2r}(1-\cos A+1-\cos B+1-\cos C)$$

$$=\frac{1}{2r}[3-(\cos A+\cos B+\cos C)]$$

$$=\frac{1}{2r}\left[3-\left(1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)\right]$$

$$=\frac{1}{2r}\left[2-\left(\frac{4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}{R}\right)\right]$$

$$=\frac{1}{2r}\left[2-\frac{r}{R}\right]=\frac{1}{r}-\frac{1}{2R}=\text{R.H.S.}$$

81. If r : R : r_1 = 2 : 5 : 12, then prove that the triangle is right angled at A.

Sol. If r : R : r_1 = 2 : 5 : 12, then r = 2k, R = 5k and r_1 = 12k for some k.

$$r_1 - r = 12k - 2k = 10k = 2(5k) = 2R$$

$$\Rightarrow 4R\sin\frac{A}{2}\left[\cos\frac{B}{2}\cos\frac{C}{2} - \sin\frac{B}{2}\sin\frac{C}{2}\right] = 2R$$

$$\Rightarrow 2\sin\frac{A}{2}\cos\left(\frac{B+C}{2}\right) = 1 \Rightarrow \sin^2\frac{A}{2} = \frac{1}{2}\left[\because \cos\left(\frac{B+C}{2}\right) = \sin\frac{A}{2}\right]$$

$$\Rightarrow \sin\frac{A}{2} = \frac{1}{\sqrt{2}} = \sin 45°$$

$$\Rightarrow \frac{A}{2} = 45° \Rightarrow A = 90°$$

Hence the triangle is right angled at A.

82. Show that r + r_3 + r_1 – r_2 = 4R cos B.

Sol. r + r_3

$$= 4R\sin\frac{C}{2}\left[\sin\frac{A}{2}\sin\frac{B}{2} + \cos\frac{A}{2}\cos\frac{B}{2}\right]$$

$$= 4R\sin\frac{C}{2}\cos\left(\frac{A-B}{2}\right)$$

$r_1 - r_2$

$$= 4R\cos\frac{C}{2}\left[\sin\frac{A}{2}\cos\frac{B}{2} - \cos\frac{A}{2}\sin\frac{B}{2}\right]$$

$$= 4R\cos\frac{C}{2}\sin\left(\frac{A-B}{2}\right)$$

$\therefore$ r + r_3 + r_1 – r_2

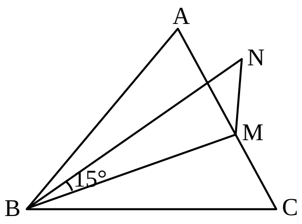
$$= 4R\left[\sin\frac{C}{2}\cos\left(\frac{A-B}{2}\right) + \cos\frac{C}{2}\sin\left(\frac{A-B}{2}\right)\right]$$

$$= 4R\sin\left(\frac{C}{2} + \frac{A}{2} - \frac{B}{2}\right) = 4R\sin\left(90° - \frac{B}{2} - \frac{B}{2}\right)$$

$$= 4R\cos B$$

83. A lamp post is situated at the middle point M of the side AC of a triangular plot ABC with BC = 7 m, CA = 8 m and AB = 9 m. Lamp post subtends an angle 15° at the point B. Find the height of the lamp post.

Sol.



MN is the height of the lamp post.

Let MN = h (?)

Given that ∠NBM = 15°

In ΔABC, $\cos C = \dfrac{b^2 + c^2 - a^2}{2abc}$

$= \dfrac{64 + 49 - 81}{2 \times 8 \times 7} = \dfrac{16 \times 2}{16 \times 7} = \dfrac{32}{112} = \dfrac{2}{7}$

$\therefore \cos C = \dfrac{2}{7}$

Let BM = x

In ΔBCM, $\cos C = \dfrac{7^2 + 4^2 - x^2}{2 \times 7 \times 4}$

$\dfrac{2}{7} = \dfrac{49 + 16 - x^2}{7 \times 8}$

$16 = 65 - x^2$

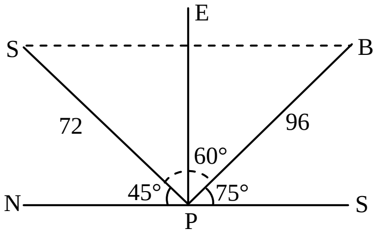
$x^2 = 65 - 16 \Rightarrow x = 7$

In ΔBMN : $\tan 15° = \dfrac{h}{x}$

$h = x \tan 15° = 7(2 - \sqrt{3})$

84. Two ships leave a port at the same time. One goes 24 km per hour in the direction N 45°E and other travels 32 km per hour in the direction S 75° E. Find the distance between the ships at the end of 3 hours.

