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for Joint Entrance Examination JEE (Advanced)

Trigonometry

G. TEWANI

MATHEMATICS

Trigonometry

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Mathematics: Trigonometry

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Preface

While the paper-setting pattern and assessment methodology have been revised many times over and newer criteria devised to help develop more aspirant-friendly engineering entrance tests, the need to standardize the selection processes and their outcomes at the national level has always been felt. The Joint Entrance Examination (JEE) to India's prestigious engineering institutions (IITs, IIITs, NITs, ISM, IISERs, and other engineering colleges) aims to serve as a common national-level engineering entrance test, thereby eliminating the need for aspiring engineers to sit through multiple entrance tests.

While the methodology and scope of an engineering entrance test are prone to change, there are two basic objectives that any test needs to serve:

1. The objective to test an aspirant's caliber, aptitude, and attitude for the engineering field and profession.
2. The need to test an aspirant's grasp and understanding of the concepts of the subjects of study and their applicability at the grassroots level.

Students appearing for various engineering entrance examinations cannot bank solely on conventional shortcut measures to crack the entrance examination. Conventional techniques alone are not enough as most of the questions asked in the examination are based on concepts rather than on just formulae. Hence, it is necessary for students appearing for joint entrance examination to not only gain a thorough knowledge and understanding of the concepts but also develop problem-solving skills to be able to relate their understanding of the subject to real-life applications based on these concepts.

This series of books is designed to help students to get an all-round grasp of the subject so as to be able to make its useful application in all its contexts. It uses a right mix of fundamental principles and concepts, illustrations which highlight the application of these concepts, and exercises for practice. The objective of each book in this series is to help students develop their problem-solving skills/accuracy, the ability to reach the crux of the matter, and the speed to get answers in limited time. These books feature all types of problems asked in the examination—be it MCQs (one or more than one correct), assertion-reason type, matching column type, comprehension type, or integer type questions. These problems have skillfully been set to help students develop a sound problem-solving methodology.

Not discounting the need for skilled and guided practice, the material in the books has been enriched with a number of fully solved concept application exercises so that every step in learning is ensured for the understanding and application of the subject. This whole series of books adopts a multi-faceted approach to mastering concepts by including a variety of exercises asked in the examination. A mix of questions helps stimulate and strengthen multi-dimensional problem-solving skills in an aspirant.

It is imperative to note that this book would be as profound and useful as you want it to be. Therefore, in order to get maximum benefit from this book, we recommend the following study plan for each chapter.

Step 1: Go through the entire opening discussion about the fundamentals and concepts.

Step 2: After learning the theory/concept, follow the illustrative examples to get an understanding of the theory/concept.

Overall the whole content of the book is an amalgamation of the theme of mathematics with ahead-of-time problems, which equips the students with the knowledge of the field and paves a confident path for them to accomplish success in the JEE.

With best wishes!

G. Tewani

CHAPTER 1

Logarithm and Its Applications

EXPONENTIAL FUNCTION

Exponential functions are perhaps the most important class of functions in mathematics. We use this type of function in the calculation of interest on investments, growth and decline rates of populations, forensics investigations, and in many other applications.

Definition

$y = f(x) = a^x$, where $a > 0$; $a \neq 1$, and $x \in R$. Here $a^x > 0$ for $\forall x \in R$. Thus, the range of the function is $(0, \infty)$.

Exponential functions always have some positive number other than 1 as the base. If you think about it, having a negative number (such as -2) as the base would not be very useful, since the even powers would give you positive answers (such as "$(-2)^2 = 4$") and the odd powers would give you negative answers (such as "$(-2)^3 = -8$"), and what would you even do with the powers that are not whole numbers? Also, having 0 or 1 as the base would be a kind of dumb, since 0 and 1 to any power are just 0 and 1, respectively; what would be the point? This is why exponentials always have something positive and other than 1 as the base.

Graphs of Exponential Function

When $a > 1$

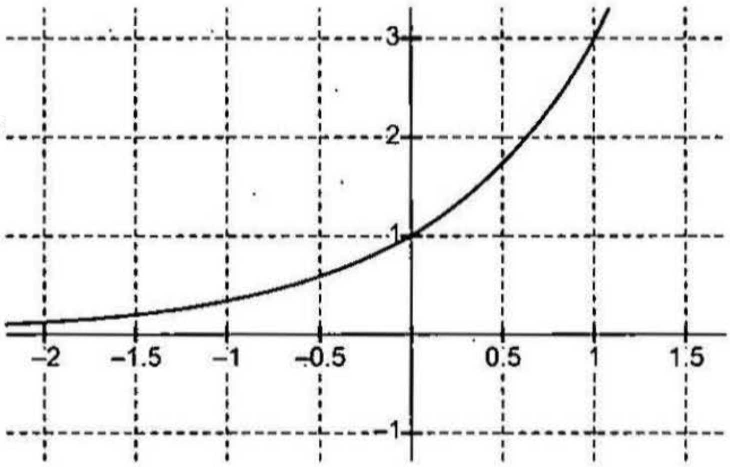


Fig. 1.1

From the graph, the function is increasing. For $x_2 > x_1 \Rightarrow a^{x_2} > a^{x_1}$.

When $0 < a < 1$,

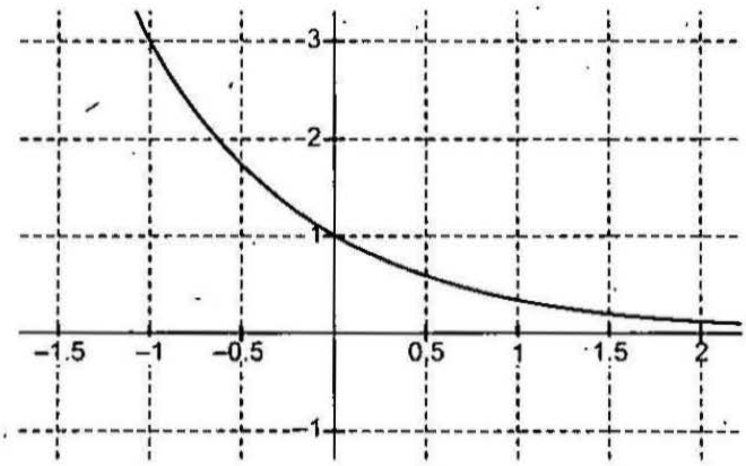


Fig. 1.2

From the graph, the function is decreasing. For $x_2 > x_1 \Rightarrow a^{x_2} < a^{x_1}$.

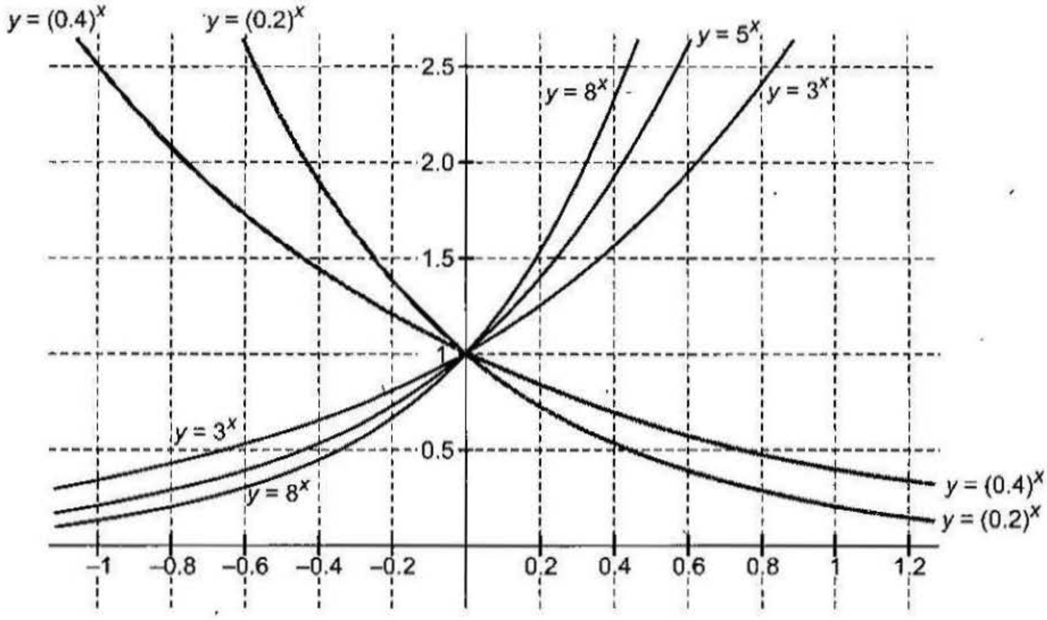


Fig. 1.3

LOGARITHMIC FUNCTION

The **logarithm** of a number to a given *base* is the *exponent* to which the base must be raised in order to produce that number. For example, the logarithm of 1000 to base 10 is 3, because 10 to the power of 3 is 1000, i.e., $10^3 = 1000$. Logarithm function is an inverse of exponential function. Hence, domain and range of the logarithmic functions are range and domain of exponential function, respectively.

Also graph of function can be obtained by taking the mirror image of the graph of the exponential function in the line $y = x$. If we consider point (x_1, y_1) on the graph of $y = a^x$, then we find point (y_1, x_1) on the graph of $y = \log_a x$.

Definition

Logarithmic function is defined as $y = \log_a x$, $a > 0$ and $a \neq 1$.
Domain : $(0, \infty)$,
Range : $(-\infty, \infty)$.

Graphs of Logarithm Function

When $a > 1$

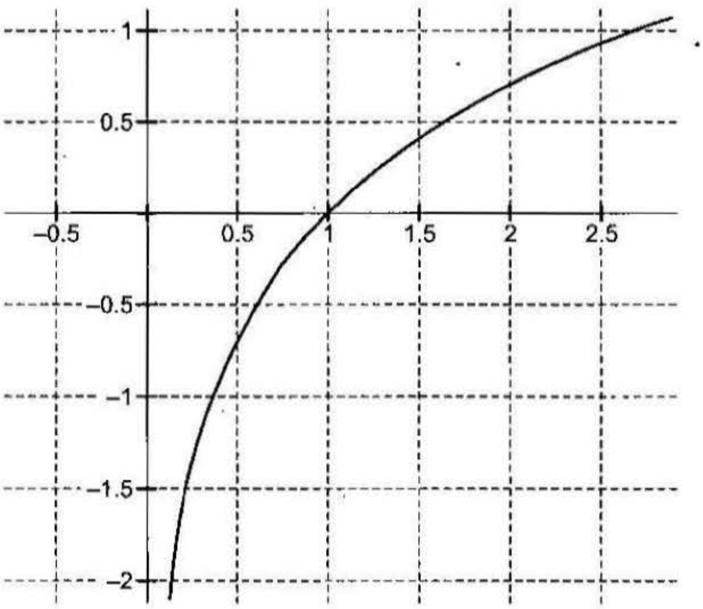


Fig. 1.4

When $a > 1$, $f(x) = \log_a x$ is an increasing function. Then for $x_2 > x_1 \Rightarrow \log_a x_2 > \log_a x_1$. Also $\log_a x_2 > x_1$ $\Rightarrow x_2 > a^{x_1}$

When $0 < a < 1$

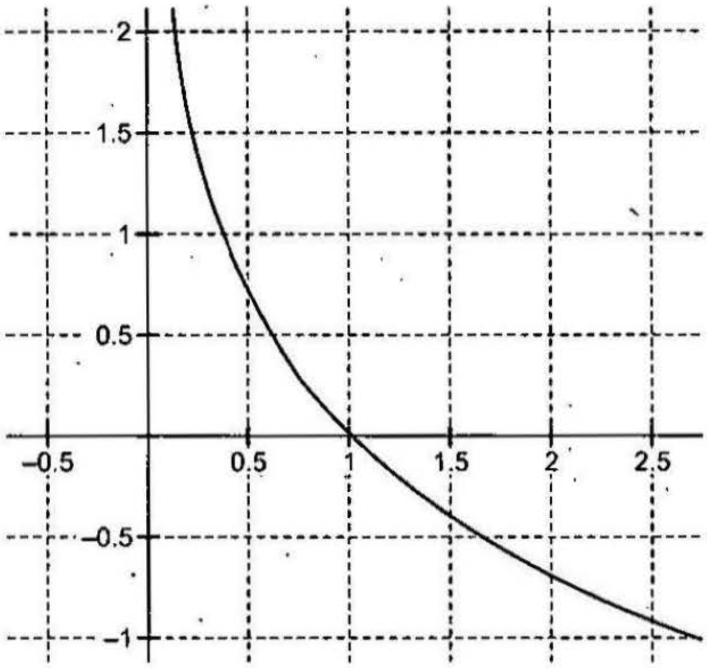


Fig. 1.5

When $0 < a < 1$, $f(x) = \log_a x$ is a decreasing function. Then for $x_2 > x_1 \Rightarrow \log_a x_2 < \log_a x_1$. Also $\log_a x_2 > x_1$ $\Rightarrow 0 < x_2 < a^{x_1}$

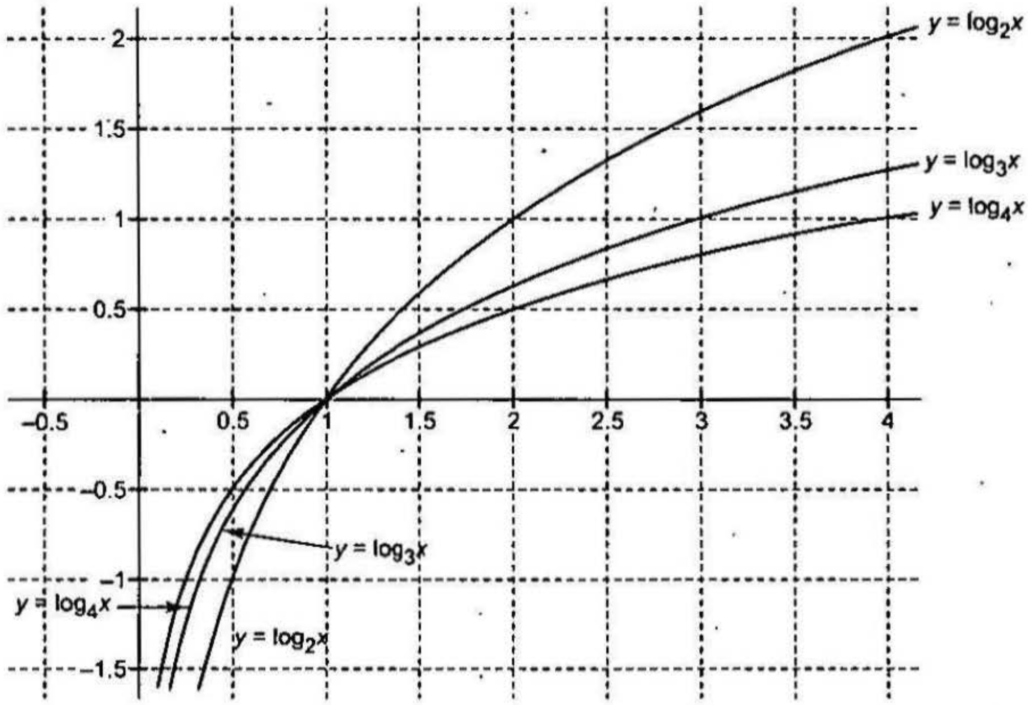


Fig. 1.6

The **natural logarithm** is the *logarithm* to the *base e*, where *e* is an *irrational* constant approximately equal to 2.718281828. Here, *e* is an irrational number. Also, *e* is defined exactly as $e = (1 + 1/m)^m$ as m increases to infinity. You can see how this definition produces *e* by inputting a large value of m like $m = 10{,}000{,}000$ to get $(1 + 1/10000000)^{10000000} = 2.7182817$ (rounded), which is very close to the actual value. The natural logarithm is generally written as $\ln(x)$, $\log_e(x)$.

