DEFINITIONS, CONCEPTS AND FORMULAE:

1. If n is an integer, $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$
2. If n is a rational number, then one of the values of $(\cos\theta + i\sin\theta)^n$ is $\cos n\theta + i\sin n\theta$.
3. If $z = r\,\text{cis}\,\theta = r\,\text{cis}\,(2k\pi + \theta)$, then $z^{1/n} = r^{1/n}\ \text{cis}\left(\frac{2k\pi+\theta}{n}\right)$ where k = 0, 1, 2, ..., n - 1.
4. If $x = \cos\theta + i\sin\theta$ then $\frac{1}{x} = \cos\theta + i\sin\theta$.

   and i) $x + \frac{1}{x} = 2\cos\theta$ ii) $x - \frac{1}{x} = 2i\sin\theta$
5. If $x = \cos\theta + i\sin\theta$ then $\frac{1}{x} = \cos\theta - i\sin\theta$ and

   i) $x^n + \frac{1}{x^n} = 2\cos n\theta$

   ii) $x^n - \frac{1}{x^n} = 2i\sin n\theta$.
6. Cube roots of unity :
   The roots of $x^3 = 1$ are called cube roots of unity which are 1, ω, ω^2 where

   $\omega = \frac{-1+i\sqrt{3}}{2},\ \omega^2 = \frac{-1-i\sqrt{3}}{2}$.
7. $\text{cis}\theta_1.\text{cis}\theta_2 = \text{cis}(\theta_1 + \theta_2)$
8. $\frac{\text{cis}\theta_1}{\text{cis}\theta_2} = \text{cis}\,(\theta_1 - \theta_2)$.
9. $\text{cis}\theta_1.\text{cis}\theta_2\,.\text{cis}\theta_3..............\text{cis}\theta_n = \text{cis}\,(\theta_1 + \theta_2 + \theta_3 + ... + \theta_n)$.

LEVEL - I (VSAQ)

1. Find the value of $(1 + i)^{16}$.

A: $(1+i)^{16} = \left[\sqrt{2}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)\right]^{16}$

$= \left(\sqrt{2}\right)^{16}\left[\cos45^0 + i\sin45^0\right]^{16}$

$= 2^8\,[\cos16(45^0) - i\sin16\,(45^0)]$

$= 256\,[\cos720^0 + i\sin720^0] = 256\,[1 - i.0] = 256$

2. Find the value of $\left(1+i\sqrt{3}\right)^3$.

A: $\left(1+i\sqrt{3}\right)^3 = \left[2\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)\right]^3 = 8\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^3$

$= 8(\cos60^0 + i\sin60^0)^3$

By applying De Moivre's theorem for an integral index.

$= 8[\cos3(60^0) + i\sin3(60^0)]$

$= 8(\cos180^0 + i\sin180^0) = 8\,[-1 + i(0)] = -8$.

3. Find the value of $(1 - i)^8$.

AIMS

A: $(1-i)^8 = \left[\sqrt{2}\left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right)\right]^8$

$= \left(\sqrt{2}\right)^8 (\cos45^0 - i\sin45^0)^8$

By applying De Moivre's theorem for an integral index.

$= 2^4\,[\cos8(45^0) - i\sin8\,(45^0)]$

$= 2^4\,[\cos360^0 - i\sin360^0] = 16\,[1 - i(0)] = 16$

4. Find the value of $\left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right)^5 - \left(\frac{\sqrt{3}}{2}-\frac{i}{2}\right)^5$.

A: $\left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right)^5 - \left(\frac{\sqrt{3}}{2}-\frac{i}{2}\right)^5$

$= (\cos30^0 + i\sin30^0)^5 - (\cos30^0 - i\sin30^0)^5$

$= \cos5(30^0) + i\sin5(30^0) - [\cos5(30^0) - i\sin5(30^0)]$

$=$

~~$\cos150^0$~~ $+ i\sin150^0 - \left[\right.$ ~~$\cos150^0$~~ $\left. - i\sin150^0\right]$

$= 2i\sin150^0 = 2i\left(\frac{1}{2}\right) = i$.

5. If A, B, C are the angles of a triangle such that x = cis A, y = cis B, z = cis C, then find xyz.

A: Given that x = cis A, y = cis B, z = cis C

Now xyz = cis A.cis B.cis C = cis(A+B+C)

= cos (A + B + C) + i sin (A + B + C)

= cos 180^0 + i sin 180^0 = - 1 + i(0) = - 1.

6. If x = cis θ, then find the value of $\left(x^6 + \frac{1}{x^6}\right)$.

A: Given that x = cos θ + i sin θ.

$\Rightarrow x^6 = (\cos\theta + i\sin\theta) = \cos 6\theta + i \sin 6\theta$.

Now, $\frac{1}{x^6} = \frac{1}{\cos 6\theta + i\sin 6\theta} = \cos 6\theta - i\sin 6\theta$.

