# 4. THEORY OF EQUATIONS

***DEFINITIONS, CONCEPTS AND FORMULAE***

1. If $\alpha, \beta, \gamma$ are the roots of $ax^3 + bx^2 + cx + d = 0$, then

   i) $s_1 = \alpha + \beta + \gamma = \frac{-b}{a}$

   ii) $s_2 = \alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a}$

   iii) $s_3 = \alpha\beta\gamma = \frac{-d}{a}$

   Required cubic equation is $x^3 - s_1x^2 + s_2x - s_3 = 0$

2. If $\alpha, \beta, \gamma, \delta$ are the roots of $ax^4 + bx^3 + cx^2 + dx + e = 0$, then

   i) $s_1 = \alpha + \beta + \gamma + \delta = \frac{-b}{a}$

   ii) $s_2 = \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = \frac{c}{a}$

   iii) $s_3 = \alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = \frac{-d}{a}$

   iv) $s_4 = \alpha\beta\gamma\delta = \frac{e}{a}$

   Required biquadratic equation is
   $x^4 - s_1x^3 + s_2x^2 - s_3x + s_4 = 0$

3. For a cubic equation, when the roots are

   i) in A.P, then they are taken as a-d, a, a+d

   ii) in G.P, then they are taken as a/r , a, ar.

   iii) in H.P, then they are taken as $\frac{1}{a-d}, \frac{1}{a}, \frac{1}{a+d}$

4. For a biquadratic equation, if the roots are

   i) in A.P, then they are taken as a-3d, a-d, a+d, a+3d

   ii) in G.P, then they are taken as $\frac{a}{r^3}, \frac{a}{r}$, ar, $ar^3$

   iii) in H.P, then they are taken as $\frac{1}{a-3d}, \frac{1}{a-d}, \frac{1}{a+d}, \frac{1}{a+3d}$

5. If $f'(\alpha) = 0$ and $f(\alpha) = 0$, then $\alpha$ is a multiple root of order 2 for the algebraic equation $f(x) = 0$.

6. If $f''(\alpha) = 0$ and $f(\alpha) = 0$, then $\alpha$ is a multiple root of order 3 for the algebraic equation $f(x) = 0$

7. In an equation with rational co-efficients, irrational roots occur in pairs of conjugate surds.

8. In an equation with real coefficients, imaginary roots occur in conjugate pairs.

9. Remainder theorem : If f(x) is a polynomial, then the remainder of f(x) when divided by x - a is f(a).

10. Factor theorem: If f(x) is a polynomial and f(a) = 0 then x - a is a factor of f(x).

11. Horner's method: If f(x) is a polynomial function of degree 5, then
    $f(x + h) = A_0x^5 + A_1x^4 + A_2x^3 + A_3x^2 + A_4x + A_5$
    where $A_1, A_2, A_3, A_4, A_5$ are the remainders of f(x) when divided by $(x - h)^5$, $(x - h)^4$, $(x - h)^3$, $(x - h)^2$, (x - h) respectively and $A_0$ is the coefficient of $x^5$ in f(x).

AIMS

12. For a cubic equation

    i) $\sum \alpha^2\beta = s_1s_2 - 3s_3$

    ii) $\sum \alpha^3\beta^3 = s_2^3 - 3s_1s_2s_3 + 3s_3^2$

13. For a biquadratic equation:

    i) $\sum \alpha^2\beta = s_1s_2 - 3s_3$

    ii) $\sum \alpha^2\beta\gamma = s_1s_3 - 4s_4$

14. Consider $f(x) = p_0x^n + p_1x^{n-1} + ..... + p_n = 0$

    i) To eliminate the second term, f(x) = 0 is transformed

    to f(x + h) = 0 where $h = \frac{-p_1}{n.p_0}$.

    ii) To eliminate the third term, f(x) = 0 is transformed to f(x + h) = 0 where h is given by
    $\frac{n(n-1)}{2}p_0h^2 + (n-1)p_1h + p_2 = 0$.

15. An equation f(x) = 0 is said to be a reciprocal equation if $\frac{1}{\alpha}$ is a root of f(x) = 0 whenever $\alpha$ is a root of f(x) = 0

16. A R.E. $f(x) = p_0x^n + p_1x^{n-1} + .....+ p_n = 0$ is said to be a reciprocal equation of first class if $p_i = p_{n-i}$ for all i.

17. A R.E. $f(x) = p_ox^n + p_1x^{n-1} + .... + p_n = 0$ is said to be a reciprocal equation of second class if $p_i = -p_{n-i}$ for all i.

18. If f(x) = 0 is a reciprocal equation first class and odd degree, then '-1' is a root of f(x) = 0.

19. If f(x) = 0 is a reciprocal equation of second class and odd degree, then '1' is a root of f(x) = 0.

20. If f(x) = 0 is a reciprocal equation of second class and even degree, then 1, -1 are roots of f(x) = 0.

21. The transformed equation whose roots are negatives of roots of f(x) = 0 is f(-x) = 0.

22. The T.E. whose roots are multiplied by $k(\neq 0)$ of the roots of f(x) = 0 is $f\left(\frac{x}{k}\right) = 0$.

23. The T.E. whose roots are reciprocals of the roots of f(x) = 0 is $f\left(\frac{1}{x}\right) = 0$.

24. The T.E. whose roots exceed by h than those of f(x) = 0 is f(x - h) = 0.

25. The T.E. whose roots are diminished by h than those of f(x) = 0 is f(x + h) = 0

26. The T.E. whose roots are squares of the roots of f(x) = 0 is obtained by eliminating square root from $f(\sqrt{x}) = 0$.

27. The T.E. whose roots are cubes of the roots of f(x) = 0 is obtained by eliminating cube root from $f(\sqrt[3]{x}) = 0$.

# LEVEL - I (VSAQ)

**1. Form a polynomial equation of lowest degree, whose roots are 1, -1, 3.**

A: Polynomial equation of lowest degree whose roots are 1, -1, 3 is (x - 1) (x + 1) (x - 3) = 0

$\Rightarrow (x^2 - 1)(x + 3) = 0$

$\Rightarrow x^3 - 3x^2 - x + 3 = 0.$

**2. Form a polynomial equation with rational coefficients and whose roots are $2 \pm \sqrt{3}, 1 \pm 2i$.**

A: Given roots are $2 \pm \sqrt{3}, 1 \pm 2i$.