The **common logarithm** is the *logarithm* with base 10. It is also known as the **decadic logarithm**, named after its base. It is indicated by $\log_{10}(x)$ On calculators, it is usually written as "log", but *mathematicians* usually mean *natural logarithm* rather than common logarithm when they write "log". To mitigate this ambiguity, the *ISO specification* is that $\log_{10}(x)$ should be **lg** (x) and $\log_e(x)$ should be **ln** (x).

FUNDAMENTAL LAWS OF LOGARITHMS

1. For $m, n, a > 0, a \neq 1$; $\log_a(mn) = \log_a m + \log_a n$

   Proof: Let $\log_a m = x$ and $\log_a n = y$.

   Then $\log_a m = x \Rightarrow a^x = m$ and, $\log_a n = y \Rightarrow a^y = n$

   $\therefore mn = a^x \cdot a^y$

   $\Rightarrow mn = a^{x+y} \Rightarrow \log_a(mn) = x + y \Rightarrow \log_a(mn) = \log_a m + \log_a n$

   In general for $x_1, x_2, \ldots, x_n$ are positive real numbers,

   $\log_a(x_1 x_2 \ldots x_n) = \log_a x_1 + \log_a x_2 + \ldots \log_a x_n$.

2. For $m, n, a > 0, a \neq 1$; $\log_a\left(\frac{m}{n}\right) = \log_a m - \log_a n$

   Proof: Let $\log_a m = x \Rightarrow a^x = m$ and $\log_a n = y \Rightarrow a^y = n$

   $\therefore \frac{m}{n} = \frac{a^x}{a^y} \Rightarrow \frac{m}{n} = a^{x-y} \Rightarrow \log_a\left(\frac{m}{n}\right) = x - y \Rightarrow \log_a\left(\frac{m}{n}\right) = \log_a m - \log_a n$

3. For $m, n, a > 0, a \neq 1$; $\log_a(m^n) = n. \log_a m$

   Proof: Let $\log_a m = x \Rightarrow a^x = m \Rightarrow (a^x)^n = m^n \Rightarrow a^{xn} = m^n \Rightarrow \log_a(m^n) = nx \Rightarrow \log_a(m^n) = n. \log_a m$

4. $\log_a 1 = 0$.

   Proof: Since $a^0 = 1$. Therefore, by definition of log, we have $\log_a 1 = 0$.

5. $\log_a a = 1$

   Proof: Since $a^1 = a$. Therefore, by definition of log, we have $\log_a a = 1$.

6. For $m, a, b > 0$ and $a \neq 1, b \neq 1$, then $\log_a m = \dfrac{\log_b m}{\log_b a}$

   Proof: Let $\log_a m = x$. Then, $a^x = m$

   Now, $a^x = m \Rightarrow \log_b(a^x) = \log_b m$ [Taking log to the base b]

   $\Rightarrow x \log_b a = \log_b m \Rightarrow \log_a m. \log_b a = \log_b m$ [$\because \log_m m = 1$]

   $\Rightarrow \log_a m = \dfrac{\log_b m}{\log_b a}$

   Replacing b by m in the above result, we get

   $\log_a m = \dfrac{\log_m m}{\log_m a} \Rightarrow \log_a m = \dfrac{1}{\log_m a}$ [$\because \log_m m = 1$]

7. For $a, n > 0$ and $a \neq 1$; $a^{\log_a n} = n$

   Proof: Let $\log_a n = x$. Then $a^x = n$. Therefore, $a^{\log_a n} = n$. [Putting the value of x in $a^x = n$]

   For example, $3^{\log_3 8} = 8$, $2^{3\log_2 5} = 2^{\log_2 5^3} = 5^3$, $5^{-2\log_5 3} = 5^{\log_5 3^{-2}} = 3^{-2} = 1/9$

8. $\log_{a^q} n^p = \dfrac{p}{q}\log_a n$, where $a, n > 0, a \neq 1$

   Proof: Let $\log_{a^q} n^p = x$ and $\log_a n = y$. Then, $(a^q)^x = n^p$ and $a^y = n$

Therefore, $a^{qx} = n^p$ and $a_y = n \Rightarrow a^{qx} = n^p$ and $(a^y)^p = n^p \Rightarrow a^{qx} = (a^y)^p \Rightarrow a^{qx} = a^{yp} \Rightarrow qx = yp \Rightarrow x = (p/2)y$

$\Rightarrow \log_{a^q} n^p = \dfrac{p}{q}\log_a n.$

9. $a^{\log_b c} = c^{\log_b a}$,

Proof: Let $a^{\log_b c} = p \Rightarrow \log_b c = \log_a p \Rightarrow \dfrac{\log c}{\log b} = \dfrac{\log p}{\log a} \Rightarrow \dfrac{\log a}{\log b} = \dfrac{\log p}{\log c} \Rightarrow \log_b a = \log_c p$

$\Rightarrow p = c^{\log_b a} \Rightarrow a^{\log_b c} = c^{\log_b a}$

Example 1.1 **Solve $(1/2)^{x^2-2x} < 1/4$.**

Sol. We have $(1/2)^{x^2-2x} < (1/2)^2$

It means $x^2 - 2x > 2$

$\Rightarrow (x-(1+\sqrt{3}))(x-(1-\sqrt{3})) > 0$

$\Rightarrow x > 1+\sqrt{3}$ or $x < 1-\sqrt{3}$

$\Rightarrow x \in (-\infty, 1-\sqrt{3}) \cup (1+\sqrt{3}, \infty)$

Example 1.2 **Solve $\dfrac{1-5^x}{7^{-x}-7} \geq 0$.**

Sol. $g(x) = \dfrac{5^x - 1}{7^{-x}-7} \leq 0$. Now $5^x - 1 = 0 \Rightarrow x = 0$ and $7^{-x} - 7 = 0 \Rightarrow x = -1$

Sign scheme of $g(x)$:

Fig. 1.7

Hence, from sign scheme of $g(x)$, $x \in (-\infty, -1) \cup [0, \infty)$.

Example 1.3 **Which of the following numbers are positive/negative?**

(i) $\log_2 7$ (ii) $\log_{0.2} 3$ (iii) log1/3(1/5)

(iv) $\log_4 3$ (v) $\log_2(\log_2 9)$

Sol. (i) Let $\log_2 7 = x \Rightarrow 7 = 2^x \Rightarrow x > 0$

(ii) Let $\log_{0.2} 3 = x \Rightarrow 3 = 0.2^x \Rightarrow x < 0$

(iii) Let $\log_{1/3}(1/5) = x \Rightarrow 1/5 = (1/3)^x \Rightarrow 5 = 3^x \Rightarrow x > 0$

(iv) Let $\log_4 3 = x \Rightarrow 3 = 4^x \Rightarrow x < 0$

(v) Let $\log_2(\log_2 9) = x \Rightarrow \log_2 9 = 2^x \Rightarrow 9 = 2^{2^x} \Rightarrow x > 0$

Example 1.4 **What is logarithm of $32\sqrt[5]{4}$ to the base $2\sqrt{2}$?**

Sol. $\log_{2\sqrt{2}} 32\sqrt[5]{4} = \log_{(2^{3/2})}(2^5 4^{\frac{1}{5}}) = \log_{(2^{3/2})}(2^{5+\frac{2}{5}}) = \dfrac{2}{3}\dfrac{27}{5}\log_2 2 = \dfrac{18}{5} = 3.6$

Example 1.5 **Find the value of $\log_5 \log_2 \log_3 \log_2 512$.**

Sol. $\log_5 \log_2 \log_3 \log_2 2^9 = \log_5 \log_2 \log_3 (9 \log_2 2)$

$= \log_5 \log_2 \log_3 3^2$ ($\log_a a = 1$ if $a > 1$ $a \neq 1$)

$= \log_5 \log_2 2 = \log_5 1 = 0.$

Example 1.6 If $\log_{\sqrt{8}} b = 3\frac{1}{3}$, then find the value of b.

Sol. $\log_{\sqrt{8}} b = 3\frac{1}{3} \Rightarrow \frac{2}{3}\log_2 b = \frac{10}{3} \Rightarrow \log_2 b = 5 \Rightarrow b = 2^5 = 32$

Example 1.7 If $n > 1$, then prove that $= \frac{1}{\log_2 n} + \frac{1}{\log_3 n} + \cdots + \frac{1}{\log_{53} n} = \frac{1}{\log_{53!} n}$.

Sol. The given expression is equal to $\log_n 2 + \log_n 3 + \cdots + \log_n 53 = \log_n (2.3 \ldots 53) = \log_n 53! = \frac{1}{\log_{53!} n}$

Example 1.8 Which is greater $x = \log_3 5$ or $y = \log_{17} 25$?

Sol. $\frac{1}{y} = \log_{25} 17 = \frac{1}{2}\log_5 17$ and $\frac{1}{x} = \log_5 3 = \frac{1}{2}\log_5 9$

$\therefore \frac{1}{y} > \frac{1}{x} \Rightarrow x > y$

Example 1.9 $y = 2^{\frac{1}{\log_x 4}}$, then find x in terms of y.

Sol. Since $y = 2^{\frac{1}{\log_x 4}}$, we get $\log_2 y = \frac{1}{\log_x 4}$ $\quad (\because x > 0, x \neq 1)$

$\Rightarrow \log_2 y = \log_4 x = \frac{1}{2}\log_2 x$

$\Rightarrow 2\log_2 y = \log_2 x$

$\Rightarrow \log_2 y^2 = \log_2 x$

$\therefore x = y^2$

Example 1.10 Find the value of $81^{(1/\log_5 3)} + 27^{\log_9 36} + 3^{4/\log_7 9}$.

Sol. $81^{(1/\log_5 3)} + 27^{\log_9 36} + 3^{4/\log_7 9} = (3^4)^{\log_3 5} + (3^3)^{\log_{3^2}(6^2)} + 3^{4\log_9 7}$

$= 3^{\log_3 5^4} + (3^3)^{\log_3 (6)} + 3^{4\log_{3^2} 7}$

$= 5^4 + 3^{\log_3 6^3} + 3^{2\log_3 7}$

$= 5^4 + 6^3 + 3^{\log_3 7^2}$

$= 625 + 216 + 7^2 = 890$

Example 1.11 Prove that number $\log_2 7$ is an irrational number.

Sol. Let $\log_2 7$ is a rational number

$\Rightarrow \log_2 7 = \frac{p}{q} \Rightarrow 7 = 2^{p/q} \Rightarrow 7^q = 2^p$ which is not possible for any integral values of p and q.

Hence, $\log_2 7$ is not rational.

Example 1.12 **Find the value of $\log_3 4 \log_4 5 \log_5 6 \log_6 7 \log_7 8 \log_8 9$.**

Sol. $\log_3 4 \log_4 5 \log_5 6 \log_6 7 \log_7 8 \log_8 9 = \frac{\log 4}{\log 3}\times\frac{\log 5}{\log 4}\times\frac{\log 6}{\log 5}\times\frac{\log 7}{\log 6}\times\frac{\log 8}{\log 7}\times\frac{\log 9}{\log 8}$

$$= \frac{\log 9}{\log 3} = \log_3 9 = 2$$

Example 1.13 **If $\log_e\left(\frac{a+b}{2}\right) = \frac{1}{2}(\log_e a + \log_e b)$, then find the relation between a and b.**

Sol. $\log_e\left(\frac{a+b}{2}\right) = \frac{1}{2}(\log_e a + \log_e b)$

$\Rightarrow \log_e\left(\frac{a+b}{2}\right) = (\log_e \sqrt{ab}) \Rightarrow \frac{a+b}{2} = \sqrt{ab}$

$\Rightarrow a + b - 2\sqrt{ab} = 0 \Rightarrow (\sqrt{a} - \sqrt{b})^2 = 0 \Rightarrow a = b$

Example 1.14 **If $a^x = b, b^y = c, c^z = a$, then find the value of xyz.**

Sol. $a^x = b, b^y = c, c^z = a \Rightarrow x = \log_a b, y = \log_b c, z = \log_c a$

$\Rightarrow xyz = (\log_a b)(\log_b c)(\log_c a) = \frac{\log b}{\log a}\frac{\log c}{\log b}\frac{\log a}{\log c} = 1$

Example 1.15 **If $\frac{x(y+z-x)}{\log x} = \frac{y(z+x-y)}{\log y} = \frac{z(x+y-z)}{\log z}$, prove that $x^y y^x = z^y y^z = x^z z^x$.**

Sol. Let $\frac{x(y+z-x)}{\log_a x} = \frac{y(z+x-y)}{\log_a y} = \frac{z(x+y-z)}{\log^a z} = k$

$\Rightarrow \log_a x = \frac{x(y+z-x)}{k} \Rightarrow x = a^{\frac{x(y+y-z)}{k}}$

Similarly, $y = a^{\frac{y(z+x-z)}{k}}$ and $z = a^{\frac{z(x+y-z)}{k}}$

Now $x^y y^x = a^{\frac{xy(y+z-x)}{k}} a^{\frac{yx(z+x-y)}{k}} = a^{\frac{xy^2+xyz-x^2y+xyz+x^2y-xy^2}{k}} = a^{\frac{2xyz}{k}}$

Similarly, $z^y y^z = x^z z^x = a^{\frac{2xyz}{k}}$.

