DEFINITIONS, CONCEPTS AND FORMULAE

1. If n is a positive integer, $(x + a)^n = {}^nC_o x^n + {}^nC_1 x^{n-1}.a + {}^nC_2 x^{n-2}.a^2 + ... + {}^nC_r x^{n-r}.a^r + ... + {}^nC_n.a^n$.
2. ${}^nC_o, {}^nC_1, {}^nC_2 ..... {}^nC_n$ are called binomid coefficients.
3. No. of terms in $(x+a)^n$ is n + 1.
4. No. of terms in $(x+y+z)^n$ is $\frac{(n+1)(n+2)}{2}$.
5. General term in $(x+a)^n$ is $T_{r+1} = {}^nC_r.x^{n-r}.a^r$.
6. In $(x+a)^n$,

   i) if n is even then the middle term is $T_{\frac{n}{2}+1}$

   ii) If n is odd, then the two middlemost terms are $T_{\frac{n+1}{2}}, T_{\frac{n+3}{2}}$
7. Term independent of 'x' or constant term is the term containing x^0.
8. In $(1+x)^n$,

   i) if $\frac{(n+1)|x|}{|x|+1}$ =p (an integer), then the two numerically greatest terms are $|T_P|$ & $|T_{P+1}|$

   ii) If $\frac{(n+1)|x|}{|x|+1} = p + f$ where p is an integer, $0 < f < 1$ then the only numerically greatest term is $|T_{p+1}|$
9. i) $C_o+C_1+C_2+.....+C_n=2^n$

   ii) $C_o+C_2+C_4+.....=C_1+C_3+C_5+.=2^{n-1}$
10. i) $a.C_o+(a+d)C_1+(a+2d)C_2+....+(a+nd)C_n=(2a+nd)2^{n-1}$

    ii) $a.C_o-(a+d)C_1+(a+2d)C_2-......$n+1 terms =0
11. i) $C_1+2.C_2x+3C_3x^2+....+n.C_n.x^{n-1}=n(1+x)^{n-1}$

    ii) $C_1+2.C_2+3.C_3+.....n.C_n=n.2^{n-1}$

    iii) $C_1-2.C_2+3C_3-....+{}^nC_n(-1)^{n-1}=0$
12. i) $C_0 + \frac{C_1}{2}x + \frac{C_2}{3}x^2 + ... + \frac{C_n}{n+1}x^n = \frac{(1+x)^{n+1}-1}{(n+1)x}$

    ii) $C_0 + \frac{C_1}{2} + \frac{C_2}{3} + ... + \frac{C_n}{n+1} = \frac{2^{n+1}-1}{n+1}$

    iii) $C_0 - \frac{C_1}{2} + \frac{C_2}{3} - .... + \frac{C_n}{n+1}(-1)^n = \frac{1}{n+1}$

    iv) $C_0 + \frac{C_2}{3} + \frac{C_4}{5} + ... = \frac{2^n}{n+1}$

    v) $\frac{C_1}{2} + \frac{C_3}{4} + \frac{C_5}{6} + ... = \frac{2^n-1}{n+1}$
13. $C_oC_r+C_1C_{r+1}+C_2C_{r+2}+.....+C_{n-r}C_n = {}^{2n}C_{n+r} = \frac{(2n)!}{(n-r)!(n+r)!}$

    ii) $C_0^2 + C_1^2 + C_2^2 + ... + C_n^2 = {}^{2n}C_n$
14. If n is a rational number and |x| < 1, then

    i) $(1+x)^n=1+nx+\frac{n(n-1)}{1.2}x^2+\frac{n(n-1)(n-2)}{1.2.3}x^3+.....\infty$

    ii) $(1-x)^{-n}=1+nx+\frac{n(n+1)}{1.2}x^2+\frac{n(n+1)(n+2)}{1.2.3}x^3+.....\infty$
15. If |x|<1 and p,q are positive integers then

    i) $(1-x)^{-p/q} = 1+\frac{p}{1!}\left(\frac{x}{q}\right)+\frac{p(p+q)}{2!}\left(\frac{x}{q}\right)^2+.......\infty$

    ii) $(1+x)^{-p/q}=1-\frac{p}{1!}\left(\frac{x}{q}\right)+\frac{p(p+q)}{2!}\left(\frac{x}{q}\right)^2-.......\infty$
16. If |x| < 1, then

    i) $(1-x)^{-1} = 1+x + x^2+x^3+.....+x^n+......\infty$

    ii) $(1+x)^{-1} = 1-x + x^2-x^3+.....+(-1)^nx^n+......\infty$

    iii) $(1-x)^{-2} = 1+2x +3x^2+.....+(n+1)x^n+......\infty$

    iv) $(1+x)^{-2} = 1-2x +3x^2-.....+(-1)^n(n+1)x^n+......\infty$

    v) $(1-x)^{-3} = 1+3x +6x^2+10x^3+..... + \frac{(n+1)(n+2)}{2}x^n+.....\infty$

    vi) $(1+x)^{-3} = 1-3x+6x^2-10x^3+.....+(-1)^n \frac{(n+1)(n+2)}{2}x^n+.....\infty$
17. If |x| < 1 and n is a rational number,

    i) $(1 + x)^n \approx 1 + \frac{n}{1!}x$ if x^2 and higher powers of x are neglected.

    ii) $(1 + x)^n \approx 1 + \frac{n}{1!}x + \frac{n(n-1)}{2!}x^2$ if x^3 and higher powers of x are neglected.

    iii) $(1 - x)^{-n} \approx 1 + \frac{n}{1!}x$ if x^2 and higher powers of x are neglected.

    iv) $(1 - x)^{-n} \approx 1 + \frac{n}{1!}x + \frac{n(n+1)}{2!}x^2$ if x^3 and higher powers of x are neglected.

LEVEL - I (VSAQ)

1. Find the number of terms in (2a + 3b + c)^5.

A: Number of terms in $(2a + 3b + C)^5$

$$= \frac{(n+1)(n+2)}{2!}$$

$$= \frac{(5+1)(n+2)}{2}$$

$$= \frac{6 \times 7}{2}$$

= 21.

2. Find the 3rd term from the end in the expansion of $\left(x^{\frac{-2}{3}} - \frac{3}{x^2}\right)^8$.

A: In $\left(x^{\frac{-2}{3}} - \frac{3}{x^2}\right)^8$, 3rd term from the end

$= T_7$

$= T_{6+1}$

$= {}^8C_6 (x^{-2/3})^{8-6} (-3/x^2)^6$

$= {}^8C_2 \times x^{-4/3} \times 36/x^{12}$

$= {}^8C_2 \times 3^6/x^{40/3}$.

3. Find the coefficient of x^{-6} in $(3x - 4/x)^{10}$

A: $T_{r+1} = {}^{10}C_r (3x)^{10-r} (-4/x)^r$

$= {}^{10}C_r . 3^{10-r} (-4)^r . x^{10-r-r}$

To get the coefficient of x^{-6},

$10 - 2r = -6$

$\Rightarrow 2r = 16$

$\Rightarrow r = 8$

Coefficient of $x^{-6} = {}^{10}C_8 . 3^2 . (-4)^8$

$= {}^{10}C_2 . 3^2 . 4^8$.

4. Find the coefficient of x^{-7} in $\left(\frac{2x^2}{3} - \frac{5}{4x^5}\right)^7$.

A: In $\left(\frac{2x^2}{3} - \frac{5}{4x^5}\right)^7$,

$$T_{r+1} = {}^7C_r . \left(\frac{2x^2}{3}\right)^{7-r} \left(\frac{-5}{4x^5}\right)^r$$

$$= {}^7C_r . \left(\frac{2}{3}\right)^{7-r} \left(\frac{-5}{4}\right)^r . x^{14-7r}$$

To get the coefficient of x^{-7},

$14 - 7r = 7$

$7r = 21$

$r = 3$

$\therefore$ Coefficient of x^{-7}

$$= {}^7C_3 \left(\frac{2}{3}\right)^{7-3} \left(\frac{-5}{4}\right)^3$$

$$= -35 \left(\frac{2^4}{3^4}\right) \left(\frac{5^3}{4^3}\right)$$

$$= \frac{-4375}{324}.$$

5. Find the term independent of in $\left(\frac{\sqrt{x}}{3} - \frac{4}{x^2}\right)^{10}$.

A: General term $T_{r+1} = {}^{10}C_r . \left(\frac{\sqrt{x}}{3}\right)^{10-r} \left(\frac{-4}{x^2}\right)^r$

$$= {}^{10}C_r \frac{(-4)^r}{3^{10-r}} . x^{\frac{10-r}{2} - 2r}$$

To get the term independent of x, $\frac{10-r}{2} - 2r = 0$.

$10 - 5r = 0 \Rightarrow r = 2$.

$\therefore$ Term independent of x

$$= {}^{10}C_2 \frac{(-4)^2}{3^8}$$

$$= \frac{45 \times 16}{3^8} = \frac{80}{729}.$$

6. Find the numerically greatest terms the expansion of (3 + 2a)^{15} when a = 5/2.

A: $(3 + 2a)^{15} = 3^{15} \left(1 + \frac{2a}{3}\right)^{15}$

$$|x| = \left|\frac{2a}{3}\right| = \left|\frac{2}{3} . \frac{5}{2}\right| = \frac{5}{3}$$

Now $\frac{(n+1)|x|}{|x|+1} = \frac{(15+1) . 5/3}{8/3} = 10$

$\therefore |T_{10}|$ and $|T_{11}|$ are numerically greatest.