Required biquadratic equation is

$[x^2 - (2+\sqrt{3} + 2-\sqrt{3})x + (2+\sqrt{3})(2-\sqrt{3})]$

$[x^2 - (1 + 2i + 1 - 2i)x + (1 + 2i)(1 - 2i)] = 0$

$\Rightarrow (x^2 - 4x + 1)(x^2 - 2x + 5) = 0$

$\Rightarrow x^4 - 2x^3 + 5x^2 - 4x^3 + 8x^2 - 20x + x^2 - 2x + 5 = 0$

$\Rightarrow x^4 - 6x^3 + 14x^2 - 22x + 5 = 0.$

**3. If -1, 2, $\alpha$ are the roots of $2x^3 + x^2 - 7x - 6 = 0$, then find $\alpha$.**

A: Given that -1, 2, $\alpha$ are the roots of $2x^3 + x^2 - 7x - 6 = 0$

$\Rightarrow s_1 = \alpha + \beta + \gamma = -b/a$

$\Rightarrow -1 + 2 + \alpha = -1/2$

$\Rightarrow \alpha = -3/2.$

**4. If the product of the roots of the equation $4x^3 + 16x^2 - 9x - a = 0$ is 9, then find a.**

A: Product of the roots of $4x^3 + 16x^2 - 9x - a = 0$ is 9

$\Rightarrow s_3 = \alpha\beta\gamma = -d/a = 9$

$\Rightarrow -(-a)/4 = 9$

$\Rightarrow a = 36.$

**5. If $\alpha$, $\beta$ and 1 are the roots of $x^3 - 2x^2 - 5x + 6 = 0$, then find $\alpha$ and $\beta$.**

A: Given $\alpha$, $\beta$, 1 are the roots of $x^3 - 2x^2 - 5x + 6 = 0$

$\Rightarrow s_1 = \alpha + \beta + 1 = 2$

$\Rightarrow \alpha + \beta = 1.$

Also $s_3 = \alpha\beta\gamma = \alpha\beta(1) = -d/a = -6/1 = -6$

$\Rightarrow \alpha\beta = -6$

By observation $\alpha = 3$, $\beta = -2$.

**6. If 1, -2, 3 are the roots of $x^3 - 2x^2 + ax + 6 = 0$, then find a.**

A: Given that 1 is a root of $x^3 - 2x^2 + ax + 6 = 0$

$\Rightarrow 1 - 2 + a + 6 = 0$

$\Rightarrow a = -5.$

**7. Solve the equation $x^3 - 3x^2 - 6x + 8 = 0$, given that the roots are in A.P.**

A: Let the roots be a- d , a, a + d.

$s_1 = a - d + a + a + d = 3$

$\Rightarrow 3a = 3 \Rightarrow a = 1$

$s_3 = (a - d)(a)(a + d) = -8$

$\Rightarrow (1 - d)(1)(1 + d) = -8$

$\Rightarrow 1 - d^2 = -8$

$\Rightarrow d^2 = 9$

$\therefore d = \pm 3$

If d = 3, the roots are 1 - 3, 1, 1 + 3

i.e - 2, 1, 4.

**8. Find $s_1, s_2, s_3$ and $s_4$ for the equation $8x^4 - 2x^3 - 27x^2 + 6x + 9 = 0$.**

A: Given equation is $8x^4 - 2x^3 - 27x^2 + 6x + 9 = 0$

Comparing this with $ax^4 + bx^3 + cx^2 + dx + e = 0$.

$s_1 = -\frac{b}{a} = \frac{2}{8} = \frac{1}{4}$

$s_2 = \frac{c}{a} = \frac{-27}{8}$

$s_3 = -\frac{d}{a} = \frac{-6}{8} = \frac{-3}{4}$

$s_4 = \frac{e}{a} = \frac{9}{8}$.

**9. Solve the equation $x^3 - 3x^2 - 16x + 48 = 0$, one root being 3.**

A: Given equation is $x^3 - 3x^2 - 16x + 48 = 0$

$s_1 = 3 + \beta + \gamma = 3$

$\Rightarrow \beta + \gamma = 0$

$s_3 = 3(\beta)(\gamma) = -48$

$\Rightarrow \beta\gamma = -16$

The quadratic equation whose roots are $\beta, \gamma$ is $x^2 - (\beta + \gamma)x + \beta\gamma = 0$

$\Rightarrow x^2 - (0)x - 16 = 0$

$x = \pm 4$.

$\therefore$ The other two roots are 4, -4.

**10. If 1, 2, 3, 4 are the roots of $x^4 + ax^3 + bx^2 + cx + d = 0$, then find the values of a, b, c, d.**

A: Given that 1, 2, 3, 4 are the roots of

$x^4 + ax^3 + bx^2 + cx + d = 0$.

$\Rightarrow a = -s_1 = -(\alpha + \beta + \gamma + \delta) = -(1 + 2 + 3 + 4) = -10$

$b = s_2 = \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta$

$= 1(2) + 1(3) + 1(4) + 2(3) + 2(4) + 3(4) = 35$

$c = -s_3 = -(\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta)$

$= -(1.2.3 + 1.2.4 + 1.3.4 + 2.3.4) = -50$

$d = s_4 = \alpha\beta\gamma\delta = 1.2.3.4 = 24$

$\therefore a = -10, b = 35, c = -50, d = 24$

**11. If $\alpha, \beta, \gamma$ are the roots of $x^3 + px^2 + qx + r = 0$, then find $\alpha^2 + \beta^2 + \gamma^2$.**

A: Given that $\alpha, \beta, \gamma$ are the roots of $x^3 + px^2 + qx + r = 0$.

Now $\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$

$= (-p)^2 - 2(q)$

$= p^2 - 2q$.

**12. If $\alpha, \beta, \gamma$ are the roots of $x^3 + px^2 + qx + r = 0$, then find the value of $\alpha^3 + \beta^3 + \gamma^3$.**

A: We know that

$\alpha^3 + \beta^3 + \gamma^3 - 3\alpha\beta\gamma = (\alpha+\beta+\gamma)[\alpha^2 + \beta^2 + \gamma^2 - \alpha\beta - \beta\gamma - \gamma\alpha]$

$\Rightarrow \alpha^3+\beta^3+\gamma^3 = (\alpha+\beta+\gamma)[(\alpha+\beta+\gamma)^2 - 3(\alpha\beta + \beta\gamma + \gamma\alpha)] + 3\alpha\beta\gamma$

$= (-p)[(-p)^2 - 3(q)] + 3(-r)$

$= 3pq - p^3 - 3r$.