Hence, $x^6 + \frac{1}{x^6} =$

$= \cos 6\theta + \cancel{i\sin 6\theta} + \cos 6\theta - \cancel{i\sin 6\theta}$

= 2cos 6θ.

7. Find the cube roots of 8.

A: Let x = $\sqrt[3]{8} \Rightarrow x^3 = 8$

$x^3 = 2^3 = (2.1)^3$

$\Rightarrow x = 2(1^{1/3})$

$= 2(1, \omega, \omega^2)$

$= 2, 2\omega, 2\omega^2$.

8. If α, β are the roots of the equation $x^2 + x + 1 = 0$, then prove that $\alpha^4 + \beta^4 + \alpha^{-1}\beta^{-1} = 0$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{3}i}{2} = \omega, \omega^2$$

A

$$\alpha^4 + \beta^4 + \alpha^{-1}\beta^{-1} = \omega^4 + (\omega^2)^4 + \frac{1}{\omega}.\frac{1}{\omega^2} = \omega + \omega^2 + 1 = 0$$

9. Simplify $\frac{(\cos\alpha + i\sin\alpha)^4}{(\sin\beta + i\cos\beta)^8}$.

A: $= \frac{(\cos\alpha + i\sin\alpha)^4}{(-i^2\sin\beta + i\cos\beta)^8} = \frac{(\cos\alpha + i\sin\alpha)^4}{[i(\cos\beta - i\sin\beta)]^8}$

$= \frac{(\cos\alpha + i\sin\alpha)^4}{i^8(\cos\beta - i\sin\beta)^8} = \frac{\cos 4\alpha + i\sin 4\alpha}{\cos 8\beta - i\sin 8\beta}$

= (cos4α + isin4α)(cos8β+isin8β)

= cos (4α + 8β) + isin (4α + 8β) = cis(4α + 8β)

10. Solve $x^4 - 1 = 0$.

A: $x^4 - 1 = 0 \Rightarrow (x^2 + 1)(x^2 - 1) = 0$

$\Rightarrow x^2 + 1 = 0$ or $x^2 - 1 = 0$

$\Rightarrow x^2 = -1$ or $x^2 = 1$

$\Rightarrow x = \sqrt{-1}$ or $x = \sqrt{1}$

$\Rightarrow x = \pm i$ or $x = \pm 1$.

11. If the cube roots of unity are 1, ω, ω², then find the roots of the equation $(x - 1)^3 + 8 = 0$.

$(x - 1)^3 = -8 = (-2)^3 = -2(1)^{1/3} = -2(1, \omega, \omega^2)$

$\Rightarrow x - 1 = -2, -2\omega, -2\omega^2$

$\therefore x = 1 - 2, 1 - 2\omega, 1 - 2\omega^2$

$= -1, 1 - 2\omega, 1 - 2\omega^2$

LEVEL - I (LAQ)

AIMS

1 If cosα + cosβ + cosγ = 0 = sinα + sinβ + sinγ, then show that

i) cos 3α + cos 3β + cos 3γ = 3cos (α+β +γ)

ii) sin 3α + sin 3β + sin 3γ =3sin (α+β+γ)

A: Given: cosα + cosβ + cosγ = 0 = sinα + sinβ + sinγ

Let a = cosα +isinα,

b = cosβ + isinβ

c = cosγ + isinγ

∴ a + b + c = (cosα +cosβ+cosγ) + i(sinα+sinβ+sinγ)

= 0 + i (0)

a+b+c = 0

$\Rightarrow a^3 + b^3 + c^3 = 3abc$

$(\cos\alpha + i\sin\alpha)^3 + (\cos\beta + i\sin\beta)^3 + (\cos\gamma + i\sin\gamma)^3$

= 3(cisα) (cisβ) (cisγ)

By applying DeMoivre's Theorem for an integral index, we get

cos 3α + i sin 3α + cos 3β + i sin 3β + cos 3γ + i sin 3γ

= 3cis(α + β + γ)

(cos 3α + cos 3β + cos 3γ) + i(sin 3α + sin 3β + sin 3γ)

= 3[cos(α + β +γ) + i sin(α + β + γ)]

Equating of the real and imaginary parts on bothsides, we get

cos 3α + cos 3β + cos 3γ = 3cos(α + β + γ)

&

sin 3α + sin 3β + sin 3γ = 3sin(α + β + γ).

2. If $\cos\alpha + \cos\beta + \cos\gamma = 0 = \sin\alpha + \sin\beta + \sin\gamma$, show that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{3}{2}$ $= \sin^2\alpha + \sin^2\beta + \sin^2\gamma$.