Sol.



P is the position of the port.

A is the position of the North-East traveled ship after 3 hours is = 72 km

Position of the South-East traveled ship after 3 hours is $3 \times 32 = 96$ km

From the data $\angle APB = 60°$

In ΔAPB,

$$\cos P = \frac{AP^2 + BP^2 - AB^2}{2AP\,BP}$$

$$\cos 60° = \frac{(72)^2 + (96)^2 - AB^2}{2 \times 72 \times 96}$$

$$\frac{1}{2} = \frac{72^2 + 96^2 - AB^2}{2 \times 72 \times 96}$$

$$1 = \frac{5184 + 9216 - AB^2}{72 \times 96}$$

$$1 = \frac{14400 - AB^2}{6912}$$

$$6912 = 14400 - AB^2$$

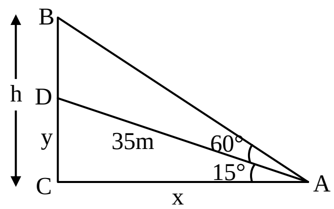
$$AB^2 = 14400 - 6912$$

$$AB^2 = 7488$$

$$AB = \sqrt{7488} = 86.53 = 86.4\,\text{km}$$

85. A tree stands vertically on the slant of the hill. From a point A on the ground 35 meters down the hill from the base of the tree, the angle of elevation of the top of the tree is 60°. If the angle of elevation of the foot of the tree from A is 15°, then find the height of the tree.

Sol.



BD is the height of the tree and A is the point of observation.

Let CD = y
AC = x

Given that, ∠CAD = 15°, ∠CAB = 60° and AD = 35 m.

In ΔCAD, $\sin 15° = \frac{y}{35}$

$y = 35\sin 15° = \frac{35(\sqrt{3}-1)}{2\sqrt{2}}$ …(1)

$\cos 15° = \frac{x}{35}$

$x = \frac{\sqrt{3}+1}{2\sqrt{2}} \times 35$ …(2)

In ΔCAB, $\tan 60° = \frac{h}{x}$

$h = x\sqrt{3} = \frac{\sqrt{3}+1}{2\sqrt{2}} \times 35 \times \sqrt{3}$

Height of the tree = h – y

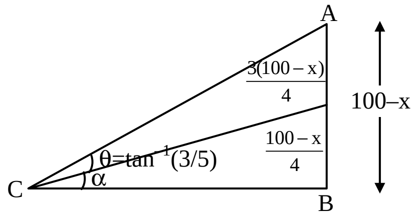
$\frac{\sqrt{3}+1}{2\sqrt{2}} \times 35\sqrt{3} - \frac{\sqrt{3}-1}{2\sqrt{2}} \times 35 =$

$= \frac{\sqrt{3}+1}{2\sqrt{2}}\left[3+\sqrt{3}-\sqrt{3}+1\right]$

$= \frac{35\times 4}{2\sqrt{2}} = 35\sqrt{2}\text{ m}$

86. The upper 3/4th portion of a vertical pole subtends an angle $Tan^{-1}3/5$ at a point in the horizontal plane through its foot and at a distance 40 m from the foot. Given that the vertical pole is at a height less than 100 m from the ground, find its height.

Sol.



AB is the height of the tree.

AD is the 3/4th part of upper part of the tree.

DB is the 1/4th lower part of the tree.

Let AB = 100 – x

C is the point of observation.

In ΔBCD,

Let $\angle DCA: \theta = \tan^{-1}\frac{3}{5} \Rightarrow \tan\theta = \frac{3}{5}$

$$\tan\alpha = \frac{100-x}{4} \times \frac{1}{40} = \frac{100-x}{160}$$

$$\tan(\theta+\alpha) = \frac{\tan\theta + \tan\alpha}{1-\tan\theta\tan\alpha}$$

$$\frac{100-x}{40} = \frac{\frac{3}{5}+\frac{100-x}{160}}{1-\frac{3}{5}\times\frac{100-x}{160}}$$

$$\frac{100-x}{40} = \frac{480+5(100-x)}{800-3(100-x)}$$

$$\frac{100-x}{40} = \frac{480+500-5x}{800-300+3x}$$

$$\frac{100-x}{40} = \frac{980-5x}{500+3x}$$

$[100-x][500+3x]=40[980-5x]$

$50000+300x-500x-3x^2=39200-200x$

$\Rightarrow 3x^2+500x-400x=50000-39200$

$3x^2=10800$

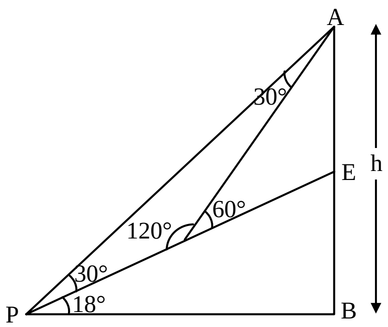
$x^2=\dfrac{10800}{3}=3600$

$x=\sqrt{3600}=60$

Height of the tree = 100 – x = 40 m.

