DEFINITIONS, CONCEPTS AND FORMULAE

1. If a, b, c are complex numbers and $a \neq 0$, then $ax^2 + bx + c = 0$ is called a quadratic equation.

2. The roots of the quadratic equation $ax^2 + bx + c = 0$ are $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

3. If α, β are roots of $ax^2 + bx + c = 0$, then $\alpha + \beta = \frac{-b}{a}$, $\alpha\beta = \frac{c}{a}$.

4. $\Delta = b^2 - 4ac$ is called discriminant of $ax^2 + bx + c = 0$.

5. If a, b, c are real, then the nature of the roots of $ax^2 + bx + c = 0$ is as follows :

   i) If $b^2 - 4ac < 0$, then the roots are imaginary and they are conjugate complex numbers.

   ii) If $b^2 - 4ac = 0$, then the roots are real and equal.

   iii) If $b^2 - 4ac > 0$, then the roots are real and distinct.

6. If a, b, c are rational, then the nature of the roots of $ax^2 + bx + c = 0$ is as follows :

   i) If $b^2 - 4ac < 0$, then the roots are imaginary and they are conjugate complex numbers.

   ii) If $b^2 - 4ac = 0$, then the roots are rational and equal.

   iii) If $b^2 - 4ac > 0$ and $b^2 - 4ac$ is a perfect square, then the roots are rational and distinct.

   iv) If $b^2 - 4ac > 0$ and $b^2 - 4ac$ is not a perfect square, then the roots are irrational and distinct. They are conjugate surds.

7. The quadratic equation whose roots are α, β is $x^2 - (\alpha + \beta)x + \alpha\beta = 0$.

8. If α, β are the roots of $ax^2+bx+c=0$, then $ax^2+bx+c=a(x - \alpha)(x - \beta)$.

9. A necessary and sufficient condition for the quadratic equations $a_1x^2 + b_1x + c_1 = 0$ and $a_2x^2 + b_2x + c_2 = 0$ to have a common root is $(c_1a_2 - c_2a_1)^2 = (a_1b_2 - a_2b_1)(b_1c_2 - b_2c_1)$.

   Here the common root is $\frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}$

10. If the roots of $ax^2 + bx + c = 0$ are imaginary (complex roots) then for $x \in R$, '$ax^2 + bx + c$' and 'a' have the same sign.

11. If the roots of $ax^2 + bx + c = 0$ are real and equal to $\alpha = \frac{-b}{2a}$, then for $\alpha \neq x \in R$, '$ax^2 + bx + c$' and 'a' have the same sign.

12. Let α, β be the real roots of $ax^2 + bx + c = 0$ and $\alpha < \beta$, then for

    i) $x \in R$, $\alpha < x < \beta \Rightarrow$ '$ax^2 + bx + c$' and 'a' have opposite signs.

    ii) $x \in R$, $x < \alpha$ or $x > \beta \Rightarrow$ 'ax^2+bx+c' and 'a' have the same sign.

13. Let $f(x) = ax^2+bx+c$ be a quadratic function.

    i) If $a > 0$, then f (x) has minimum at $x = \frac{-b}{2a}$ and the minimum value = $\frac{4ac - b^2}{4a}$.

    ii) If $a < 0$, then f (x) has maximum at $x = \frac{-b}{2a}$ and the maximum value = $\frac{4ac - b^2}{4a}$.

AIMS

14. Let α, β be the roots of $ax^2+bx+c = 0$, then the equation whose roots are

    i) $\frac{1}{\alpha}$ and $\frac{1}{\beta}$ is $f\left(\frac{1}{x}\right) = 0$

    ii) $\alpha + k$ and $\beta + k$ is $f(x - k) = 0$

    iii) $\alpha - k$ and $\beta - k$ is $f(x + k) = 0$

    iv) $-\alpha$ and $-\beta$ is $f(-x) = 0$

    v) $k\alpha$ and $k\beta$ is $f\left(\frac{x}{k}\right) = 0$

15. If $ax^2 + bx + c$ is a quadratic expression, then $ax^2 + bx + c > 0$ or $ax^2 + bx + c \geq 0$ or $ax^2 + bx + c < 0$ or $ax^2 + bx + c \leq 0$ are called a 'quadratic inequations'.

LEVEL - I (VSAQ)

1. Form a quadratic equation whose roots are -3 ± 5i.

A: The quadratic equation whose roots are

$-3 + 5i$ and $-3 - 5i$ is $x^2 - (\alpha + \beta)x + \alpha\beta = 0$

$\Rightarrow x^2 - (-3 + 5i - 3 - 5i)\,x + (-3 + 5i)(-3 - 5i) = 0$

$\because (a + ib)(a - ib) = a^2 + b^2$

$\Rightarrow x^2 + 6x + 34 = 0.$

2. Obtain a quadratic equation whose roots are $\frac{p-q}{p+q}, \frac{-(p+q)}{p-q}$.

A: The quadratic equation whose roots are $\frac{p-q}{p+q}, \frac{-(p+q)}{p-q}$ is

$$x^2 - \left[\frac{p-q}{p+q} - \frac{(p+q)}{p-q}\right]x + \left(\frac{p-q}{p+q}\right)\left\{\frac{-(p+q)}{p-q}\right\} = 0$$

$$\Rightarrow x^2 - \left\{\frac{(p-q)^2 - (p+q)^2}{p^2 - q^2}\right\}x - 1 = 0$$

$\Rightarrow (p^2 - q^2)x^2 + 4pqx - (p^2 - q^2) = 0$

3. Find the quadratic equation, the sum of whose roots is 7 and the sum of the squares of the roots is 25.

A: Let α, β be the roots of the required quadratic equation.