$|T_{10}| = {}^{15}C_9\, 3^6 \left(2 . \frac{5}{2}\right)^9 = {}^{15}C_9 . 3^6 . 5^9$

$|T_{11}| = {}^{15}C_{10}\, 3^5 \left(2 . \frac{5}{2}\right)^{10} = {}^{15}C_0 . 3^5 . 3^{10}$

and $|T_{10}| = |T_{11}|$.

7. Find the numerically greatest term in the expansion of $(3x+ 5y)^{12}$ when x = 1/2, y = 4/3.

A: $(3x + 5y)^{12} = (3x)^{12} \left(1+\frac{5y}{3x}\right)^{12}$

$| x | = \left|\frac{5}{3}.\frac{4}{3}.\frac{2}{1}\right| = \frac{40}{9}$

Now $\frac{(n+1)|x|}{|x|+1} = \frac{13x\frac{40}{9}}{\frac{49}{9}} = \frac{520}{9} = 10.4$

$\therefore$ Numerically greatest term

$= | T_{10+1} |$

$= | {}^{12}C_{10} (3.½)^{12-10} (5.4/3)^{10} |$

$= {}^{12}C_2 . (3/2)^2 (20/3)^{10}$

8. If the coefficients of $(2r + 4)^{th}$ and $(r - 2)^{nd}$ terms in the expansion of $(1 + x)^{18}$ are equal, find r

A: In $(1 + x)^{18}$, $T_{2r+4} = T_{(2r+3)+1}$

$= {}^{18}C_{2r+3}$

$T_{r-2} = T_{(r-3)+1}$

$= {}^{18}C_{r-3}$

But ${}^{18}C_{2r+3} = {}^{18}C_{r-3}$

$\Rightarrow$ r = s   n = r + s

$\Rightarrow$ 2r + 3 = r - 3   18 = 2r + 3 + r - 3

$\Rightarrow$ r = -6   18 = 3r

is not possible   r = 6

$\therefore$ r = 6.

9. If ${}^{22}C_r$ is the largest binomial coefficient in the expansion of $(1 + x)^{22}$ find the value of ${}^{13}C_r$.

A: $= {}^nC_{n/2}$ if n is even

$= {}^{22}C_{11}$

r = 11.

Now ${}^{13}C_r = {}^{13}C_{11}$

$= {}^{13}C_2$

$= \frac{13 \times 12}{2}$

= 78.

10. If $(1 + 3x - 2x^2)^{10} = a_0+ a_1 x + a_2x^2 + ... a_{20} x^{20}$, then prove that (i) $a_0 + a_1 + a_2 + ... + a_{20} = 2^{10}$

(ii) $a_0 - a_1 + a_2 - ..... + a_{20} = 4^{10}$

A: $(1 + 3x - 2x^2)^{10} = a_0 + a_1x + a_2x^2 + ... + a_{20} x^{20}$

Put x = 1 in the above relation,

$a_0 + a_1.1 + a_2 - 1^2 + ... fa_{20} .1^{20} = (1 + 3 - 2)^{10}$

$\Rightarrow a^0 + a_1 + a_2 + ... = a_{20} = 2^{10}$

Put x = -1 in the given relation,

$a_0 + a_1 (-1) + a_2 (-1)^2 + ... + a_{20} (-1)^{20} = (1 - 3 - 2)^{10}$

$a_0 - a_1 + a_2 - .... + a_{20} = 4^{10}$

11. Obtain the values of x for which the binomial expansion of $(2 + 3x)^{-2/3}$ is valid.

A: $(2 + 3x)^{-2/3} + 2^{-2/3} (1 + 3x/x)^{-2/3}$

The above expansion is valid if

$| 3x/x | < 1$

$\Rightarrow | x | < 2/3$

$\Rightarrow x \in \left(\frac{-2}{3},\frac{2}{3}\right)$.

12. Find the values of x for which the binomial expansion $(7 + 3x)^{-5}$ is valid.

A: $(7 + 3x)^{-5} = 7^{-5} \left(1+\frac{3x}{7}\right)^{-5}$

The avove expansion is valid if $\left|\frac{3x}{7}\right| < 1$

$\Rightarrow |x| < \frac{7}{3}$

$\Rightarrow x \in \left(\frac{-7}{3},\frac{7}{3}\right)$.

13. If C_r denote ${}^nC_r$, then prove that $a C_0 +(a+d) C_1 + (a+2d) C_2 + ...+ (a+nd) C_n = 2^{n-1}$.

A: Let

$S = a C_0 + (a+d) C_1 + (a+2d) C_2 + ...+ (a+nd)C_n$ ---(1)

$\because C_r = C_{n-r}$

$S = (a +nd) C_0 + [a + (n - 1)d] C_1 + [a + (n - 2)d] C_2 + ... + a C_n$ ------- (2)

$(1) + (2) \Rightarrow 2S = (2a + nd) C_0 + (2a + nd) C_1 + (2a + nd) C_2 + ... + (2a + nd) C_n$.

$\Rightarrow 2S = (2a + nd) [C_0 + C_1 + C_2 + ... + C_n]$

$\Rightarrow 2S = (2a + 2d)^{2n}$

$\therefore S = (2a + nd) 2^{n-1}$.

1. State and prove 'Binomial theorem' for a positive integral index.

A: If n is a positive integer, then

$(x+a)^n = {}^nC_o x^n + {}^nC_1 x^{n-1}.a + {}^nC_2 x^{n-2}.a^2 + ... + {}^nC_r x^{n-r}.a^r + ... + {}^nC_n.a^n.$

Let S(n) be the given statement.

If n = 1, LHS = $(x + a)^1 = x + a$

RHS = ${}^1C_0\, x^1 + {}^1C_1\, x^{1-1} . a = x + a$

$\therefore$ LHS = RHS.

Thus S(1) is true

Assume that S(k) is true.

$\therefore (x + a)^k = {}^kC_0 x^k + {}^kC_1 x^{k-1} . a + {}^kC_2 x^{k-2} . a^2 + ...+ + {}^kC_r x^{k-r} . a^r + ... + {}^kC_k a^k.$

Now $(x + a)^{k+1} = (x + a)(x + a)^k$

$= (x + a) [{}^kC_0 x^k + {}^kC_1 x^{k-1} . a + {}^kC_2 x^{k-2} . a^2 + ...+ + {}^kC_r x^{k-r} . a^r + ... + {}^kC_k a^k]$

$= {}^kC_0 x^{k+1} + {}^kC_1 x^k . a + {}^kC_2 x^{k-1} . a^2 + ...+ + {}^kC_r x^{k-r+1} . a^r + ... + {}^kC_k .x. a^k$

$+ {}^kC_0 x^k .a + {}^kC_1 x^{k-1} . a^2 + {}^kC_r x^{k-r} . a^{r+1} + ...+ {}^kC_k a^{k+1}$

$= x^{k+1} + ({}^kC_1 + {}^kC_0)x^k . a + ({}^kC_2 + {}^kC_1) x^{k-1} . a^2 + ... + ({}^kC_r + {}^kC_{r-1}) x^{k+1-r} . a^r + ... + a^{k+1}$

$\because {}^nC_r + {}^nC_{n-r} = {}^{n+1}C_r$

$= {}^{k+1}C_0 x^{k+1} + {}^{k+1}C_1 x^k . a + {}^{k+1}C_2 x^{k-1} . a^2 + ...+ + {}^{k+1}C_r . x^{k+1-r} . a^r + ... + {}^{k+1}C_{k+1} a^{r+1}$

$\therefore$ S(k + 1) is also true.

Hence, by the principle of mathematical induction S(n) is true for all n $\in$ N.

2. If 2nd, 3rd and 4th terms in the expansion of $(a + x)^n$ are respectively 240, 720, 1080, find a, x, m.

A: In $(a + x)^n$, $T_2 = T_{1+1}$

$= {}^nC_1 . a^{n-1} . x = 240$ --------- (1)

$T_3 = T_{2+1}$

$= {}^nC_2 . a^{n-2} . x^2 = 720$ ------- (2)

$T_4 = T_{3+1}$

$= {}^nC_3\, a^{n-3} . x^3 = 1080$ ------- (3)

$\frac{\boxed{2}}{\boxed{1}} \Rightarrow \frac{{}^nC_2\, a^{n-2}\, x^2}{{}^nC_1\, a^{n-1}\, x} = \frac{720}{240}$

$\Rightarrow \frac{n(n-1)}{2} \frac{a^n}{a^2} \frac{x^2\, a}{a^n x . n} = 3$

$\Rightarrow \frac{n(n-1)x}{2a} = 3$

$\Rightarrow (n - 1)x = 6a$ --------- (4)

$\frac{\boxed{3}}{\boxed{2}} \Rightarrow \frac{{}^nC_3\, a^{n-3}\, x^3}{{}^nC_2\, a^{n-2}\, x^2} = \frac{1080}{72}$

$\Rightarrow \frac{n(n-1)(n-2)}{6} \frac{a^n}{a^3} \frac{x^3 . 2 .a^2}{n(n-1)a^n x^2} = \frac{1080}{72}$

$\Rightarrow \frac{(n-2)x}{3a} = \frac{3}{2}$

$\Rightarrow (n - 2)(2x) = 9a$ ------- (5)

$\frac{\boxed{5}}{\boxed{4}} \Rightarrow \frac{(n-2)(2x)}{(n-1)x} = \frac{9a}{6a}$

$\Rightarrow 4n - 8 = 3n - 3$

$\Rightarrow \therefore n = 5$

From (4), $4x = 6a$

$x = \frac{3a}{2}$ -------- (6)

Now (1) becomes ${}^5C_1\, a^4\, x = 240$

$\Rightarrow 5a^4 . \frac{3a}{2} = 240$

$\Rightarrow a^5 = 32 = 2^5$

$\therefore a = 2$

From (6) $x = \frac{3a}{2} = \frac{3(2)}{2} = 3$

$\therefore$ n = 5, a = 2, x = 3.