Example 1.16 **Which of the following is greater: $m = (\log_2 5)^2$ or $n = \log_2 20$?**

Sol. $m - n = (\log_2 5)^2 - [\log_2 5 + 2]$
Let $\log_2 5 = x \Rightarrow m - n = x^2 - x - 2 = (x-2)(x+1) = (\log_2 5 - 2)(\log_2 5 + 1) > 0$
Hence, $m > n$.

Example 1.17 **If $\log_{12} 27 = a$, then find $\log_6 16$ in terms of a.**

a. $2\left(\frac{3-a}{3+a}\right)$ **b.** $3\left(\frac{3-a}{3+a}\right)$ **c.** $4\left(\frac{3-a}{3+a}\right)$ **d.** $5\left(\frac{3-a}{3+a}\right)$

Sol. Since $a = \log_{12} 27 = \log_{12}(3)^3 = 3\log_{12} 3$, we get

$$\frac{3}{\log_3 12} = \frac{3}{1+\log_3 4} = \frac{3}{1+2\log_3 2}$$

$$\therefore \log_3 2 = \frac{3-a}{2a}$$

Then, $\log_6 16 = \log_6 2^4 = 4\log_6 2 = \dfrac{4}{\log_2 6} = \dfrac{4}{1+\log_2 3} = \dfrac{4}{1+\dfrac{2a}{3-a}} = 4\left(\dfrac{3-a}{3+a}\right)$

Example 1.18 **Simplify** $\dfrac{1}{1+\log_a bc} + \dfrac{1}{1+\log_b ca} + \dfrac{1}{1+\log_c ab}$.

Sol. $\dfrac{1}{1+\log_a bc} + \dfrac{1}{1+\log_b ca} + \dfrac{1}{1+\log_c ab}$

$$= \frac{\log a}{\log a + \log b + \log c} + \frac{\log b}{\log a + \log b + \log c} + \frac{\log c}{\log a + \log b + \log c} = 1$$

Example 1.19 **If $y^2 = xz$ and $a^x = b^y = c^z$, then prove that $\log_b a = \log_c b$.**

Sol. $a^x = b^y = c^z \Rightarrow x\log a = y\log b = z\log c$

$$\therefore \frac{y}{x} = \frac{z}{y} \Rightarrow \frac{\log a}{\log b} = \frac{\log b}{\log c}$$

$$\Rightarrow \log_b a = \log_c b$$

Example 1.20 **Suppose $x, y, z > 0$ and not equal to 1 and $\log x + \log y + \log z = 0$. Find the value of**

$$x^{\frac{1}{\log y}+\frac{1}{\log z}} \times y^{\frac{1}{\log z}+\frac{1}{\log x}} \times z^{\frac{1}{\log x}+\frac{1}{\log y}} \quad \textbf{(base 10)}$$

Sol. Let $K = x^{\frac{1}{\log y}+\frac{1}{\log z}} \times y^{\frac{1}{\log z}+\frac{1}{\log x}} \times z^{\frac{1}{\log x}+\frac{1}{\log y}}$

$$\log K = \log x\left[\frac{1}{\log y} + \frac{1}{\log z}\right] + \log y\left[\frac{1}{\log z} + \frac{1}{\log x}\right] + \log z\left[\frac{1}{\log x} + \frac{1}{\log y}\right] \quad \text{(i)}$$

Putting $\log x + \log y + \log z = 0$ (given), we get

$$\frac{\log x}{\log y} + \frac{\log z}{\log y} = -1; \frac{\log y}{\log x} + \frac{\log z}{\log x} = -1 \text{ and } \frac{\log x}{\log z} + \frac{\log y}{\log z} = -1$$

Therefore, R.H.S. of Eq. (i) $= -3 \Rightarrow \log_{10} K = -3 \Rightarrow K = 10^{-3}$

Example 1.21 **Prove that** $\dfrac{2^{\log_{2^{1/4}} x} - 3^{\log_{27}(x^2+1)^3} - 2x}{7^{4\log_{49} x} - x - 1} > 0; \forall x \in R.$

Sol. $y = \dfrac{2^{\log_{2^{1/4}} x} - 3^{\log_{27}(x^2+1)^3} - 2x}{7^{4\log_{49} x} - x - 1}$

$$= \frac{x^4 - (x^2+2x+1)}{x^2 - x - 1}$$

$= x^2 + x + 1$

$= (x+1/2)^2 + 3/4 > 0; \forall x \in R$

Concept Application Exercise 1.1

1. Prove that $\log_7 \log_7 \sqrt{7\sqrt{(7\sqrt{7})}} = 1 - 3\log_7 2$.
2. Solve for x and y : $y^x = x^y$; $x = 2y$.
3. Find the value of $3^{2\log_9 3}$.
4. If $\log_{10} x = y$, then find $\log_{1000} x^2$ in terms of y.
5. If $\log_7 2 = m$, then find $\log_{49} 28$ in terms of m.
6. Find the value of $\sqrt{(\log_{0.5}^2 4)}$.
7. Find the value of $7\log\left(\frac{16}{15}\right) + 5\log\left(\frac{25}{24}\right) + 3\log\left(\frac{81}{80}\right)$.
8. If $a^2 + b^2 = 7ab$, prove that $\log\left(\frac{a+b}{3}\right) = \frac{1}{2}(\log a + \log b)$.
9. Prove the following identities:

   (i) $\frac{\log_a n}{\log_{ab} n} = 1 + \log_a b$ (ii) $\log_{ab} x = \frac{\log_a x \log_b x}{\log_a x + \log_b x}$
10. If $\log_a (ab) = x$, then evaluate $\log_b (ab)$ in terms of x.
11. Compute $\log_{ab} (\sqrt[3]{a} / \sqrt{b})$ if $\log_{ab} a = 4$.
12. If $a^x = b^y = c^z = d^w$, show that $\log_a (bcd) = x\left(\frac{1}{y} + \frac{1}{z} + \frac{1}{w}\right)$.
13. Solve for x: $11^{4x-5} \cdot 3^{2x} = 5^{3-x} \cdot 7^{-x}$.
14. If $\log_b n = 2$ and $\log_n 2b = 2$, then find the value of b.
15. Suppose that a and b are positive real numbers such that $\log_{27}a + \log_9 b = 7/2$ and $\log_{27}b + \log_9 a = 2/3$. Then find the value of ab.

LOGARITHMIC EQUATIONS

While solving logarithmic equations, we tend to simplify the equation. Solving equation after simplification may give some roots which are not defining all the terms in the initial equation. Thus, while solving equations involving logarithmic function, we must take care of domain of the equation.

Example 1.22 **Solve $\log_4 8 + \log_4(x+3) - \log_4(x-1) = 2$.**

Sol. $\log_4 8 + \log_4(x+3) - \log_4(x-1) = 2$

$$\Rightarrow \log_4 \frac{8(x+3)}{x-1} = 2 \Rightarrow \frac{8(x+3)}{x-1} = 4^2$$

$$\Rightarrow x + 3 = 2x - 2 \Rightarrow x = 5$$

Also for $x = 5$ all terms of the equation are defined.

Example 1.23 **Solve $\log(-x) = 2\log(x+1)$.**

Sol. By definition, $x < 0$ and $x + 1 > 0$. $\Rightarrow -1 < x < 0$

Now $\log(-x) = 2\log(x+1) \Rightarrow -x = (x+1)^2 \Rightarrow x^2 + 3x + 1 = 0$

$$\Rightarrow x = \frac{-3+\sqrt{5}}{2}, \frac{-3-\sqrt{5}}{2}$$

Hence, $x = \frac{-3+\sqrt{5}}{2}$ is the only solution.

Example 1.24 **Solve $\log_2(3x-2) = \log_{1/2} x$.**

Sol. $\log_2(3x-2) = \log_{1/2} x = \dfrac{\log_2 x}{\log_2 2^{-1}} = \log_2 x^{-1}$

$\Rightarrow 3x - 2 = x^{-1} \Rightarrow 3x^2 - 2x = 1 \Rightarrow x = 1$ or $x = -1/3$. But $\log_2(3x-2)$ and $\log_{1/2} x$ are meaningful if $x > 2/3$.

Hence, $x = 1$.

Example 1.25 **Solve $2^{x+2}27^{x/(x-1)} = 9$.**

Sol. Taking log of both sides, we have $(x+2)\log 2 + \dfrac{x}{x-1}\log 27 = \log 9$

$$\Rightarrow (x+2)\log 2 + \frac{x}{x-1} 3\log 3 = 2\log 3$$

$$\Rightarrow (x+2)\log 2 + \left(\frac{3x}{x-1} - 2\right)\log 3 = 0$$

$$\Rightarrow (x+2)\left[\log 2 + \frac{\log 3}{x-1}\right] = 0$$

$$\Rightarrow x = -2 \text{ or } x - 1 = -\frac{\log 3}{\log 2}$$

$$\Rightarrow x = -2, 1 - \frac{\log 3}{\log 2}.$$

Example 1.26 **Solve $\log_2(4 \times 3^x - 6) - \log_2(9^x - 6) = 1$.**

Sol. $\log_2(4 \times 3^x - 6) - \log_2(9^x - 6) = 1$

$$\Rightarrow \log_2 \frac{4 \times 3^x - 6}{9^x - 6} = 1$$

$$\Rightarrow \frac{4 \times 3^x - 6}{9^x - 6} = 2$$

$\Rightarrow 4y - 6 = 2y^2 - 12$ (putting $3^x = y$)

$\Rightarrow y^2 - 2y - 3 = 0$

$\Rightarrow y = -1, 3$

$\Rightarrow 3^x = 3$

$\Rightarrow x = 1$

Example 1.27 **Solve $6(\log_x 2 - \log_4 x) + 7 = 0$.**

Sol. $6(\log_x 2 - \log_4 x) + 7 = 0$

$$\Rightarrow 6\left(\log_x 2 - \frac{1}{2}\log_2 x\right) + 7 = 0$$

$$\Rightarrow 6\left(\frac{1}{y} - \frac{y}{2}\right) + 7 = 0 \text{ (where } y = \log_2 x)$$

$$\Rightarrow 6\left(\frac{2-y^2}{2y}\right) + 7 = 0$$

$$\Rightarrow 3\left(\frac{2-y^2}{y}\right) + 7 = 0$$

$$\Rightarrow 6 - 3y^2 + 7y = 0$$
$$\Rightarrow 3y^2 - 7y - 6 = 0$$
$$\Rightarrow 3y^2 + 2y - 9y - 6 = 0$$
$$\Rightarrow (y-3)(3y+2) = 0$$
$$\Rightarrow y = 3 \text{ or } y = -2/3$$
$$\Rightarrow \log_2 x = 3 \text{ or } -2/3$$
$$\Rightarrow x = 8 \text{ or } x = 2^{-2/3}$$

Example 1.28 **Find the number of solution to equation $\log_2(x+5) = 6 - x$.**

Sol. Here, $x + 5 = 2^{6-x}$.

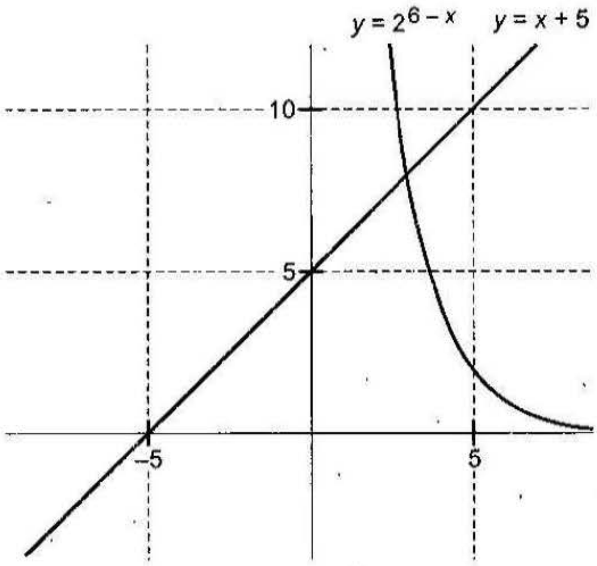


Fig. 1.8

Now graph of $y = x + 5$ and $y = 2^{6-x}$ intersect only once.
Hence, there is only one solution.

Example 1.29 **Solve $4^{\log_2 \log x} = \log x - (\log x)^2 + 1$ (base is e).**

Sol. $\log_2 \log x$ is meaningful if $x > 1$.

Since $4^{\log_2 \log x} = 2^{2\log_2 \log x} = (2^{\log_2 \log x})^2 = (\log x)^2$ $(a^{\log_a x} = x, a > 0, a \neq 1)$

So the given equation reduces to $2(\log x)^2 - \log x - 1 = 0$.

Therefore, $\log x = 1$, $\log x = 1/2$. But for $x > 1$,

$\log x > 0$ so $\log \log x = 1$, i.e., $x = e$.