Given that $\alpha + \beta = 7$, $\alpha^2 + \beta^2 = 25$.

$(\alpha + \beta)^2 = 7^2$

$\Rightarrow \alpha^2 + \beta^2 + 2\alpha\beta = 49$

$\Rightarrow 25 + 2\alpha\beta = 49$

$\Rightarrow 2\alpha\beta = 49 - 25 = 24$

$\Rightarrow \alpha\beta = 12$

$\therefore$ Required quadratic equation is

$x^2 - (\alpha + \beta)x + \alpha\beta = 0$

$\Rightarrow x^2 - 7x + 12 = 0.$

4. If α, β are the roots of the equation $ax^2 + bx + c = 0$. Find the value of $\frac{1}{\alpha} + \frac{1}{\beta}$.

A: α, β are the roots of $ax^2 + bx + c = 0$

$$\Rightarrow \alpha + \beta = \frac{-b}{a}, \alpha\beta = \frac{c}{a}$$

$$\text{Now } \frac{1}{\alpha} + \frac{1}{\beta} = \frac{\beta + \alpha}{\alpha\beta} = \frac{-b/a}{c/a} = \frac{-b}{c}.$$

5. If α and β are the roots of the equation $ax^2 + bx + c = 0$, then find the value of $\frac{1}{\alpha^2} + \frac{1}{\beta^2}$.

A: α, β are the roots of $ax^2 + bx + c = 0$.

$$\Rightarrow \alpha + \beta = \frac{-b}{a}, \alpha\beta = \frac{c}{a}$$

$$\text{Now } \frac{1}{\alpha^2} + \frac{1}{\beta^2} = \frac{\beta^2 + \alpha^2}{(\alpha\beta)^2}$$

$$= \frac{(\alpha + \beta)^2 - 2\alpha\beta}{(\alpha\beta)^2}$$

$$= \frac{(-b/a)^2 - 2c/a}{(c/a)^2}$$

AIMS

$$= \frac{b^2 - 2ac}{a^2} \cdot \frac{a^2}{c^2}$$

$$= \frac{b^2 - 2ac}{c^2}.$$

6. If α and β are the roots of the equation $x^2 + x + 1 = 0$, find the value of $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$.

A: α and β are the roots of $x^2 + x + 1 = 0$.

$\Rightarrow \alpha + \beta = -b/a = -1$; $\alpha\beta = c/a = 1$

$$\text{Now } \frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha^2 + \beta^2}{\alpha\beta}$$

$$= \frac{(\alpha + \beta)^2 - 2\alpha\beta}{\alpha\beta}$$

$$= \frac{(-1)^2 - 2(1)}{1}$$

$= -1.$

7. If α and β are the roots of the equation $2x^2 + 3x + 6 = 0$, find the quadratic equation whose roots are α^3 and β^3.

A: α, β are the roots of $2x^2 + 3x + 6 = 0$

$\Rightarrow \alpha + \beta = \frac{-b}{a} = \frac{-3}{2}$; $\alpha\beta = \frac{c}{a} = \frac{6}{2} = 3$

$\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$

$= \frac{-27}{8} + \frac{27}{2}$

$= \frac{-27 + 108}{8}$

$= \frac{81}{8}$

$\alpha^3\beta^3 = 3^3 = 27$

Required quadratic equation is

$x^2 - (\alpha^3 + \beta^3)x + \alpha^3 \beta^3 = 0$

$x^2 - \frac{81}{8}x + 27 = 0$

$8x^2 - 81x + 216 = 0.$

8. If the equation $x^2 - 15 - m(2x - 8) = 0$ has equal roots, then find the values of m.

A: Given equation is $x^2 - 2mx + (8m - 15) = 0$

since it has equal roots $b^2 - 4ac = 0$

$\Rightarrow (-2m)^2 - 4(1)(8m - 15) = 0$

$\Rightarrow 4m^2 - 4(8m - 15) = 0$ $\quad \div 4$

$\Rightarrow m^2 - 8m + 15 = 0$

$\Rightarrow (m - 3)(m - 5) = 0$

$\therefore$ m = 3 or 5.

9. If $(m + 1)x^2 + 2(m + 3)x + m + 8 = 0$ has equal roots, find m.

A: Given equation is $(m + 1)x^2 + 2(m + 3)x + (m + 8) = 0$

Since it has equal roots $b^2 - 4ac = 0$

$\Rightarrow \{2(m+3)\}^2 - 4(m + 1)(m + 8) = 0$ $\quad \div 4$

$\Rightarrow m^2 + 6m + 9 - (m^2 + 9m + 8) = 0$

$\Rightarrow -3m + 1 = 0$

$\Rightarrow m = 1/3.$

10. Prove that the roots of $(x - a)(x - b) = h^2$ are always real.

A: Given equation is $(x - a)(x - b) = h^2$

$\Rightarrow x^2 - (a + b)x + (ab - h^2) = 0$

Its discriminant

$= \{-(a + b)\}^2 - 4(1)(ab - h^2)$

$= a^2 + b^2 + 2ab - 4ab + 4h^2$

$= (a - b)^2 + (2h)^2$

≥ 0

Hence the roots of the given equation are always real.