3. If the coefficient of x^{10} in the expansion of $\left(ax^2 + \frac{1}{bx}\right)^{11}$ is equal to the coefficient of x^{-10} in the expansion of $\left(ax - \frac{1}{bx^2}\right)^{11}$, find the relation between and a and b, where a and b

are real numbers.

A: In $\left(ax^2 + \frac{1}{bx^2}\right)^{11}$,

$T_{r+1} = {}^{11}C_r . (ax^2)^{11-r} \left(\frac{1}{bx}\right)^r$

$= {}^{11}C_r . \frac{a^{11-r}}{b^r} . x^{22-3r}$

To get the coefficient of x^{10},

$22 - 3r = 10$

$12 = 3r$

$r = 4$

$\therefore$ Coefficient of x^{10} in $\left(ax^2 + \frac{1}{bx^2}\right)^{11} = {}^{11}C_4 . \frac{a^7}{b^4}$

In $\left(ax^2 - \frac{1}{bx^2}\right)^{11}$, $T_{r+1} = {}^{11}C_r . (ax)^{11-r} \left(\frac{-1}{bx^2}\right)^r$

$= {}^{11}C_r .(-1)^r . \frac{a^{11-r}}{b^r} . x^{11-3r}$

To get the coefficient of x^{-10},

$11 - 3r = -10$

$21 = 3r$

$r = 7.$

Thus, coefficient of x^{-10} in $\left(ax - \frac{1}{bx^2}\right)^{11}$ is

$= (-1)^7 \, {}^{11}C_7 . \frac{a^4}{b^7}$

But ${}^{11}C_4 . \frac{a^7}{b^4} = -{}^{11}C_7 . \frac{a^4}{b^7}$ $\quad \because {}^nC_r = {}^nC_{n-r}$

$\Rightarrow a^3 = \frac{-1}{b^3}$

$\Rightarrow (ab^3) = -1$

$\therefore ab = -1.$ $\quad (\because a, b \in R)$

4. If $(7 + 4\sqrt{3})^n = I + f$ where I and n are positive integers and $0 < f < 1$, then show that
i) I is an odd positive integer
ii) $(I + f)(1 - f) = 1$.

A: i) Given that I, n are positive integers.

$0 < f < 1$ and $(7 + 4\sqrt{3})^n = I + f$.

Clearly $0 < 7 - 4\sqrt{3} < 1$

$\Rightarrow 0 < (7 - 4\sqrt{3})^n < 1$

Let $(7 - 4\sqrt{3})^n = x$

$\therefore \quad 0 < x < 1$

$\underline{0 < f < 1}$ given

on addition $0 < f + x < 2$ -------- (1)

Now $(I + f) + x$

$= (7 + 4\sqrt{3})^n + (7 - 4\sqrt{3})^n$

$= {}^nC_0 . 7^n + {}^nC_1 7^{n-1} (4\sqrt{3}) + {}^nC_2 7^{n-2} . (4\sqrt{3})^2 + ...$
$+ {}^nC_n (4\sqrt{3})^n + {}^nC_0 7^n - {}^nC_1 . 7^{n-1} (4\sqrt{3})$
$+ {}^nC_2 7^{n-2} (4\sqrt{3})^2 - ..... + {}^nC_2 (-4\sqrt{3})^n.$

$= 2[{}^nC_0 . 7^n + {}^nC_2 . 7^{n-2} (4\sqrt{3})^2 + ...]$

$= 2$ (some integer)

$\therefore$ I + f + x = an even integer

$\Rightarrow$ f + x = an even integer - (I)

$\Rightarrow$ f + x = some integer ------(2)

Combining (1), (2) the only possibility left is f + x = 1.

Now I + f + x = an even integer

I + 1 = an even integer

$\Rightarrow$ I = an even integer - 1

$\Rightarrow$ I is an odd integer

ii) $(I + f)(1 - f) = (I + f)x$

$= (7 + 4\sqrt{3})^n + (7 - 4\sqrt{3})^n$

$= (49 - 48)^n$

$= 1^n$

$= 1.$

5. If R, n are positive integers, n is odd, $0 < F < 1$ and if $(5\sqrt{5} + 11)^n = R + F$, then prove that
i) R is an even integer
ii) $(R + F)F = 4^n$.

A: i) Given that R, n are positive integers, n is odd, $0 < F < 1$ and $R + F = (5\sqrt{5} + 11)^n$.

Clearly $0 < 5\sqrt{5} - 11 < 1$

$\Rightarrow 0 < (5\sqrt{5} - 11)^n < 1$

Let $(5\sqrt{5} - 11)^n = x$

$\therefore \quad \Rightarrow 0 < x < 1$

$\Rightarrow 0 > -x > -1$

$\Rightarrow -1 < -x < 0$

Also $\underline{0 < F < 1}$

on addition $-1 < F - x < 1$ -------- (1)

Now $(R + F) - x = (5\sqrt{5} + 11)^n - (5\sqrt{5} - 11)^n$

$= {}^nC_0 . (5\sqrt{5})^n + {}^nC_1 (5\sqrt{5})^{n-1} (11) + {}^nC_2 (5\sqrt{5})^{n-2} (11)^2 + ... + {}^nC_n (-11)^n$

$= -\{{}^nC_0 . (5\sqrt{5})^n - {}^nC_1 (5\sqrt{5})^{n-1} (11) + {}^nC_2 (5\sqrt{5})^{n-2} (11)^2 - .... + {}^nC_n (-11)^n\}$

$= 2\{{}^nC_1 (5\sqrt{5})^{n-1} (11) + {}^nC_3 (5\sqrt{5})^{n-3} (11^3) + ...\}$

= 2 (some integer)

$\therefore$ R + F - x = an even integer

$\Rightarrow$ F - x = an even integer - (R)

$\Rightarrow$ F - x = some integer ------(2)

Combing (1), (2) the only possibility left is F - x = 0.

F - x = 0

$\Rightarrow$ F = x

Now R + F - x = an even integer

$\Rightarrow$ R + 0 = an even integer

So R is an even integer

ii) $(R + F)F = (R + F)x$

$= (5\sqrt{5} + 11)^n + (5\sqrt{5} - 11)^n$

$= (125 - 121)^n$

$= 4^n$.

6. If P and Q are sum of odd terms and sum of even terms respectively in the expansion of $(x + a)^n$, then prove that

i) $P^2 - Q^2 = (x^2 - a^2)^n$.

ii) $4PQ = (x + a)^{2n} - (x - a)^{2n}$.

A: Now

$(x+a)^n = {}^nC_o x^n + {}^nC_1 x^{n-1}.a + {}^nC_2 x^{n-2}.a^2 + ... + {}^nC_3 x^{n-3}.a^3 + ...$

Given that $P = T_1 + T_3 + ...$

$= {}^nC_0\, x^n + {}^nC_2\, x^{n-2} . a^2 + ...$

$Q = T_2 + T_4 + ...$

$= {}^nC_1 x^{n-1}.a + {}^nC_3 x^{n-3}.a^3 + ...$

Now $P + Q = (x + a)^n$ and $P - Q = (x - a)^n$.

i) $P^2 - Q^2 = (P + Q)(P - Q)$

$= (x + a)^n (x - a)^n$

$= [(x + a)(x - a)]^n$

$= (x^2 - a^2)^n$.

ii) $4PQ = (P + Q)^2 - (P - Q)^2$

$= \{(x + a)^n\}^2 - \{(x - a)^n\}^2$

$= (x + a)^{2n} - (x - a)^{2n}$.

7. With usual notation, prove that
$C_0 + C_2 + C_4 + ... = C_1 + C_3 + C_5 + ... = 2^{n-1}$.

A: We know that

$(1 + x)^n = {}^nC_o + {}^nC_1 x + {}^nC_2 x^2 + {}^nC_3 x^3 + ... + {}^nC_n . x^n$.