Example 1.30 Solve $4\log_{x/2}\left(\sqrt{x}\right)+2\log_{4x}\left(x^2\right)=3\log_{2x}\left(x^3\right)$

Sol. $\dfrac{4\log_2\sqrt{x}}{\log_2(x/2)}+\dfrac{2\log_2(x^2)}{\log_2(4x)}=\dfrac{3\log_2(x^3)}{\log_2(2x)}$

$$\Rightarrow \frac{4\times\frac{1}{2}\log_2(x)}{\log_2 x-1}+\frac{4\log_2(x)}{2+\log_2(x)}=\frac{9\log_2(x)}{1+\log_2(x)}$$

Let $\log_2 x=t$, given equation reduces to

$$\frac{2t}{t-1}+\frac{4t}{t+2}=\frac{9t}{t+1}$$

$$\Rightarrow t=0 \text{ or } \frac{2}{t-1}+\frac{4}{t+2}=\frac{9}{t+1}$$

$$\Rightarrow \frac{2t+4+4t-4}{(t-1)(t+2)}=\frac{9}{t+1}$$

$\Rightarrow t^2+t-6=0$

$\Rightarrow (t+3)(t-2)=0$

$\Rightarrow t=0, 2 \text{ or } -3$

$\Rightarrow x=1, 4, 1/8$

Example 1.31 Solve $4^{\log_9 x}-6x^{\log_9 2}+2^{\log_3 27}=0.$

Sol. Let $2^{\log_9 x}=y$, we get $y^2-6y+8=0$

$\Rightarrow y=4 \text{ or } 2$

If $2^{\log_9 x}=2^2 \Rightarrow \log_9 x=2 \Rightarrow x=81$

If $2^{\log_9 x}=2^1 \Rightarrow \log_9 x=1 \Rightarrow x=9$

Concept Application Exercise 1.2

1. Solve $\log_2(25^{x+3}-1)=2+\log_2(5^{x+3}+1)$.
2. Solve $\log_4(2\times 4^{x-2}-1)+4=2x$.
3. Solve $\log_5(5^{1/x}+125)=\log_5 6+1+\dfrac{1}{2x}$.
4. Solve $\log_4(x-1)=\log_2(x-3)$.
5. Solve $\log_6 9-\log_9 27+\log_8 x=\log_{64}x-\log_6 4$.
6. Solve $\log_2\left(2\sqrt{17-2x}\right)=1-\log_{1/2}(x-1)$.
7. Solve $3\log_x 4+2\log_{4x}4+3\log_{16x}4=0$.
8. Solve $(\log_3 x)(\log_5 9)-\log_x 25+\log_3 2=\log_3 54$.
9. Solve $(x^{\log_{10}3})^2-(3^{\log_{10}x})-2=0$.

LOGARITHMIC INEQUALITIES

Standard Logarithmic Inequalities

1. If $\log_a x > \log_a y \Rightarrow \begin{cases} x > y, \text{ if } a > 1 \\ 0 < x < y, \text{ if } 0 < a < 1 \end{cases}$

2. If $\log_a x > y \Rightarrow \begin{cases} x > a^y, \text{ if } a > 1 \\ 0 < x < a^y, \text{ if } 0 < a < 1 \end{cases}$

3. $\log_a x > 0 \Rightarrow x > 1$ and $a > 1$
   or $0 < x < 1$ and $0 < a < 1$

Frequently Used Inequalities

1. $(x-a)(x-b) < 0 \ (a < b) \Rightarrow a < x < b$
2. $(x-a)(x-b) > 0 \ (a < b) \Rightarrow x < a$ or $x > b$
3. $|x| < a \Rightarrow -a < x < a$
4. $|x| > a \Rightarrow x < -a$ or $x > a$

Example 1.32 **Solve $\log_2(x-1) > 4$.**

Sol. $\log_2(x-1) > 4 \Rightarrow x - 1 > 2^4 \Rightarrow x > 17$

Example 1.33 **Solve $\log_3(x-2) \le 2$.**

Sol. $\log_3(x-2) \le 2 \Rightarrow 0 < x - 2 \le 3^2 \Rightarrow 2 < x \le 11$

Example 1.34 **Solve $\log_{0.3}(x^2 - x + 1) > 0$.**

Sol. $\log_{0.3}(x^2 - x + 1) > 0 \Rightarrow 0 < x^2 - x + 1 < (0.3)^0$

$\Rightarrow 0 < x^2 - x + 1 < 1 \Rightarrow x^2 - x + 1 > 0$ and $x^2 - x < 0 \Rightarrow x(x-1) < 0$

$\Rightarrow 0 < x < 1$ (as $x^2 - x + 1 = (x - 1/2)^2 + 3/4 > 0$ for all real x)

Example 1.35 **Solve $1 < \log_2(x-2) \le 2$.**

Sol. $1 < \log_2(x-2) \le 2 \Rightarrow 2^1 < x - 2 \le 2^2$

$\Rightarrow 4 < x < 6$

Example 1.36 **Solve $\log_2|x-1| < 1$.**

Sol. $\log_2|x-1| < 1 \Rightarrow 0 < |x-1| < 2^1$

$\Rightarrow -2 < x - 1 < 2$ and $x - 1 \ne 0$

$\Rightarrow -1 < x < 3$ and $x \ne 1$

$\Rightarrow x \in (-1, 3) - \{1\}$

Example 1.37 **Solve $\log_{0.2}|x-3| \ge 0$.**

Sol. $\log_{0.2}|x-3| \ge 0 \Rightarrow 0 < |x-3| \le (0.2)^0$

$\Rightarrow 0 < |x-3| \le 1 \Rightarrow -1 < x - 3 \le 1$ and $x - 3 \ne 0$

$\Rightarrow 2 < x \le 4$ and $x \ne 3 \Rightarrow x \in (2, 4] - \{3\}$

Example 1.38 Solve $\log_2 \frac{x-1}{x-2} > 0$.

Sol. $\log_2 \frac{x-1}{x-2} > 0 \Rightarrow \frac{x-1}{x-2} > 2^0$

$\Rightarrow \frac{x-1}{x-2} > 1 \Rightarrow \frac{x-1}{x-2} - 1 > 0$

$\Rightarrow \frac{x-1-x+2}{x-2} > 0 \Rightarrow \frac{1}{x-2} > 0 \Rightarrow x > 2$

Example 1.39 Solve $\log_{0.5} \frac{3-x}{x+2} < 0$.

Sol. $\log_{0.5} \frac{3-x}{x+2} < 0 \Rightarrow \frac{3-x}{x+2} > (0.5)^0$

$\Rightarrow \frac{3-x}{x+2} > 1 \Rightarrow \frac{3-x}{x+2} - 1 > 0$

$\Rightarrow \frac{3-x-x-2}{x+2} > 0 \Rightarrow \frac{2x-1}{x+2} < 0$

$\Rightarrow -2 < x < 1/2$

Example 1.40 Solve $\log_3(2x^2 + 6x - 5) > 1$.

Sol. $\log_3(2x^2 + 6x - 5) > 1 \Rightarrow 2x^2 + 6x - 5 > 3^1$

$\Rightarrow 2x^2 + 6x - 8 > 0 \Rightarrow x^2 + 3x - 4 > 0$

$\Rightarrow (x-1)(x+4) > 0 \Rightarrow x < -4$ or $x > 1$

Example 1.41 Solve $\log_{0.04}(x-1) \ge \log_{0.2}(x-1)$.

Sol. $\log_{0.04}(x-1) \ge \log_{0.2}(x-1)$

$\Rightarrow \log_{(0.2^2)}(x-1) \ge \log_{0.2}(x-1)$

$\Rightarrow \frac{1}{2}\log_{0.2}(x-1) \ge \log_{0.2}(x-1)$

$\Rightarrow \log_{0.2}(x-1) \ge 2\log_{0.2}(x-1)$

$\Rightarrow \log_{0.2}(x-1) \ge \log_{0.2}(x-1)^2$

$\Rightarrow (x-1) \le (x-1)^2$

$\Rightarrow (x-1)^2 - (x-1) \ge 0$

$\Rightarrow (x-1)(x-1-1) \ge 0$

$\Rightarrow (x-1)(x-2) \ge 0$

$\Rightarrow x \le 1$ or $x \ge 2$

Also, $x > 1$; hence, $x \ge 2$.

Example 1.42 Solve $\log_{(x+3)}(x^2 - x) < 1$.

Sol. $\log_{x+3}(x^2 - x) < 1$

$x(x-1) > 0 \Rightarrow x > 1$ or $x < 0$ (i)

If $x + 3 > 1 \Rightarrow x > -2$

then $x^2 - x < x + 3$

$\Rightarrow x^2 - 2x - 3 < 0$

$\Rightarrow (x-3)(x+1)<0$

Hence, $x \in (-1, 0) \cup (1, 3)$

If $0 < x+3 < 1 -3 < x < -2$, then (i)

$x^2 - x > x + 3$

$\Rightarrow x^2 - 2x - 3 > 0$

$\Rightarrow (x-3)(x+1) > 0$ (ii)

$\Rightarrow x \in (-3, -2)$

Example 1.43 Solve $2\log_3 x - 4\log_x 27 \le 5$ $(x > 1)$.

Sol. Let $\log_3 x = y \Rightarrow x = 3y$ (i)

Therefore, the given inequality $2\log_3 x - 12\log_x 3 \le 5$

$$\Rightarrow 2y - \frac{12}{y} \le 5$$

$\Rightarrow 2y^2 - 5y - 12 \le 0$ (as $x > 1 \Rightarrow y > 0$)

$\Rightarrow (2y+3)(y-4) \le 0$

$$\Rightarrow y \in \left[-\frac{3}{2}, 4\right] \Rightarrow -\frac{3}{2} \le \log_3 x \le 4$$

$\Rightarrow 3^{-3/2} \le x \le 81$

Concept Application Exercise 1.3

1. Solve $\log_3|x| > 2$.
2. Solve $\log_2 \dfrac{x-4}{2x+5} < 1$.
3. Solve $\log_{10}(x^2 - 2x - 2) \le 0$.
4. Let $f(x) = \sqrt{\log_{10} x^2}$. Find the set of all values of x for which $f(x)$ is real.
5. Solve $2^{\log_2(x-1)} > x + 5$.
6. Solve $\log_2|4 - 5x| > 2$.
7. Solve $\log_{0.2} \dfrac{x+2}{x} \le 1$.
8. Solve $\log_{1/2}(x^2 - 6x + 12) \ge -2$.
9. Solve $\log_{1-x}(x-2) \ge -1$.
10. Solve $\log_3(x+2)(x+4) + \log_{1/3}(x+2) < \dfrac{1}{2}\log_{\sqrt{3}} 7$.
11. Solve $\log_x(x^2 - 1) \le 0$.

FINDING LOGARITHM

To calculate the logarithm of any positive number in decimal form, we always express the given positive number in decimal form as the product of an integral power of 10 and a number between 1 and 10, i.e., any positive number k in decimal form is written in the form as

$$K = m \times 10^p,$$

where p is an integer and $1 \le m < 10$. This is called the standard from of k.

Characteristic and Mantissa of a Logarithm

Let n be a positive real number and let $m \times 10^p$ be the standard fom of n. Then $n = m \times 10^p$ where p is an integer and m is a real number between 1 and 10, i.e., $1 \le m < 10$

$$\Rightarrow \quad \log_{10} n = \log_{10}(m \times 10^p)$$
$$= \log_{10} m + \log_{10} 10^p$$
$$= \log_{10} m + p \log_{10} 10$$
$$= p + \log_{10} m$$

Here p is an integer and $1 \le m < 10$. Now, $1 \le m < 10$

$\Rightarrow \quad \log_{10} 1 \le \log_{10} m < \log_{10} 10$

$\Rightarrow \quad 0 \le \log_{10} m < 1$.

Thus, the logarithm of positive real number n consists of two parts:

(i) The integral part p, which is positive, negative or zero, is called characteristic.
(ii) The decimal part $\log m$, which is a real number between 0 and 1, is called mantissa.

Thus, $\log n$ = Characteristic + Mantissa.

Note that it is only the characteristic that changes when the decimal point is moved. An advantage of using the base 10 is thus revealed: if the characteristic is known, the decimal point may easily be placed. If the number is known, the characteristic may be determined by inspection; that is, by observing the location of the decimal point.

Although an understanding of the relation of the characteristic to the powers of 10 is necessary for thorough comprehension of logarithms, the characteristic may be determined mechanically by the application of the following rules:

1. For a number greater than 1, the characteristic is positive and is one less than the number of digits to the left of the decimal point in the number.
2. For a positive number less than 1, the characteristic is negative and has an absolute value one more than the number of zeros between the decimal point and the first non-zero digit of the number.

Example 1.44 **Write the characteristic of each of the following numbers by using their standard forms:**

(i) 1235.5 **(ii) 346.41** **(iii) 62.723** **(iv) 7.12345**
(v) 0.35792 **(vi) 0.034239** **(viii) 0.002385** **(viii) 0.0009468**

Sol.

Number	Standard Form	Characteristic
1235.5	1.2355×10^3	3
346.41	3.4641×10^2	2
62.723	6.2723×10^1	1
7.12345	7.12345×10^0	0
0.35792	3.5792×10^{-1}	-1
0.034239	3.4239×10^{-2}	-2
0.002385	2.385×10^{-3}	-3
0.0009468	9.468×10^{-4}	-4

Mantissa of the Logarithm of a Given Number

The logarithm table is used to find the mantissa of logarithms of numbers. It contains 90 rows and 20 column.

Every row beings with a two-digit number 10, 11, 12, ..., 98, 99 and every column is headed by a one-digit number 0, 1, 2, 3, ..., 9. On the right of the table, we have a big column which is divided into 9 sub-columns headed by the digit 1, 2, 3, ..., 9. This column is called the column of mean differences.

Note that the position of the decimal point in a number is immaterial for finding the mantissa. To find the mantissa of a number, we consider first four digits from the left most side of the number. If the number in the decimal form is less than one and it has four or more consecutive zeros to the right of the decimal point, then its mantissa is calculated with the help of the number formed by digits beginning with the first non-zero digit. For example , to find the mantissa of 0.000032059, we consider the number 3205. If the given number has only one digit, we replace it by a two-digit number obtained by adjoining zero to the right of the number. Thus, 2 is to be replaced by 20 for finding the mantissa.

Significant Digits

The digits used to compute the mantissa of a given number are called its significant digits.

Example 1.45 **Write the significant digits in each of the following numbers to compute the mantissa of their logarithms :**

(i) 3.239 **(ii) 8** **(iii) 0.9** **(iv) 0.02**
(v) 0.0367 **(vi) 89** **(vii) 0.0003** **(viii) 0.00075**

Sol.

Number	Significant digits to find the mantissa of its logarithm
3.239	3239
8	80
0.9	90
0.02	20
0.0367	367
89	89
0.0003	30
0.00075	75

NEGATIVE CHARACTERISTICS

When a characteristic is negative, such as –2, we do not perform the subtraction, since this would involve a negative mantissa. There are several ways of indicating a negative characteristic. Mantissas as presented in the table in the appendix are always positive and the sign of the characteristic is indicated separately. For example, where $\log 0.023 = \bar{2}.36173$, the bar over the 2 indicates that only the characteristic is negative, that is, the logarithm is $-2 + 0.36173$.

Example 1.46 **Find the mantissa of the logarithm of the number 5395.**

Sol. To find the mantissa of log 5395, we first look into the row starting with 53. In this row, look at the number in the column headed by 9. The number is 7316.

