DEFINITIONS, CONCEPTS AND FORMULAE:

1) If a, b R and $i = \sqrt{-1}$, then a + ib is a complex number.
2) Conjugate of z = a + ib is $\bar{z}$ = a - ib.
3) Additive inverse of z = a + ib is -z = -a - ib.
4) Multiplicative inverse of z = a + ib is $z^{-1} = \dfrac{a - ib}{a^2 + b^2}$.
5) $\sqrt{a+ib} = \pm\left(\sqrt{\dfrac{\sqrt{a^2+b^2}+a}{2}} + i\sqrt{\dfrac{\sqrt{a^2+b^2}-a}{2}}\right)$
6) Modulus of z = a + ib is $|z| = \sqrt{a^2+b^2}$
7) Mod - amplitude form of a + ib = r(cos q + i sin q)

   where $r = \sqrt{a^2+b^2}$, cos q = $\frac{a}{r}$

   sin q = $\frac{b}{r}$, q Î [-p, p]
8) e^{iq} = cos q + i sin q

   e^{-iq} = cos q - i sin q
9) If $z_1 = r_1$ cis q_1, $z_2 = r_2$ cis q_2 then

   (i) $z_1z_2 = r_1r_2$ cis $(q_1 + q_2)$

   (ii) $z_1/z_2 = r_1/r_2$ cis$(q_1 - q_2)$
10) cis q_1 . cis q_2 . cis q_3 = cis $(q_1+q_2+q_3)$
11) $\dfrac{1}{\cos\theta + i\sin\theta}$ = cos q - i sin q.
12) The cube roots of unity are

    1, $w = \dfrac{-1+\sqrt{3}i}{2}$, $w^2 = \dfrac{-1-\sqrt{3}i}{2}$

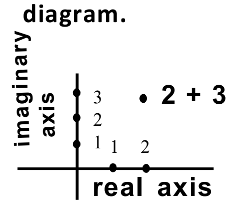
    $w^3 = 1$; $1 + w + w^2 = 0$
13) The fourth roots of unity are 1, -1, i, -i.
14) The n^{th} roots of unity are 1, a, a^2 ..... a^{n-1} where a = cis $\left(\frac{2\pi}{n}\right)$

    i) Sum of n^{th} roots of unity is 0.

    ii) Product of n^{th} roots of unity is $(-1)^{n-1}$.
15) Arg $(z_1 z_2)$ = Arg z_1 + Arg z_2.
16) Arg $\left(\dfrac{z_1}{z_2}\right)$ = Arg z_1 - Arg z_2.

(VSAQ)

1. Represent the complex number 2 + 3i in Argand diagram.



2 Find the real and imaginary parts of the complex number $\dfrac{a-ib}{a+ib}$.

A: $\dfrac{a-ib}{a+ib} = \left(\dfrac{a-ib}{a+ib}\right)\left(\dfrac{a-ib}{a-ib}\right)$

$= \dfrac{(a^2-b^2)+(-2ab)i}{a^2+b^2}$

Real part $= \dfrac{a^2-b^2}{a^2+b^2}$, imaginary part $= \dfrac{-2ab}{a^2+b^2}$.

3. If $(a + ib)^2 = x + iy$, find $x^2 + y^2$.

A: Given that $(a + ib)^2 = x + iy$

Now, $|a + ib|^2 = |x + iy|$

$\Rightarrow \left(\sqrt{a^2+b^2}\right)^2 = \sqrt{x^2+y^2} \Rightarrow a^2+b^2 = \sqrt{x^2+y^2}$

$\Rightarrow \boxed{x^2+y^2 = (a^2+b^2)^2}$

4. Find the square roots of - 5 + 12i.

A: We know that

$\sqrt{a+ib} = \pm\left[\sqrt{\dfrac{\sqrt{a^2+b^2}+a}{2}} + i\sqrt{\dfrac{\sqrt{a^2+b^2}-a}{2}}\right]$

Here a = -5, b = 12

$\sqrt{-5+12i} = \pm\left[\sqrt{\dfrac{\sqrt{25+144}+(-5)}{2}} + i\sqrt{\dfrac{\sqrt{25+144}-(-5)}{2}}\right]$

$= \pm\left[\sqrt{\dfrac{13-5}{2}} + i\sqrt{\dfrac{13+5}{2}}\right]$

$= \pm(2 + 3i)$.