A: Let $a=\cos\alpha + i\sin\alpha$, $b=\cos\beta + i\sin\beta$, $c=\cos\gamma + i\sin\gamma$

$\Rightarrow a+b+c=(\cos\alpha+\cos\beta+\cos\gamma)+i(\sin\alpha+\sin\beta+\sin\gamma)$

$= 0 + i.0$

$= 0.$

On squaring, we get

$(a+b+c)^2 = 0$

$\Rightarrow a^2 + b^2 + c^2 + 2(ab + bc + ca) = 0$

$\Rightarrow a^2 + b^2 + c^2 = -2(ab + bc + ca)$

$\Rightarrow cis2\alpha + cis2\beta + cis2\gamma = -2abc\left(\frac{1}{c}+\frac{1}{a}+\frac{1}{b}\right)$

$= -2abc[\cos\gamma - i\sin\gamma+\cos\alpha - i\sin\alpha + \cos\beta - i\sin\beta]$

$=-2abc[\cos\alpha+\cos\beta+\cos\gamma)- i(\sin\alpha+\sin\beta+\sin\gamma)]$

$= -2abc[0-i(0)]$

$= 0.$

$\Rightarrow \cos2\alpha + i\sin2\alpha+\cos2\beta+ i\sin2\beta+\cos2\gamma+ i\sin2\gamma = 0$

$\Rightarrow (\cos2\alpha+\cos2\beta+\cos2\gamma)+i(\sin2\alpha+\sin2\beta+\sin2\gamma)=0+i(0)$

Equating the real parts, we get

$\cos2\alpha + \cos2\beta + \cos2\gamma = 0$ ....(1)

$\Rightarrow 2\cos^2\alpha -1 + 2\cos^2\beta -1 + 2\cos^2\gamma-1 = 0$

$\Rightarrow 2(\cos^2\alpha + \cos^2\beta + \cos^2\gamma) = 3$

$\therefore \cos^2\alpha + \cos^2\beta + \cos^2\beta = \frac{3}{2}$

Also from (1) $\cos2\alpha + \cos2\beta + \cos2\gamma = 0$

$\Rightarrow 1-2\sin^2\alpha + 1-2\sin^2\beta + 1-2\sin^2\gamma = 0$

$\Rightarrow 3 = 2(\sin^2\alpha + \sin^2\beta + \sin^2\gamma)$

$\therefore \sin^2\alpha + \sin^2\beta + \sin^2\gamma = \frac{3}{2}$.

3. Prove that $(1+\cos\theta + i\sin\theta)^n + (1+\cos\theta-i\sin\theta)^n = 2^{n+1}\cos^n\left(\frac{\theta}{2}\right)\cos\left(\frac{n\theta}{2}\right)$.

A: Now $(1+\cos\theta + i\sin\theta)^n + (1+\cos\theta - i\sin\theta)^n$

$$=\left(2\cos^2\frac{\theta}{2}+i\,2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)^n+\left(2\cos^2\frac{\theta}{2}-i\,2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)^n$$

$$=\left[2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)\right]^n+\left[2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2}-i\sin\frac{\theta}{2}\right)\right]^n$$

By applying DeMoivre's theorem for an integral index

$$=2^n\cos^n\left(\frac{\theta}{2}\right)\left[\cos\frac{n\theta}{2}+i\sin\frac{n\theta}{2}\right]$$
$$+2^n\cos^n\left(\frac{\theta}{2}\right)\left[\cos\frac{n\theta}{2}-i\sin\frac{n\theta}{2}\right]$$

$$=2^n\cos^n\left(\frac{\theta}{2}\right)\left[\cos\frac{n\theta}{2}+i\sin\frac{n\theta}{2}+\cos\frac{n\theta}{2}-i\sin\frac{n\theta}{2}\right]$$

$$=2^n\cos^n\left(\frac{\theta}{2}\right).2\cos\left(\frac{n\theta}{2}\right)$$

$$=2^{n+1}\cos^n\left(\frac{\theta}{2}\right)\cos\left(\frac{n\theta}{2}\right).$$

4. In n is a positive integer, show that $(1+i)^n + (1-i)^n = 2^{\frac{n+2}{2}}\cos\left(\frac{n\pi}{4}\right)$.

A: $1+i = \sqrt{2}\left(\frac{1}{\sqrt{2}}+i.\frac{1}{\sqrt{2}}\right)$

$= \sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$

AIMS

$1 - i = \sqrt{2}\left(\cos\frac{\pi}{4}-i\sin\frac{\pi}{4}\right)$

$(1+i)^n + (1 - i)^n$

$$=\left[\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)\right]^n+\left[\sqrt{2}\left(\cos\frac{\pi}{4}-i\sin\frac{\pi}{4}\right)\right]^n$$

$$=\sqrt{2}^n\left(\cos\frac{n\pi}{4}+i\sin\frac{n\pi}{4}\right)+\sqrt{2}^n\left(\cos\frac{n\pi}{4}-i\sin\frac{n\pi}{4}\right)$$

By applying DeMoivre's theorem for a +ve integer

$$=2^{\frac{n}{2}}\left[\cos\frac{n\pi}{4}+i\sin\frac{n\pi}{4}+\cos\frac{n\pi}{4}-i\sin\frac{n\pi}{4}\right]$$

$$=2^{\frac{n}{2}}.2\cos\left(\frac{n\pi}{4}\right)$$

$$=2^{\frac{n}{2}+1}.\cos\left(\frac{n\pi}{4}\right)$$

$$=2^{\frac{n+2}{2}}\cos\left(\frac{n\pi}{4}\right)$$

5. If α, β are the roots of the equation $x^2 - 2x + 4 = 0$. Show that $\alpha^n + \beta^n = 2^{n+1}\cos\left(\frac{n\pi}{3}\right)$.