	0	1	2	3	4	5	6	7	8	9	Mean Differences								
											1	2	3	4	5	6	7	8	9
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	2	3	4	5	6	7	7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6	7

Fig. 1.9

Now, move to the column of mean differences and look under the column headed by 5 in the row corresponding to 53. We see the number 4 there. Add this number 4 to 7316 to get 7320. This is the required mantissa of log 5395.

If we wish to find the log 5395, then we compute its characteristic also.

Clearly, the characteristic is 3. So, log 5395 = 3.7320.

Example 1.47 **Find the mantissa of the logarithm of the number 0.002359.**

Sol. The first four digits beginning with the first non-zero digit on the right of the decimal point form the number 2359. To find the mantissa of log (0.002359), we first look in the row starting with 23. In this row, look at the number in the column headed by 5. The number is 3711.

	0	1	2	3	4	5	6	7	8	9	Mean Differences								
											1	2	3	4	5	6	7	8	9
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	15	18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15

Fig. 1.10

Now, move to the column of mean difference and look under the column headed by 9 the row corresponding to 23. We see the number 17 there.

Add this number to 3711. We get the number 3728. This is the required mantissa of log (0.002359).

Mantissa of log 23.598, log 2.3598 and 0.023598 is the same (only characteristic are different).

Example 1.48 **Use logarithm tables to find the logarithm of the following numbers:**

(i) 25795 **(ii) 25.795**

Sol.

(i) The characteristic of the logarithm of 25795 is 4.
To find the mantissa of the logarithm of 25795, we take the first four digits.
The number formed by the first four digits is 2579. Now, we look in the row starting with 25. In this row, look at the number in the column headed by 7. The number is 4099. Now, move to the column of mean differences and look under the column headed by 9 in the row corresponding to 25. We see that the number there is 15.
Add this number to 4099. We get the number 4114. This is the required mantissa. Hence, log (25795) = 4.4114

(ii) The characteristic of the logarithm of 25.795 is 1, because there are two digits to the left of the decimal point. The mantissa is the same as in the above question. Hence, log 25.795 = 1.4114.

Similarly, $\log 2.5795 = 0.4114$. and $\log(0.25795) = -1 + 0.4114 = \bar{1}.4114$
Here $-1 + 04114$ cannot be written as -1.4114, as -1.4114 is a negative number of magnitude 1.4114, whereas $-1 + 0.4114$ is equal to -05886. In order to avoid this confusion, we write $\bar{1}$ for -1 and thus $\log(0.25795) = \bar{1}.4114$.

ANTILOGARITHM

The positive number n is called the antilogarithm of a number m if $\log n = m$. If n is antilogarithm of m, we write $n = \text{antilog } m$. For example,

(i) $\log 100 = 2 \quad \Leftrightarrow \quad \text{antilog } 2 = 100$

(ii) $\log 431.5 = 2.6350 \quad \Leftrightarrow \quad \text{antilog}(2.6350) = 431.5$

(iii) $\log 0.1257 = 1.993 \quad \Leftrightarrow \quad \text{antilog}(1.993) = 0.1257$

To find the antilog of a given number, we use the antilogarithm tables given at the end of the book. To find n, when $\log n$ is given, we use only the mantissa part. The characteristic is used only in determining the number of digits in the integral part or the number of zeros on the right side of the decimal point in the required number.

To Find Antilog of a Number

Step I: Determine whether the decimal part of the given number is positive or negative. If it is negative, make it positive by adding 1 to the decimal part and by subtracting 1 from the integral part.
For example, in -2.5983, the decimal part is -0.5983 which is negative. So, write

$$\begin{aligned}-2.5983 &= -2-0.5983\\ &= -2-1+1-0.5983\\ &= -3+0.4017\\ &= \bar{3}.4017\end{aligned}$$

Step II: In the antilogarithm table, look into the row containing the first two digits in the decimal part of the given number.

Step III: In the row obtained in step II, look at the number in the column headed by the third digit in the decimal part.

Step IV: In the row chosen in step III, move in the column of mean differences and look at the number in the column headed by the fourth digit in the decimal part. Add this number to number obtained in step III.

Step V: Obtain the integral part (Characteristic) of the given number.
If the characteristic is positive and is equal to n, then insert decimal point after $(n + 1)$ digits in the number obtained in step IV.

If $n > 4$, then write zeros on the right side to get $(n + 1)$ digits.
If the characteristic is negative and is equal to $-n$ or $\bar{n}$, then on the right side of decimal point write $(n - 1)$ consecutive zeros and then write the number obtained in step IV.

Example 1.49 **Find the antilogarithm of each of the following :**

(i) 2.7523 **(ii) 3.7523** **(iii) 5.7523** **(iv) 0.7523**

(v) $\bar{1}.7523$ **(vi) $\bar{2}.7523$** **(vii) $\bar{3}.7523$**

Sol.

(i) The mantissa of 2.7523 is positive and is equal to 0.7523.
Now, look into the row starting 0.75. In this row, look at the number in the column headed by 2. The number is 5649. Now in the same row move in the column of mean differences and look at the number in the column headed by 3. The number there is 4. Add this number to 5649 to get 5653. The characteristic is 2. So, the decimal point is put after 3 digits to get 565.3.
Hence, antilog (2.7523) = 565. 3.

(ii) The mantissa of 3.7523 is the same as the mantissa of the number in (i), but the characteristic is 3.
Hence, antilog (3.7523) = 5653.0.

(iii) The mantissa of 5.7523 is the same as the mantissa of the number in (i), but the characteristic is 5.
Hence, antilog (5.7523) = 565300.0.

(iv) Proceeding as above, we have antilog (0.7523) = 5.653.

(v) In this case, the characteristic is $\bar{1}$, i.e., -1.

Hence, antilog $(\bar{1}.7523) = 0.5653$.

(vi) In this case, the characteristic is $\bar{2}$, i.e., -2. So, we write one zero on the right side of the decimal point.

Hence, antilog $(\bar{2}.7523) = 0.05653$.

(vii) Proceeding as above, antilog $(\bar{3}.7523) = 0.005653$.

Example 1.50 **Evaluate $\sqrt[3]{72.3}$, if $\log 0.723 = \bar{1}.8591$.**

Sol. Let $x = \sqrt[3]{72.3}$.

Then, $\log x = (72.3)^{1/3} \Rightarrow \log x = \dfrac{1}{3} \log 72.3$

$\Rightarrow \log x = \dfrac{1}{3} \times 1.8591 \Rightarrow \log x = 0.6197$

$\Rightarrow x = \text{antilog}\,(0.6197)$
$\Rightarrow x = 4.166$ (using antilog table)

Example 1.51 **Using logarithms, find the value of 6.45 × 981.4.**

Sol. Let $x = 6.45 \times 981.4$,
Then, $\lg x = \log(6.45 \times 981.4)$
$= \log 6.45 + \log 981.4$
$= 0.8096 + 2.9919$ (using log table)
$= 3.8015$
$\therefore x = \text{antilog}\,(3.8015) = 6331$ (using antilog table)

Example 1.52 **Let $x = (0.15)^{20}$. Find the characteristic and mantissa of the logarithm of x to the base 10. Assume $\log_{10}2 = 0.301$ and $\log_3 10 = 0.477$.**

Sol. $\log x = \log(0.15)^{20} = 20 \log\left(\frac{15}{100}\right)$

$= 20[\log 15 - 2]$
$= 20[\log 3 + \log 5 - 2]$

$= 20[\log 3 + 1 - \log 2 - 2] \quad \left(\because \log_{10} 5 = \log_{10} \frac{10}{2}\right)$

$= 20[-1 + \log 3 - \log 2]$
$= 20[-1 + 0.477 - 0.301]$

$= -20 \times 0.824 = -16.48 = \overline{17}.52$

Hence, characteristic $= -17$ and mantissa $= 0.52$.

Example 1.53 **In the 2001 census, the population of India was found to be 8.7×10^7. If the population increases at the rate of 2.5% every year, what would be the population in 2011 ?**

Sol. Here, $P_0 = 8.7 \times 10^7$, $r = 2.5$ and $n = 10$.

Let P be the population in 2011.

Then, $P = P_0\left(1 + \frac{r}{100}\right)^n$

$= 8.7 \times 10^7\left(1 + \frac{2.5}{100}\right)^{10}$

$= 8.7 \times 10^7 (1.025)^{10}$

Taking log of both sides, we get

$\log P = \log[8.7 \times 10^7 (1.025)^{10}]$

$= \log 8.7 + \log 10^7 + \log (1.025)^{10}$

$= \log 8.7 + 7 \log 10 + 10 \log (1.025)$

$= 0.9395 + 7 + 0.1070$

$= 8.0465$

$\Rightarrow \quad P = \text{antilog}(8.0465) = 1.113 \times 10^8$ (using antilog table)

Example 1.54 **Find the compound interest on ₹ 12000 for 10 years at the rate of 12% per annum compounded annually.**

Sol. We know that the amount A at the end of n years at the rate of r% per annum when the interest is compounded annually is given by

$A = P\left(1 + \frac{r}{100}\right)^n$.

Here, $P =$ ₹ 12000, $r = 12$ and $n = 10$.

$\therefore A =$ ₹ $\left[12000\left(1 + \frac{12}{100}\right)^{10}\right]$

$$= ₹\left[12000\left(1+\frac{3}{25}\right)^{10}\right]$$

$$= ₹\left[12000\left(\frac{25+3}{25}\right)^{10}\right]$$

$$= ₹\left[12000\left(\frac{28}{25}\right)^{10}\right]$$

Now, $A = ₹\,12000\left(\frac{28}{25}\right)^{10}$

$\Rightarrow \log A = \log 12000 + 10(\log 28 - \log 25)$

$= 4.0792 + 10(1.4472 - 1.3979)$

$= 4.0792 + 0.493 \Rightarrow \log A = 4.5722$

$\Rightarrow A = \text{antilog}(4.5722) = 37350.$

So, the amount after 10 years is ₹ 37350.

Hence, compound interest = ₹ (37350 – 12000) = ₹ 25350.

Example 1.55 **If P is the number of natural numbers whose logarithms to the base 10 have the characteristic p and Q is the number of natural numbers logarithms of whose reciprocals to the base 10 have the characteristic $-q$, then find the value of $\log_{10}P - \log_{10}Q$.**

Sol. $10^p \le P < 10^{p+1} \Rightarrow P = 10^{p+1} - 10^p \Rightarrow P = 9 \times 10^p$

Similarly, $10^{q-1} < Q \le 10^q \Rightarrow Q = 10^q - 10^{q-1} = 10^{q-1}(10-1) = 9 \times 10^{q-1}$

$\therefore \log_{10}P - \log_{10}Q = \log_{10}(P/Q) = \log_{10}10^{p-q+1} = p - q + 1$

Example 1.56 **Let L denote antilog$_{32}$ 0.6 and M denote the number of positive integers which have the characteristic 4, when the base of log is 5 and N denote the value of $49^{(1-\log_7 2)} + 5^{-\log_5 4}$. Find the value of LM/N.**

Sol. $L = \text{antilog}_{32} 0.6 = (32)^{6/10} = 2^{(5\times6)/10} = 2^3 = 8$

M = Integer from 625 to 3125 = 2500

$$N = 49^{(1-\log_7 2)} + 5^{-\log_5 4}$$

$$= 49 \times 7^{-2\log_7 2} + 5^{-\log_5 4}$$

$$= 49 \times \frac{1}{4} + \frac{1}{4} = \frac{50}{4} = \frac{25}{2}$$

$$\therefore \frac{LM}{N} = \frac{8 \times 2500 \times 2}{25} = 1600$$

EXERCISES

Subjective Type

Solutions on page 1.30

1. If $x = \log_{2a} a$, $y = \log_{3a} 2a$, $z = \log_{4a} 3a$, prove that $1 + xyz = 2yz$.
2. Solve the equations for x and y: $(3x)^{\log 3} = (4y)^{\log 4}$, $4^{\log x} = 3^{\log y}$.
3. If $a = \log_{12} 18$, $b = \log_{24} 54$, then find the value of $ab + 5(a - b)$.

4. If $y = a^{\frac{1}{1-\log_a x}}$ and $z = a^{\frac{1}{1-\log_a y}}$, then prove that $x = a^{\frac{1}{1-\log_a z}}$.
5. Solve $\log_x 2 \log_{2x} 2 = \log_{4x} 2$.
6. Let a, b, c, d be positive integers such that $\log_a b = 3/2$ and $\log_c d = 5/4$. If $(a-c) = 9$, then find the value of $(b-d)$.
7. Solve $\sqrt{\log(-x)} = \log\sqrt{x^2}$ (base is 10).
8. If $a \ge b > 1$, then find the largest possible value of the expression $\log_a(a/b) + \log_b(b/a)$.
9. Solve $3^{(\log_9 x)^2 - \frac{9}{2}\log_9 x + 5} = 3\sqrt{3}$.
10. Solve the inequality $\sqrt{\log_2\left(\dfrac{2x-3}{x-1}\right)} < 1$.
11. Find the number of solutions of equation $2^x + 3^x + 4^x - 5^x = 0$.
12. Solve $x^{\log_y x} = 2$ and $y^{\log_x y} = 16$.
13. Solve $\log_{2x} 2 + \log_4 2x = -3/2$.
14. Solve for x: $(2x)^{\log_b 2} = (3x)^{\log_b 3}$.
15. If $\log_b a \log_c a + \log_a b \log_c b + \log_a c \log_b c = 3$ (where a, b, c are different positive real numbers $\neq 1$), then find the value of abc.
16. If $\dfrac{\log_a N}{\log_c N} = \dfrac{\log_a N - \log_b N}{\log_b N - \log_c N}$, where $N > 0$ and $N \neq 1$, $a, b, c > 0$ and not equal to 1, then prove that $b^2 = ac$.
17. Given a and b are positive numbers satisfying $4(\log_{10} a)^2 + (\log_2 b)^2 = 1$, then find the range of values of a and b.