87. Let an object be placed at some height h cm and let P and Q be two points of observation which are at a distance 10 cm apart on a line inclined at angle 15° to the horizontal. If the angles of elevation of the object from P and Q are 30° and 60° respectively then find h.

Sol.



A is the position of the object.

Given that AB = h cm

P and Q are points of observation.

Given that, PQ = 10 cm

We have,

$\angle BPE = 15°$, $\angle EPA = 30°$, $\angle EQA = 60°$

In ΔPQA,

P = 30°, Q = 120° and A = 30°

$\therefore$ By sine rule,

$$\frac{AP}{\sin 120°}=\frac{PQ}{\sin 30°}$$

$$\frac{AP}{\sin(180°-60°)}=\frac{10}{1/2}$$

$$\frac{AP}{\sin 60^\circ} = 20^\circ \Rightarrow \frac{AP}{\sqrt{3}/2} = 20^\circ$$

$$AP = 20 \times \frac{\sqrt{3}}{2} = 10\sqrt{3}$$

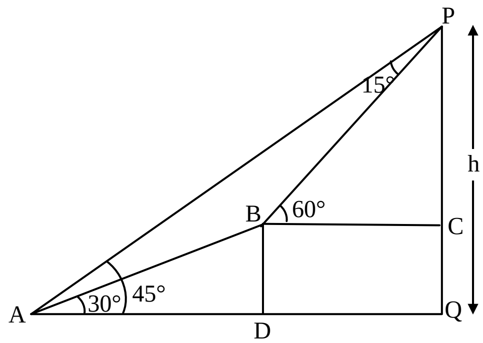
In $\Delta PBA,\ \sin 45^\circ = \frac{AB}{AP}$

$$\frac{1}{\sqrt{2}} = \frac{h}{10\sqrt{3}}$$

$$h = \frac{10\sqrt{3}}{\sqrt{2}} = \frac{5\cdot 2\cdot\sqrt{3}}{\sqrt{2}} = 5\sqrt{2}\sqrt{3} = 5\sqrt{6}\,\text{cm}$$

88. The angle of elevation of the top point P of the vertical tower PQ of height h from a point A is 45● and from a point B is 60●, where B is a point at a distance 30 meters from the point A measured along the line AB which makes an angle 30● with AQ. Find the height of the tower.

Sol.



In the figure

PQ = h, $\angle PAQ = 45^\circ$

$\angle BAQ = 30^\circ$ and $\angle PBC = 60^\circ$

Also, AB = 30 m

$\therefore\ \angle BAP = \angle APB = 15^\circ$

This gives, BP = AB = 30 and

h = PC + CD = BP sin 60° + AB sin 30°

$= 15\sqrt{3} + 15 = 15(\sqrt{3}+1)$ meters.

89**. Theorem : - In a triangle ABC prove that**

(i) $\sin\frac{A}{2} = \sqrt{\frac{(S-b)(S-c)}{bc}}$ **(ii)** $\cos\frac{A}{2} = \sqrt{\frac{S(S-a)}{bc}}$

(iii) $\tan\frac{A}{2} = \sqrt{\frac{(S-b)(S-C)}{S(S-a)}} = \frac{\Delta}{S(S-a)} = \frac{(S-b)(S-C)}{\Delta}$

(iv) $\cot A/2 = \sqrt{\frac{S(S-a)}{(S-b)(S-c)}} = \frac{\Delta}{(S-b)(S-c)} = \frac{S(S-a)}{\Delta}$

Proof (i)

From cosine rule we know that

$a^2 + b^2 + c^2 = 2bc\cos A \Rightarrow 2bc\cos = b^2 + c^2 - a^2$

$\cos A = \frac{b^2+c^2-a^2}{2bc}$

We know that $2\sin^2\frac{A}{2} = 1 - \cos A$

$$= 1 - \frac{b^2+c^2-a^2}{2bc} = \frac{2bc - b^2 - c^2 + a^2}{2bc}$$

$\therefore 2\sin^2\frac{A}{2} = \frac{a^2 - \{b^2 + c^2 - 2bc\}}{2bc} \Rightarrow \sin^2\frac{A}{2} = \frac{a^2 - (b-c)^2}{4bc}$