A: Given equation is $x^2 - 2x +4 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{2 \pm \sqrt{4 - 16}}{2}$$

$$= \frac{2 \pm \sqrt{12\, i^2}}{2} \qquad \because \cos\theta - i\sin\theta$$

$$= \frac{2 \pm 2\sqrt{3}\ i}{2} \qquad = \frac{1}{\cos\theta + i\sin\theta}$$

$$= 1 \pm \sqrt{3}\ i$$

Let $\alpha = 1 + \sqrt{3}\ i$, $\beta = 1 - \sqrt{3}\ i$

Now $\alpha^n + \beta^n = (1 + \sqrt{3}\ i)^n + (1 - \sqrt{3}\ i)^n$

$$= \left[2\left(\frac{1}{2} + i.\frac{\sqrt{3}}{2}\right)\right]^n + \left[2\left(\frac{1}{2} - i.\frac{\sqrt{3}}{2}\right)\right]^n$$

$$= \left[2\left(\cos\frac{\pi}{3} + i.\sin\frac{\pi}{3}\right)\right]^n + \left[2\left(\cos\frac{\pi}{3} - i.\sin\frac{\pi}{3}\right)\right]^n$$

$$= 2^n\left(\cos\frac{n\pi}{3} + i.\sin\frac{n\pi}{3}\right) + 2^n\left(\cos\frac{n\pi}{3} - i.\sin\frac{n\pi}{3}\right)$$

$$= 2^n\left[\cos\frac{n\pi}{3} + i.\sin\frac{n\pi}{3} + \cos\frac{n\pi}{3} - i.\sin\frac{n\pi}{3}\right]$$

$$= 2^n.2\cos\frac{n\pi}{3}$$

$$= 2^{n+1}\cos\frac{n\pi}{3}$$

6. If n is an integer, Show that

$(1 + i)^{2n} + (1 - i)^{2n} = 2^{n+1}\cos\frac{n\pi}{2}$.

A: $1 + i = \sqrt{2}\left(\frac{1}{\sqrt{2}} + i.\frac{1}{\sqrt{2}}\right)$

$$= \sqrt{2}\left(\cos\frac{\pi}{4} + i.\sin\frac{\pi}{4}\right)$$

$$\Rightarrow 1 - i = \sqrt{2}\left(\cos\frac{\pi}{4} - i.\sin\frac{\pi}{4}\right)$$

Now $(1 + i)^{2n} + (1 + i)^{2n}$

$$= \left[\sqrt{2}\left(\cos\frac{\pi}{4} + i.\sin\frac{\pi}{4}\right)\right]^{2n} + \left[\sqrt{2}\left(\cos\frac{\pi}{4} - i.\sin\frac{\pi}{4}\right)\right]^{2n}$$

$$= \left(\sqrt{2}\right)^{2n}\left(\cos\frac{\pi}{4} + i.\sin\frac{\pi}{4}\right)^{2n} + \left(\sqrt{2}\right)^{2n}\left(\cos\frac{\pi}{4} - i.\sin\frac{\pi}{4}\right)^{2n}$$

Using the De Moivre's Theorem for an integral index

$$= 2^n\left(\cos 2n\frac{\pi}{4} + i.\sin 2n\frac{\pi}{4}\right) + 2^n\left(\cos 2n\frac{\pi}{4} - i.\sin 2n\frac{\pi}{4}\right)$$

$$= 2^n\left(\cos\frac{n\pi}{2} + i.\sin\frac{n\pi}{2} + \cos\frac{n\pi}{2} - i.\sin\frac{n\pi}{2}\right)$$

$$= 2^n.\ 2\cos\frac{n\pi}{2}$$

$$= 2^{n+1}\cos\frac{n\pi}{2}.$$

7. If n is an integer and z = cisθ, then show that

$\frac{Z^{2n} - 1}{Z^{2n} + 1} = i\tan n\theta$.

A: Given: $Z = cis\theta$

$= \cos\theta + i\sin\theta$.