Objective Type

Solutions on page 1.35

1. $\log_4 18$ is
   a. a rational number **b.** an irrational number **c.** a prime number **d.** none of these
2. If $\log_4 5 = a$ and $\log_5 6 = b$, then $\log_3 2$ is equal to
   a. $\dfrac{1}{2a+1}$ **b.** $\dfrac{1}{2b+1}$ **c.** $2ab+1$ **d.** $\dfrac{1}{2ab-1}$
3. The value of x satisfying $\sqrt{3}^{-4+2\log_{\sqrt{5}} x} = 1/9$ is
   a. 2 **b.** 3 **c.** 4 **d.** none of these
4. $x^{\log_5 x} > 5$ implies
   a. $x \in (0, \infty)$ **b.** $x \in (0, 1/5) \cup (5, \infty)$ **c.** $x \in (1, \infty)$ **d.** $x \in (1, 2)$
5. The number $N = 6\log_{10} 2 + \log_{10} 31$ lies between two successive integers whose sum is equal to
   a. 5 **b.** 7 **c.** 9 **d.** 10
6. The value of $49^{(1-\log_7 2)} + 5^{-\log_5 4}$ is
   a. 27/2 **b.** 25/2 **c.** 625/16 **d.** none of these

7. If $\sqrt{\log_2 x} - 0.5 = \log_2 \sqrt{x}$, then x equals

a. odd integer b. prime number c. composite number d. irrational

8. If $\log_y x + \log_x y = 1$, $x^2 + y = 12$, then the value of xy is

a. 9 b. 12 c. 15 d. 21

9. If $\log_a 3 = 2$ and $\log_b 8 = 3$, then $\log_a b$ is

a. $\log_3 2$ b. $\log_2 3$ c. $\log_3 4$ d. $\log_4 3$

10. If $(4)^{\log_9 3} + (9)^{\log_2 4} = (10)^{\log_x 83}$, then x is equal to

a. 2 b. 3 c. 10 d. 30

11. The value of $\log ab - \log|b| =$

a. $\log a$ b. $\log |a|$ c. $-\log a$ d. none of these

12. If $(x+1)^{\log_{10}(x+1)} = 100(x+1)$, then

a. all the roots are positive real numbers. b. all the roots lie in the interval (0, 100)

c. all the roots lie in the interval [−1, 99] d. none of these

13. If a, b, c are distinct positive numbers different from 1 such that $(\log_b a \log_c a - \log_a a) + (\log_a b \log_c b - \log_b b) + (\log_a c \log_b c - \log_c c) = 0$, then abc is

a. 0 b. e c. 1 d. none of these

14. Given that log (2) = 0.3010..., the number of digits in the number 2000^{2000} is

a. 6601 b. 6602 c. 6603 d. 6604

15. The value of $\dfrac{\log_2 24}{\log_{96} 2} - \dfrac{\log_2 192}{\log_{12} 2}$ is

a. 3 b. 0 c. 2 d. 1

16. The set of all values of x satisfying $x^{\log_x (1-x)^2} = 9$ is

a. a subset of R containing N b. a subset of R containing Z (set of all integers)

c. is a finite set containing at least two elements d. a finite set

17. If $\ln\left(\dfrac{a+b}{3}\right) = \left(\dfrac{\ln a + \ln b}{2}\right)$, then $\dfrac{a}{b} + \dfrac{b}{a}$ is equal to

a. 1 b. 3 c. 5 d. 7

18. The value of b for which the equation $2 \log_{1/25}(bx + 28) = -\log_5 (12 - 4x - x^2)$ has coincident roots if

a. $b = -12$ b. $b = 4$ c. $b = 4$ or $b = -12$ d. $b = -4$ or $b = 12$

19. If $S = \{x \in N : 2 + \log_2 \sqrt{x+1} > 1 - \log_{1/2} \sqrt{4 - x^2}\}$, then

a. $S = \{1\}$ b. $S = Z$ c. $S = N$ d. none of these

20. If $a^4 \cdot b^5 = 1$, then the value of $\log_a(a^5 b^4)$ equals

a. 9/5 b. 4 c. 5 d. 8/5

21. If the equation $2^x + 4^y = 2^y + 4^x$ is solved for y in terms of x, where $x < 0$, then the sum of the solutions is

a. $x \log_2(1 - 2^x)$ b. $x + \log_2(1 - 2^x)$ c. $\log_2(1 - 2^x)$ d. $x \log_2(2^x + 1)$

22. The minimum value of the expression $2\log_{10}x - \log_x 0.01$, where $x > 1$, is

a. 2 **b.** 0.1 **c.** 4 **d.** 1

23. The value of $3^{\log_4 5} - 5^{\log_4 3}$ is

a. 0 **b.** 1 **c.** 2 **d.** none of these

24. If a, b, c are consecutive positive integers and $\log(1 + ac) = 2K$, then the value of K is

a. $\log b$ **b.** $\log a$ **c.** 2 **d.** 1

25. If $\log_2 x + \log_x 2 = \dfrac{10}{3} = \log_2 y + \log_y 2$ and $x \neq y$ then $x + y =$

a. 2 **b.** 65/8 **c.** 37/6 **d.** none of these

26. If $\log_{10}\left[\dfrac{1}{2^x + x - 1}\right] = x[\log_{10} 5 - 1]$, then $x =$

a. 4 **b.** 3 **c.** 2 **d.** 1

27. If $S = \{x \in R : (\log_{0.6} 0.216)\log_5(5 - 2x) \leq 0\}$, then S is equal to

a. $[2.5, \infty)$ **b.** $[2, 2.5)$ **c.** $(2, 2.5)$ **d.** $(0, 2.5)$

28. Solution set of the inequality $\log_3(x + 2)(x + 4) + \log_{1/3}(x + 2) < (1/2)\log_{\sqrt{3}} 7$ is

a. $(-2, -1)$ **b.** $(-2, 3)$ **c.** $(-1, 3)$ **d.** $(3, \infty)$

29. If $\log_3\{5 + 4\log_3(x - 1)\} = 2$, then x is equal to

a. 2 **b.** 4 **c.** 8 **d.** $\log_2 16$

30. If $2x^{\log_4 3} + 3^{\log_4 x} = 27$, then x is equal to

a. 2 **b.** 4 **c.** 8 **d.** 16

31. Equation $\log_4(3 - x) + \log_{0.25}(3 + x) = \log_4(1 - x) + \log_{0.25}(2x + 1)$ has

a. only one prime solution **b.** two real solutions

c. no real solution **d.** none of these

32. The value of $\dfrac{1 + 2\log_3 2}{(1 + \log_3 2)^2} + (\log_6 2)^2$ is

a. 2 **b.** 3 **c.** 4 **d.** 1

33. Product of roots of the equation $\dfrac{\log_8(8/x^2)}{(\log_8 x)^2} = 3$ is

a. 1 **b.** 1/2 **c.** 1/3 **d.** 1/4

34. Let $a > 1$ be a real number. Then the number of roots equation $a^{2\log_2 x} = 5 + 4x^{\log_2 a}$ has

a. 2 **b.** infinite **c.** 0 **d.** 1

35. If $(21.4)^a = (0.00214)^b = 100$, then the value of $\dfrac{1}{a} - \dfrac{1}{b}$ is

a. 0 **b.** 1 **c.** 2 **d.** 4

36. The solution set of the inequality $\log_{10}(x^2-16) \le \log_{10}(4x-11)$ is
a. $(4, \infty)$ b. $(4, 5]$ c. $(11/4, \infty)$ d. $(11/4, 5)$

37. The number of roots of the equation $\log_{3\sqrt{x}} x + \log_{3x} \sqrt{x} = 0$ is
a. 1 b. 2 c. 3 d. 0

38. The set of all x satisfying the equation $x^{\log_3 x^2 + (\log_3 x)^2 - 10} = 1/x^2$ is
a. $\{1, 9\}$ b. $\{1, 9, 1/81\}$ c. $\{1, 4, 1/81\}$ d. $\{9, 1/81\}$

39. If $xy^2 = 4$ and $\log_3(\log_2 x) + \log_{1/3}(\log_{1/2} y) = 1$, then x equals
a. 4 b. 8 c. 16 d. 64

40. If $2^{x+y} = 6^y$ and $3^{x-1} = 2^{y+1}$, then the value of $(\log 3 - \log 2)/(x-y)$ is
a. 1 b. $\log_2 3 - \log_3 2$ c. $\log(3/2)$ d. none of these

41. If $\log_2 x + \log_2 y \ge 6$, then the least value of $x + y$ is
a. 4 b. 8 c. 16 d. 32

42. If $f(x) = \log\left(\frac{1+x}{1-x}\right)$, then
a. $f(x_1) . f(x_2) = f(x_1 + x_2)$
b. $f(x+2) - 2f(x+1) + f(x) = 0$
c. $f(x) + f(x+1) = f(x^2 + x)$
d. $f(x_1) + f(x_2) = f\left(\frac{x_1 + x_2}{1 + x_1 x_2}\right)$

43. Which of the following is not the solution of $\log_x\left(\frac{5}{2} - \frac{1}{x}\right) > \left(\frac{5}{2} - \frac{1}{x}\right)$?
a. $\left(\frac{2}{5}, \frac{1}{2}\right)$ b. $(1, 2)$ c. $\left(\frac{2}{5}, \frac{3}{4}\right)$ d. none of these

44. If x_1 and x_2 are the roots of the equation $e^2 \cdot x^{\ln x} = x^3$ with $x_1 > x_2$, then
a. $x_1 = 2x_2$ b. $x_1 = x_2^2$ c. $2x_1 = x_2^2$ d. $x_1^2 = x_2^3$

45. The number of real values of the parameter k for which $(\log_{16} x)^2 - \log_{16} x + \log_{16} k = 0$ with real coefficients will have exactly one solution is
a. 2 b. 1 c. 4 d. none of these

Multiple Correct Answers Type

Solutions on page 1.41

Each question has four choices a, b, c, and d, out of which one or more answers are correct.

1. For $a > 0, \ne 1$, the roots of the equation $\log_{ax} a + \log_x a^2 + \log_{a^2 x} a^3 = 0$ are given by
a. $a^{-4/3}$ b. $a^{-3/4}$ c. a d. $a^{-1/2}$

2. The real solutions of the equation $2^{x+2} . 5^{6-x} = 10^{x^2}$ is/are
a. 1 b. 2 c. $-\log_{10}(250)$ d. $\log_{10} 4 - 3$

3. If $\frac{\log x}{b-c} = \frac{\log y}{c-a} = \frac{\log z}{a-b}$, then which of the following is/are true?
a. $xyz = 1$ b. $x^a y^b z^c = 1$ c. $x^{b+c} y^{c+a} z^{a+b} = 1$ d. $xyz = x^a y^b z^c$

4. If $\log_k x \cdot \log_5 k = \log_x 5$, $k \neq 1$, $k > 0$, then x is equal to

a. k **b.** $1/5$ **c.** 5 **d.** none of these

5. If $p, q \in N$ satisfy the equation $x^{\sqrt{x}} = \left(\sqrt{x}\right)^x$; then p and q are

a. relatively prime **b.** twin prime

c. coprime **d.** if $\log_q p$ is defined, then $\log_p q$ is not and vice versa

6. Which of the following, when simplified, reduces to unity?

a. $\log_{10}5 \cdot \log_{10}20 + (\log_{10}2)^2$ **b.** $\dfrac{2\log 2 + \log 3}{\log 48 - \log 4}$

c. $-\log_5 \log_3 \sqrt{\sqrt[5]{9}}$ **d.** $\dfrac{1}{6}\log_{\frac{\sqrt{3}}{2}}\left(\dfrac{64}{27}\right)$

7. If $\log_a x = b$ for permissible values of a and x, then identify the statement (s) which can be correct.

a. If a and b are two irrational numbers, then x can be rational.

b. If a is rational and b is irrational, then x can be rational.

c. If a is irrational and b is rational, then x can be rational.

d. If a is rational and b is rational, then x can be rational.

8. The equation $\log_{x+1}(x-0.5) = \log_{x-0.5}(x+1)$ has

a. two real solutions **b.** no prime solution **c.** one integral solution **d.** no irrational solution

9. The equation $\sqrt{1+\log_x \sqrt{27}}\ \log_3 x + 1 = 0$ has

a. no integral solution **b.** one irrational solution **c.** two real solutions **d.** no prime solution

10. If $\log_{1/2}(4-x) \geq \log_{1/2}2 - \log_{1/2}(x-1)$, then x belongs to

a. $(1, 2]$ **b.** $[3, 4)$ **c.** $(1, 3]$ **d.** $[1, 4)$

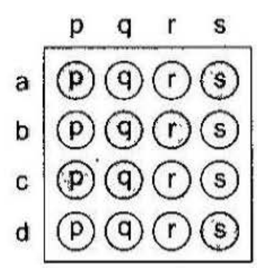
11. If the equation $x^{\log_a x^2} = \dfrac{x^{k-2}}{a^k}$, $a \neq 0$, has exactly one solution for x, then the value of k is/are

a. $6+4\sqrt{2}$ **b.** $2+6\sqrt{3}$ **c.** $6-4\sqrt{2}$ **d.** $2-6\sqrt{3}$

Matrix-Match Type

Solutions on page 1.44

Each question contains statements given in two columns which have to be matched.
Statements (a, b, c, d) in column I have to be matched with statements (p, q, r, s) in column II. If the correct matches are a-p, a-s, b-q, b-r, c-p, c-q and d-s, then the correctly bubbled 4 × 4 matrix should be as follows:



1.

Column I	Column II
a. The smallest integer greater than $\frac{1}{\log_3 \pi} + \frac{1}{\log_4 \pi}$ is	p. 10
b. Let $3^a = 4, 4^b = 5, 5^c = 6, 6^d = 7, 7^e = 8$ and $8^f = 9$. Then the value of the product $(abcdef)$ is	q. 3
c. Characteristic of the logarithm of 2008 to the base 2 is	r. 1
d. If $\log_2(\log_2(\log_3 x)) = \log_2(\log_3(\log_2 y)) = 0$, then the value of $(x - y)$ is	s. 2

2.

Column I	Column II
a. The value of $\log_2 \log_2 \log_4 256 + 2 \log \sqrt{2}^2$ is	p. 1
b. If $\log_3(5x - 2) - 2\log_3\sqrt{3x+1} = 1 - \log_3 4$, then $x =$	q. 6
c. Product of roots of the equation $7^{\log_7(x^2 - 4x + 5)} = (x - 1)$ is	r. 3
d. Number of integers satisfying $\log_2 \sqrt{x} - 2(\log_{1/4} x)^2\ x + 1 > 0$ are	s. 5

3.