5. Find the square roots of 7 + 24i.

A: We know that

$$\sqrt{a+ib}=\pm\left[\sqrt{\frac{\sqrt{a^2+b^2}+a}{2}}+i\sqrt{\frac{\sqrt{a^2+b^2}-a}{2}}\right]\ [\because b>0]$$

$$\therefore \sqrt{7+24i}=\pm\left[\sqrt{\frac{\sqrt{7^2+24^2}+7}{2}}+i\sqrt{\frac{\sqrt{7^2+24^2}-7}{2}}\right]$$

$$=\pm\left[\sqrt{\frac{\sqrt{625}+7}{2}}+i\sqrt{\frac{\sqrt{625}-7}{2}}\right]$$

$$=\pm\left[\sqrt{\frac{25+7}{2}}+i\sqrt{\frac{25-7}{2}}\right]$$

$$=\pm\left[\sqrt{\frac{32}{2}}+i\sqrt{\frac{18}{2}}\right]$$

$$=\pm\left[\sqrt{16}+i\sqrt{9}\right]=\pm(4+3i)$$

6. Find the complex conjugate of (2 + 5i) (- 4 + 6i).

A: $(2 + 5i)(-4 + 6i) = -8 + 12i - 20i + 30i^2$

$= -7 - 8i - 30 = -38 - 8i$

Hence, its complex conjugate is $-38 + 8i$.

7. Show that $z_1 = \frac{2+11i}{25}$, $z_2 = \frac{-2+i}{(1-2i)^2}$ are conjugate to each other.

A: $z_2 = \frac{-2+i}{(1-2i)^2}$

$$= \frac{-2+i}{1-4-4i}$$

$$= \frac{-2+i}{-3-4i}$$

$$= \frac{2-i}{3+4i}\times\frac{3-4i}{3-4i}$$

$$= \frac{6-8i-3i-4}{9+16} \qquad \because (a+ib)(a-ib) = a^2+b^2$$

$$= \frac{2-11i}{25}$$

z_1 and z_2 are conjugate to each other.

8. Find the additive inverse of $(\sqrt{3}, 5)$.

A: $(\sqrt{3}, 5) = \sqrt{3} + 5i$

Its additive inverse $= -(\sqrt{3}, 5) = -\sqrt{3} - 5i$

$= (-\sqrt{3}, -5)$.

9. Write the multiplicative inverse of (7, 24).

A: $(7, 24) = 7 + 24i$.

Multiplicative inverse of $7 + 24i = \frac{1}{7+24i}$.

$$= \frac{1}{7+24i}\times\frac{7-24i}{7-24i} = \frac{7-24i}{7^2-24^2i^2} = \frac{7-24i}{49+576}.$$

$$= \frac{7-24i}{625} = \frac{7}{625} - i\frac{24}{625} = \left(\frac{7}{625}, -\frac{24}{625}\right).$$

10. If $x+iy = \frac{1}{1+\cos\theta+i\sin\theta}$, show that $4x^2 - 1 = 0$.

A: Now $x+iy = \frac{1}{1+\cos\theta+i\sin\theta}$

$$= \frac{1}{2\cos^2\frac{\theta}{2}+i\,2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}$$

$$= \frac{1}{2\cos\frac{\theta}{2}\left[\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right]}\times\frac{\cos\frac{\theta}{2}-i\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}-i\sin\frac{\theta}{2}}$$

$$= \frac{\cos\frac{\theta}{2}-i\sin\frac{\theta}{2}}{2\cos\frac{\theta}{2}\left[\cos^2\frac{\theta}{2}+\sin^2\frac{\theta}{2}\right]}$$

$$= \frac{\cos\frac{\theta}{2}-i\sin\frac{\theta}{2}}{2\cos\frac{\theta}{2}}$$

$$= \frac{1}{2} - i\frac{1}{2}\tan\frac{\theta}{2}$$

Equating the real parts on both sides,

$x = \frac{1}{2}$

$2x = 1$ S.O. B

$4x^2 = 1$

$4x^2 - 1 = 0$.

11. If z = 2 - 3i, show tht $z^2 - 4z + 13 = 0$.

A: Given that z = 2 - 3i

Þ z - 2 = -3i

Squaring on both sides,

$(z - 2)^2 = (-3i)^2$

Þ $z^2 - 4z + 4 = -9$

Þ $z^2 - 4z + 13 = 0$.

12. Find the least positive integer n, satisfying $\left(\frac{1+i}{1-i}\right)^n = 1$.

A: Given that $\left(\frac{1+i}{1-i}\right)^n = 1 \Rightarrow \left(\frac{1+i}{1-i} \times \frac{1+i}{1+i}\right)^n = 1$

$$\Rightarrow \left[\frac{(1+i)^2}{1^2 - i^2}\right]^n = 1 \Rightarrow \left(\frac{1+i^2+2i}{1+1}\right)^n = 1$$

$$\left(\frac{\not{1} - \not{1} + 2i}{1+1}\right)^n = 1 \Rightarrow \left[\frac{\not{2}i}{\not{2}}\right]^n = 1 \Rightarrow i^n = 1$$

n = {4, 8, 12,............¥}

\ Required least positive integer is 4.