Now $\frac{Z^{2n} - 1}{Z^{2n} + 1} = \frac{(\cos\theta + i\sin\theta)^{2n} - 1}{(\cos\theta - i\sin\theta)^{2n} + 1}$

$$= \frac{\cos 2n\theta + i\sin 2n\theta - 1}{\cos 2n\theta - i\sin 2n\theta + 1}$$

By applying DeMoivre's Theorem for an integral index

$$= \frac{i\sin 2n\theta - (1 - \cos 2n\theta)}{i\sin 2n\theta + (1 + \cos 2n\theta)}$$

$$= \frac{i\, 2\sin n\theta\cos n\theta - 2\sin^2 n\theta}{i\, 2\sin n\theta\cos n\theta + 2\cos^2 n\theta}$$

$$= \frac{i\, 2\sin n\theta\cos n\theta + i^2\, 2\sin^2 n\theta}{i\, 2\sin n\theta\cos n\theta + 2\cos^2 n\theta}$$

$$= \frac{2i\sin n\theta[\cos n\theta + i\sin n\theta]}{2\cos n\theta[\cos n\theta + i\sin n\theta]}$$

$$= i\tan n\theta.$$

8. If n is a positive integer, show that $(p+iq)^{1/n} + (p - iq)^{1/n}$

$= 2(p^2+q^2)^{1/2n}\cos\left[\frac{1}{n}\arctan\frac{q}{p}\right]$.

A: Given: n is a positive integer

Now $(p+iq)^{1/n} + (p - iq)^{1/n}$

$$= \left[\sqrt{p^2 + q^2}\left\{\frac{p}{\sqrt{p^2 + q^2}} + i\frac{q}{\sqrt{p^2 + q^2}}\right\}\right]^{\frac{1}{n}}$$

$$+ \left[\sqrt{p^2 + q^2}\left\{\frac{p}{\sqrt{p^2 + q^2}} - i\frac{q}{\sqrt{p^2 + q^2}}\right\}\right]^{\frac{1}{n}}$$

By applying DeMoivre's theorem for a rational index, we get one value as

$(p^2+q^2)^{1/2n}$ $[(\cos\alpha + i\sin\alpha)^{1/n} + (\cos\alpha - i\sin\alpha)^{1/n}]$

Where $\cos\alpha = \frac{p}{\sqrt{p^2+q^2}}$, $\sin\alpha = \frac{q}{\sqrt{p^2+q^2}}$

$$= \left(p^2+q^2\right)^{\frac{1}{2n}} \left[\cos\frac{\alpha}{n} + i\sin\frac{\alpha}{n} + \cos\frac{\alpha}{n} - i\sin\frac{\alpha}{n}\right]$$

$$= \left(p^2+q^2\right)^{\frac{1}{2n}} 2\cos\left(\frac{1}{n}\alpha\right)$$

$$= \left(p^2+q^2\right)^{\frac{1}{2n}} 2\cos\left(\frac{1}{n}\text{Tan}^{-1}\frac{q}{p}\right)$$

$$= \left(p^2+q^2\right)^{\frac{1}{2n}} 2\cos\left(\frac{1}{n}\text{arcTan}\left(\frac{q}{p}\right)\right)$$

9. Show that one value of

$$\left[\frac{1+\sin\frac{\pi}{8}+i\cos\frac{\pi}{8}}{1+\sin\frac{\pi}{8}-i\cos\frac{\pi}{8}}\right]^{\frac{8}{3}} = -1$$

A: Consider $\frac{1+\sin\frac{\pi}{8}+i\cos\frac{\pi}{8}}{1+\sin\frac{\pi}{8}-i\cos\frac{\pi}{8}}$

$$= \frac{1+\cos\left(\frac{\pi}{2}-\frac{\pi}{8}\right)+i\sin\left(\frac{\pi}{2}-\frac{\pi}{8}\right)}{1+\cos\left(\frac{\pi}{2}-\frac{\pi}{8}\right)-i\sin\left(\frac{\pi}{2}-\frac{\pi}{8}\right)}$$

$$= \frac{1+\cos\frac{3\pi}{8}+i\sin\frac{3\pi}{8}}{1+\cos\frac{3\pi}{8}-i\sin\frac{3\pi}{8}}$$

$$= \frac{2\cos^2\frac{3\pi}{16}+i\,2\sin\frac{3\pi}{16}\cos\frac{3\pi}{16}}{2\cos^2\frac{3\pi}{16}-i\,2\sin\frac{3\pi}{16}\cos\frac{3\pi}{16}}$$

$$= \frac{2\cos\frac{3\pi}{16}\left(\cos\frac{3\pi}{16}+i\sin\frac{3\pi}{16}\right)}{2\cos\frac{3\pi}{16}\left(\cos\frac{3\pi}{16}-i\sin\frac{3\pi}{16}\right)}$$

$$= \left(\cos\frac{3\pi}{16}+i\sin\frac{3\pi}{16}\right)^2$$

By applying DeMoivre's Theorem for an integral index $= \cos\frac{3\pi}{8}+i\sin\frac{3\pi}{8}$

Now by applying DeMoivre's Theorem for a rational index, then one value of

$$\left[\frac{1+\sin\frac{\pi}{8}+i\cos\frac{\pi}{8}}{1+\sin\frac{\pi}{8}-i\cos\frac{\pi}{8}}\right]^{\frac{8}{3}}.$$

$= \text{one value of } \left(\cos\frac{3\pi}{8}+i\sin\frac{3\pi}{8}\right)^{\frac{8}{3}}$

$= \cos\pi + i\sin\pi$

$= -1 + i(0)$

$= -1.$

10. Solve the equation $x^9 - x^5 + x^4 - 1 = 0$.

A: Given equation is $x^9 - x^5 + x^4 - 1 = 0$.