Column I	Column II
a. $2^{\log_{(2\sqrt{2})} 15}$ is	p. rational
b. $\sqrt[3]{\left(5^{1/\log_7 5} + \frac{1}{\sqrt{(-\log_{10} 0.1)}}\right)}$ is	q. irrational
c. $\log_3 5 \cdot \log_{25} 27$ is	r. composite
d. Product of roots of equation $x^{\log_{10} x} = 100x$ is	s. prime

Integer Type

Solutions on page 1.45

1. If $\log_a b = 2$; $\log_b c = 2$ and $\log_3 c = 3 + \log_3 a$, then the value of $c/(ab)$ is ________.
2. The value of $(\log_{10}2)^3 + \log_{10}8 \cdot \log_{10}5 + (\log_{10}5)^3$ is ________.
3. If $\log_4 A = \log_6 B = \log_9(A + B)$, then $\left[4\dfrac{B}{A}\right]$ (where $[\cdot]$ represents the greatest integer function) equals ________.
4. Integral value of x which satisfies the equation $\log_6 54 + \log_x 16 = \log_{\sqrt{2}} x - \log_{36}\dfrac{4}{9}$ is ________.
5. If $a = \log_{245}175$ and $b = \log_{1715}875$, then the value of $\dfrac{1-ab}{a-b}$ is ________.
6. The difference of roots of the equation $(\log_{27}x^3)^2 = \log_{27}x^6$ is ________.
7. Sum of all integral values of x satisfying the inequality $(3^{5/2\log_3(12-3x)}) - (3^{\log_2 x}) > 32$ is ________.
8. The least integer greater than $\log_2 15 \cdot \log_{1/6} 2 \cdot \log_3 1/6$ is ________.
9. The reciprocal of $\dfrac{2}{\log_4(2000)^6} + \dfrac{3}{\log_5(2000)^6}$ is ________.
10. Sum of integers satisfying $\sqrt{\log_2 x - 1} - 1/2\log_2(x^3) + 2 > 0$ is ________.
11. Number of integers satisfying the inequality $\log_{1/2}|x-3| > -1$ is ________.
12. Number of integers ≤ 10 satisfying the inequality $2\log_{1/2}(x-1) \le \dfrac{1}{3} - \dfrac{1}{\log_{x^2-x} 8}$ is ________.
13. The value of $\log_{\left(\sqrt{3+2\sqrt{2}} + \sqrt{3-2\sqrt{2}}\right)} 2^9$ is ________.

ANSWERS AND SOLUTIONS

Subjective Type

1. $1 + xyz = 1 + (\log_{2a} a)(\log_{3a} 2a)(\log_{4a} 3a)$

$$= 1 + \frac{\log a}{\log 2a}\,\frac{\log 2a}{\log 3a}\,\frac{\log 3a}{\log 4a}$$

$$= 1 + \frac{\log a}{\log 4a}$$

$$= \log_{4a}4a + \log_{4a}a$$
$$= \log_{4a}4a^2$$
$$= 2\log_{4a}2a$$
$$= 2(\log_{3a}2a)(\log_{4a}3a) = 2yz$$

2. $(3x)^{\log 3} = (4y)^{\log 4}, 4^{\log x} = 3^{\log y}$

$\Rightarrow (\log 3)(\log 3x) = (\log 4)(\log 4y)$ and $(\log x)(\log 4) = (\log y)(\log 3)$

$\Rightarrow (\log 3)(\log 3 + \log x) = (\log 4)(\log 4 + \log y)$ and $(\log x)(\log 4) = (\log y)(\log 3)$

$\Rightarrow (\log 3)(\log 3 + p) = (\log 4)(\log 4 + q)$ and $p(\log 4) = q(\log 3)$ (where $p = \log x$ and $q = \log y$)

$$\Rightarrow (\log 3)\left(\log 3 + \frac{q \log 3}{\log 4}\right) = \log 4(\log 4 + q) \text{ (eliminating } p)$$

$$\Rightarrow (\log 3)^2 - (\log 4)^2 = \frac{(\log 4)^2 - (\log 3)^2}{\log 4} q$$

$\Rightarrow q = -\log 4 \Rightarrow \log y = \log 4^{-1} \Rightarrow y = 1/4$

Now $p(\log 4) = q(\log 3)$

$\Rightarrow p(\log 4) = -(\log 4)(\log 3)$

$\Rightarrow p = -\log 3$

$\Rightarrow \log x = \log 3^{-1}$

$\Rightarrow x = 1/3$

3. We have $a = \log_{12} 18 = \dfrac{\log_2 18}{\log_2 12} = \dfrac{1 + 2\log_2 3}{2 + \log_2 3}$ and $b = \log_{24} 54 = \dfrac{\log_2 54}{\log_2 24} = \dfrac{1 + 3\log_2 3}{3 + \log_2 3}$

Putting $x = \log_2 3$, we have

$$ab + 5(a - b) = \frac{1+2x}{2+x} \cdot \frac{1+3x}{3+x} + 5\left(\frac{1+2x}{2+x} - \frac{1+3x}{3+x}\right)$$

$$= \frac{6x^2 + 5x + 1 + 5(-x^2 + 1)}{(x+2)(x+3)}$$

$$= \frac{x^2 + 5x + 6}{(x+2)(x+3)} = 1$$

4. $\log_a y = \dfrac{1}{1 - \log_a x}$, therefore, $1 - \log_a y = 1 - \dfrac{1}{1 - \log_a x} = \dfrac{-\log_a x}{1 - \log_a x}$

or $\dfrac{1}{1 - \log_a y} = \dfrac{1 - \log_a x}{-\log_a x}$ (i)

But $z = a^{\frac{1}{1 - \log_a y}} \Rightarrow \log_a z = \dfrac{1}{1 - \log_a y} = -\dfrac{1}{\log_a x} + 1$

$\Rightarrow \dfrac{1}{\log_a x} = 1 - \log_a z \Rightarrow \log_a x = \dfrac{1}{1 - \log_a z}$

$\Rightarrow x = a^{\frac{1}{1 - \log_a z}}$

5. Since $\log_x 2 \log_{2x} 2 = \log_{4x} 2$, we have $x > 0$, $2x > 0$ and $4x > 0$ and $x \neq 1$, $2x \neq 1$, $4x \neq 1$

$\Rightarrow x > 0$ and $x \neq 1, \dfrac{1}{2}, \dfrac{1}{4}$

Then, $\dfrac{1}{\log_2 x} \cdot \dfrac{1}{\log_2 2x} = \dfrac{1}{\log_2 4x}$

$\Rightarrow \log_2 x \,.\, \log_2 2x = \log_2 4x$

$\Rightarrow \log_2 x(1 + \log_2 x) = (2 + \log_2 x)$

$\Rightarrow (\log_2 x)^2 = 2 \Rightarrow \log_2 x = \pm\sqrt{2}$

$\therefore x = 2^{\pm\sqrt{2}}$, i.e., $\{2^{-\sqrt{2}}, 2^{\sqrt{2}}\}$

6. $b = a^{3/2}$ and $d = c^{5/4}$
Let $a = x^2$ and $c = y^4$, $x, y \in N$
$\Rightarrow b = x^3$; $d = y^5$
Given $a - c = 9$, then $x^2 - y^4 = 9$
$\Rightarrow (x - y^2)(x + y^2) = 9$. Hence, $x - y^2 = 1$ and $x + y^2 = 9$.
(No other combination in the set of + ve integers will be possible.)
$x = 5$ and $y = 2$. Therefore, $b - d = x^3 - y^5 = 125 - 32 = 93$.

7. Since the equation can be satisfied only for $x < 0$, hence $\sqrt{x^2} = |x| = -x$. That is,

$\sqrt{\log(-x)} = \log(-x) \Rightarrow \log(-x) = [\log(-x)]^2$

$\Rightarrow \log(-x)[1 - \log(-x)] = 0$

if $\log(-x) = 0 \Rightarrow -x = 1 \Rightarrow x = -1$

if $\log_{10}(-x) = 1 \Rightarrow -x = 10 \Rightarrow x = -10$

8. Let $x = \log_a(a/b) + \log_b(b/a) = \log_a a - \log_a b + \log_b b - \log_b a = 2 - (\log_b a + \log_a b)$

$$= -\left(\sqrt{\log_b a} - \sqrt{\log_a b}\right)^2 \le 0$$

Hence, the maximum value is 0.

9. We have $(\log_9 x)^2 - \dfrac{9}{2}\log_9 x + 5 = \dfrac{3}{2}$ (taking log on both sides to the base 3)

Putting $\log_9 x = y$, we have $y^2 - \dfrac{9}{2}y + 5 = \dfrac{3}{2}$

$\Rightarrow 2y^2 - 9y + 7 = 0$, i.e., $(2y - 7)(y - 1) = 0$
$\Rightarrow y = 7/2, 1$
Therefore, either $\log_9 x = 1$ or $\log_9 x = 7/2$
i.e., either $x = 9$ or $x = 9^{7/2} = 3^7$

10. Inequality is true if

$$0 \le \log_2\left(\frac{2x-3}{x-1}\right) < 1 \Rightarrow 1 \le \frac{2x-3}{x-1} < 2$$

Now $\dfrac{2x-3}{x-1} - 2 < 0 \Rightarrow \dfrac{2x-3-2x+2}{x-1} < 0$

$$\Rightarrow \frac{-1}{x-1} < 0 \Rightarrow \frac{1}{x-1} > 0 \Rightarrow x > 1 \qquad \text{(i)}$$

and $\dfrac{2x-3}{x-1} \geq 1 \Rightarrow \dfrac{2x-3}{x-1} - 1 \geq 0$

$\Rightarrow \dfrac{2x-3-x+1}{x-1} \geq 0 \Rightarrow \dfrac{x-2}{x-1} \geq 0 \Rightarrow x \geq 2$ or $x < 1$ (ii)

Taking intersection of Eqs. (i) and (ii), we have $x \geq 2$.

11. $2^x + 3^x + 4^x - 5^x = 0 \Rightarrow 2^x + 3^x + 4^x = 5^x$

$\Rightarrow \left(\dfrac{2}{5}\right)^x + \left(\dfrac{3}{5}\right)^x + \left(\dfrac{4}{5}\right)^x = 1$

Now the number of solutions of the equation is equal to number of times.

$y = \left(\dfrac{2}{5}\right)^x + \left(\dfrac{3}{5}\right)^x + \left(\dfrac{4}{5}\right)^x$ and $y = 1$ intersect.

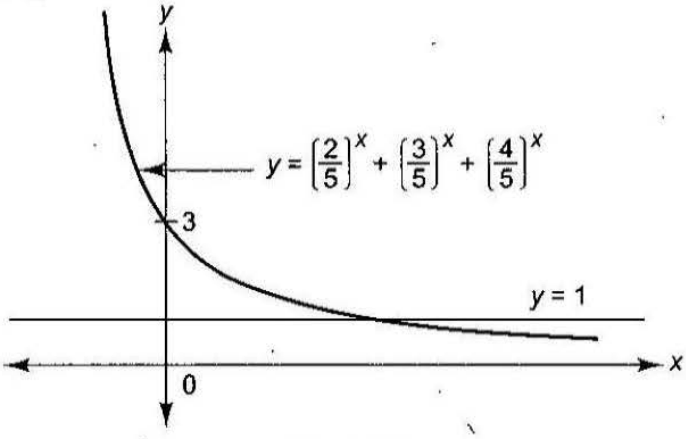


Fig. 1.11

From the graph, equation has only one solution.

12. Let $\log_y x = t \Rightarrow x = y^t$ (i)

Also $x^t = 2$ and $y^{1/t} = 2^4$

$\Rightarrow x = 2^{1/t}$ (ii)

$\Rightarrow y = 2^{4t}$ (iii)

Putting the values of x and y in Eq. (i), we have

$2^{1/t} = 2^{4t^2} \Rightarrow 4t^3 = 1$

$\therefore t = \left(\dfrac{1}{4}\right)^{1/3}$ (iv)

Using Eq. (iv) in Eq. (ii), we get $x = (2)^{(4)^{1/3}} = 2^{\sqrt[3]{4}}$.

Using Eq. (iv) in Eq. (iii), we get $y = (2)^{(4)^{4/3}}$.

13. Given equation is

$$\frac{1}{1+\log_2 x} + \frac{\log_2 2x}{2} = -\frac{3}{2}$$

$$\Rightarrow \frac{1}{1+\log_2 x} + \frac{1+\log_2 x}{2} = -\frac{3}{2}$$

Let $1+\log_2 x = y \Rightarrow \frac{1}{y}+\frac{y}{2}=-\frac{3}{2}$

$\Rightarrow 2+y^2+3y=0 \Rightarrow y=-1$ or -2

$\Rightarrow 1+\log_2 x=-1$ or -2

$\Rightarrow \log_2 x=-2$ or -3

$\Rightarrow x=2^{-2}$ or 2^{-3}

14. $(2x)^{\log_b 2}=(3x)^{\log_b 3}$

$\Rightarrow \log_b 2[\log 2+\log x]=\log_b 3[\log 3+\log x]$

$\Rightarrow (\log_b 2)(\log 2)-\log_b 3\cdot\log 3=(\log_b 3-\log_b 2)\log x$

$$\Rightarrow \frac{\log 2}{\log b}\cdot\log 2-\frac{\log 3}{\log b}\cdot\log 3=\left(\frac{\log 3}{\log b}-\frac{\log 2}{\log b}\right)\log x$$

$$\Rightarrow \frac{(\log 2)^2-(\log 3)^2}{\log b}=\left(\frac{\log 3-\log 2}{\log b}\right)\log x$$

$\Rightarrow \log x=-(\log 3+\log 2)=\log(6)^{-1}$

$\Rightarrow x=1/6$

15. $\log_b a\,.\,\log_c a+\log_a b\,.\,\log_c b+\log_a c\,.\,\log_b c=3$

$$\Rightarrow \frac{\log a}{\log b}\frac{\log a}{\log c}+\frac{\log b}{\log a}\frac{\log b}{\log c}+\frac{\log c}{\log a}\frac{\log c}{\log b}=3$$

$\Rightarrow (\log a)^3+(\log b)^3+(\log c)^3=3(\log a)(\log b)(\log c)$

$\Rightarrow \log a+\log b+\log c=0$ (as a, b, c are different)