11. If $x^2 - 6x + 5 = 0$ and $x^2 - 12x + p = 0$ have a common root, then find p.

A: $x^2 - 6x + 5 = 0$

$\Rightarrow (x - 1)(x - 5) = 0$

$\Rightarrow x = 1, 5$

$\Rightarrow$ If x = 1, $1 - 12 + p = 0 \Rightarrow p = 11$

$\Rightarrow$ If x = 5, $25 - 60 + p = 0 \Rightarrow p = 35$

$\therefore$ p = 11 or 35

AIMS

12. If the quadratic equations $ax^2 + 2bx + c = 0$ and $ax^2 + 2cx + b = 0$, $(b \neq c)$ have a common root, then show that $a + 4b + 4c = 0$.

A: Let α be the common root of given two equations.

$a\alpha^2 + 2b\alpha + c = 0$

$a\alpha^2 + 2c\alpha + b = 0$

on subtraction $2(b - c)\alpha - (b - c) = 0$

$2\alpha - 1 = 0 \qquad \because b - c \neq 0$

$\alpha = 1/2$

$\Rightarrow a(1/2)^2 + 2b(1/2) + c = 0$

$\Rightarrow a + 4b + 4c = 0.$

13. For what values of x, the expression $3x^2 + 4x + 4$ is positive.

A: Given expression is $3x^2 + 4x + 4$

Consider $3x^2 + 4x + 4 = 0$

Roots are $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$= \frac{-4 \pm \sqrt{16 - 4(3)(4)}}{2(3)}$$

$$= \frac{-4 \pm \sqrt{-32}}{6}$$

$$= \frac{-4 \pm 4\sqrt{2}\, i}{6}$$

which are complex numbers.

Thus, $\forall\, x \in R$, $3x^2 + 4x + 4$ is positive.

14. For what values of x, the expression $15 + 4x - 3x^2$ is negative.

A: Given exprssion is $15 + 4x - 3x^2$.

Here $a = -3 < 0$.

Consider $15 + 4x - 3x^2 = 0$

$\Rightarrow 3x^2 - 4x - 15 = 0$

$\Rightarrow 3x^2 - 9x + 5x - 15 = 0$

$\Rightarrow 3x (x - 3) + 5(x - 3) = 0$

$\Rightarrow (3x + 5) (x - 3) = 0$

$\Rightarrow \alpha = -5/3,\ \beta = 3 \quad \because \alpha < \beta$

Thus for $x \in R$ and $x < -5/3$ or $x > 3$, then $15 + 4x - 3x^2$ is negative.

15. Find the maximum value of $2x - 7 - 5x^2$ for $x \in R$.

A: Comparing $2x - 7 - 5x^2$ with $ax^2 + bx + c$, we get $a = -5$, $b = 2$, $c = -7$.
Maximum value of $2x - 7 - 5x^2$

$$= \frac{4ac - b^2}{4a}$$

$$= \frac{4(-5)(-7) - 2^2}{4(-5)}$$

$$= \frac{140 - 4}{-20}$$

$$= \frac{136}{-20}$$

$$= -\frac{34}{5}.$$

LEVEL - I (SAQ)

1. Determine the range of the experssion $\frac{x^2 + x + 1}{x^2 - x + 1}$, $x \in R$.

A: Let $\frac{x^2 + x + 1}{x^2 - x + 1} = y$

$\Rightarrow x^2 + x + 1 = y x^2 - yx + y$

$\Rightarrow (y - 1)x^2 - (y + 1)x + (y - 1) = 0.$

For $x \in R$, $b^2 - 4ac \geq 0$

$\Rightarrow \{- (y + 1)\}^2 - 4(y - 1) (y - 1) \geq 0$

$\Rightarrow -3y^2 + 10y - 3 \geq 0$ $\qquad \times (-1)$

$\Rightarrow 3y^2 - 10y + 3 \leq 0$

$\Rightarrow 3y^2 - y - 9y + 3 \leq 0$

$\Rightarrow y(3y - 1) - 3(3y - 1) \leq 0$

$\Rightarrow (3y - 1) (y - 3) \leq 0$ $\qquad \div 3$

$\Rightarrow \left(y - \frac{1}{3}\right) (y - 3) \leq 0$

$\Rightarrow y \in \left[\frac{1}{3}, 3\right]$

AIMS

Hence the range of $\frac{x^2 + x + 1}{x^2 - x + 1}$ is $\left[\frac{1}{3}, 3\right]$.