**13. If $\alpha, \beta, \gamma$ are the roots of $x^3 - 2x^2 + 3x - 4 = 0$, then find $\Sigma\alpha^2\beta^2$**

A: Given that $\alpha, \beta, \gamma$ are the roots of $x^3 - 2x^2 + 3x - 4 = 0$

$\Sigma\alpha^2\beta^2 = \alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2$

$= (\alpha\beta + \beta\gamma + \gamma\alpha)^2 - 2(\alpha\beta.\beta\gamma + \beta\gamma.\gamma\alpha + \gamma\alpha.\alpha\beta)$

$= (\alpha\beta + \beta\gamma + \gamma\alpha)^2 - 2\alpha\beta\gamma(\beta + \gamma + \alpha)$

$= 3^2 - 2(4)(2)$

$= 9 - 16$

$= -7$.

**14. Find the quotient and remainder, when $2x^5 - 3x^4 + 5x^3 - 3x^2 + 7x - 9$ divided by $x^2 - x - 3$.**

A: By synthetic division, dividing

$2x^5 - 3x^4 + 5x^3 - 3x^2 + 7x - 9$ is by $x^2 - x - 3$

| | | | | | | |
|---|---|---|---|---|---|---|
| | 2 | -3 | 5 | -3 | 7 | -9 |
| 1 | 0 | 2 | -1 | 10 | 4 | 0 |
| 3 | 0 | 0 | 6 | -3 | 30 | 12 |
| | 2 | -1 | 10 | 4 | 41 | 3 |

Required quotient is $2x^3 - x^2 + 10x + 4$ and the remainder is $41x + 3$.

**15. Find the polynomial equation of degree 4 whose roots are negatives of the roots of $x^4 - 6x^3 + 7x^2 - 2x + 1 = 0$**

A: Required transformed equation is f(-x) = 0

$\Rightarrow (-x)^4 - 6(-x)^3 + 7(-x)^2 - 2(-x) + 1 = 0$

$\Rightarrow x^4 + 6x^3 + 7x^2 + 2x + 1 = 0.$

**16. Find the algebraic equation whose roots are 3 times the roots of $x^3 + 2x^2 - 4x + 1 = 0$.**

A: Given equation is $f(x) = x^3 + 2x^2 - 4x + 1 = 0$.

Required transformed equation is f (x/3) = 0

$\Rightarrow \frac{x^3}{27} + \frac{2x^2}{9} - \frac{4x}{3} + 1 = 0$

$\Rightarrow x^3 + 6x^2 - 36x + 27 = 0.$

**17. If $\alpha$, $\beta$, $\gamma$ are the roots of the equation $x^3 + 2x^2 - 4x - 3 = 0$, find the equation whose roots are $\frac{\alpha}{3}, \frac{\beta}{3}, \frac{\gamma}{3}$.**

A: Given equation is $f(x) = x^3 + 2x^2 - 4x - 3 = 0$

Required transformed equation is f(3x) = 0

$\Rightarrow 27x^3 + 2(9x^2) - 4(3x) - 3 = 0$

$\Rightarrow 9x^3 + 6x^2 - 4x - 1 = 0.$

**18. Find the equation whose roots are squares of the roots of $x^3 + 3x^2 - 7x + 6 = 0$.**

A: Given equation is $f(x) = x^3 + 3x^2 - 7x + 6 = 0$.

Required transformed equation is $f(\sqrt{x}) = 0$

$\Rightarrow (\sqrt{x})^3 + 3(\sqrt{x})^2 - 7\sqrt{x} + 6 = 0$

$\Rightarrow x\sqrt{x} + 3x - 7\sqrt{x} + 6 = 0$

$\Rightarrow \sqrt{x}(x - 7) = -(3x + 6)$

Squaring on both sides,

$\Rightarrow x(x^2 - 14x + 49) = 9x^2 + 36x + 36$

$\Rightarrow x^3 - 14x^2 + 49x - 9x^2 - 36x - 36 = 0$

$\Rightarrow x^3 - 23x^2 + 13x - 36 = 0.$

# LEVEL - I (LAQ)

**1. Solve $4x^3 - 24x^2 + 23x + 18 = 0$, given that the roots are in A.P.**

A: Given that the roots of $4x^3 - 24x^2 + 23x + 18 = 0$ are in A.P.

Let the roots be a - d, a, a + d.

Sum of the roots $a - d + a + a + d = \frac{-b}{a}$

$\Rightarrow 3a = \frac{24}{4}$

$\Rightarrow a = \frac{6}{3} = 2.$

Product of the roots $(a - d)(a)(a + d) = \frac{-d}{a}$

$\Rightarrow (2 - d)(2)(2 + d) = \frac{-18}{4}$

$\Rightarrow 4 - d^2 = \frac{-9}{4}$

AIMS

$\Rightarrow d^2 = 4 + \frac{9}{4} = \frac{25}{4}$

$\Rightarrow d = \pm \frac{5}{2}$

Hence the roots of the given equation are

$2 - \frac{5}{2}, 2, 2 + \frac{5}{2}$

$= \frac{-1}{2}, 2, \frac{9}{2}.$

**2. If the roots of the equation $x^3 + 3px^2 + 3qx + r = 0$ are in A.P., then show that $2p^3 - 3pq + r = 0$.**

A: Given that the roots of $x^3 + 3px^2 + 3qx + r = 0$ ---- (1) are in A.P.

Let the roots be a - d, a, a + d.

Sum of the roots $a - d + a + a + d = \frac{-b}{a}$

$\Rightarrow 3a = -3p$

$\Rightarrow a = -p$

Substituting x = -p in (1), we get

$(-p)^3 + 3p(-p)^2 + 3q(-p) + r = 0$

$\Rightarrow -p^3 + 3p^3 - 3pq + r = 0$

$\Rightarrow 2p^3 - 3pq + r = 0.$

**3. Solve the equation $x^3 - 7x^2 + 14x - 8 = 0$, given that the roots are in geometric progression.**

A: Given that the roots of $x^3 - 7x^2 + 14x - 8 = 0$ are in G.P.