$\sin^2\frac{A}{2} = \frac{(a+b-c)(a-b+c)}{4bc}$

$\because a+b+c = 2S$ we have $2S - 2C = a+b-c$

$\therefore \sin^2\frac{A}{2} = \frac{2(S-C)\,2(S-b)}{4bc} \Rightarrow \sin\frac{A}{2} = \sqrt{\frac{(S-b)(S-C)}{bc}}$

Proof (ii)

$2\cos^2\frac{A}{2} = 1 + \cos A = 1 + \frac{b^2+c^2-a^2}{2bc}$

$2\cos^2\frac{A}{2} = \frac{2bc + b^2 + c^2 - a^2}{2bc} \Rightarrow 2\cos^2\frac{A}{2}\,\frac{(b+c)^2 - a^2}{2bc}$

$\cos^2\frac{A}{2} = \frac{(b+c-a)(b+c+a)}{4bc}$

Since $a+b+c = 2S$; $2S - 2a = b+c-a$

$\therefore \cos^2\frac{A}{2} = \frac{2(S-a)\,2S}{4bc} \Rightarrow \cos\frac{A}{2} = \sqrt{\frac{S(S-a)}{bc}}$

Proof (iii) $\tan\frac{A}{2}=\frac{\sin\frac{A}{2}}{\cos\frac{A}{2}}=\sqrt{\frac{\frac{(S-b)(S-c)}{b-c}}{\frac{S(S-a)}{b-c}}}=\sqrt{\frac{(S-b)(S-c)}{S(S-a)}}$

$$\tan\frac{A}{2}=\sqrt{\frac{(S-b)(S-c)}{S(S-a)}\frac{(S-b)(S-c)}{(S-b)(S-c)}}=\frac{(S-b)(S-c)}{\sqrt{S(S-a)(S-b)(S-c)}}=\frac{(S-b)(S-c)}{\Delta}$$

$$\tan\frac{A}{2}\sqrt{\frac{(S-b)(S-c)}{S(S-a)}\times\frac{S(S-a)}{S(S-a)}}=\frac{\sqrt{S(S-a)(S-b)(S-c)}}{S(S-a)}=\frac{\Delta}{S(S-a)}$$

Proof of (iv)

By taking reciprocal of $\tan A/2$ we get $\cot A/2$

List of formulae related to half angles

$\sin\frac{A}{2}\sqrt{\frac{(S-b)(S-c)}{bc}}$ $\qquad$ $\sin B/2=\sqrt{\frac{(S-c)(S-a)}{ac}}$ $\qquad$ $\sin\frac{C}{2}=\sqrt{\frac{(S-a)(S-b)}{ab}}$

$\cos\frac{A}{2}=\sqrt{\frac{S(S-a)}{bc}}$ $\qquad$ $\cos\frac{B}{2}=\sqrt{\frac{S(S-b)}{ac}}$ $\qquad$ $\cos\frac{C}{2}\sqrt{\frac{S(S-c)}{ab}}$

$$\tan\frac{A}{2}=\sqrt{\frac{(S-b)(S-C)}{S(S-a)}}=\frac{\Delta}{S(S-a)}=\frac{(S-b)(S-c)}{\Delta}$$

$$\tan\frac{B}{2}=\sqrt{\frac{(S-c)(S-a)}{S(S-b)}}=\frac{\Delta}{S(S-b)}=\frac{(S-c)(S-a)}{\Delta}$$

$$\tan\frac{C}{2}=\sqrt{\frac{(S-a)(S-b)}{S(S-c)}}=\frac{\Delta}{S(S-c)}=\frac{(S-c)(S-a)}{\Delta}$$

$$\cot\frac{A}{2}=\sqrt{\frac{S(S-a)}{(S-b)(S-c)}}=\frac{\Delta}{(S-b)(S-c)}=\frac{S(S-a)}{\Delta}$$

$$\cot\frac{B}{2}=\sqrt{\frac{S(S-b)}{(S-a)(S-c)}}=\frac{\Delta}{(S-a)(S-c)}=\frac{S(S-b)}{\Delta}$$

$$\cot\frac{C}{2}=\sqrt{\frac{S(S-C)}{(S-a)(S-b)}}=\frac{\Delta}{(S-a)(S-b)}=\frac{S(S-C)}{\Delta}$$