2. Find the range of $\frac{2x^2 - 6x + 5}{x^2 - 3x + 2}$ if $x \in R$.

A: Let $\frac{2x^2 - 6x + 5}{x^2 - 3x + 2} = y$

$\Rightarrow 2x^2 - 6x + 5 = yx^2 - 3yx + 2y$

$\Rightarrow (y - 2) x^2 + 3(2 - y)x + (2y - 5) = 0$

For $x \in R$, $b^2 - 4ac \geq 0$

$\Rightarrow \{3(2 - y)\}^2 - 4(y - 2) (2y - 5) \geq 0$

$\Rightarrow (y - 2) [9(y - 2) - 4(2y - 5)] \geq 0$

$\Rightarrow (y - 2) [9y - 18 - 8y + 20] \geq 0$

$\Rightarrow (y - 2) (y + 2) \geq 0$

$\Rightarrow [y - (-2)] (y - 2) \geq 0.$

Hence the range of $\frac{2x^2 - 6x + 5}{x^2 - 3x + 2}$ is $(-\infty, -2] \cup [2, \infty)$.

3. If x is a real number, find the range $\frac{x+2}{2x^2+3x+6}$.

A: Let $\frac{x+2}{2x^2+3x+6} = y$

$\Rightarrow x + 2 = 2yx^2 + 3yx + 6y$

$\Rightarrow 2yx^2 + (3y - 1)x + 2(3y - 1) = 0$

For $x \in R$, $b^2 - 4ac \geq 0$

$\Rightarrow (3y - 1)^2 - 4(2y)(2)(3y - 1) \geq 0$

$\Rightarrow (3y - 1)[3y - 1 - 16y] \geq 0$

$\Rightarrow (3y - 1)(-13y - 1) \geq 0$ x (-1)

$\Rightarrow (3y - 1)(13y + 1) \leq 0$ ÷3(13)

$\Rightarrow \left[y - \left(\frac{-1}{13}\right)\right]\left(y - \frac{1}{3}\right) \leq 0$

$\Rightarrow y \in \left[\frac{-1}{13}, \frac{1}{3}\right]$

$\therefore$ Range of $\frac{x+2}{2x^2+3x+6}$ is $\left[\frac{-1}{13}, \frac{1}{3}\right]$.

4. Prove that $\frac{1}{3x+1} + \frac{1}{x+1} - \frac{1}{(3x+1)(x+1)}$ does not lie between 1 and 4, if x is real.

A: $\frac{1}{3x+1} + \frac{1}{x+1} - \frac{1}{(3x+1)(x+1)} = \frac{x+1+3x+1-1}{(3x+1)(x+1)}$

$= \frac{4x+1}{(3x+1)(x+1)}$

Let $\frac{4x+1}{3x^2+4x+1} = y$

$\Rightarrow 4x + 1 = 3yx^2 + 4yx + y$

$\Rightarrow 3yx^2 + 4(y - 1)x + (y - 1) = 0$

For $x \in R$, $b^2 - 4ac \geq 0$

$\Rightarrow \{(4(y - 1)\}^2 - 4(3y)(y - 1) \geq 0$ ÷4

$\Rightarrow (y - 1)[4(y - 1) - 3y] \geq 0$

$\Rightarrow (y - 1)(y - 4) \geq 0$

$\Rightarrow y \in (-\infty, 1] \cup [4, \infty)$.

Hence the given expression does not lie between 1 and 4.

5. If x is real, show that the values of the expression $\frac{x^2+34x-71}{x^2+2x-7}$ do not lie between 5 and 9.

A: Let $\frac{x^2+34x-71}{x^2+2x-7} = y$

$\Rightarrow x^2 + 34x - 71 = yx^2 + 2yx - 7y$

$\Rightarrow (y - 1)x^2 + 2(y - 17)x + (71 - 7y) = 0$

For $x \in R$, $b^2 - 4ac \geq 0$

$\Rightarrow \{2(y - 17)\}^2 - 4(y - 1)(71 - 7y) \geq 0$

$\Rightarrow 4(y^2 - 34y + 289) + 4(7y^2 - 78y + 71) \geq 0$

$\Rightarrow y^2 - 34y + 289 + 7y^2 - 78y + 71 \geq 0$

$\Rightarrow 8y^2 - 112y + 360 \geq 0$ ÷ 8

$\Rightarrow y^2 - 14y + 45 \geq 0$

$\Rightarrow y^2 - 9y - 5y + 45 \geq 0$

$\Rightarrow y(y - 9) - 5(y - 9) \geq 0$

$\Rightarrow (y - 5)(y - 9) \geq 0$

$\Rightarrow y \in (-\infty, 5] \cup [9, \infty)$

Hence the values of $\frac{x^2+34x-71}{x^2+2x-7}$ do not lie between 5 and 9.

AIMS

6. Show that $\frac{x}{x^2-5x+9}$ lies between $\frac{-1}{11}$, 1.

A: Let $\frac{x}{x^2-5x+9} = y$

$\Rightarrow x = yx^2 - 5yx + 9y$

$\Rightarrow yx^2 - (5y + 1)x + 9y = 0$

For $x \in R$, $b^2 - 4ac \geq 0$

$\Rightarrow \{-(5y + 1)\}^2 - 4(y)(9y) \geq 0$

$\Rightarrow 25y^2 + 10y + 1 - 36y^2 \geq 0$

$\Rightarrow -11y^2 + 10y + 1 \geq 0$ x (-1)

$\Rightarrow 11y^2 - 10y - 1 \leq 0$

$\Rightarrow 11y^2 - 11y + y - 1 \leq 0$

$\Rightarrow 11y(y - 1) + 1(y - 1) \leq 0$

$\Rightarrow (11y + 1)(y - 1) \leq 0$ ÷ 11

$\Rightarrow (11y + 1)(y - 1) \leq 0$

$\Rightarrow \left[y - \left(\frac{-1}{11}\right)\right](y - 1) \leq 0$

$\Rightarrow y \in \left[\frac{-1}{11}, 1\right]$

Hence $\frac{x}{x^2 - 5x + 9}$ lies between $\frac{-1}{11}$, 1.