$\Rightarrow (1 + x)^n = C_o + C_1 x + C_2 x^2 + C_3 x^3 + ... + C_n x^n$ ---- (1)

Put x = 1 in (1), we get

$C_0 + C_1(1) + C_2(1^2) + C_3 (1^3) + ... + C_n (1^n) = (1 + 1)^n$

$\therefore C_0 + C_1 + C_2 + ... + C_n = 2^n$ ----- (2)

Put x = -1 in (1), we get

$C_0 + C_1(-1) + C_2(-1)^2 + C_3(-1)^3 + ... + C_n (-1)^n = (1 - 1)^n$

$\therefore C_0 - C_1 + C_2 - C_3 + ... + (-1)^n C_n = 0$ ----- (3)

$(2) + (3) \Rightarrow 2[C_0 + C_2 + C_4 + ...] = 2^n + 0 = 2^n$

$\Rightarrow C_0 + C_2 + C_4 + ... = \frac{2^n}{2} = 2^{n-1}$ ------- (4)

$(2) - (3) \Rightarrow 2[C_1 + C_3 + C_5 + ...] = 2^n - 0 = 2^n$

$\Rightarrow C_1 + C_3 + C_5 + ... = \frac{2^n}{2} = 2^{n-1}$ ------- (5)

Combing (4) & (5), we get

$C_0 + C_2 + C_4 + ... = C_1 + C_3 + C_5 + ... = 2^{n-1}$.

8. With usual notation, prove that

$C_1 + 2C_2 x + 3 C_3 x^2 + ... + n C_n x^{n-1} = n(1 + x)^{n-1}$

Deduce that $C_1 + 2C_2 + 3C_3 + ... + n C_n = n . 2^{n-1}$.

A: We know that

$(1 + x)^n = C_o + C_1 x + C_2 x^2 + C_3 x^3 + ... + C_n x^n$.

Differentiating w.r.t. x,

$\frac{d}{dx} [C_o + C_1 x + C_2 x^2 + ... + C_n x^n] = \frac{d}{dx} (1 + x)^n$.

$\Rightarrow \frac{d}{dx} (C_0) + C_1 \frac{d}{dx} (x) + C_2 \frac{d}{dx} (x^2) + ... + C_n . nx^{n-1}$

$= \frac{d}{dx}(1+x)^n$

$\Rightarrow 0 + C_1 . 1 + C_2 . 2x + C_3 . 3x^2 + ...+ C_n . nx^{n-1}$

$= n(1+x)^{n-1}$

$\Rightarrow C_1 + 2C_2x + 3C_3 x^2 + ... + {}^nC_n x^{n-1}$

$= n(1+x)^{n-1}$ ---------(1)

Put x = 1 in (1), we get

$C_1 + 2C_2 (1) + 3C_3 (1^2) + ... + n C_n (1^{n-1}) = n(1+1)^{n-1}$

$\Rightarrow C_1 + 2C_2 + 3C_3 + ... n C_n = n(2^{n-1})$.

9. Prove that $C_0 + \frac{C_1}{2}x + \frac{C_2}{3}x^2 + ...+ \frac{C_n}{n+1}x^n = \frac{(1+x)^{n+1}-1}{(n+1)x}$

Deduce that $\frac{C_1}{2} + \frac{C_3}{4} + \frac{C_5}{6} + ... = \frac{2^n-1}{n+1}$.

A: Now $C_0 + \frac{C_1}{2}x + \frac{C_2}{3}x^2 + ... + \frac{C_n}{n+1}x^n$

$= {}^nC_0 + \frac{{}^nC_1}{2}x + \frac{{}^nC_2}{3}x^2 + ... + \frac{{}^nC_n}{n+1}x^n$

$= 1 + \frac{n}{1.2}x + \frac{n(n-1)}{1.2.3}x^2 + ... + \frac{1}{n+1}x^n$

$= \frac{1}{(n+1)x}\left[(n+1)x + \frac{(n+1)n}{2!}x^2 + \frac{(n+1)n(n-1)}{3!}x^3 + ...+ \frac{n+1}{n+1}x^{n+1}\right]$

$= \frac{1}{(n+1)x}[{}^{n+1}C_1 x + {}^{n+1}C_2 x^2 + {}^{n+1}C_3 x^3 + ... + 1.x^{n+1}]$

$= \frac{1}{(n+1)x}[{}^{n+1}C_0 + {}^{n+1}C_1 x + {}^{n+1}C_2 x^2 + ...+ {}^{n+1}C_{n+1} x^{n+1} - 1]$

$= \frac{(1+x)^{n+1}-1}{(n+1)x}$

$\therefore C_0 + \frac{C_1}{2}x + \frac{C_2}{3}x^2 + ...+ \frac{C_n}{n+1}x^n = \frac{(1+x)^{n+1}-1}{(n+1)x}$ ----(1)

Put x = 1 in (1), we get

$C_0 + \frac{C_1}{2}(1) + \frac{C_2}{3}(1^2) + ... + \frac{C_n}{n+1}(1^n) = \frac{(1+1)^{n+1}-1}{(n+1)(1)}$

$\Rightarrow C_0 + \frac{C_1}{2} + \frac{C_2}{3} + ... + \frac{C_n}{n+1} = \frac{2^{n+1}-1}{n+1}$ ------ (2)

Put x = -1 in (1), we get

$C_0 + \frac{C_1}{2}(-1) + \frac{C_2}{3}(-1^2) + ... + \frac{C_n}{n+1}(-1)^n$

$= \frac{(1-1)^{n+1}-1}{(n+1)(-1)}$

$C_0 - \frac{C_1}{2} + \frac{C_2}{3} - .... + (-1)^n \frac{C_n}{n+1} = \frac{1}{n+1}$ ------ (3)

$(2) - (3) \Rightarrow 2\left[\frac{C_1}{2} + \frac{C_3}{4} + \frac{C_5}{6} + ...\right] = \frac{2^{n+1}-1-1}{n+1}$

$\Rightarrow 2\left[\frac{C_1}{2} + \frac{C_3}{4} + \frac{C_5}{6} + ...\right] = \frac{2(2^n-1)}{n+1}$

$\therefore \frac{C_1}{2} + \frac{C_3}{4} + \frac{C_5}{6} + ... = \frac{2^n-1}{n+1}$.

10. Prove that
$C_0 C_r + C_1 C_{r+1} + C_2 C_{r+2} + ...+ C_{n-r} C_n = {}^{2n}C_{n+r}$.
Deduce $C_0^2 + C_1^2 + C_2^2 + ... + C_n^2 = {}^{2n}C_n$.

A: We know that

$(1+x)^n = C_o + C_1x^2 + ...+ C_{n-r}x^{n-r} + ...+ C_n x^n$ -------- (1)

Also $(x+1)^n = C_o x^n + C_1 x^{n-1} + ...+ C_r x^{n-r} + C_{r+1} x^{n-(r+1)}$
$+ C_{r+2} x^{n-(r+2)} + ...+ C_n$ -------- (2)

Multiplying (1) & (2) and equating the coefficient of x^{n-r} on bothsides,

$C_0 + C_r + C_1 C_{r+1} + C_2 C_{r+2} + ... + C_{n-r} C_n$

= coe. of x^{n-r} in $(1+x)^n (x+1)^n$

= coe. of x^{n-r} in $(1+x)^n (1+x)^n$

= coe. of x^{n-r} in $(1+x)^{2n}$

In $(1+x)2n$
$T_{r+1} = {}^{2n}C_r . x^r$

$= {}^{2n}C_{n-r}$

$= {}^{2n}C_{2n-(n-r)}$

$= {}^{2n}C_{n+r}$

Put r = 0 in the above relation, we get

$C_0 + C_0 + C_1 C_{0+1} + C_2 C_{0+2} + ... + C_{n-0} C_n = {}^{2n}C_{n+0}$

$\therefore C_0^2 + C_1^2 + C_2^2 + ... + C_n^2 = {}^{2n}C_n$.

11. Prove that $(C_0 + C_1)(C_1 + C_2)(C_2 + C_3) .... (C_{n-1} + C_n) = \frac{(n+1)^n}{n!} C_0 C_1 C_2 .... C_n$.

A: Now $(C_0 + C_1)(C_1 + C_2)(C_2 + C_3) .... (C_{n-1} + C_n)$

$= C_0\left(1+\frac{C_1}{C_0}\right)C_1\left(1+\frac{C_2}{C_1}\right)C_2\left(1+\frac{C_3}{C_2}\right).....C_{n-1}\left(1+\frac{C_n}{C_{n-1}}\right)$

$= C_0 C_1 C_2 .... C_{n-1}\left(1+\frac{n}{1}\right)\left(1+\frac{n(n-1)}{2}\frac{1}{n}\right)\left(1+\frac{n(n-1)(n-2)}{6}\frac{2}{n(n-1)}\right) .... \left(1+\frac{1}{n}\right)$

$= C_0 C_1 C_2 .... C_{n-1} . 1\left(\frac{1+n}{1}\right)\left(1+\frac{n-1}{2}\right)\left(1+\frac{n-2}{3}\right) .... \left(1+\frac{1}{n}\right)$

$= C_0 C_1 C_2 ...... C_{n-1} C_n \left(\frac{1+n}{1}\right)\left(\frac{1+n}{2}\right)\left(\frac{1+n}{3}\right)..... \left(\frac{1+n}{n}\right)$

$= C_0 C_1 C_2 ...... C_{n-1} C_n \frac{(1+n)^n}{n!}$.

12. If the coefficeints of r^{th}, $(r+1)^{th}$, $(r+2)^{nd}$ terms in the expansion of $(1+x)^n$ are in A.P., then show that $n^2 - (4r+1)n + 4r^2 - 2 = 0$.