13. If z = (cos q, sin q) then find $z - \frac{1}{z}$

A: Given that z = (cos q, sin q) = cos q + i sin q

then $\frac{1}{z} = \frac{1}{\cos\theta + i\sin\theta} \times \frac{\cos\theta - i\sin\theta}{\cos\theta - i\sin\theta}$

$$= \frac{\cos\theta - i\sin\theta}{(\cos\theta)^2 - (i\sin\theta)^2} = \frac{\cos\theta - i\sin\theta}{\cos^2\theta + \sin^2\theta}$$

= cosq - i sin q

$$\therefore z - \frac{1}{z} = \not{\cos\theta} + i\sin\theta - (\not{\cos\theta} - i\sin\theta)$$

$$\boxed{\Rightarrow z - \frac{1}{z} = 2i\sin\theta}$$

14. If $z_1 = -1$, $z_2 = -i$ then find Arg ($z_1 z_2$)

A: Given that $z_1 = -1$, $z_2 = -i$

then z_1 = cisp, z_2 = $\text{cis}\left(-\frac{\pi}{2}\right)$

\ Arg ($z_1 z_2$) = Arg z_1 + Arg z_2.

$$= \pi + \left(-\frac{\pi}{2}\right) = \frac{\pi}{2}.$$

15. If $z_1 = -1$, $z_2 = i$ then find $\text{Arg}\left(\frac{z_1}{z_2}\right)$.

A: Given that $z_1 = -1$, $z_2 = i$

then z_1 = cisp, z_2 = $\text{cis}\frac{\pi}{2}$

\ $\text{Arg}\left(\frac{z_1}{z_2}\right)$ = Arg z_1 - Arg z_2.

$$= \pi - \frac{\pi}{2} = \frac{\pi}{2}.$$

16. If Arg $\bar{z}_1$ and Arg z_2 are $\frac{\pi}{5}$ and $\frac{\pi}{3}$ respectively, find Arg z_1 + Arg z_2.

A: Given Arg $\bar{z}_1 = \frac{\pi}{5}$ and $z_2 = \frac{\pi}{3}$

$\Rightarrow$ Arg $z_1 = \frac{-\pi}{5}$ and Arg $z_2 = \frac{\pi}{3}$

\ Arg z_1 + Arg z_2 = $\frac{-\pi}{5} + \frac{\pi}{3} = \frac{= 3\pi + 5\pi}{15} = \frac{2\pi}{5}$.

$$\therefore \boxed{\text{Arg } z_1 + \text{Arg } z_2 = \frac{2\pi}{5}}$$

17. Find the modulus and amplitude form of the complex number $1 + \sqrt{3}i$.

A: Let x + iy = $1 + \sqrt{3}i$

Here x = 1, y = $\sqrt{3}$

Now, r = $\sqrt{x^2 + y^2} = \sqrt{1+3} = \sqrt{4} = 2$

$$\cos\theta = \frac{x}{r} \Rightarrow \cos\theta = \frac{1}{2}$$

Hence, $\sin\theta = \frac{y}{r} \Rightarrow \sin\theta = \frac{\sqrt{3}}{2}$

\ 'q' lies in I quadrant and q = $\frac{\pi}{3}$ Î (- p, p]

\ Modulus amplitude form of $1 + \sqrt{3}i$

= r (cosq + isinq) = 2[cos$\frac{\pi}{3}$ + isin $\frac{\pi}{3}$].

18. Express $-1-i\sqrt{3}$ in polar form.

A: Let x + iy = $-1-i\sqrt{3}$

Here x = -1, y = $-\sqrt{3}$

Now, r = $\sqrt{x^2+y^2}=\sqrt{1+3}=\sqrt{4}=2$

$$\left.\begin{aligned}\cos\theta = \frac{x}{r} \Rightarrow \cos\theta = \frac{-1}{2}\\ \sin\theta = \frac{y}{r} \Rightarrow \sin\theta = \frac{-\sqrt{3}}{2}\end{aligned}\right\}$$

\ 'q' lies in III quadrant and q = $-\left(\pi-\frac{\pi}{3}\right) = -\frac{2\pi}{3}$.

\ Polar form of $-1-i\sqrt{3}$ = r (cosq + isinq)

= 2{cos$\left(-\frac{2\pi}{3}\right)$ + isin$\left(-\frac{2\pi}{3}\right)$].