Let the roots be $\frac{a}{r}$, a, ar

$s_3 = \left(\frac{a}{r}\right)(a)(ar) = \frac{-d}{a}$

$\Rightarrow a^3 = 8$

$\Rightarrow a = 2$

$s_1 = \frac{a}{r} + a + ar = \frac{-b}{a}$

$\Rightarrow 2\left(r + \frac{1}{r} + 1\right) = 7$

$\Rightarrow r + \frac{1}{r} + 1 = \frac{7}{2}$

$\Rightarrow r + \frac{1}{r} = \frac{7}{2} - 1 = \frac{5}{2}$

$\Rightarrow r + \frac{1}{r} = 2 + \frac{1}{2}$ by observation

$\therefore r = 2$ or $\frac{1}{2}$

If r = 2, the required roots are $\frac{2}{2}$, 2, 2(2)

= 1, 2, 4.

**4. Show that the condition that the roots of $x^3 + 3px^2 + 3qx + r = 0$ may be in G.P. is $p^3r = q^3$.**

A: Given that the roots of $x^3 + 3px^2 + 3qx + r = 0$ ---- (1) are in G.P.

Let the roots be $\frac{\alpha}{\beta}$, $\alpha$, $\alpha\beta$.

$\left(\frac{\alpha}{\beta}\right)(\alpha)(\alpha\beta) = \frac{-d}{a}$

$\Rightarrow \alpha^3 = -r$

$\Rightarrow \alpha^3 + r = 0$ ------- (2)

substituting $x = \alpha$ in (1),

$\alpha^3 + 3p\alpha^2 + 3q\alpha + r = 0$

$\Rightarrow (\alpha^3 + r) + 3\alpha(p\alpha + q) = 0$

$\Rightarrow 0 + 3\alpha(p\alpha + q) = 0$

$\Rightarrow p\alpha + q = 0 \quad \because \alpha \neq 0$

$\Rightarrow p\alpha = -q$

Cubing on both sides,

$p^3\alpha^3 = -q^3$

$\Rightarrow p^3(-r) = -q^3$

$\Rightarrow p^3r = q^3$ is the required condition.

**5. Solve the equation $15x^3 - 23x^2 + 9x - 1 = 0$, given that the roots are in H.P.**

A: Given that the roots of

$f(x) = 15x^3 - 23x^2 + 9x - 1 = 0$ --------- (1) are in H.P.

$\Rightarrow$ Roots of $f\left(\frac{1}{x}\right) = 0$ are in A.P.

$f\left(\frac{1}{x}\right) = \frac{15}{x^3} - \frac{23}{x^2} + \frac{9}{x} - 1 = 0$

$\Rightarrow 15 - 23x + 9x^2 - x^3 = 0$

$\Rightarrow x^3 - 9x^2 + 23x - 15 = 0$ -------- (2)

AIMS

Let the roots of (2) be a - d, a, a + d.

$s_1 = a - d + a + a + d = \frac{-b}{a}$

$\Rightarrow 3a = 9$

$\Rightarrow a = 3$

$s_3 = (a - d)(a)(a + d) = \frac{-d}{a}$

$\Rightarrow 3(9 - d^2) = 15$

$\Rightarrow 9 - d^2 = \frac{15}{3} = 5$

$\Rightarrow d^2 = 4$

$\Rightarrow d = \pm 2$

If d = 2, the roots of (2), are 3 - 2, 3, 3 + 2

= 1, 3, 5.

Hence the roots of the given equation are 1, $\frac{1}{3}$, $\frac{1}{5}$.

**6. Given that the roots of $x^3 + 3px^2 + 3qx + r = 0$ are in H.P, show that $2q^3 = 4(3pq - r)$.**

A: Let $f(x) = x^3 + 3px^2 + 3qx + r = 0$

Given that roots of f(x) = 0 are in H.P.

$\Rightarrow$ Roots of $f\left(\frac{1}{x}\right) = 0$ are in A.P.

$f\left(\frac{1}{x}\right)=0 \Rightarrow \frac{1}{x^3}+\frac{3p}{x^2}+\frac{3q}{x}+r=0$.

$\Rightarrow rx^3+3qx^2+3px+1=0$

Let the roots of this equation be a - d, a, a + d.

sum = a - d + a + a + d = $-\frac{3q}{r}$.

$\Rightarrow 3a=-\frac{3q}{r}$.

$\Rightarrow a=\frac{-q}{r}$.

Since a is root, $ra^3 + 3qa^2 + 3pa + 1 = 0$.

$\Rightarrow r\left(\frac{-q}{r}\right)^3+3q\left(\frac{-q}{r}\right)^2+3p\left(\frac{-q}{r}\right)+1=0$

$\Rightarrow \frac{-q^3}{r^2}+\frac{3q^3}{r^2}-\frac{3pq}{r}+1=0$

$=\frac{2q^3-3pqr+r^2}{r^2}=0$

$\Rightarrow 2q^3 - 3pqr + r^2 = 0$

$\Rightarrow 2q^3 = 3pqr - r^2$.

$\therefore 2q^3 = r(3pq - r)$.

**7. Solve the equation $18x^3 + 81x^2 + 121x + 60 = 0$, given that one root is equal to half the sum of the remaining roots.**

A: Let $\alpha, \beta, \gamma$ be the roots such that $\beta = \frac{\alpha+\gamma}{2}$

$\Rightarrow \alpha+\gamma = 2\beta$

Now $s_1 = \alpha+\beta+\gamma = \frac{-b}{a}$

$\Rightarrow 2\beta + \beta = \frac{-81}{18}$

$\Rightarrow 3\beta = \frac{-9}{2}$

$\Rightarrow \beta = \frac{-3}{2}$

$\therefore \alpha+\gamma = -3$

Also $s_3 = \alpha\beta\gamma = \frac{-d}{a}$

$\Rightarrow \alpha\gamma\left(\frac{-3}{2}\right) = \frac{-60}{18}$

$\Rightarrow \alpha\gamma = \frac{20}{9}$

The quadratic equation, whose roots are $\alpha, \gamma$ is $x^2 - (\alpha+\gamma)x + \alpha\gamma = 0$.

$\Rightarrow x^2 - (-3)x + \frac{20}{9} = 0$

$\Rightarrow 9x^2 + 27x + 20 = 0$

$\Rightarrow 9x^2 + 12x + 15x + 20 = 0$

$\Rightarrow 3x(3x + 4) + 5(3x + 4) = 0$

$\Rightarrow (3x + 4)(3x + 5) = 0$

$\Rightarrow x = \frac{-4}{3}, \frac{-5}{3}$

P = 9. 20
= 3.3.4.5
= 12 . 15

Hence, the required roots of the given Cubic equation are $\frac{-4}{3}, \frac{-3}{2}, \frac{-5}{3}$.