A: In $(1+x)^n$,

$T_r = T_{(r-1)+1} = {}^nC_{r-1} x^{r-1}$

$T_{r+1} = {}^nC_r x^r$

$T_{r+2} = T_{(r+1)+1} = {}^nC_{r+1} x^{r+1}$

It is given that ${}^nC_{r-1}$, ${}^nC_r$, ${}^nC_{r+1}$ are in A.P.

$\Rightarrow 2 . {}^nC_r = {}^nC_{r-1} + {}^nC_{r+1}$

$\Rightarrow 2 . \frac{n!}{(n-r)!\,r!} = \frac{n!}{[n-(r-1)]!\,(r-1)!} + \frac{n!}{[n-(r+1)]!\,(r+1)!}$

$\Rightarrow \frac{2}{(n-r)[n-(r+1)]!\,r(r-1)!} = \frac{2}{[n-(r-1)](n-r)[n-(r+1)]!\,(r-1)!} + \frac{1}{[n-(r+1)]!\,(r+1)\,r(r-1)!}$

$\Rightarrow \frac{2}{(n-r)r} = \frac{1}{(n-r+1)(n-r)} + \frac{1}{(r+1)r}$

$\Rightarrow \frac{2}{(n-r)r} = \frac{(r+1)r + (n-r+1)(n-r)}{(n-r+1)(n-r)(r+1)r}$

$\Rightarrow 2(n-r+1)(r+1) = (r+1)r + (n-r+1)(n-r)$

$\Rightarrow 2(nr + n - r^2 - \not{r} + \not{r} + 1) = r^2 + \not{r} + n^2 - 2nr + r^2 + n - \not{r}$

$\Rightarrow 0 = n^2 - 4nr - n + 4r^2 - 2$

$\Rightarrow n^2 - (4r+1)n + (4r^2 - 2) = 0$.

13. If the coefficients of 4 consecutive terms in the expansion of $(1+x)^n$ are a_1, a_2, a_3, a_4 respectively, then show that $\frac{a_1}{a_1+a_2} + \frac{a_3}{a_3+a_4} = \frac{2a_2}{a_2+a_3}$.

A: In $(1+x)^n$,

$T_r = T_{(r-1)+1}$

$= {}^nC_{r-1} . x^{r-1}$

$T_{r+1} = {}^nC_r . x^r$

$T_{r+2} = T_{(r+1)+1}$

$= {}^nC_{r+1} . x^{r+1}$

$T_{r+3} = T_{(r+2)+1}$

$= {}^nC_{r+2} . x^{r+2}$

Given that $a_1 = {}^nC_{r-1}$, $a_2 = {}^nC_r$,

$a_3 = {}^nC_{r+1}$, $a_4 = {}^nC_{r+2}$

$LHS = \frac{a_1}{a_1+a_2} + \frac{a_3}{a_3+a_4}$

$$= \frac{^nC_{r-1}}{^nC_{r-1} + {}^nC_r} + \frac{^nC_{r+1}}{^nC_{r+1} + {}^nC_{r+2}}$$

$$\because {}^nC_{r-1} + {}^nC_r = {}^{n+1}C_r$$

$$= \frac{^nC_{r+1}}{^{n+1}C_r} + \frac{^nC_{r+1}}{^{n+1}C_{r+2}}$$

$$= \frac{n!}{[n-(r-1)]!(r-1)!} \cdot \frac{(n+1-r)r!}{(n+1)!} + \frac{n!}{[n-(r+1)]!(r+1)!} \cdot \frac{[n+1-(r+2)]!(r+2)!}{(n+1)!}$$

$$= \frac{r}{n+1} + \frac{r+2}{n+1}$$

$$= \frac{2(r+1)}{n+1} \text{ ------ (1)}$$

$$\text{RHS} = \frac{2a_2}{a_2 + a_3}$$

$$= \frac{2.\,^nC_r}{^nC_r + {}^nC_{r+1}}$$

$$= \frac{2.\,^nC_r}{^{n+1}C_{r+1}}$$

$$= \frac{2.n!}{(n-r)!r!} \times \frac{[n+1-(r+1)]!(r+1)!}{(n+1)!}$$

$$= \frac{2(r+1)}{n+1} \text{ ------- (2)}$$

From (1) & (2),

$$\frac{a_1}{a_1 + a_2} + \frac{a_3}{a_3 + a_4} = \frac{2a_2}{a_2 + a_3}.$$

14. If n is a positive integer, prove that

$$\sum_{r=1}^{n} r^3 \left(\frac{^nC_r}{^nC_{r-1}}\right)^2 = \frac{n(n+1)^2(n+2)}{12}.$$

A: Now $\sum_{r=1}^{n} r^3 \left(\frac{^nC_r}{^nC_{r-1}}\right)^2$

$$= \sum_{r=1}^{n} r^3 \left[\frac{n!}{(n-r)!\,r!} \times \frac{[n-(r-1)]!(r-1)!}{n!}\right]^2$$

$$= \sum_{r=1}^{n} r^3 \left(\frac{n-r+1}{r}\right)^2$$

$$= \sum_{r=1}^{n} r[(n+1)-r]^2$$

$$= \sum_{r=1}^{n} r[(n+1)^2 - 2(n+1)r + r^2]$$

$$= \sum_{r=1}^{n} [(n+1)^2 r - 2(n+1)r^2 + r^3]$$

$$= (n+1)^2 \sum_{r=1}^{n} r - 2(n+1)\sum_{r=1}^{n} r^2 + \sum_{r=1}^{n} r^3$$

$$= (n+1)^2 \frac{n(n+1)}{2} - 2(n+1)\frac{n(n+1)(2n+1)}{6} + \frac{n^2(n+1)^2}{4}$$

$$= \frac{n(n+1)^2}{12}[6n+1) - 4(2n+1) + 3n]$$

$$= \frac{n(n+1)^2}{12}[6n + 6 - 8n - 4 + 3n]$$

$$= \frac{n(n+1)^2}{12}[n+2]$$

$$= \frac{n(n+1)^2(n+2)}{12}.$$

15. Show that for any non - zero rational number

$$x,\ 1 + \frac{x}{2} + \frac{x(x-1)}{2.4} + \frac{x(x-1)(x-2)}{2.4.6} + ...\infty$$

$$= 1 + \frac{x}{3} + \frac{x(x+1)}{3.6} + \frac{x(x+1)(x+2)}{3.6.9} + ...\infty.$$

A: LHS $= 1 + \frac{x}{2} + \frac{x(x-1)}{2.4} + \frac{x(x-1)(x-2)}{2.4.6} + ...\infty$

$$= 1+\frac{x}{1!}\left(\frac{1}{2}\right)+\frac{x(x-1)}{2!}\left(\frac{1}{2}\right)^2+\frac{x(x-1)(x-2)}{3!}\left(\frac{1}{2}\right)^3+........\infty$$

$$\because (1-x)^{-n} = 1+\frac{n}{1!}x+\frac{n(n+1)}{2!}x^2+.....\infty$$

$$= \left(1-\frac{1}{3}\right)^{-x}$$

$$= \left(\frac{2}{3}\right)^{-x}$$

$$= \left(\frac{3}{2}\right)^{x} \quad \text{----- (2)}$$

From (1) & (2),

$$1+\frac{x}{2}+\frac{x(x-1)}{2.4}+\frac{x(x-1)(x-2)}{2.4.6}+.......\infty$$

$$= 1+\frac{x}{3}+\frac{x(x+1)}{3.6}+\frac{x(x+1)(x+2)}{3.6.9}+.......\infty.$$

16.Find the sum to infinite series

$$\frac{7}{5}\left[1+\frac{1}{10^2}+\frac{1.3}{1.2}+\left(\frac{1}{10^4}\right)+\frac{1.3.5}{1.2.3}+\left(\frac{1}{10^6}\right)+.......\infty\right].$$

A: Comparing the infinite series with

$$(1-x)^{-p/q} = 1+\frac{p}{1!}\left(\frac{x}{q}\right)+\frac{p(p+q)}{2!}\left(\frac{x}{q}\right)^2+........\infty$$

$$\frac{7}{5}\left[1+\frac{1}{1!}+\left(\frac{1}{100}\right)+\frac{1.3}{2!}+\left(\frac{1}{100}\right)^2+\frac{1.3.5}{3!}+\left(\frac{1}{100}\right)^3+.......\infty\right]$$

Here p = 1, p + q = 3

q = 3 - 1 = 2

$$\frac{x}{q} = \frac{1}{100}$$

$$\Rightarrow x = \frac{2}{100} = \frac{1}{50}$$

$$= \frac{7}{5}[1-x]^{-p/q}$$

$$= \frac{7}{5}\left(1-\frac{1}{50}\right)^{-1/2}$$

$$= \frac{7}{5}\left(\frac{49}{50}\right)^{-1/2}$$

$$= \frac{7}{5}\sqrt{\frac{50}{49}}$$

$$= \frac{7}{5}\left(\frac{5\sqrt{2}}{7}\right)$$

$$= \sqrt{2}.$$

17.Find the sum of infinite series

$$\frac{3}{4}+\frac{3.5}{4.8}+\frac{3.5.7}{4.8.12}+.......\infty.$$

A: Now $\frac{3}{4}+\frac{3.5}{4.8}+\frac{3.5.7}{4.8.12}+.......\infty$

$$= 1+\frac{3}{1!}\left(\frac{1}{4}\right)+\frac{3.5}{2!}\left(\frac{1}{4}\right)^2+\frac{3.5.7}{3!}\left(\frac{1}{4}\right)^3+.......\infty$$