19. If $\sqrt{3}+i=r(\cos\theta+i\sin\theta)$. Find the value of q.

A: $a+ib=\sqrt{a^2+b^2}\left(\frac{a^2}{\sqrt{a^2+b^2}}+i\frac{b}{\sqrt{a^2+b^2}}\right)$

$\sqrt{3}+i=2\left(\frac{\sqrt{3}}{2}+i\frac{1}{2}\right)$

$=2\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$

$\therefore \theta=\frac{\pi}{6}$.

20. If $(\cos 2\alpha+i\sin 2\alpha)(\cos 2\beta+i\sin 2\beta)$

= cos q + i sin q find the value of q.

A: Given that $(\cos 2\alpha+i\sin 2\alpha)(\cos 2\beta+i\sin 2\beta)$

= cos q + i sin q

(cis 2a) (cis 2b) = cos q + i sin q.

cis (2a + 2b) = cis q

$\therefore \boxed{\theta=2\alpha+2\beta}$

21. If z = x + iy = cisa.cisb, then find the value of $x^2 + y^2$.

A: Given that x + iy

= (cosa + i sina) (cosb + i sinb)

= (cosacosb + sinasinb) + i (sinacosb + cosasinb)

i.e., x + iy = cos(a + b) + i sin(a + b)

$|x+iy| = \sqrt{\cos^2(\alpha+\beta)+\sin^2(\alpha+\beta)}$

$\Rightarrow \sqrt{x^2+y^2}=1 \Rightarrow \boxed{x^2+y^2=1}$

22. If $\left(\sqrt{3}+i\right)^{100}=2^{99}(a+ib)$, show that $a^2+b^2=4$.

A: Given that $\left(\sqrt{3}+i\right)^{100}=2^{99}(a+ib)$

$\Rightarrow \left|\sqrt{3}+i\right|^{100}=\left|2^{99}(a+ib)\right|$

$\Rightarrow \left(\sqrt{\left(\sqrt{3}\right)^2+(1)^2}\right)^{100}=2^{99}\sqrt{a^2+b^2}$

$\Rightarrow 2^{100}=2^{99}\sqrt{a^2+b^2}$

$\Rightarrow 2=\sqrt{a^2+b^2} \Rightarrow \boxed{a^2+b^2=4}$

23. If z = x + iy and |z| = 2, find the locus of z.

A: Given that z = x + iy and

|z| = 2

|x + iy| = 2

$\sqrt{x^2+y^2}=2$

$x^2 + y^2 = 4$

so, the locus of z is $x^2 + y^2 = 4$.

24. If the amplitude of z - 1 is $\frac{\pi}{2}$, find the locus of z.

A: Let z = x + iy

z - 1 = x + iy - 1

= (x - 1) + iy

Given that amplitude of z - 1 is $\frac{\pi}{2}$

$\tan^{-1}\left(\frac{y}{x-1}\right)=\frac{\pi}{2}$

$\frac{y}{x-1}=\tan\frac{\pi}{2}=\frac{1}{0}$

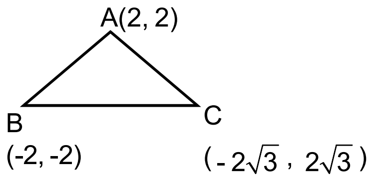
x - 1 = 0

Locus of z is x = 1.

(SAQ)

1. Show that the triangle formed by the points in the Argand plane represented by 2 + 2i, -2 - 2i, $-2\sqrt{3} + 2\sqrt{3}\,i$ is an equilateral triangle.

A: Let A(2, 2), B(-2, -2) C($-2\sqrt{3}$, $2\sqrt{3}$) represent the given complex numbers in the Argand diagram.



AB = $\sqrt{(2+2)^2 + (2+2)^2}$

= $\sqrt{16+16}$

= $\sqrt{32}$

BC = $\sqrt{(2\sqrt{3} - 2)^2 + (-2 - 2\sqrt{3})^2}$

= $\sqrt{12 - 8\cancel{\sqrt{3}} + 4 + 4 + 8\cancel{\sqrt{3}} + 12}$

= $\sqrt{32}$

CA = $\sqrt{(2 + 2\sqrt{3})^2 + (2 - 2\sqrt{3})^2}$

= $\sqrt{4 + 8\cancel{\sqrt{3}} + 12 + 4 - 8\cancel{\sqrt{3}} + 12}$

= $\sqrt{32}$

$\because$ AB = BC = CA, so DABC is an equilateral triangle.

2. If z = x + iy and if the point P in the Argand diagram represents z, find the locus of z satisfying the equation |z - 2 - 3i| = 5.