7. If x is real, find the maximum and minimum values of the expression $\frac{x^2 + 14x + 9}{x^2 + 2x + 3}$.

A: Let $\frac{x^2 + 14x + 9}{x^2 + 2x + 3} = y$

$\Rightarrow x^2 + 14x + 9 = y x^2 + 2yx + 3y$

$\Rightarrow (y - 1)x^2 + 2(y - 7)x + (3y - 9) = 0$

For $x \in R$, $b^2 - 4ac \geq 0$

$\Rightarrow \{2(y - 7)\}^2 - 4(y - 1)(3y - 9) \geq 0$

$\Rightarrow y^2 - 14y + 49 - (3y^2 - 12y + 9) \geq 0$

$\Rightarrow -2y^2 - 2y + 40 \geq 0 \qquad \div -2$

$\Rightarrow y^2 + y - 20 \leq 0$

$\Rightarrow y^2 + 5y - 4y - 20 \leq 0$

$\Rightarrow y(y + 5) - 4(y + 5) \leq 0$

$\Rightarrow (y + 5)(y - 4) \leq 0$

$\Rightarrow y \in [-5, 4]$

$\therefore$ Maximum value of $\frac{x^2 + 14x + 9}{x^2 + 2x + 3}$ is 4 and the minimum value is -5.

8. Solve $4^{x-1} - 3.2^{x-1} + 2 = 0$.

A. Given $4^{x-1} - 3.2^{x-1} + 2 = 0$

$\Rightarrow \frac{4^x}{4} - \frac{3.2^x}{2} + 2 = 0$

Let $2^x = t$

$\Rightarrow 4^x = 2^{2x} = t^2$

The above equation becomes

$\Rightarrow \frac{t^2}{4} - \frac{3.t}{2} + 2 = 0$

$\Rightarrow t^2 - 6t + 8 = 0$

$\Rightarrow t^2 - 4t - 2t + 8 = 0$

$\Rightarrow t(t - 4) - 2(t - 4) = 0$

$(t - 2)(t - 4) = 0$

$t = 2$ (or) 4

<u>**Case (i):**</u> If t = 2 — <u>**Case - (ii):**</u> If t = 4

$2^x = 2$ — $2^x = 2^2$

$\therefore x = 1$ — $\therefore x = 2$

$\therefore x = 1, 2$

9. If the roots of $ax^2 + bx + c = 0$ are imaginary, show that for all $x \in R$, '$ax^2 + bx + c$' and 'a' have the same sign.

A: Given that the roots of $ax^2 + bx + c = 0$ are imaginary.

$\Rightarrow b^2 - 4ac < 0$

$4ac - b^2 > 0$ ------ (1)

Consider $\frac{ax^2 + bx + c}{a}$

AIMS

$= x^2 + \frac{b}{a}x + \frac{c}{a}$

$= x^2 + 2x.\frac{b}{2a} + \left(\frac{b}{2a}\right)^2 + \frac{c}{a} - \left(\frac{b}{2a}\right)^2$

$= \left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}$ from (1)

$\geq 0 \qquad > 0$

> 0

$\therefore$ For all $x \in R$, '$ax^2 + bx + c$' and 'a' have the same sign.

10. Let α, β be the real roots of $ax^2 + bx + c = 0$ where $\alpha < \beta$, then prove the following.

i) for $\alpha < x < \beta$; '$ax^2 + bx + c$' and 'a' have opposite signs.

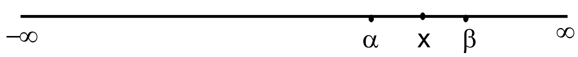
ii) for $x < \alpha$ or $x > \beta$; '$ax^2 + bx + c$' and 'a' have the same sign.

A: Given that α, β are the real roots of $ax^2 + bx + c = 0$ with $\alpha < \beta$.

$\Rightarrow ax^2 + bx + c = a(x - \alpha)(x - \beta)$ $\div a$

$\Rightarrow \dfrac{ax^2 + bx + c}{a} = (x - \alpha)(x - \beta)$ -------- (1)

i) Suppose $x \in R$ and $\alpha < x < \beta$



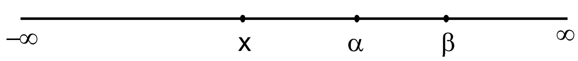
Now $x - \alpha > 0$ and $x - \beta < 0$

$\Rightarrow (x - \alpha)(x - \beta) < 0$

$\Rightarrow \dfrac{ax^2 + bx + c}{a} < 0$ from (1)

Thus for $x \in R$ and $\alpha < x < \beta$, then '$ax^2 + bx + c$' and 'a' have opposite signs.

ii) Suppose $x \in R$ and $x < \alpha$

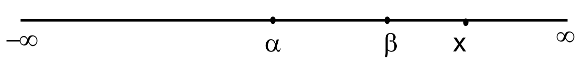


Now $x - \alpha < 0$ and $x - \beta < 0$

$\Rightarrow (x - \alpha)(x - \beta) > 0$

$\Rightarrow \dfrac{ax^2 + bx + c}{a} > 0$ from (1)

Suppose $x \in R$ and $x > \beta$



Now $x - \alpha > 0$ and $x - \beta > 0$

$\Rightarrow (x - \alpha)(x - \beta) > 0$

$\Rightarrow \dfrac{ax^2 + bx + c}{a} > 0$ from (1)

Thus for $x \in R$, $x < \alpha$ or $x > \beta$, then '$ax^2 + bx + c$' and 'a' have the same sign.