Comparing the infinite series with

$$(1-x)^{-p/q} = 1+\frac{p}{1!}\left(\frac{x}{q}\right)+\frac{p(p+q)}{2!}\left(\frac{x}{q}\right)^2+........\infty$$

Here p = 3

$p + q = 5 \Rightarrow q = 2$

$$\frac{x}{q} = \frac{1}{4} \Rightarrow x = \frac{2}{4} = \frac{1}{2}$$

$\therefore$ Sum of the given infinite series

$= (1-x)^{-p/q} - 1$

$$= \left(1-\frac{1}{2}\right)^{-3/2} - 1$$

$$= \left(\frac{1}{2}\right)^{-3/2}$$

$= 2^{3/2} - 1$

$= 2\sqrt{2} - 1.$

18.If $x = \frac{1}{5} + \frac{1.3}{5.10} + \frac{1.3.5}{5.10.15} + ...\infty$, then find the value of $3x^2 + 6x$.

A: Given $x = \frac{1}{5} + \frac{1.3}{5.10} + \frac{1.3.5}{5.10.15} + ...\infty$

$$\Rightarrow x = \frac{1}{1!}\left(\frac{1}{5}\right) + \frac{1.3}{2!}\left(\frac{1}{5}\right)^2 + \frac{1.3.5}{3!}\left(\frac{1}{5}\right)^3 + ...\infty$$

$$\Rightarrow x + 1 = 1 + \frac{1}{1!}\left(\frac{1}{5}\right) + \frac{1.3}{2!}\left(\frac{1}{5}\right)^2 + ...\infty$$

Comparing this with

$$(1 - y)^{-p/q} = 1 + \frac{p}{1!}\left(\frac{y}{q}\right) + \frac{p(p+q)}{2!}\left(\frac{y}{q}\right)^2 + ...\infty$$

Here p = s

$p + q = 3 \Rightarrow q = 2$

$\frac{y}{q} = \frac{1}{5} \Rightarrow y = \frac{2}{5}$

$\therefore x + 1 = (1 - y)^{-p/1}$

$= \left(1 - \frac{2}{5}\right)^{-1/2}$

$= \left(\frac{3}{5}\right)^{-1/2}$

$x + 1 = \sqrt{\frac{5}{3}}$

Squaring on both sides,

$x^2 + 2x + 1 = \frac{5}{3}$

$\Rightarrow 3x^2 + 6x + 3 = 5$

$\Rightarrow 3x^2 + 6x = 5 - 3 = 2.$

19.If $x = \frac{1.3}{3.6} + \frac{1.3.5}{3.6.9} + \frac{1.3.5.7}{3.6.9.12} + ...\infty$, then prove that $9x^2 + 24x = 11$.

A: Given $x = \frac{1.3}{3.6} + \frac{1.3.5}{3.6.9} + \frac{1.3.5.7}{3.6.9.12} + ...\infty$

$$\Rightarrow x = \frac{1.3}{2!}\left(\frac{1}{3}\right)^2 + \frac{1.3.5}{3!}\left(\frac{1}{3}\right)^3 + ...\infty$$

$$\Rightarrow x + 1 + \frac{1}{1!}\left(\frac{1}{3}\right) = 1 + \frac{1}{1!}\left(\frac{1}{3}\right) + \frac{1.3}{2!}\left(\frac{1}{3}\right)^2 + \frac{1.3.5}{3!}\left(\frac{1}{3}\right)^3 + ...\infty$$

Comparing RHS with

$$(1 - y)^{-p/q} = 1 + \frac{p}{1!}\left(\frac{y}{q}\right) + \frac{p(p+q)}{2!}\left(\frac{y}{q}\right)^2 + ...\infty$$

Here p = 1

$p + q = 3 \Rightarrow q = 2$

$\frac{y}{q} = \frac{1}{3} \Rightarrow y = \frac{2}{3}$

$\therefore x + \frac{4}{3} = (1 - y)^{-p/q}$

$= \left(1 - \frac{2}{3}\right)^{-1/2}$

$= \left(\frac{1}{3}\right)^{-1/2}$

$\frac{3x + 4}{3} = \sqrt{3}$

$\Rightarrow 3x + 4 = 3\sqrt{3}$

Squaring on both sides,

$9x^2 + 24x + 16 = 27$

$\therefore 9x^2 + 24x = 11.$

20.Find the sum of the infinite series $\frac{3.5}{5.10} + \frac{3.5.7}{5.10.15} + \frac{3.5.7.9}{5.10.15.20} + ...\infty$.

A: Now $\frac{3.5}{5.10} + \frac{3.5.7}{5.10.15} + \frac{3.5.7.9}{5.10.15.20} + ...\infty$

$$= \frac{3.5}{2!}\left(\frac{1}{5}\right)^2 + \frac{3.5.7}{3!}\left(\frac{1}{5}\right)^2 + ...\infty$$

$$= 1+\frac{3}{1!}\left(\frac{1}{5}\right)+\frac{3.5}{2!}+\left(\frac{1}{5}\right)^2 + ...\infty - \left(1+\frac{3}{5}\right)$$

Comparing this with

$$(1-x)^{-p/q} = 1+\frac{p}{1!}\left(\frac{x}{q}\right)+\frac{p(p+q)}{2!}\left(\frac{x}{q}\right)^2 + ...\infty$$

Here p = 3

$p + q = 5 \Rightarrow q = 2$

$$\frac{x}{q} = \frac{1}{5} \Rightarrow x = \frac{2}{5}$$

Sum of the given infinite series

$$= (1-x)^{-p/q} - \frac{1}{5}$$

$$= \left(1-\frac{2}{5}\right)^{-3/2} - \frac{8}{5}$$

$$= \left(\frac{3}{5}\right)^{-3/2} - \frac{8}{5}$$

$$= \frac{5\sqrt{5}}{3\sqrt{3}} - \frac{8}{5}.$$

LEVEL - II (VSAQ)

1. Find the number of terms with non-zero coefficients in $(4x - 7y)^{49} + (4x - 7y)^{49}$.

A: The number of terms in the expansion $(x + y)^n + (x - y)^n$ when 'n'is odd is

$$\frac{n+1}{2} = \frac{49+1}{2} = 25.$$

2. Write down and simplify 6^{th} term in $\left(\frac{2x}{3}+\frac{3y}{2}\right)^9$.

A: 6^{th} term = $T_6 = T_{5+1.}$

$$= {}^9C_5\left(\frac{2x}{3}\right)^{9-5}\left(\frac{3y}{2}\right)^5 = {}^9C_5\left(\frac{2x}{3}\right)^4\left(\frac{3y}{2}\right)^5$$

$$= 126.\left[\frac{3}{2}\right]x^4y^5 = 189\, x^4y^5.$$

3. If A and B are coefficients of x^n in the expansion of $(1 + x)^{2n}$ and $(1 + x)^{2n-1}$ respectively, then find the value of $\frac{A}{B}$.

A: A = coefficient of x^n in $(1 + x)^{2n} = {}^{2n}C_n$.

B = Coefficient of x^n in $(1 + x)^{2n-1} = {}^{2n-1}C_n$.

$$\frac{A}{B} = \frac{{}^{2n}C_n}{{}^{2n-1}C_n} = \frac{(2n)!}{(2n-n)!n!} x \frac{(2n-1-n)!n!}{(2n-1)!}$$

$$= \frac{(2n)!}{(n!)^2} x \frac{(n-1)!n!}{(2n-1)!} = \frac{(2n)(2n-1)!}{n!n(n-1)!} x \frac{(n-1)n!}{(2n-1)!}$$

$$= \frac{2n}{n} = 2.$$

4. Find the largest binomial coefficient(s) in the expansion of $(1 + x)^{24}$.

A: Here n = 24, an even integer.

Hence there is only one largest binomial coefficient, that is ${}^nC_{\frac{n}{2}} = {}^{24}C_{12}$.

5. Find the largest binomial coefficient(s) in the expansion of $(1 + x)^{19}$.

A: Here n = 19 (odd).

$\therefore$ The largest binomial coefficients are

$${}^nC_{\frac{n-1}{2}}, {}^nC_{\frac{n+1}{2}} = {}^{19}C_9, {}^{19}C_{10}.$$

$$\left[\text{Note that } {}^{19}C_9 = {}^{19}C_{10}\right].$$

6. Find the middle terms in the expansion of $\left(4a+\frac{3}{2}b\right)^{11}$.

A: Given expansion is $\left(4a+\frac{3}{2}b\right)^{11}$.