A: Given that z = x + iy is any point on the locus.

and |z - 2- 3i| = 5.

| x + iy - 2 - 3i | = 5.

| (x - 2) + i (y - 3)| = 5.

$(x - 2)^2 + (y - 3)^2 = 5^2$

$x^2 - 4x + 4 + y^2 - 6y + 9 - 25 = 0.$

Locus of P is $x^2 + y^2 - 4x - 6y - 12 = 0.$

3. Show that the four points in the Argand plane represented by the complex numbers 2 + i, 4 + 3i, 2 + 5i, 3i are the vertices of a square.

A: Let the given four points in the Argand plane be A, B, C, D with A = (2, 1), B = (4, 3), C = (2,5), D = (0, 3).

$\therefore AB = \sqrt{(4-2)^2 + (3-1)^2} = \sqrt{4+4} = 8 = 2\sqrt{2}$

$BC = \sqrt{(2-4)^2 + (5-3)^2} = \sqrt{4+4} = 2\sqrt{2}$

$CD = \sqrt{(0-2)^2 + (3-5)^2} = \sqrt{4+4} = 2\sqrt{2}$

$DA = \sqrt{(2-0)^2 + (1-3)^2} = \sqrt{4+4} = 2\sqrt{2}$

$\therefore AB = BC = CD = DA$

Now $AC = \sqrt{(2-2)^2 + (5-1)^2} = \sqrt{0+16} = 4$

$BD = \sqrt{(0-4)^2 + (3-3)^2} = \sqrt{16+0} = 4$

AC = BD.

Hence, the given four complex numbers are the vertices of a square.

4. Show that the points in the Argand plane represented by the complex numbers -2 + 7i, $\frac{-3}{2} + \frac{1}{2}i$, 4 - 3i, $\frac{7}{2}(1+i)$ are the vertices of rhombus.

A: Let the given four complex numbers in the Argand plane be A, B, C, D with A = (-2, 7), B = $\left(\frac{-3}{2}, \frac{1}{2}\right)$, C = (4, - 3), D = $\left(\frac{7}{2}, \frac{7}{2}\right)$.

$AB = \sqrt{\left(\frac{-3}{2}+2\right)^2 + \left(\frac{1}{2}-7\right)^2} = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{-13}{2}\right)^2}$

$= \sqrt{\frac{1+169}{4}} = \frac{\sqrt{170}}{2}$

$BC = \sqrt{\left(4+\frac{3}{2}\right)^2 + \left(-3-\frac{1}{2}\right)^2} = \sqrt{\left(\frac{11}{2}\right)^2 + \left(\frac{-7}{2}\right)^2}$

$= \sqrt{\frac{121+49}{4}} = \frac{\sqrt{170}}{2}$

$CD = \sqrt{\left(\frac{7}{2}-4\right)^2 + \left(\frac{7}{2}+3\right)^2} == \sqrt{\left(\frac{-1}{2}\right)^2 + \left(\frac{13}{2}\right)^2}$

$= \sqrt{\frac{1+169}{4}} = \frac{\sqrt{170}}{2}$

$DA = \sqrt{\left(-2-\frac{7}{2}\right)^2 + \left(\frac{7}{2}-7\right)^2} == \sqrt{\left(\frac{-11}{2}\right)^2 + \left(\frac{-7}{2}\right)^2}$

$= \sqrt{\frac{121+49}{4}} = \frac{\sqrt{170}}{2}$

$$\therefore AB = BC = CD = DA$$

$$\text{Also } AC = \sqrt{(4+2)^2 + (-3-7)^2} = \sqrt{36+100}$$

$$= \sqrt{136} = \sqrt{4 \times 34} = 2\sqrt{34}$$

$$BD = \sqrt{\left(\frac{7}{2}+\frac{3}{2}\right)^2 + \left(\frac{7}{2}+\frac{1}{2}\right)^2} = \sqrt{25+9} = \sqrt{34}$$

\ AC ¹ BD.

Hence, given four complex numbers are the vertices of a rhombus.

5. If z = 3 - 5i, then show that $z^3 - 10z^2 + 58z - 136 = 0$.

A: Given z = 3 - 5i

z - 3 = - 5i

$(z - 3)^2 = (- 5i)^2$.

$z^2 - 6z + 9 = - 25$.

$z^2 - 6z + 34 = 0$.

Now $z^3 - 10z^2 + 58z - 136$

$= z(z^2 - 6z + 34) - 4z^2 + 24z - 136$.

$= z(0) - 4 (z^2 - 6z + 34)$

= 0 - 4 (0)

= 0.

6. Find the real values of q in order that $\frac{3+2i\sin\theta}{1-2i\sin\theta}$.

i) real number

ii) purely imaginary number.