**8. Solve $x^3 - 9x^2 + 14x + 24 = 0$ given that two of the roots are in the ratio 3 : 2.**

A: Given that two of the roots of $x^3 - 9x^2 + 14x + 24 =$ 0 are in the ratio 3 : 2. AIMS

Let the roots be $3\alpha, 2\alpha, \gamma$.

$s_1 = 3\alpha + 2\alpha + \gamma = 9$

$\Rightarrow \gamma = 9 - 5\alpha$ -------- (1)

$s_2 = (3\alpha)(2\alpha) + (2\alpha)(\gamma) + \gamma(3\alpha) = 14$

$\Rightarrow 6\alpha^2 + 2\alpha\gamma + 3\alpha\gamma = 14$

$\Rightarrow 6\alpha^2 + 5\alpha\gamma = 14$ ------ (2)

$s_3 = (3\alpha)(2\alpha)(\gamma) = -24$

$\Rightarrow 6\alpha^2\gamma = -24$

$\Rightarrow \alpha^2\gamma = -4$ -------- (3)

From (1) & (2), $6\alpha^2 + 5\alpha(9 - 5\alpha) = 14$

$\Rightarrow 6\alpha^2 + 45\alpha - 25\alpha^2 - 14 = 0$

$\Rightarrow -19\alpha^2 + 45\alpha - 14 = 0$ P = 19.14

$\Rightarrow 19\alpha^2 - 45\alpha + 14 = 0$ = 19 . 2. 7

$\Rightarrow 19\alpha^2 - 38\alpha - 7\alpha + 14 = 0$ = 38 . 7

$\Rightarrow 19\alpha(\alpha - 2) - 7(\alpha - 2) = 0$

$\Rightarrow (19\alpha - 7)(\alpha - 2) = 0$

$\Rightarrow \alpha = 2, \frac{7}{19}$

If $\alpha = 2$, then $\gamma = 9 - 5(2) = -1$.

Substituting $\alpha, \gamma$ values in LHS of (3),

$\alpha^2\gamma = 2^2(-1)$

$= -4$

= RHS, which is satisfied.

Whereas $\alpha = \frac{7}{19}$ does not satisfy equation (3).

Hence, the required roots of given cubic equation are 3(2), 2(2), -1

= 6, 4, -1.

**9. Solve the equation $x^4 - 2x^3 + 4x^2 + 6x - 21 = 0$, the sum of two roots being zero.**

A: Sum of two roots of $x^4 - 2x^3 + 4x^2 + 6x - 21 = 0$ is zero. Let the roots $\alpha, -\alpha, \gamma, \delta$

$s_1 = \alpha - \alpha + \gamma + \delta = 2$

$\gamma + \delta = 2$

$s_3 = \alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = \frac{-d}{a}$

$\Rightarrow -\alpha^2\gamma - \alpha^2\delta + \alpha\gamma\delta - \alpha\gamma\delta = -6$

$\Rightarrow \alpha^2(\gamma + \delta) = 6$

$\Rightarrow \alpha^2 = 3$

$\alpha = \pm\sqrt{3}$.

$s_4 = (\alpha)(-\alpha)(\gamma)(\delta) = \frac{e}{a}$

$\Rightarrow -3\gamma\delta = -21$

$\Rightarrow \gamma\delta = 7$.

The quadratic equation whose roots are $\gamma, \delta$ is

$x^2 - (\gamma + \delta)x + \gamma\delta = 0$

$\Rightarrow x^2 - 2x + 7 = 0$

$\Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \frac{2 \pm \sqrt{(-2)^2 - 4(1)(7)}}{2(1)}$

$= \frac{2 \pm \sqrt{-24}}{2}$

$= \frac{2 \pm 2\sqrt{6}\,i}{2}$

$= 1 \pm \sqrt{6}\,i$

Hence, the required roots of the given equation are $\sqrt{3}, -\sqrt{3}, 1+\sqrt{6}\,i, 1-\sqrt{6}\,i$.

**10. Solve the equation $x^4 - 5x^3 + 5x^2 + 5x - 6 = 0$, the product of two roots being 3.**

A: Product of two roots of $x^4 - 5x^3 + 5x^2 + 5x - 6 = 0$ is 3.
Let $\alpha, \beta, \gamma, \delta$ be the roots with $\alpha\beta = 6$.

$s_4 = \alpha\beta\gamma\delta = \frac{e}{a}$

$3\gamma\delta = -6$

$\gamma\delta = -2$ ------- (1)

$s_1 = \alpha + \beta + \gamma + \delta = \frac{-b}{a} = 5$ ------ (2)

$s_3 = \alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = \frac{-d}{a}$

$\Rightarrow \alpha\beta(\gamma + \delta) + \gamma\delta(\alpha + \beta) = -5$

$\Rightarrow 3(\gamma + \delta) - 2(\alpha + \beta) = -5$

$\Rightarrow 3(\alpha + \beta + \gamma + \delta) - 5(\alpha + \beta) = -5$

$\Rightarrow 3(5) - 5(\alpha + \beta) = -5$

$\Rightarrow 15 + 5 = 5(\alpha + \beta)$

$\Rightarrow \alpha + \beta = 4, \gamma + \delta = 5 - 4 = 1.$

AIMS

| | |
|---|---|
| $\alpha + \beta = 4, \alpha\beta = 3$ | $\gamma + \delta = 1, \gamma\delta = -2$ |
| $x^2 - (\alpha + \beta)x + \alpha\beta = 0$ | $x^2 - (\gamma + \delta)x + \gamma\delta = 0$ |
| $x^2 - 4x + 3 = 0.$ | $x^2 - x - 2 = 0$ |
| $(x - 1)(x - 3) = 0$ | $x^2 - 2x + x - 2 = 0$ |
| $\alpha = 1, \beta = 3$ | $x(x - 2) + 1(x - 2) = 0$ |
| | $(x + 1)(x - 2) = 0$ |
| | $\gamma = -1, \delta = 2$ |

Hence the required roots of the given biquadratic equation are 1, 3, -1, 2.

**11. Solve the equation $x^4 + 4x^3 - 2x^2 - 12x + 9 = 0$, if it has two pairs of equal roots.**

A: Given that $x^4 + 4x^3 - 2x^2 - 12x + 9 = 0$ has two pairs of equal roots.
Let the roots be $\alpha, \alpha, \beta, \beta$.