Here $\boxed{n=1}$, odd

So, middle terms are $T_{\frac{11+1}{2}}, T_{\frac{11+3}{2}} = T_6, T_7$.

$$T_6 = T_{5+1} = {}^{11}C_5(4a)^{11-5}\left(\frac{3}{5}b\right)^5$$

$$= {}^{11}C_5 4^6.a^6.\left(\frac{3}{2}\right)^5.b^2 = {}^{11}C_5.4^6.\left(\frac{3}{2}\right)^5 a^6 b^6$$

$T_7 = T_{6+1} = {}^{11}C_6 (4a)^{11-6}\left(\frac{3}{5}b\right)^6$

$= {}^{11}C_5 4^5.a^5.\left(\frac{3}{2}\right)^6 . b^6 = {}^{11}C_6.4^5.\left(\frac{3}{2}\right)^6 a^5 b^6$.

7. Find the middle term in the expansion of $\left(\frac{3x}{7} - 2y\right)^{10}$.

A: Here $\boxed{n = 10}$, even

So, middle term $= T_{\frac{10}{2}+1} = T_{5+1}$.

$\therefore T_{5+1} = {}^{10}C_5.\left(\frac{3x}{7}\right)^{10-5}.\ (-2y)^5$

$= -{}^{10}C_5.\left(\frac{3x}{7}\right)^5.(2y)^5 == -{}^{10}C_5.\left(\frac{6}{7}\right)^5 x^5y^5$.

8. Find the coefficient of x^7 in $\left[\frac{3x^2}{7} + \frac{4}{5x^3}\right]^{11}$.

A: Given expansion is $\left[\frac{3x^2}{7} + \frac{4}{5x^3}\right]^{11}$.

General term $T_{r+1} = {}^nC_r.x^{n-4}.a^r$.

$= {}^{11}C_r.\left(\frac{3x^2}{7}\right)^{11-r}.\left(\frac{4}{5x^3}\right)^r$.

$= {}^{11}C_r.\left(\frac{3}{7}\right)^{11-r}.x^{22-2r}\left(\frac{4}{5}\right)^r.x^{-3r}$

$= {}^{11}C_r.\left(\frac{3}{7}\right)^{11-r}.\left(\frac{4}{5}\right)^r.x^{22-5r}$

take 22 - 5r = 7 $\Rightarrow$ 5r = 15 $\Rightarrow \boxed{r = 3}$

Coefficient of x^7

is ${}^{11}C_3.\left(\frac{3}{7}\right)^{11-3}.\left(\frac{4}{5}\right)^3 = {}^{11}C_3.\left(\frac{3}{7}\right)^8.\left(\frac{4}{5}\right)^3$.

9. Find the term independent of x in the expansion of $\left(\frac{3}{\sqrt[3]{x}} + 5\sqrt{x}\right)^{25}$.

A: General term $T_{r+1} = {}^nC_r\, x^{n-r} a^r$.

$= {}^{25}C_r\left(\frac{3}{\sqrt[3]{x}}\right)^{25-r}\left(5\sqrt{x}\right)^r = {}^{25}C_r\, 3^{25-r}\left(\frac{1}{x^{\frac{25-r}{3}}}\right)\left(5\sqrt{x}\right)^r$

$= {}^{25}C_r 3^{25-r}5^r x^{-\left(\frac{25-r}{3}\right)+\frac{r}{2}}$

take $\frac{-25+r}{3} + \frac{r}{2} = 0$ then

$-50 + 2r + 3r = 0 \Rightarrow \boxed{r = 10}$.

$\therefore$ The term independent of x is

$T_{11} = {}^{25}C_{10}\, 3^{25-10}\, 5^{10} = {}^{25}C_{10} 3^{15} 5^{10}$.

10. Find the term independent of x in the expansion of $\left(4x^3 + \frac{7}{x^2}\right)^{14}$.

A: General term $T_{r+1} = {}^nC_r\, x^{n-r} a^r$.

$T_{r+1} = {}^{14}C_r(4x^3)^{14-r}\left(\frac{7}{x^2}\right)^r = {}^{14}C_r 4^{14-r}7^r x^{42-5r}$.

If 42 - 5r = 0 then r = $\frac{42}{5}$.

Which is not possible.

$\therefore$ The term indepenent of x is ‘0’.

11. Prove that $C_0 + 2.C_1 + 2^2.c_2 + ...+ 2^n . C_n = 3^n$.

A: We know that $C_0 + C_1.x + C_2.x^2 + ...+ C_n.x_n = (1+x)^n$.

Let $\boxed{x = 2}$

then we get

$\boxed{C_0 + 2.C_1 + 2^2.C_2 + ... + 2^n.C_n = 3^n}$.

12. Find the sum of $3.C_0 + 6.C_1 + 12.C_2 + ...3.2^n C_n$.

A: take $3.C_0 + 6.C_1 + 12.C_2 + ...3.2^n C_n$.

$= 3.C_0 + 3.2.C_1 + 3.2^2.C_2 + ... + 3.2^n C_n$

$= 3\left[C_0 + 2.C_1 + 2^2.C_2 + ... + 2^n C_n\right]$

$= 3\left[(1+2)^n\right] = 3.3^n = 3^{n+1}$.

13. Prove that

$\frac{C_1}{C_0} + 2.\frac{C_2}{C_1} + 3.\frac{C_3}{C_2} + ... + n.\frac{C_n}{C_{n-1}} = \frac{n(n+1)}{2}$.

A: L.H.S. = $\frac{C_1}{C_0}+2.\frac{C_2}{C_1}+3.\frac{C_3}{C_2}+.........+n.\frac{C_n}{C_{n-1}}$

$=\frac{n}{1}+2\frac{\frac{n(n-1)}{2}}{n}+3\frac{\frac{n(n-1)(n-2)}{3!}}{\frac{n(n-1)}{2}}+......+n\frac{1}{n}$

= n + (n - 1) + (n - 2) + ...+ 2 + 1.

= 1 + 2 + 3 ....+ n = $\frac{n(n+1)}{2}$ = R.H.S.

14. Find the range of x for which the binomial expansion $(3-4x)^{3/4}$ is valid.

A: $(3-4x)^{3/4}$

$=\left[3\left(1-\frac{4x}{3}\right)\right]^{3/4}=3^{3/4}\left(1-\frac{4x}{3}\right)^{3/4}$

The expansion is valid when $\left|\frac{-4x}{3}\right|<1$

$\Rightarrow |x|<\frac{3}{4}\Rightarrow \boxed{\frac{-3}{4}<x<\frac{3}{4}}$ (or) $\boxed{x\in\left(\frac{-3}{4},\frac{3}{4}\right)}$.

15. Write the first 3 terms in the expansion of $(8-5x)^{2/3}$.

A: $(8-5x)^{2/3}=8^{2/3}\left(1-\frac{5x}{8}\right)^{2/3}=4\left(1-\frac{5x}{8}\right)^{2/3}$.

$=4\left[1+\frac{2}{3}\left(-\frac{5x}{8}\right)+\frac{\frac{2}{3}\left(\frac{2}{3}-1\right)}{2!}\left(-\frac{5x}{8}\right)^2+.........\infty\right]$

$=4\left[1-\frac{5x}{12}-\frac{25x^2}{576}......\infty\right]$

Therefore, the first three terms in the expansion of $(8-5x)^{2/3}$ are $4, \frac{-5x}{12}, \frac{-25x^2}{576}$.

16. Write the first 3 terms in the expansion of $(2-7x)^{-3/4}$.

A: $(2-7x)^{-3/4}=2^{-3/4}\left(1-\frac{7x}{2}\right)^{-3/4}$.

$=2^{-3/4}\left[1+\left(-\frac{3}{4}\right)\left(-\frac{7x}{2}\right)+\frac{\left(-\frac{3}{4}\right)\left(-\frac{3}{4}-1\right)}{2!}\left(-\frac{7x}{2}\right)^2+.........\infty\right]$

$=2^{-3/4}\left[1+\frac{21x}{8}+\frac{1029x^2}{128}+.....\infty\right]$

The first 3 terms are

$2^{-3/4},\ 2^{-3/4}x\frac{21x}{8},\ 2^{-3/4}x\frac{1029x^2}{128}$.

17. Find an approximate value of the following corrected to 4 decimal places $\sqrt[3]{1002}-\sqrt[3]{998}$.

A: $\sqrt[3]{1002}-\sqrt[3]{998}=(1002)^{1/3}-(998)^{1/3}$.

$=(1000+2)^{1/3}-(1000-2)^{1/3}$

$=(1000)^{1/3}-(1000-2)^{1/3}$.

$=(1000)^{1/3}\left[1+\frac{2}{1000}\right]^{1/3}-(1000)^{1/3}\left[1-\frac{2}{1000}\right]^{1/3}$

$=10\left[(1+0.002)^{1/3}-(1-0.002)^{1/3}\right]$

$=10\left[\left(1+\frac{1}{3}(0.002)+\frac{\frac{1}{3}\left(-\frac{2}{3}\right)}{2!}(0.002)^2+.....\right)-\left(1-\frac{1}{3}(0.002)+\frac{\frac{1}{3}\left(-\frac{2}{3}\right)}{2!}(0.002)^2+.....\right)\right]$

$\approx\frac{0.04}{3}\approx 0.0133.$

18. Expand $5\sqrt{5}$ in increasing powers of $\frac{4}{5}$.

A: $5\sqrt{5}=5.5^{\frac{1}{2}}=5.5^{\frac{3}{2}}=\left(\frac{1}{5}\right)^{-3/2}=\left(1-\frac{4}{5}\right)^{\frac{-3}{2}}$.