$x^5(x^4-1) + 1(x^4-1) = 0$

$(x^5+1)(x^4-1) = 0$

Now $x^5 + 1 = 0$

$x^5 = -1$

$= \cos\pi + i\sin\pi$

$= \text{cis}\,\pi$

$= \text{cis}(2k\pi + \pi),\ k \in z$

$= \text{cis}(2k+1)\pi,\ k \in z$

AIMS

$\therefore\ x = [\text{cis}(2k+1)\pi]^{1/5}$

$= \text{cis}(2k+1)\frac{\pi}{5}$ where k = 0, 1, 2, 3, 4.

$= \text{cis}\frac{\pi}{5},\ \text{cis}\frac{3\pi}{5},\ \text{cis}\pi,\ \text{cis}\frac{7\pi}{5},\ \text{cis}\frac{9\pi}{5}$

Also $x^4 - 1 = 0$

$(x^2-1)(x^2+1) = 0$

$x = \pm 1, \pm i$

Hence the required roots are

$\pm 1, \pm i, \text{cis}\frac{\pi}{5}, \text{cis}\frac{3\pi}{5}, \text{cis}\pi, \text{cis}\frac{7\pi}{5}, \text{cis}\frac{9\pi}{5}$.

LEVEL - II (VSAQ)

1. If 1, ω, ω^2 are the cube roots of unity, then prove that $\frac{1}{2+\omega}+\frac{1}{1+2\omega}=\frac{1}{1+\omega}$.

$$\frac{1}{2+\omega}+\frac{1}{1+2\omega}=\frac{1+2\omega+2+\omega}{2+4\omega+\omega+2\omega^2}$$

$$=\frac{3(1+\omega)}{2(1+\omega+\omega^2)+3\omega}=\frac{3(1+\omega)}{3\omega}=\frac{(1+\omega)^2}{\omega(1+\omega)}=\frac{(1+\omega+\omega^2)+\omega}{\omega(1+\omega)}$$

$$=\frac{\omega}{\omega(1+\omega)}=\frac{1}{1+\omega}.$$

2. Prove that $-\omega$ and $-\omega^2$ are the roots of $z^2 - z + 1 = 0$, where ω and ω^2 are the complex cube roots of unity.

A: $z^2 - z + 1 = 0$.

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{1 \pm \sqrt{(-1)^2 - 4(1)(1)}}{2(1)}$$

$$= \frac{1 \pm \sqrt{3i^2}}{2}$$

$$= \frac{1 - \sqrt{3}i}{2}, \frac{1 + \sqrt{3}i}{2}$$

$$= -\left(\frac{-1 + \sqrt{3}i}{2}\right), -\left(\frac{-1 - \sqrt{3}i}{2}\right)$$

$$= -\omega, -\omega^2.$$

3. If 1, ω, ω^2 are the cube roots of unity, find the value of $(1 - \omega + \omega^2)^3$.

$$(1 - \omega + \omega^2)^3 = \left[\left(1 + \omega^2\right) - \omega\right]^3 \quad \because 1 + \omega + \omega^2 = 0$$

$$= (-\omega - \omega)^3$$

$$= (-2\omega)^3$$

$$= -8\omega^3$$

$$= -8 \cdot$$

4. **If 1, ω, ω^2 are the cube roots of unity, find the value of $(1 - \omega)(1 - \omega^2)(1 - \omega^4)(1 - \omega^8)$.**

A: $(1 - \omega)(1 - \omega^2)(1 - \omega^4)(1 - \omega^8)$

$= (1 - \omega)(1 - \omega^2)(1 - \omega)(1 - \omega^2)$

$= [(1 - \omega)(1 - \omega^2)]^2$

$= [1 - \omega - \omega^2 + \omega^3]^2$

$= [1 - (\omega + \omega^2) + 1]^2$

$= [2 - (-1)]^2$

$= 3^2$

$= 9.$

LEVEL - II (LAQ)

1. Find all the roots of $x^{11} - x^7 + x^4 - 1 = 0$.

A: Given equation is $x^{11} - x^7 + x^4 - 1 = 0$

$$\Rightarrow x^7\left(x^4 - 1\right) + 1\left(x^4 - 1\right) = 0$$

$$\Rightarrow \left(x^4 - 1\right)\left(x^7 + 1\right) = 0$$

Now $x^4 - 1 = 0$.

$$\Rightarrow x^4 = 1 = \text{cis}0 = \text{cis}(0 + 2k\pi) = \text{cis}2k\pi$$

$$\Rightarrow x = (\text{cis}2k\pi)^{\frac{1}{4}} = \text{cis}\frac{2k\pi}{4}, k = 0, 1, 2, 3$$

$$= \text{cis}\frac{k\pi}{2}, k = 0, 1, 2, 3$$

$$= \text{cis}0, \text{cis}\frac{\pi}{2}, \text{cis}\pi, \text{cis}\frac{3\pi}{2}$$