11. If the expression $\dfrac{x - p}{x^2 - 3x + 2}$ takes all real values for $x \in R$, then find the bounds for p.

A: Let $\dfrac{x - p}{x^2 - 3x + 2} = y$

$\Rightarrow x - p = yx^2 - 3yx + 2y$

$\Rightarrow yx^2 - (3y + 1)x + (2y + p) = 0$

For $x \in R$, $b^2 - 4ac \geq 0$

$\Rightarrow \{-(3y + 1)\}^2 - 4(y)(2y + p) \geq 0$

$\Rightarrow 9y^2 + 6y + 1 - 8y^2 - 4yp \geq 0$

$\Rightarrow y^2 - 2(2p - 3)y + 1 \geq 0$

Here, coefficient of $y^2 = 1 > 0$

So, the roots of $y^2 - 2(2p - 3)y + 1 = 0$ are imaginary or real and equal $\Rightarrow b^2 - 4ac \leq 0$

$\Rightarrow \{-2(2p - 3)\}^2 - 4(1)(1) \leq 0$ $\div 4$

$\Rightarrow 4p^2 - 12p + 9 - 1 \leq 0$

$\Rightarrow 4p^2 - 12p + 8 \leq 0$ $\div 4$

$\Rightarrow p^2 - 3p + 2 \leq 0$

$\Rightarrow (p - 1)(p - 2) \leq 0$

$\Rightarrow p \in [1, 2]$

But $\dfrac{x - p}{x^2 - 3x + 2}$ is not defined for p = 1, 2.

$\therefore p \in (1, 2)$.

LEVEL - II (VSAQ)

1. Find the nature of the roots of $3x^2 + 7x + 2 = 0$.

A: Given equation is $3x^2 + 7x + 2 = 0$.

Now, $\Delta = b^2 - 4ac = (7)^2 - 4(3)(2)$

$= 49 - 24 = 25 = 5^2 > 0$.

$\therefore$ Roots are rational and not equal.

2. If α, β are the roots of the equation $ax^2 + bx + c = 0$, then find the value of $\dfrac{\alpha^2 + \beta^2}{\alpha^{-2} + \beta^{-2}}$.

A: Clearly, $\alpha + \beta = \dfrac{-b}{a}$, $\alpha\beta = \dfrac{c}{a}$.

Now $\dfrac{\alpha^2 + \beta^2}{\alpha^{-2} + \beta^{-2}} = \dfrac{\alpha^2 + \beta^2}{\dfrac{1}{\dfrac{1}{\alpha^2} + \dfrac{1}{\beta^2}}}$

$= \dfrac{\alpha^2 + \beta^2}{\dfrac{\beta^2 + \alpha^2}{\alpha^2\beta^2}} = (\alpha\beta)^2 = \left(\dfrac{c}{a}\right)^2 = \dfrac{c^2}{a^2}$.

3. If α, β are the roots of $ax^2 + bx + c = 0$, find the values of $\alpha^2 + \beta^2$ and $\alpha^3 + \beta^3$.

A: If α, β are the roots of $ax^2 + bx + c = 0$ then

$$\alpha + \beta = \frac{-b}{a}, \alpha\beta = \frac{c}{a}$$

(i) $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

$$= \left(\frac{-b}{a}\right)^2 - 2\left(\frac{c}{a}\right) = \frac{b^2 - 2ac}{a^2}.$$

(ii) $\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$

$$= \left(\frac{-b}{a}\right)^3 - 3\left(\frac{c}{a}\right)\left(\frac{-b}{a}\right)$$

$$= \frac{b^3}{a^3} + \frac{3bc}{a^2} = \frac{3abc - b^3}{a^3}.$$

4. Find a quadratic equation, the sum of whose roots is 1 and sum of the squares of roots is 13.

A: Let a, b be the roots of required equation then

$\alpha + \beta = 1, \alpha^2 + \beta^2 = 13$

We have, $(\alpha + \beta)^2 = \alpha^2 + \beta^2 + 2\alpha\beta$

$\Rightarrow (1)^2 = 13 + 2\alpha\beta$

$\Rightarrow 1 - 13 = 2\alpha\beta \Rightarrow 2\alpha\beta = -12$

$\Rightarrow \boxed{\alpha\beta = -6}$

$\therefore$ Required equation : $x^2 - (\alpha + \beta)x + \alpha\beta = 0$

$$\Rightarrow x^2 - (1)x - 6 = 0 \Rightarrow \boxed{x^2 - x - 6 = 0}.$$

5. If $x^2 + bx + c = 0$ and $x^2 + cx + b = 0$ $(b \neq c)$ have a common root then show that $1 + b + c = 0$.

A: Let α be the common root of the given equations

then $\alpha^2 + b\alpha + c = 0$ ....(1)

and $\alpha^2 + c\alpha + b = 0$ ....(2)