Formula :

$(1-x)^{-n}=1+nx+\frac{n(n+1)}{2!}.x^2+...............\infty$

$=1+\frac{3}{2}\left(\frac{4}{5}\right)+\frac{\frac{3}{2}\left(\frac{3}{2}+1\right)}{2!}\left(\frac{4}{5}\right)^2+.................\infty$

$=1+\frac{3}{2}\left(\frac{4}{5}\right)+\frac{3.5}{2^2.2!}\left(\frac{4}{5}\right)^2+....................\infty$.

LEVEL - II (LAQ)

1. If the coefficients of x^9, x^{10}, x^{11} in the expansion of $(1+x)^n$ are in A. P. Then prove that $n^2 - 41n+398 = 0$

A. Given expansion $(1 +x)^n$

The coefficient of x^r in the expansion of $(1 + x)^n$ is ^nC_r Given that coefficients of x^9, x^{10}, x^{11} are in A. P.

$\Rightarrow 2\ {}^nC_{10} = {}^nC_9 + {}^nC_{11}$ $\because$ a,b, c are in A. P. $\Rightarrow$ 2b = a + c

$$\Rightarrow 2\frac{n!}{(n-10)!10!} = \frac{n!}{(n-9)!9!} + \frac{n!}{(n-11)!11!}$$

$$\Rightarrow 2\frac{2}{(n-10)10} = \frac{1}{(n-9)(n-10)} + \frac{1}{11 \text{ x } 10}$$

$$\Rightarrow \frac{2}{(n-10)10} = \frac{110+(n-9)(n-10)}{(n-9)(n-10) \text{ x } 11 \text{x } 10}$$

$$\Rightarrow 22(n-9) = 110 + n^2 + 19n + 90$$

$$\Rightarrow 22n - 198 = 200 + n^2 - 19n$$

$$\Rightarrow n^2 - 41n + 318 = 0 \cdot$$

2. Show that the middle term in the expansion of $(1 + x)^{2n}$ is $\frac{1.3.5......(2n-1)}{n!}(2x)^n$.

A: Given expansion is $(1 + x)^{2n}$

Middle term $T_{\frac{2n}{2}+1} = T_{n+1}$

$T_{n+1} = {}^{2n}C_n \,.\, 1^{2n-x} \,.\, x^n$

$$= \frac{(2n)!}{(2n-n)!\,n!}x^n$$

$$= \frac{(2n)!}{n!\,n!}x^n$$

$$= \frac{1.2.3.4.......(2n-2)(2n-1)(2n)}{1.2.......(n-1)\,n\,(n!)} \,.\, x^n$$

$$= \frac{1.3.5......(2n-1)}{n!} \,.\, 2^n \,.\, x^n$$

$$= \frac{1.3.5......(2n-1)}{n!}(2x)^n.$$

3. Prove that

$({}^{2n}C_0)^2 - ({}^{2n}C_1)^2 + ({}^{2n}C_2)^2 - ....+ ({}^{2n}C_{2n})^2 = (-1)^n\ {}^{2n}C_n$.

A: We know that

$(1 + x)^{2n} = {}^{2n}C_o + {}^{2n}C_1x + {}^{2n}C_2x^2 + ...+ {}^{2n}C_{2n}x^{2n}$

Also $(x + 1)^{2n} = {}^{2n}C_ox^{2n} - {}^{2n}C_1x^{2n-1} + {}^{2n}C_2x^{2n-2} - .... + {}^{2n}C_{2n}$

Multiplying the above two expansions and equating the coefficient of x^{2n}, we get

$({}^{2n}C_0)^2 - ({}^{2n}C_1)^2 + ({}^{2n}C_2)^2 - ....+ ({}^{2n}C_{2n})^2$

= coe. of x^{2n} in $(1 + x)^{2n}(x - 1)^{2n}$

= coe. of x^{2n} in $(x + 1)^{2n}(x - 1)^{2n}$

= coe. of x^{2n} in $(x^2 - 1)^{2n}$ In $(x^2 - 1)2n$, $T_{r+1} = {}^{2n}C_r(x^2)^{2n-r}(-1)^r$ Here r = n

$= {}^{2n}C_n(-1)^n$.

4. Find the sum of the infinite series

$$\mathbf{1+\frac{2}{3}.\frac{1}{2}+\frac{2}{3}.\frac{5}{6}\left(\frac{1}{2}\right)^2+\frac{2}{3}.\frac{5}{6}.\frac{8}{9}\left(\frac{1}{2}\right)^3+......\infty}.$$

A: Now $1+\frac{2}{3}.\frac{1}{2}+\frac{2}{3}.\frac{5}{6}\left(\frac{1}{2}\right)^2+\frac{2}{3}.\frac{5}{6}.\frac{8}{9}\left(\frac{1}{2}\right)^3+......\infty$

$$= 1+2\left(\frac{1}{6}\right)+\frac{2.5}{2!}\left(\frac{1}{6}\right)^2+\frac{2.5.8}{3!}\left(\frac{1}{6}\right)^3+......\infty$$

Comparing this with

$$(1-x)^{-p/q} = 1+\frac{p}{1!}\left(\frac{x}{q}\right)+\frac{p(p+q)}{2!}\left(\frac{x}{q}\right)^2+........\infty$$

Here p = s

$p + q = 5 \Rightarrow q = 2$

$\frac{x}{q} = \frac{1}{4} \Rightarrow x = \frac{2}{4} = \frac{1}{2}$

$\therefore$ Sum of the given infinite series

$= (1 - x)^{-p/q}$

$$= \left(1-\frac{1}{2}\right)^{-2/3}$$

$$= \left(\frac{1}{2}\right)^{-2/3}$$

$= 2^{2/3}$

$= \sqrt[3]{4}$.

5.If $x = \frac{5}{2!\,3} + \frac{5.7}{3!\,3^2} + \frac{5.7.9}{4!\,3^3} + ...\infty$, then find the value of $x^2 + 4x$.

A: Given $x = \frac{5}{2!\,3} + \frac{5.7}{3!\,3^2} + \frac{5.7.9}{4!\,3^3} + ...\infty$

$$\Rightarrow x = \frac{3.5}{2!}\left(\frac{1}{3}\right)^2 + \frac{3\,.5.7}{3!} + \left(\frac{1}{3}\right)^3 + ...\infty$$

$$\Rightarrow x + 2 = 1 + \frac{3}{1!}\left(\frac{1}{3}\right) + \frac{3.5}{2!}\left(\frac{1}{3}\right)^2 + \frac{3.5.7}{3!}\left(\frac{1}{3}\right)^3 + ...\infty$$

Comparing RHS with

$$(1 - y)^{-p/q} = 1 + \frac{p}{1!}\left(\frac{y}{q}\right) + \frac{p(p+q)}{2!}\left(\frac{y}{q}\right)^2 + ...\infty$$

Here p = 3

$p + q = 5 \Rightarrow q = 2$

$\frac{y}{q} = \frac{1}{3} \Rightarrow y = \frac{2}{3}$

$\therefore x + 2 = (1 - y)^{-p/q}$

$$= \left(1 - \frac{2}{3}\right)^{-3/2}$$

$$= \left(\frac{1}{3}\right)^{-3/2}$$

$x + 2 = 3^{3/2}$

Squaring on both sides

$x^2 + 4x + 4 = 27$

$\therefore x^2 + 4x = 23.$

6. Find the sum of the infinite series $\frac{3}{4.8} - \frac{3.5}{4.8.12} + \frac{3.5.7}{4.8.12.16} - ......\infty$.

A: Given $\frac{3}{4.8} - \frac{3.5}{4.8.12} + \frac{3.5.7}{4.8.12.16} - ......\infty$

$$= \frac{1.3}{4.8} - \frac{1.3.5}{4.8.12} + \frac{1.3.5.7}{4.8.12.16} - ......\infty$$

$$= \frac{1.3}{2!}\left(\frac{1}{4}\right)^2 - \frac{1.3.5}{3!}\left(\frac{1}{4}\right)^2 + ...\infty$$

$$= 1 - \frac{1}{1!}\left(\frac{1}{4}\right) + \frac{1.3}{2!}\left(\frac{1}{4}\right)^2 - \frac{1.3.5}{3!}\left(\frac{1}{4}\right)^3 ....\infty - \left(1 - \frac{1}{4}\right)$$

Comparing the infinite series with

$$(1 + x)^{-p/q} = 1 - \frac{p}{1!}\left(\frac{x}{q}\right) + \frac{p(p+q)}{2!}\left(\frac{x}{q}\right)^2 - ....\infty$$

Here p = 1

$p + q = 3 \Rightarrow q = 2$

$\frac{x}{q} = \frac{1}{4} \Rightarrow x = \frac{2}{4} = \frac{1}{2}$

Sum of the given infinite series

$$= (1 + x)^{-p/q} - \frac{3}{4}$$

$$= \left(1 + \frac{1}{2}\right)^{-1/2} - \frac{3}{4}$$

$$= \left(\frac{3}{2}\right)^{-1/2} - \frac{3}{4}$$

$$= \sqrt{\frac{2}{3}} - \frac{3}{4}.$$