$s_1 = \alpha + \alpha + \beta + \beta = -4$

$\Rightarrow 2(\alpha + \beta) = -4$

$\Rightarrow \alpha + \beta = -2$

$s_3 = \alpha^2\beta + \alpha^2\beta + \alpha\beta^2 + \alpha\beta^2 = 12$

$\Rightarrow 2\alpha\beta(\alpha + \beta) = 12$

$\Rightarrow \alpha\beta(\alpha + \beta) = 6$

$\Rightarrow \alpha\beta(-2) = 6$

$\Rightarrow \alpha\beta = -3$

The quadratic equation whose roots are $\alpha, \beta$ is

$x^2 - (\alpha + \beta) x + \alpha\beta = 0$

$\Rightarrow x^2 + 2x - 3 = 0$

$\Rightarrow x^2 + 3x - x - 3 = 0$

$\Rightarrow x(x + 3) -1 (x + 3) = 0.$

$\Rightarrow (x + 3) (x - 1) = 0$

Hence the required roots of the given biqudratic equation are -3, -3, 1, 1.

**12. Find the roots of $x^4 - 16x^3 + 86x^2 - 176x + 105 = 0$.**

A: Given equation is

$f(x) = x^4 - 16x^3 + 86x^2 - 176x + 105 = 0$

Now f(1) = 1 - 16 + 86 - 176 + 105

= 192 - 192

= 0.

So, 1 is a root of f(x) = 0.

By Synthetic division,

| | | | | | |
|---|---|---|---|---|---|
| | 1 | -16 | 86 | -176 | 105 |
| 1 | 0 | 1 | -15 | 71 | -105 |
| | 1 | -15 | 71 | -105 | 0 |

Now, the given equation can be written as

$(x - 1) (x^3 - 15x^2 + 71x - 105) = 0$

Let $g(x) = x^3 - 15x^2 + 71x - 105$

$g(1) \neq 0, g(2) \neq 0, g(3) = 0.$

So, 3 is a root of g(x) =0.

| | | | | |
|---|---|---|---|---|
| | 1 | -15 | 71 | -105 |
| 3 | 0 | 3 | -36 | 105 |
| | 1 | -12 | 35 | 0 |

Now f(x) = 0 can be written as

$(x - 1) (x - 3) (x^2 - 12x + 35) = 0$

$x^2 - 12x + 35 = 0$

$(x - 5) (x - 7) = 0$

x = 5, 7.

Hence, the required roots of the given biquadratic equation are 1, 3, 5, 7.

**13. Solve the equation $x^4 + 2x^3 - 5x^2 + 6x + 2 = 0$, given that one root of it is 1 + i.**

A: For the equation $x^4 + 2x^3 - 5x^2 + 6x + 2 = 0$, one root is 1 + i.

So, the other root is 1 - i.

The quadratic equation whose roots are 1 + i, 1 - i is $x^2 - (1 + i + 1 - i) x + (1 + i) (1 - i) = 0$

$\Rightarrow x^2 - 2x + 2 = 0$

By synthetic division,

| | | | | | |
|---|---|---|---|---|---|
| | 1 | 2 | -5 | 6 | 2 |
| 2 | 0 | 2 | 8 | 2 | 0 |
| -2 | 0 | 0 | -2 | -8 | -2 |
| | 1 | 4 | 1 | 0 | 0 |

Remainder is 0,

Quotient is $x^2 + 4x + 1$.

Now the given equation can be written as

$(x^2 - 2x + 2) (x^2 + 4x + 1) = 0$

AIMS

$x^2 + 4x + 1 = 0$

$x = \frac{-4 \pm \sqrt{16 - 4}}{2}$

$= \frac{-4 \pm 2\sqrt{3}}{2}$

$= -2 \pm \sqrt{3}$

Hence the required roots of the given biquadratic equation are 1 + i, 1 - i, $-2 + \sqrt{3}$, $-2 - \sqrt{3}$.

**14. Find the multiple roots of the equation $x^5 - 3x^4 - 5x^3 + 27x^2 - 32x + 12 = 0$.**

A: Given equation is

$f(x) = x^5 - 3x^4 - 5x^3 + 27x^2 - 32x + 12 = 0$

Differentiating f(x) w.r.t. x,

$f'(x) = 5x^4 - 12x^3 - 15x^2 + 54x - 32$

$f'(1) = 5 - 12 - 15 + 54 - 32$

= 59 - 59

= 0.

Also f(1) = 1 - 3 - 5 + 27 - 32 + 12

= 40 - 40

= 0.

Since $f'(1) = 0$, f(1) = 0, so 1 is a multiple root of order 2 for the given equation.

By Synthetic division

| | | | | | | |
|---|---|---|---|---|---|---|
| | 1 | -3 | -5 | 27 | -32 | 12 |
| 1 | 0 | 1 | -2 | -7 | 20 | -12 |
| | 1 | -2 | -7 | 20 | -12 | 0 |
| 1 | 0 | 1 | -1 | -8 | 12 | |
| | 1 | -1 | -8 | 12 | 0 | |

Now the given equation can be written as

$(x - 1)(x - 1)(x^3 - x^2 - 8x + 12) = 0$

Let $g(x) = x^3 - x^2 - 8x + 12$

differentiating w.r.t. x,

$g'(x) = 3x^2 - 2x - 8$

Now $g'(1) \neq 0$

$g'(2) = 12 - 4 - 8 = 0$

Also $g(2) = 8 - 4 - 16 + 12 = 0$.

Since $g'(2) = 0$, $g(2) = 0$, so 2 is a multiple root of order 2 for the cubic equation $x^3 - x^2 - 8x + 12 = 0$

$s_1 = 2 + 2 + \gamma = \frac{-b}{a} = 1$

$\gamma = 1 - 4 = -3$

Hence the required roots of the given 5th degree equation are 1, 1, 2, 2, -3.

**15. Find the algebraic equation of degree 5 whose roots are the translates of roots of $x^5 + 4x^3 - x^2 + 11 = 0$ by -3.**

A: Given equation is $f(x) = x^5 + 4x^3 - x^2 + 11 = 0$.

Here the roots are translated by '-3', so the transformed equation is $f(x + 3) = 0$.