Solving (1) & (2)

$$\cancel{\alpha^2} + b\alpha + c = 0 - \left(\cancel{\alpha^2} + c\alpha + b\right) = 0$$

$\Rightarrow (b - c)\alpha + (c - b) = 0$

$\Rightarrow (b - c)\alpha = (b - c) \Rightarrow \boxed{\alpha = 1}$

Substitute in (1) $\Rightarrow (1)^2 + b(1) + c = 0$

$\Rightarrow \boxed{1 + b + c = 0}$.

6. If the equations $x^2 + ax + b = 0$ and $x^2 + cx + d = 0$ have a common root and the first equation has equal roots then prove that $2(b + d) = ac$.

A:Let α be the common root.

Then $\alpha^2 + c\alpha + d = 0 \rightarrow (1)$

Also, $x^2 + ax + b = 0$ has equal roots.

$\Rightarrow \alpha + \alpha = -a, \alpha\alpha = b \Rightarrow \alpha = -a/2, \alpha^2 = b$.

$(1) \Rightarrow b + c(-a/2) + d = 0$.

$\Rightarrow b + d = ac/2$

$\Rightarrow \boxed{2(b + d) = ac}$.

7. Determine the sign of the expression $x^2 - 5x + 6$.

A: (i) Take $x^2 - 5x + 6 > 0 \Rightarrow (x - 2)(x - 3) > 0$.

$\Rightarrow$ for $x < 2, x > 3$ the expression is positive.

(ii) Take $x^2 - 5x + 6 < 0 \Rightarrow (x - 2)(x - 3) < 0$.

$\Rightarrow$ for $2 < x < 3$ the expression is negative.

8. Find the maximum or minimum of the expression $ax^2 + bx + a$ ($a, b \in R$ and $a \neq 0$).

A: Case: (i) Suppose $a > 0$

$\Rightarrow$ the expression has absolute minimum at

$$x = \frac{-b}{2a}$$

That minimum value is

$$= \frac{4a(a) - b^2}{4a} = \frac{4a^2 - b^2}{4a}$$

Case: (ii) Suppose $a < 0$

AIMS

$\Rightarrow$ the expression has absolute maximum at

$$x = \frac{-b}{2a}$$

That maximum value is

$$= \frac{4a(a) - b^2}{4a} = \frac{4a^2 - b^2}{4a}.$$

9.Find the maximum or minimum of the expression $3x^2 + 2x + 11$.

A: Given expression is $3x^2 + 2x + 11$

compare with $ax^2 + bx + c = 0$

then we get $a = 3 > 0, b = 2, c = 11$

Since $a > 0$

$\Rightarrow$ the expression has absolute minimum at

$$x = \frac{-b}{2a} = \frac{-2}{2(3)} = \frac{-1}{3}$$

That minimum value $= \dfrac{4ac - b^2}{4a} = \dfrac{4(3)(11) - 2^2}{4(3)}$

$$= \frac{132 - 4}{12} = \frac{128}{12} = \frac{32}{3}.$$

LEVEL - II (SAQ)

1. Solve $2x^4 + x^3 - 11x^2 + x + 2 = 0$

A. Given equation $2x^4 + x^3 - 11x^2 + x + 2 = 0$

$\Rightarrow \div$ by x^2

$$\Rightarrow 2x^2 + x - 11 + \frac{1}{x} + \frac{2}{x^2} = 0$$

$$\Rightarrow 2\left(x^2 + \frac{1}{x^2}\right) + \left(x + \frac{1}{x}\right) - 11 = 0$$

Let $x + \frac{1}{x} = z,\ x^2 + \frac{1}{x^2} = z^2 - 2$

The above equation becomes

$$\Rightarrow 2\left(z^2 - 2\right) + z - 11 = 0$$

$2z^2 - 4 + z - 11 = 0$

$\Rightarrow 2z^2 + 6z - 5z - 15 = 0$

$2z(z + 3) - 5(z + 3) = 0$

$(z + 3)(2z - 5) = 0$

$z = -3,\ 5/2$

Case - i: If $z = -3$

$$x + \frac{1}{x} = -3$$

$x^2 + 3x + 1 = 0$

$$x = \frac{-3 \pm \sqrt{9-4}}{2}$$

$$x = \frac{-3 \pm \sqrt{5}}{2}$$

Case - ii: If $z = 5/2$

$$x + \frac{1}{x} = \frac{5}{2}$$

$\Rightarrow 2x2 - 5x + 2 = 0$

$\Rightarrow 2x2 - 4x - x + 2 = 0$

$\Rightarrow 2x(x-2)-1(x-2) = 0$

$(x - 2)(2x - 1) = 0$

$x = 2,\ 1/2$

$\therefore$ The roots are $\left\{\frac{-3 \pm \sqrt{5}}{2}, 2, \frac{1}{2}\right\}$