A: $\frac{3+2i\sin\theta}{1-2i\sin\theta}$

$$= \frac{3+2i\sin\theta}{1-2i\sin\theta} \times \frac{1+2i\sin\theta}{1+2i\sin\theta}$$

$$= \frac{3+6i\sin\theta+2i\sin\theta-4\sin^2\theta}{1+4\sin^2\theta}$$

$$= \frac{3-4\sin^2\theta}{1+4\sin^2\theta} + i\frac{8\sin\theta}{1+4\sin^2\theta}$$

i) $\frac{3+2i\sin\theta}{1-2i\sin\theta}$ is a real number

its imaginary part is zero.

$\frac{8\sin\theta}{1+4\sin^2\theta}$

sin q = 0.

General solution is q = np, n Î Z.

ii) $\frac{3+2i\sin\theta}{1-2i\sin\theta}$ is a purely imaginary number

its real part is zero.

$$\Rightarrow \frac{3-4\sin^2\theta}{1+4\sin^2\theta} = 0$$

Þ 3 - 4sin² q = 0.

$$\Rightarrow \sin^2\theta = \frac{3}{4} = \left(\frac{\sqrt{3}}{2}\right)^2 = \sin^2\frac{\pi}{3}.$$

General solution is $\theta = n\pi \pm \frac{\pi}{3}, n \in Z$.

7. If the point P denotes the complex number z = x + iy in the Argand plane and if $\frac{z-i}{z-1}$ is a purely imaginary number, find the locus of P.

A: Given z = x + iy is any point on the locus.

$$\frac{z-i}{z-1} = \frac{x+iy-i}{x+iy-1}$$

$$= \frac{x+i(y-1)}{(x-1)+iy} \times \frac{(x-1)-iy}{(x-1)-iy}$$

$$= \frac{x(x-1)+i(x-1)(y-1)-ixy+y(y-1)}{(x-1)^2+y^2}$$

$$= \frac{(x^2+y^2-x-y)+i(\cancel{xy}-x-y+1-\cancel{xy})}{(x-1)^2+y^2}$$

$\frac{z-i}{z-1}$ is purely imaginary.

its real part is zero.

$$\Rightarrow \frac{x^2+y^2-x-y}{(x-1)^2+y^2} = 0.$$

$x^2 + y^2 - x - y = 0$ and (x, y) ¹ (1, 0)

The locus of P is the circle $x^2 + y^2 - x - y = 0$ excluding the point (1, 0).

8. The points P, Q denote the complex numbers z_1, z_2 in the Argand diagram. O is the origin. If $z_1\bar{z}_2 + \bar{z}_1 z_2 = 0$ then show that $\angle POQ = 90^0$.

A: Let $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$.

Now $z_1\bar{z}_2 + \bar{z}_1 z_2 = 0$.

$$\Rightarrow (x_1+iy_1)(x_2-iy_2)+(x_1-iy_1)(x_2+iy_2)=0$$

$$\Rightarrow x_1x_2 - ix_1y_2 + ix_2y_1 + y_1y_2 + x_1x_2 + ix_1y_2 - ix_2y_1 + y_1y_2 = 0$$

$$\Rightarrow 2(x_1x_2 + y_1y_2) = 0$$

$$\Rightarrow x_1x_2 + y_1y_2 = 0$$

$$\Rightarrow y_1y_2 = -x_1x_2$$

$$\Rightarrow \left(\frac{y_1-0}{x_1-0}\right)\left(\frac{y_2-0}{x_2-0}\right) = -1$$

Slope of OP x slope of OQ = -1.

$\Rightarrow OP \perp OQ$

$\underline{|POQ} = 90^0$.

9. Show that the points in the Argand diagram represented by the complex numbers z_1, z_2, z_3 are collinear if and only if there exist three real numbers p, q, r not all zero satisfying $pz_1 + qz_2 + rz_3 = 0$ and p + q + r = 0.

Now $pz_1 + qz_2 + rz_3 = 0$ and p + q + r = 0.

Û $pz_1 + qz_2 = -rz_3 = 0$ and p + q = - r.

$\Leftrightarrow \frac{pz_1 + qz_2}{-r} = z_3$ and $p + q = -r$

$\Leftrightarrow \frac{pz_1 + qz_2}{p+q} = z_3$

Û z_3 divides the line segment printing z_1, z_2 in the ratio q : p.

Û z_1, z_2, z_3 are collinear.

10. If $\frac{z_3 - z_1}{z_2 - z_1}$ is a real number, show that the points represented by the complex numbers z_1, z_2, z_3 are collinear.

A: Let $\frac{z_3 - z_1}{z_2 - z_1}$ = k(a real number)

$z_3 - z_1 = kz_2 - kz_1$

$(k - 1)z_1 = kz_2 - z_3$.