By Horner's method,

| | | | | | | |
|---|---|---|---|---|---|---|
| | 1 | 0 | 4 | -1 | 0 | 11 |
| 3 | 0 | 3 | 9 | 39 | 114 | 342 |
| | 1 | 3 | 13 | 38 | 114 | $353 = A_5$ |
| 3 | 0 | 3 | 18 | 93 | 393 | |
| | 1 | 6 | 31 | 131 | $507 = A_4$ | |
| 3 | 0 | 3 | 27 | 174 | | |
| | 1 | 9 | 58 | $305 = A_3$ | | |
| 3 | 0 | 3 | 36 | | | |
| | 1 | 12 | $94 = A_2$ | | | |
| 3 | 0 | 3 | | | | |
| | $1 = A_0$ | $15 = A_1$ | | | | |

$\therefore$ Required transformed equation is

$x^5 + 15x^4 + 94x^3 + 305x^2 + 507x + 353 = 0$.

**16. Transform $x^4 + 4x^3 + 2x^2 - 4x - 2 = 0$ into another equation in which the coefficient of second highest power of x is zero and find the transformed equation.**

A: Given equation is $f(x) = x^4 + 4x^3 + 2x^2 - 4x - 2 = 0$.

Comparing this with $p_0x^4 + p_1x^3 + p_2x^2 + p_3x + p_4 = 0$.

Here $p_0 = 1$, $p_1 = 4$, $n = 4$ (degree)

To eliminate the second term, $f(x) = 0$ is transformed to $f(x + h) = 0$ where h is given by

$h = \frac{-p_1}{n.p_0} = \frac{-4}{4(1)} = -1$

$\therefore$ Required transformed equation is $f(x - 1) = 0$.

By Horner's method

$f(x - 1) = A_0x^4 + A_1x^3 + A_2x^2 + A_3x + A_4$

| | | | | | |
|---|---|---|---|---|---|
| | 1 | 4 | 2 | -4 | -2 |
| -1 | 0 | -1 | -3 | 1 | 3 |
| | 1 | 3 | -1 | -1 | $1 = A_4$ |
| -1 | 0 | -1 | -2 | 3 | |
| | 1 | 2 | -3 | $0 = A_3$ | |
| -1 | 0 | -1 | -1 | | |
| | 1 | 1 | $-4 = A_2$ | | |
| -1 | 0 | -1 | | | |
| | $1 = A_0$ | $0 = A_1$ | | | |

AIMS

$\therefore$ Required transformed equation is $x^4 - 4x^2 + 1 = 0$.

**17. Solve $2x^5 + x^4 - 12x^3 - 12x^2 + x + 2 = 0$.**

A: Given equation is $2x^5 + x^4 - 12x^3 - 12x^2 + x + 2 = 0$.

It is a reciprocal equation of first class and odd degree. So -1 is a root of it.

By Synthetic division,

| | | | | | | |
|---|---|---|---|---|---|---|
| | 2 | 1 | -12 | -12 | 1 | 2 |
| -1 | 0 | -2 | 1 | 11 | 1 | -2 |
| | 2 | -1 | -11 | -1 | 2 | 0 |

Now the given equation can be written as

$(x + 1)(2x^4 - x^3 - 11x^2 - x + 2) = 0$.

Consider $2x^4 - x^3 - 11x^2 - x + 2 = 0$. $\div x^2$

$\Rightarrow 2x^2 - x - 11 - \frac{1}{x} + \frac{2}{x^2} = 0$.

$\Rightarrow 2\left(x^2 + \frac{1}{x^2}\right) - \left(x + \frac{1}{x}\right) - 11 = 0$.

Put $x + \frac{1}{x} = z \Rightarrow x^2 + \frac{1}{x^2} = z^2 - 2$.

Now the above equation becomes,

$\Rightarrow 2(z^2 - 2) - z - 11 = 0.$
$\Rightarrow 2z^2 - z - 15 = 0.$

$\Rightarrow 2z^2 - 6z + 5z - 15 = 0.$

$\Rightarrow 2z(z - 3) + 5(z - 3) = 0.$

| | |
|---|---|
| $z - 3 = 0$ | $2z + 5 = 0.$ |
| $x + \frac{1}{x} - 3 = 0.$ | $2\left(x + \frac{1}{x}\right) + 5 = 0.$ |
| $x^2 - 3x + 1 = 0.$ | $2x^2 + 5x + 2 = 0.$ |
| $x = \frac{3 \pm \sqrt{9 - 4}}{2}$ | $2x^2 + 4x + x + 2 = 0.$ |
| $= \frac{3 \pm \sqrt{5}}{2}$ | $2x(x + 2) + 1(x + 2) = 0.$ |
| | $(x + 2)(2x + 1) = 0.$ |
| | $x = \frac{-1}{2}, -2$ |

Hence the roots of the given 5th degree equation are $-1, \frac{3 \pm \sqrt{5}}{2}, \frac{-1}{2}, -2$.

**18. Solve the equation $6x^6 - 25x^5 + 31x^4 - 31x^2 + 25x - 6 = 0$.**

A: This is a reciprocal equation of 2nd class and even degree. So 1, -1 are the roots of it.

By Synthetic division,

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| | 6 | -25 | 31 | 0 | -31 | 25 | -6 |
| 1 | 0 | 6 | -19 | 12 | 12 | -19 | 6 |
| | 6 | -19 | 12 | 12 | -19 | 6 | 0 |
| -1 | 0 | - 6 | 25 | -37 | 25 | -6 | |
| | 6 | - 25 | 37 | -25 | 6 | 0 | |

Now the given equation can be written as
$(x - 1)(x + 1)(6x^4 - 25x^3 + 37x^2 - 25x + 6) = 0.$

Consider $6x^4 - 25x^3 + 37x^2 - 25x + 6 = 0$ $\div x^2$

$\Rightarrow 6x^2 - 25x + 37 - \frac{25}{x} + \frac{6}{x^2} = 0.$

$6\left(x^2 + \frac{1}{x^2}\right) - 25\left(x + \frac{1}{x}\right) + 37 = 0.$

Put $x + \frac{1}{x} = z \Rightarrow x^2 + \frac{1}{x^2} = z^2 - 2$.

Now the above equation becomes,

$\Rightarrow 6(z^2 - 2) - 25z + 37 = 0.$

$\Rightarrow 6z^2 - 25z + 25 = 0.$

$\Rightarrow 6z^2 - 15z - 10z + 25 = 0.$

$\Rightarrow 3z(2z - 5) - 5(2z - 5) = 0.$

$\Rightarrow (3z - 5)(2z - 5) = 0.$

| | |
|---|---|
| $3z - 5 = 0.$ | $2z - 5 = 0.$ |
| $3\left(x + \frac{1}{x}\right) - 5 = 0.$ | $2\left(x + \frac{1}{x}\right) - 5 = 0.$ |
| $3x^2 - 5x + 3 = 0.$ | $2x^2 - 5x + 2 = 0.$ |
| $x = \frac{5 \pm \sqrt{25 - 36}}{2(3)}.$ | $2x^2 - 4x - x + 2 = 0.$ |
| $= \frac{5 \pm \sqrt{11}i}{6}.$ | $2x(x - 2) - 1(x - 2) = 0.$ |
| | $(x - 2)(2x - 1) = 0.$ |
| | $x = 2, \frac{1}{2}.$ |

AIMS

Hence the required roots of the given 6th degree reciprocal equation are $-1, 1, \frac{5 \pm \sqrt{11}i}{6}, 2, \frac{1}{2}$.