AIMS

Also $x^7 + 1 = 0$

$x^7 = -1 = \text{cis}\pi = \text{cis}(\pi + 2k\pi)$

$x = [\text{cis}\,(2k+1)\,\pi]^{1/7}$

$$= \text{cis}(2k + 1)\frac{\pi}{7}, \; k = 0, 1, 2, 3, 4, 5, 6$$

$$= \text{cis}\frac{\pi}{7}, \text{cis}\frac{3\pi}{7}, \text{cis}\frac{5\pi}{7}, \text{cis}\frac{7\pi}{7}, \text{cis}\frac{9\pi}{7}, \text{cis}\frac{11\pi}{7}, \text{cis}\frac{13\pi}{7}.$$

Hence the required roots of the given equation are

$$= \text{cis}0, \text{cis}\frac{\pi}{2}, \text{cis}\pi, \text{cis}\frac{3\pi}{2}, \text{cis}\frac{\pi}{7}, \text{cis}\frac{3\pi}{7}, \text{cis}\frac{5\pi}{7},$$

$$\text{cis}\frac{7\pi}{7}, \text{cis}\frac{9\pi}{7}, \text{cis}\frac{11\pi}{7}, \text{cis}\frac{13\pi}{7}$$

2. If $(1+x)^n = a_0 + a_1x + a_2x^2 + \ldots\ldots\ldots\ldots a_nx^n$, then show that

i) $a_0 - a_2 + a_4 \ldots\ldots\ldots\ldots = 2^{\frac{n}{2}}\cos\frac{n\pi}{4}$.

ii) $a_1 - a_3 + a_5 \ldots\ldots\ldots\ldots = 2^{\frac{n}{2}}\sin\frac{n\pi}{4}$.

Now $(1 + x)^n = a_0 + a_1x + a_2x^2 + \ldots\ldots\ldots + a_nx^n$.

Put x = i, then

$$a_0 + a_1i + a_2i^2 + \ldots\ldots\ldots + a_ni^n = (1 + i)^n$$

$$= \left[\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right]^n.$$

By applying De Moivre's theorem for an integral index

$$a_0 + a_1 i - a_2 - a_3 i + a_4 + ... = 2^{\frac{n}{2}}\left(\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}\right)$$

$$(a_0 - a_2 + a_4 ....) + i(a_1 - a_3 + a_5 ....)$$

$$= 2^{\frac{n}{2}}\cos\frac{n\pi}{4} + i2^{\frac{n}{2}}\sin\frac{n\pi}{4}.$$

Equating real and imaginary parts both sides.

$$\therefore\ a_0 - a_2 + a_4 .... = 2^{\frac{n}{2}}\cos\frac{n\pi}{4}$$

$$a_1 - a_3 + a_5 .... = 2^{\frac{n}{2}}\sin\frac{n\pi}{4}.$$

3. If $z^2 + z + 1 = 0$, where z is a complex number, prove that $\left(z+\frac{1}{z}\right)^2 + \left(z^2+\frac{1}{z^2}\right)^2 + \left(z^3+\frac{1}{z^3}\right)^2 +$

$$\left(z^4+\frac{1}{z^4}\right)^2 + \left(z^5+\frac{1}{z^5}\right)^2 + \left(z^6+\frac{1}{z^6}\right)^2 = 12.$$

Now $z^2 + z + 1 = 0$.

$$\Rightarrow z = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{3}i}{2} = \omega, \omega^2$$

Taking $z = \omega$, we get

$$\left(z+\frac{1}{z}\right)^2 + \left(z^2+\frac{1}{z^2}\right)^2 + \left(z^3+\frac{1}{z^3}\right)^2 + \left(z^4+\frac{1}{z^4}\right)^2 + \left(z^5+\frac{1}{z^5}\right)^2 + \left(z^6+\frac{1}{z^6}\right)^2$$

$$= \left(\omega+\frac{1}{\omega}\right)^2 + \left(\omega^2+\frac{1}{\omega^2}\right)^2 + \left(\omega^3+\frac{1}{\omega^3}\right)^2 + \left(\omega^4+\frac{1}{\omega^4}\right)^2 + \left(\omega^5+\frac{1}{\omega^5}\right)^2 + \left(\omega^6+\frac{1}{\omega^6}\right)^2$$

$$= \left(\omega+\frac{\omega^3}{\omega}\right)^2 + \left(\omega^2+\frac{\omega^3}{\omega^2}\right)^2 + \left(1+\frac{1}{1}\right)^2 + \left(\omega+\frac{\omega^3}{\omega}\right)^2 + \left(\omega^2+\frac{\omega^3}{\omega^2}\right)^2 + \left(1+\frac{1}{1}\right)^2$$

$$= \left(\omega+\omega^2\right)^2 + \left(\omega^2+\omega\right)^2 + 2^2 + \left(\omega+\omega^2\right)^2 + \left(\omega^2+\omega\right)^2 + 2^2$$

$$= (-1)^2 + (-1)^2 + 4 + (-1)^2 + (-1)^2 + 4$$

$$= 12.$$

4. State and prove De Moivre's Theorem for an integral index.

A: De Moivre's Theorem for an integral index:

For any real number θ and any integer n,

$(\cos\theta + i\sin\theta)^n = \cos n\theta + i \sin n\theta$.