$\Rightarrow z_1 = \frac{kz_2 - z_3}{k-1}$.

Þ z_1 divides the line segment joining z_2, z_3 in the ratio 1 : k externally.

Hence z_1, z_2, z_3 are collinear.

EXTRA QUESTIONS

1. Write z = $-\sqrt{7} + \sqrt{21}\,i$ in the polar form.

A: Let z = x + iy

x + iy = $-\sqrt{7} + \sqrt{21}\,i$

$= \sqrt{28}\left[\frac{-\sqrt{7}}{\sqrt{28}} + \frac{\sqrt{21}}{\sqrt{28}}i\right]$

$= 2\sqrt{7}\left[\frac{-1}{2} + \frac{\sqrt{3}}{2}i\right]$

$= 2\sqrt{7}\left[\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right]$.

2. If a = cos a + i sin a and b = cos b + i sin b, then find $\frac{1}{2}\left(ab + \frac{1}{ab}\right)$.

A: Now ab = (cis a) (cis b)

= cis(a + b)

= cos (a + b) + i sin (a + b)

Þ $\frac{1}{ab} = \frac{1}{\cos(\alpha+\beta) + i\sin(\alpha+\beta)} \times \frac{\cos(\alpha+\beta) - i\sin(\alpha+\beta)}{\cos(\alpha+\beta) + i\sin(\alpha+\beta)}$

$= \frac{\cos(\alpha+\beta) - i\sin(\alpha+\beta)}{1}$

= cos (a + b) - i sin (a + b)

$ab + \frac{1}{ab}$ = cos(a + b) + i sin $(\alpha+\beta)$

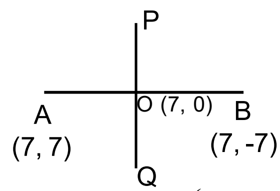
+ cos (a + b) - i sin $(\alpha+\beta)$

= 2 cos (a + b)

$\frac{1}{2}\left(ab + \frac{1}{ab}\right)$ = cos (a + b).

3. Find the equation of the perpendicular bisector of the line segment joining the points 7 + 7i, 7 - 7i in the Argand diagram.

A: A(7, 7), B(7, -7) represent given two complex numbers in the Argand diagram.



Mid point on AB = $\left(\frac{7+7}{2}, \frac{7-7}{2}\right)$ = (7, 0)

Slope of $\overline{AB}$ = $\frac{-7-7}{7-7} = \frac{-14}{0}$ = -¥

Slope of $\overline{PQ}$ = 0 ($\because \overline{AB} \perp \overline{PQ}$)

\ Equation of PQ is

y - 0 = 0 (x - 7)

Þ y = 0.

4. Show that the complex numbers z satisfying $z^2 + \bar{z}^2 = 2$ constitute a hyperbola.

A: Let z = x + iy.

Now $z^2 + \bar{z}^2 = 2$

$\Rightarrow (x+iy)^2 + (x-iy)^2 = 2$

$\Rightarrow x^2 + \cancel{2ixy} + i^2y^2 + x^2 - \cancel{2ixy} + i^2y^2 = 2$

$\Rightarrow 2(x^2 - y^2 = 2)$.

$\Rightarrow x^2 - y^2 = 1$ which is a hyperbola.

5. If (1- i) (2-i) (3-i).......(1-ni) = x - iy, prove that 2.5.10......(1+n²) = x² + y².

A: Given (1- i) (2-i) (3-i).......(1-ni) = x - iy.

Taking modulus on both sides.

$|1-i|\ |2-i|\ |3-i|.......|1-ni| = |x-iy|$

$\Rightarrow \sqrt{1+1}\sqrt{4+1}\sqrt{9+1}.......\sqrt{1+n^2} = \sqrt{x^2+y^2}$

squaring on both sides, we get

$2.5.10......(1+n^2) = x^2+y^2$.

6. If $z = 2 - i\sqrt{7}$, show that $3z^3 - 4z^2 + z + 88 = 0$.

A: Given $z = 2 - i\sqrt{7}$

$z - 2 = -i\sqrt{7}$.

$(z-2)^2 = (-i\sqrt{7})^2$

$z^2 - 4z + 4 = -7$

$z^2 - 4z + 11 = 0$.

Now $3z^3 - 4z^2 + z + 88$.

$= 3z(z^2 - 4z + 11) + 8z^2 - 32z + 88$.

$= 3z(0) + 8(z^2 - 4z + 11)$

$= 0 + 8(0)$.

$= 0$.