# LEVEL - II (VSAQ)

**1. Form the monic polynomial equation of degree 3 whose roots are 2, 3 and 6.**

A: The polynomial equation whose roots 2, 3, 6 is $(x - 2)(x-3)(x - 6) = 0.$

$\Rightarrow (x^2 - 5x + 6)(x - 6) = 0$

$\Rightarrow x^3 - 6x^2 - 5x^2 + 30x + 6x - 36 = 0$

$\Rightarrow x^3 - 11^2 + 36x - 36 = 0.$

**2. If 1, 1, $\alpha$ are the roots of $x^3 - 6x^2 + 9x - 4 = 0$ then find '$\alpha$'.**

A: Given equation is $x^3 - 6x^2 + 9x - 4 = 0$

Given that 1, 1, $\alpha$ are the roots of given equation

then $s_1 = 1 + 1 + \alpha = - p_1$

$\Rightarrow 2 + \alpha = -(-6)$

$\Rightarrow \alpha = 6 - 2$

$\Rightarrow \boxed{\alpha = 4}$ .

**3. If a, b, c are the roots of $x^3 - px^2 + qx - r = 0$ then find $\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}$.**

A: Given equation is $x^3 - px^2 + qx - r = 0$.........(1).

Given that 'a, b, c' are the roots of (1), then

a + b + c = p, ab + bc + ca =q, abc = r.

$$\therefore \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=\frac{b^2c^2+a^2c^2+a^2b^2}{a^2b^2c^2}$$

$$=\frac{(bc+ac+ab)^2-2abc(a+b+c)}{(abc)^2}=\frac{q^2-2rp}{r^2}.$$

**4. If $\alpha, \beta, \gamma$ are the roots of $x^3 - ax^2 + bx + c = 0$, then find $\Sigma\alpha^2\beta + \Sigma\alpha\beta^2$.**

A:Given that $\alpha, \beta, \gamma$ are the roots of $x^3 + ax^2 + bx + c = 0$ then

$s_1 \Rightarrow \alpha+\beta+\gamma = -a$

$s_2 \Rightarrow \alpha\beta+\beta\gamma+\gamma\alpha = b$

$s_3 \Rightarrow \alpha\beta\gamma = -c$

$\Sigma\alpha^2\beta+\Sigma\alpha\beta^2 = \alpha^2\beta+\beta^2\gamma+\gamma^2\alpha+\alpha\beta^2+\beta\gamma^2+\gamma\alpha^2.$

$=(\alpha+\beta+\gamma)(\alpha\beta+\beta\gamma+\gamma\alpha)-3\alpha\beta\gamma$

= (-a) (b) - 3 (-c)

$= s_1s_2-3s_3$ = 3c - ab.

**5. Find the equation whose roots are the cubes of the roots of $x^3 + 3x^2 + 2 = 0$.**

A: Let $f(x) = x^3 + 3x^2 + 2$.

Formula : The equation whose roots are the cubes of the roots of f(x) = 0 is $f\left(\sqrt[3]{x}\right)=0$.

$$\Rightarrow\left(\sqrt[3]{x}\right)^3+3\left(\sqrt[3]{x}\right)^2+2=0\Rightarrow x+3\left(x^{2/3}\right)+2=0.$$

$\Rightarrow (x+2)=-3x^{2/3}$ [cubing on both sides]

$$\Rightarrow (x+2)^3=\left(-3x^{2/3}\right)^3$$

$= x^3 + 2^3 + 3(x^2)(2) + 3x(2^2) = -27x^2$

$= x^3 + 8 + 6x^2 + 12x + 27x^2 = 0$

$$\Rightarrow \boxed{x^3+33x^2+12x+8=0}$$

# LEVEL - II (LAQ)

**1. Solve the equation $x^5 - 5x^4 + 9x^3 - 9x^2 + 5x - 1 = 0$**

A: Given equation $x^5 - 5x^4 + 9x^3 - 9x^2 + 5x - 1 = 0$. This is a reciprocal equation of 2nd class and odd degree.

So, 1 is a root of this equation.

Therefore 'x - 1' is a factor of it

By synthetic division.

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 1 | –5 | 9 | –9 | 5 | –1 |
| | 0 | 1 | –4 | 5 | –4 | 1 |
| | 1 | –4 | 5 | –4 | 1 | 0 |

Now the given equation can be written as $(x-1)\left(x^4-4x^3+5x^2-4x+1\right)=0$

consider, $x^4-4x^3+5x^2-4x+1=0$

On dividing this equation by $x^2$, we get

$$x^2-4x+5-\frac{4}{x}+\frac{1}{x^2}=0$$

AIMS

i.e. $\left(x^2+\frac{1}{x^2}\right)-4\left(x+\frac{1}{x}\right)+5=0$.

Put $x+\frac{1}{x}=z\Rightarrow x^2+\frac{1}{x^2}=z^2-2$

Now the above equation becomes

$\Rightarrow z^2 - 2 - 4z + 5 = 0$

$\Rightarrow z^2 - 4z + 3 = 0$

$\Rightarrow (z - 3)(z - 1) = 0$

$\Rightarrow$ z = 1 or 3

**<u>Case: 1.</u>** If z = 1

$\Rightarrow x+\frac{1}{x}=1$

$\Rightarrow x^2 - x + 1 = 0$

$\Rightarrow x=\frac{1\pm\sqrt{1-4}}{2}$

$\Rightarrow \frac{1\pm\sqrt{-3}}{2}$

$\Rightarrow \frac{1\pm i\sqrt{3}}{2}$

**<u>Case: 2</u>** If z = 3

$\Rightarrow x+\frac{1}{x}=3$

$\Rightarrow x^2-3x+1=0$

$\therefore x=\frac{3\pm\sqrt{9-4}}{2}$

$\Rightarrow \frac{3\pm\sqrt{5}}{2}$