2. Solve $\sqrt{\frac{x}{x-3}} + \sqrt{\frac{x-3}{x}} = \frac{5}{2}$ when $x \neq 0, x \neq 3$.

A. Let $\sqrt{\frac{x}{x-3}} = z$

The above equation becomes

$$z + \frac{1}{z} = \frac{5}{2}$$

$$\Rightarrow \frac{z^2+1}{z} = \frac{5}{2}$$

$\Rightarrow 2z^2 - 5z + 2 = 0$

$\Rightarrow 2z^2 - 4z - z + 2 = 0$

$\Rightarrow 2z(z - 2) - 1(z - 2) = 0$

$\Rightarrow (z - 2)(2z - 1) = 0$

$z = 2,\ 1/2$

Case - i: If $z = 2$

$$\Rightarrow \sqrt{\frac{x}{x-3}} = 2$$

$$\Rightarrow \frac{x}{x-3} = 4$$

$\Rightarrow x = 4x - 12$

$\Rightarrow 12 = 3x$

$x = 4$

Case - ii: If $z = 1/2$

$$\Rightarrow \sqrt{\frac{x}{x-3}} = \frac{1}{2}$$

$$\Rightarrow \frac{x}{x-3} = \frac{1}{2}$$

$\Rightarrow 4x = x - 3$

$\Rightarrow 3x = 3$

$\Rightarrow x = 1$

$\therefore$ The roots are {1, 4}.

3. Suppose that a, b, c $\in$ R, a $\neq$ 0 and $f(x) = ax^2 + bx + c$

i) If a > 0, then show that f has minimum at $x = \frac{-b}{2a}$ and the minimum value of f is $\frac{4ac - b^2}{4a}$.

ii) If a < 0, then show that f has maximum at $x = \frac{-b}{2a}$ and the maximum value of f is $\frac{4ac - b^2}{4a}$ (AIMS)

A: Given quadratic function is $f(x) = ax^2 + bx + c$.

Differentiating w.r.t. x successively for two times,

$f'(x) = 2ax + b$

$f''(x) = 2a$

For f(x) to be maximum or minimum, $f'(x) = 0$

$\Rightarrow 2ax + b = 0$

$$\Rightarrow x = \frac{-b}{2a}$$

If a > 0, then $f''(x) > 0$ and hence f has minimum at $x = \frac{-b}{2a}$ and the minimum value of f

$$= a\left(\frac{-b}{2a}\right)^2 + b\left(\frac{-b}{2a}\right) + c$$

$$= \frac{b^2 - 2b^2 + 4ac}{4a}$$

$$= \frac{4ac - b^2}{4a}$$

If a < 0, then $f''(x) < 0$ and hence f has maximum at $x = \frac{-b}{2a}$ and the maximum value of f

$$= a\left(\frac{-b}{2a}\right)^2 + b\left(\frac{-b}{2a}\right) + c$$

$$= \frac{4ac - b^2}{4a}.$$

4. Find set of values of x for which the inequalities $x^2 - 3x - 10 < 0$, $10x - x^2 - 16 > 0$ hold simultaneously.

A: Consider $x^2 - 3x - 10 < 0$

$\Rightarrow x^2 - 5x + 2x - 10 < 0$

$\Rightarrow x(x - 5) + 2(x - 5) < 0$

$\Rightarrow (x + 2)(x - 5) < 0$

$\Rightarrow [x - (-2)](x - 5) < 0$

$\Rightarrow x \in (-2, 5)$

Now $10x - x^2 - 16 > 0$

$\Rightarrow x^2 - 10x + 16 < 0$

$\Rightarrow x^2 - 8x - 2x + 16 < 0$

$\Rightarrow x(x - 8) - 2(x - 8) < 0$

$\Rightarrow (x - 2)(x - 8) < 0$

$\Rightarrow x \in (2, 8)$

Required solution set = $(-2, 5) \cap (2, 8)$

= (2, 5).

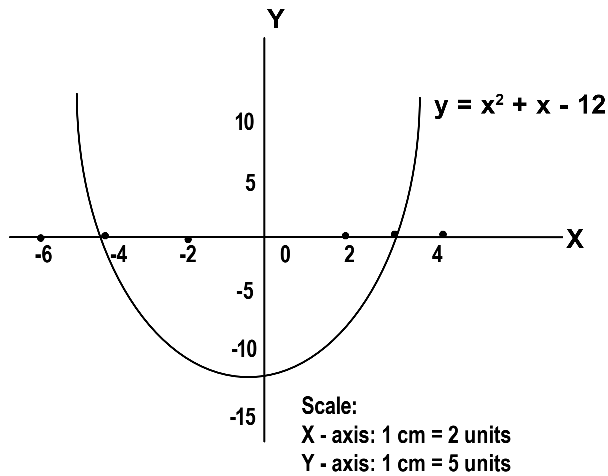
5. Find the solution set of $x^2 + x - 12 \leq 0$ by graphical method.

A: Consider $y = x^2 + x - 12$

$= (x + 4)(x - 3)$

Table for $y = x^2 + x - 12$

x	-5	-4	0	3	4
$y = x^2 + x - 12$	8	0	-12	0	8



$\therefore$ From the graph of $y = x^2 + x - 12$, the solution set of $x^2 + x - 12 \leq 0$ is [-4, 3].

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