7. If $(x-iy)^{1/3} = a - ib$, then show that $\frac{x}{a} + \frac{y}{b} = 4(a^2 - b^2)$.

A: Given that $(x-iy)^{1/3} = a - ib$

cubing on both sides,

$x - iy = (a-ib)^3$

$x - iy = a^3 - 3a^2 ib + 3a i^2b^2 - i^3b^3$

$x - iy = (a^3 - 3ab^2) - i(3a^2b - b^3)$

equating the real and imaginary parts,

$x = a^3 - 3ab^2$; $y = 3a^2b - b^3$

Now $\frac{x}{a} + \frac{y}{b} = \frac{a^3 - 3ab^2}{a} + \frac{3a^2b - b^3}{b}$

$= a^2 - 3b^2 + 3a^2 - b^2$

$= 4a^2 - 4b^2$

$= 4(a^2 - b^2)$.

8. If the real part of $\frac{z+1}{z+i}$ is 1, find the locus of z.

A: Let $z = x + iy$

Now $\frac{z+1}{z+i} = \frac{x+iy+1}{x+iy+i}$

$= \frac{(x+1)+iy}{x+i(y+1)} \cdot \frac{x-i(y+1)}{x-i(y+1)}$

$= \frac{x(x+1) - (x+1)(y+1)i + xyi + y(y+1)}{x^2+(y+1)^2}$

Given that real part of $\frac{z+1}{z+i}$ is 1

$\Rightarrow \frac{x^2+x+y^2+y}{x^2+y^2+2y+1} = 1$

$\Rightarrow \cancel{x^2} + \cancel{y^2} + x + y = \cancel{x^2} + \cancel{y^2} + 2y + 1$

$\Rightarrow x - y = 1$

Hence, the locus of z is $x - y = 1$.

9. Determine the locus of z, $z \neq 2i$, such that $\text{Re}\left(\frac{z-4}{z-2i}\right) = 0$.

A: Let $z = x + iy$

Now $\frac{z-4}{z-2i}$

$= \frac{x+iy-4}{x+iy-2i}$

$= \frac{(x-4)+iy}{x+i(y-2)} \times \frac{x-i(y-2)}{x-i(y-2)}$

$= \frac{x^2-4x+y^2-2y+i(xy-xy+2x+4y-8)}{x^2+(y-2)^2}$

Real part of $\frac{z-4}{z-2i} = \frac{x^2+y^2-4x-2y}{x^2+(y-2)^2} = 0$

$x^2 = y^2 - 4x - 2y = 0$ and $(x, y) \neq (0, 2)$

$(x-2)^2 + (y-1)^2 = 5$ and $(x, y) \neq (0, 2)$.

Hence, the required locus is a circle with centre (2, 1) and radius $\sqrt{5}$ except the point (0,2).

10. If the amplitude of $\left(\frac{z-2}{z-6i}\right) = \frac{\pi}{2}$, find its locus.

A: Let $z = x + iy$

Now $\frac{z-2}{z-6i}$.

$= \frac{x+iy-2}{x+iy-6i}$

$= \frac{(x-2)+iy}{x+i(y-6)} \times \frac{x-i(y-6)}{x-i(y-6)}$

$= \frac{x(x-2)+y(y-6)+i[xy-(x-2)(y-6)]}{x^2+(y-6)^2} = a + ib$ (say)

Then $a = \frac{x^2+y^2-2x-6y}{x^2+(y-6)}$ and $b = \frac{6x+2y-12}{x^2+(y-6)^2}$

But amplitude of a + ib = $\frac{\pi}{2}$

a = 0 ad b ³ 0.

$x^2 + y^2 - 2x - 6y = 0$ and 2(3x + y - 6) ³ 0.

Hence, locus is the arc of the circle $x^2 + y^2 - 2x - 6y = 0$ intercepted by the diameter 3x + y - 6 = 0 not containing the origin and excluding the points (0, 6) and (2, 0).

11. Find the real values of x and y if $\frac{x-1}{3+i} + \frac{y-1}{3-i} = i$.

A: Given $\frac{x-1}{3+i} + \frac{y-1}{3-i} = i$

$$\Rightarrow \frac{(x-1)(3-i)+(3+i)(y-1)}{(3+i)(3-i)} = i$$

3x - ix - 3 + i + 3y - 3 + iy - i = 10i

(3x + 3y - 6) + i (-x + y) = 0 + 10i

Equating the real and imaginary parts, 3x + 3y - 6 = 0 and - x + y = 10.

$$\Rightarrow x + y = 2$$
$$-x + y = 10$$
$$2y = 12$$

y = 6

x + y - 2 = 0 Þ x + 6 - 2 = 0 Þ x = - 4

\ x = - 4, y = 6.