Part1: Let n be a positive integer. We prove the theorem by using the principle of mathematical induction.

Let P(n) be the statement:

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

If n =1, LHS = $(\cos\theta + i\sin\theta)^1$

$= \cos\theta + i\sin\theta$

RHS = $\cos 1\theta + i\sin 1\theta$

$= \cos\theta + i\sin\theta$

∴ LHS = RHS

Thus P(1) is TRUE.

Assume that P(k) is true.

$\Rightarrow (\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta$

Multiplying bothsides by $\cos\theta + i\sin\theta$, we get

$(\cos\theta + i\sin\theta)^{k+1} = (\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta)$

$= \cos k\theta\cos\theta + i\sin k\theta\cos\theta + i\cos k\theta\sin\theta + i^2\sin k\theta\sin\theta$

$= \cos(k\theta + \theta) + i\sin(k\theta + \theta)$

$= \cos(k+1)\theta + i\sin(k+1)\theta$.

∴ P(k+1) is TRUE

By induction, P(n) is true for all positive integers n.

i.e. $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ for all $n \in z^+$.

Part 2: If n = 0, LHS = $(\cos\theta + i\sin\theta)^0$

$= 1$

RHS = $\cos 0\theta + i\sin 0\theta$

$= 1$

∴ LHS = RHS

If n =1, the statement is TRUE.

Part 3: Let n be a negative integer and n = -m, where $m \in z^+$

So for m, part 1 is applicable.

Now $(\cos\theta + i\sin\theta)^n = (\cos\theta + i\sin\theta)^{-m}$

$$= \frac{1}{(\cos\theta + i\sin\theta)^m}$$

$$= \frac{1}{\cos m\theta + i\sin m\theta} \quad \text{from Part 1}$$

$= \cos m\theta - i\sin m\theta$

$= \cos(-m)\theta + i\sin(-m)\theta$

$= \cos n\theta + i\sin n\theta$.

5. If 1, ω, ω^2 are the cube roots of unity, prove that
i) $(1-\omega+\omega^2)^6 + (1-\omega^2+\omega)^6 = 128 = (1-\omega+\omega^2)^7 + (1+\omega-\omega^2)^7$
ii) $(a + b)(a\omega + b\omega^2)(a\omega^2 + b\omega) = a^3 + b^3$.

A: Given that 1, ω, ω^2 are the cube roots of unity,

then $1 + \omega + \omega^2 = 0$ and $\omega^3 = 1$

i) $(1 - \omega + \omega^2)^6 + (1 - \omega + \omega^2)^6$

$= (-\omega - \omega)^6 + (-\omega^2 - \omega^2)^6$

$= (-2\omega)^6 + (-2\omega^2)^6$

$= (-2)^6 [\omega^6 + \omega^{12}]$

$= 64(1 + 1)$

$= 128$

$(1 - \omega + \omega^2)^7 + (1 + \omega - \omega^2)^7$

$= (-\omega - \omega)^7 + (-\omega^2 - \omega^2)^7$

$= (-2\omega)^7 + (-2\omega^2)^7$

$= (-2)^7 [\omega^7 + \omega^{14}]$

$=(-128)(\omega + \omega^2)$

$= (-128)(-1)$

$= 128.$

ii) $(a + b)(a\omega + b\omega^2)(a\omega^2 + b\omega)$

$= (a + b)(a^2\omega^3 + ab\,\omega^2 + ab\omega^4 + b^2\,\omega^3)$

$= (a + b)[a^2 + ab(\omega^2 + \omega) + b^2]$

$= (a + b)[a^2 + ab(-1) + b^2]$

$= (a + b)(a^2 - ab + b^2)$

$= a^3 + b^3$

6. Find all the values of $(1 + i)^{2/3}$.

A: Now $1+i = \sqrt{2}\left(\frac{1}{\sqrt{2}} + i.\frac{1}{\sqrt{2}}\right)$

$= \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

$= \sqrt{2}\,\text{cis}\,\frac{\pi}{4}$

$= \sqrt{2}\,\text{cis}\left(2k\pi + \frac{\pi}{4}\right), \quad k \in z$

$\therefore (1+i)^{\frac{2}{3}} = \sqrt{2}^{\frac{2}{3}}\left[\text{cis}\left(8k+1\right)\frac{\pi}{4}\right]^{\frac{2}{3}}$

$= 2^{\frac{1}{3}}\,\text{cis}\left(8k+1\right).\frac{2}{3}.\frac{\pi}{4}, \quad k = 0, 1, 2$

$= 2^{\frac{1}{3}}\,\text{cis}\left(8k+1\right)\frac{\pi}{6}, \quad k = 0, 1, 2$