$\Rightarrow \log abc=0 \Rightarrow abc=1$

16. $\dfrac{\log_a N}{\log_c N}=\dfrac{\log_a N-\log_b N}{\log_b N-\log_c N}$

$$\Rightarrow \frac{\log_N c}{\log_N a}=\frac{\frac{1}{\log_N a}-\frac{1}{\log_N b}}{\frac{1}{\log_N b}-\frac{1}{\log_N c}}$$

$$\Rightarrow \frac{\log_N c}{\log_N a}=\frac{\log_N c}{\log_N a}\times\frac{\log_N b-\log_N a}{\log_N c-\log_N b}$$

$$\Rightarrow \frac{\log_N b-\log_N a}{\log_N c-\log_N b}=1$$

$\Rightarrow \log_N b-\log_N a=\log_N c-\log_N b$

$\Rightarrow b/a=c/b$

$\Rightarrow b^2=ac$

17. $(\log_2 b)^2=1-(2\log_{10} a)^2\ge 0$

$\Rightarrow (2\log_{10} a)^2-1\le 0$

$\Rightarrow (2\log_{10} a-1)(2\log_{10} a+1)\le 0$

$$\Rightarrow \log_{10}a \in \left[-\frac{1}{2}, \frac{1}{2}\right]; a \in \left[\frac{1}{\sqrt{10}}, \sqrt{10}\right]$$

Similarly, $(\log_{10}a)^2 = \dfrac{1-(\log_{10}b)^2}{4} \ge 0 \Rightarrow (\log_{10}b)^2 - 1 \le 0$

$\Rightarrow (\log_{10}b - 1)(\log_{10}b + 1) \le 0$

$\Rightarrow \log_{10}b \in [-1, 1] \Rightarrow b \in \left[\dfrac{1}{10}, 10\right]$

Objective Type

1. b. Let $\log_4 18 = p/q$, where $p, q \in I$

$\Rightarrow \log_4 9 + \log_4 2 = \dfrac{p}{q} \Rightarrow \dfrac{1}{2} \times 2\log_2 3 + \dfrac{1}{2} = \dfrac{p}{q} \Rightarrow \log_2 3 = \dfrac{p}{q} - \dfrac{1}{2} = \dfrac{m}{n}$ (say)

where $m, n \in I$ and $n \ne 0 \Rightarrow 3 = (2)^{m/n} \Rightarrow 3^n = 2^m$ (possible only when $m = n = 0$ which is not true)
Hence, $\log_4 18$ is an irrational number.

2. d. Here, $5 = 4^a$ and $6 = 5^b$.
Let $\log_3 2 = x$, then $2 = 3^x$.
Now, $6 = 5^b = (4^a)^b = 4^{ab}$ or $3 = 2^{2ab-1}$
Therefore, $2 = (2^{2ab-1})^x = 2^{x(2ab-1)} \Rightarrow x(2ab-1) = 1$.

3. d. $3^{-2}3^{\log_{\sqrt{5}} x} = 3^{-2} \Rightarrow 3^{\log_{\sqrt{5}} x} = 1 \Rightarrow \log_{\sqrt{5}} x = 0 \Rightarrow x = 1$

4. b. Taking logarithm with base 5, we have

$x^{\log_5 x} > 5 \Rightarrow (\log_5 x)(\log_5 x) > 1 \Rightarrow (\log_5 x - 1)(\log_5 x + 1) > 0 \Rightarrow \log_5 x > 1$ or $\log_5 x < -1 \Rightarrow x > 5$ or $x < 1/5$

Also we must have $x > 0$. Thus, $x \in (0, 1/5) \cup (5, \infty)$.

5. b. $N = \log_{10} 64 + \log_{10} 31 = \log_{10} 1984$. Therefore, $3 < N < 4 \Rightarrow 7$.

6. b. $49^{(1-\log_7 2)} + 5^{-\log_5 4} = 49 \times 7^{-2\log_7 2} + 5^{-\log_5 4} = 49 \times \dfrac{1}{4} + \dfrac{1}{4} = \dfrac{50}{4} = \dfrac{25}{2}$

7. b. $\sqrt{\log_2 x} - 0.5 = \log_2 \sqrt{x}$

$\Rightarrow \sqrt{\log_2 x} - 0.5 = 0.5\log_2 x \Rightarrow y - 0.5 = 0.5y^2 \Rightarrow y^2 - 2y + 1 = 0 \Rightarrow y = 1 \Rightarrow \log_2 x = 1 \Rightarrow x = 2$

8. a. Let $t = \log_y x (x, y > 0$, and $\ne 1)$, then $t + \dfrac{1}{t} = 2$ or $(t-1)^2 = 0$

$\therefore t = \log_y x = 1$, i.e., $x = y$. We get $x^2 + x - 12 = 0\ x = -4, 3$.
$x = 3$ only (-4 rejected)

9. c. $\log_b 8 = 3 \Rightarrow 3\log_b 2 = 3 \Rightarrow \log_b 2 = 1$
$\log_a b = \log_2 b \,.\, \log_a 2 = \log_2 b.\, \log_3 2.\, \log_a 3 = 1.\log_3 2.2 = 2\log_3 2 = \log_3 4$

10. c. $(4)^{\log_9 3} + (9)^{\log_2 4} = (10)^{\log_x 83}$

$\Rightarrow (4)^{\frac{1}{2}\log_3 3} + (9)^{2\log_2 2} = (10)^{\log_x 83} \Rightarrow 2 + 81 = (10)^{\log_x 83} \Rightarrow 83 = (10)^{\log_x 83} \Rightarrow x = 10$

11. b. log ab is defined if $ab > 0$ or a and b have the same sign.

Case (i): $a, b > 0$

$\Rightarrow \log ab - \log|b| = \log a + \log b - \log b = \log a$ (i)

Case (ii): $a. b < 0$

$\Rightarrow \log ab - \log|b| = \log(-a) + \log(-b) - \log(-b) = \log(-a)$ (ii)

From Eqs. (i) and (ii), we have $\log ab - \log|b| = \log|a|$.

12. c. $(x+1)^{\log_{10}(x+1)} = 100(x+1) \Rightarrow \log_{10}(x+1)^{\log_{10}(x+1)} = \log_{10}(100(x+1))$

$\log_{10}(x+1)\log_{10}(x+1) = 2 + \log_{10}(x+1)$

Let $\log_{10}(x+1) = y$

$\Rightarrow y^2 - y - 2 = 0 \Rightarrow y = 2$ or $-1 \Rightarrow \log_{10}(x+1) = 2, -1 \Rightarrow x+1 = 100, 1/10 \Rightarrow x = 99$ or $-9/10$

13. c. $(\log_b a \log_c a - 1) + (\log_a b . \log_c b - 1) + (\log_a c \log_b c - 1) = 0$

$$\Rightarrow \frac{\log a}{\log b}\cdot\frac{\log a}{\log c} + \frac{\log b}{\log a}\cdot\frac{\log b}{\log c} + \frac{\log c}{\log a}\cdot\frac{\log c}{\log b} = 3$$

$\Rightarrow (\log a)^3 + (\log b)^3 + (\log c)^3 = 3 \log a \log b \log c$

$\Rightarrow (\log a + \log b + \log c) = 0$ [$\because$ If $a^3 + b^3 + c^3 - 3abc = 0$, then $a + b + c = 0$ if $a \neq b \neq c$]

$\Rightarrow \log abc = \log 0 \Rightarrow abc = 1$

14. c. Let $x = 2000^{2000}$

$\log x = 2000 \log_{10}(2000) = 2000(\log_{10}2 + 3) = 2000(3.3010) = 6602$

Therefore, the number of digits is 6603.

15. a. Let $\log_2 12 = a$, then $\dfrac{1}{\log_{96} 2} = \log_2 96 = \log_2 2^3 \times 12 = 3 + a$

$\log_2 24 = 1 + a \Rightarrow \log_2 192 = \log_2(16 \times 12) = 4 + a$ and $\dfrac{1}{\log_{12} 2} = \log_2 12 = a.$

Therefore, the given expression $= (1 + a)(3 + a) - (4 + a)a = 3.$

16. d. Taking logarithm of both the sides with base 3, we have

$$\log_x(1-x)^2 \log_3 x = 2 \Rightarrow \frac{\log_3(1-x)^2}{\log_3 x}\log_3 x = 2$$

$\Rightarrow \log_3(1-x)^2 = 2 \Rightarrow (1-x)^2 = 9$ (clearly $x \neq 3$)

$\Rightarrow x = 4, -2$. But $x > 0$, hence the solution set is $\{4\}$.

17. d. $\ln\left(\dfrac{a+b}{3}\right) = \dfrac{\ln ab}{2} = \ln\sqrt{ab} \Rightarrow \dfrac{a+b}{3} = \sqrt{ab} \Rightarrow a^2 + 2ab + b^2 = 9ab \Rightarrow \dfrac{a}{b} + 2 + \dfrac{b}{a} = 9$

$\therefore \dfrac{a}{b} + \dfrac{b}{a} = 7$

18. c. $\dfrac{2\log_5(bx+28)}{\log_5(1/5)^2} = -\log_5(12-4x-x^2) \Rightarrow bx + 28 = 12 - 4x - x^2 \Rightarrow x^2 + (b+4)x + 16 = 0$

For coincident roots, $D = 0 \Rightarrow (b+4)^2 = 4(16) \Rightarrow b + 4 = \pm 8$

19. a. $2+\log_2\sqrt{x+1}>1-\log_{1/2}\sqrt{4-x^2}$

$\Rightarrow 1+\log_2\sqrt{x+1}-\log_2\sqrt{4-x^2}>0$

$\Rightarrow \log_2 2+\log_2\sqrt{x+1}-\log_2\sqrt{4-x^2}>0$

$\Rightarrow \log_2\frac{2\sqrt{x+1}}{\sqrt{4-x^2}}>0 \Rightarrow \frac{2\sqrt{x+1}}{\sqrt{4-x^2}}>1$

$\Rightarrow 4(x+1)^2>4-x^2 \Rightarrow 4x^2+8x+4>4-x^2 \Rightarrow 5x^2+8x>0 \Rightarrow x>0$ (i)

Also $x+1>0$ and $4-x^2>0$

$\Rightarrow x>-1$ and $-2<x<2$ (ii)

From Eqs. (i) and (ii), $0<x<2$

$\Rightarrow x=1$ as $x\in N$

20. a. Given $4\log_a a+5\log_a b=0 \Rightarrow \log_a b=-4/5$ (i)

Now $\log_a(a^5b^4)=5+4\log_a b=5+4\left(-\frac{4}{5}\right)=5-\frac{16}{5}=\frac{9}{5}$

21. b. $2^{2y}-2^y+2^x(1-2^x)=0$

Putting $2^y=t$, we get

$t^2-t+2^x(1-2^x)=0$ where $t_1=2^{y_1}$ and $t_2=2^{y_2}$

$t_1t_2=2^x(1-2^x)$

$2^{y_1+y_2}=2^x(1-2^x)$

$y_1+y_2=x+\log_2(1-2^x)$

22. c. $2\log_{10}x-\frac{\log_{10}0.01}{\log_{10}x}, x>1$

$\Rightarrow 2\log_{10}x+\frac{2}{\log_{10}x}=2\left[\frac{1}{y}+y\right]\ge 4$

(where $\log_{10}x=y$)

23. a. Let $3^{\log_4 5}=a \Rightarrow \log_4 5=\log_3 a \Rightarrow \frac{\log 5}{\log 4}=\frac{\log a}{\log 3}$

$\Rightarrow \frac{\log a}{\log 5}=\frac{\log 3}{\log 4} \Rightarrow \log_5 a=\log_4 3$

$\Rightarrow a=5^{\log_4 3} \Rightarrow 3^{\log_4 5}-5^{\log_4 3}=0$

24. a. Let $a=x-1, b=x, c=x+1$

Now $\log(1+ac)=\log[1+(x-1)(x+1)]=\log x^2=2\log x=2\log b \Rightarrow K=\log b$

25. d. $\frac{10}{3}=3+\frac{1}{3}$. The given equation is of the form $p+\frac{1}{p}=3+\frac{1}{3}=q+\frac{1}{q}$, where $p\ne q$ as $x\ne y$

$\Rightarrow \log_2 x=3, \log_2 y=1/3 \Rightarrow x=2^3, y=2^{1/3} \Rightarrow x+y=8+2^{1/3}$.

26. d. R.H.S. $= x = [\log_{10} 5 - \log_{10} 10] = x \log_{10} \dfrac{5}{10} = \log_{10} \dfrac{1}{2^x} \Rightarrow \dfrac{1}{2^x + x - 1} = \dfrac{1}{2^x}$, therefore $x - 1 = 0$ or $x = 1$.

27. b. $(\log_{(0.6)} (0.6)^3) \log_5(5 - 2x) \le 0 \Rightarrow 5 - 2x \le 1 \Rightarrow x \ge 2$ (i)
Also, $5 - 2x > 0$ (ii)
From Eqs. (i) and (ii), we have x $[2, 2.5)$

28. b. $(x + 2)(x + 4) > 0$ and $x + 2 > 0$
$\Rightarrow x > -2$
Now the inequality can be written as $\log_3 (x + 2)(x + 4) - \log_3(x + 2) < \log_3 7$
$\Rightarrow \log_3 (x + 4) < \log_3 7 \Rightarrow x + 4 < 7$ or $x < 3$

29. b. We must have $x - 1 > 0 \Rightarrow x > 1$ (i)
and $5 + 4 \log_3 (x - 1) > 0 \Rightarrow 4 \log_3 (x - 1) > -5$

$\Rightarrow \log_3 (x - 1) > -\dfrac{5}{4}$

$\Rightarrow x - 1 > 3^{-5/4} \Rightarrow x > 1 + 3^{-5/4}$ (ii)
From Eqs. (i) and (ii), we get $x > 1 + 3^{-5/4}$. Therefore, $5 + 4 \log_3 (x - 1) = 9 \Rightarrow 4 \log_3 (x - 1) = 4$
$\Rightarrow \log_3 (x - 1) = 1 \Rightarrow x - 1 = 3 \Rightarrow x = 4$

30. d. $2\, x^{\log_4 3} + 3^{\log_4 x} = 27 \Rightarrow 2.\, 3^{\log_4 x} + 3^{\log_4 x} = 27 \Rightarrow 3^{\log_4 x} = 9 = 3^2 \Rightarrow \log_4 x = 2$, therefore $x = 4^2 = 16$.

31. d. $\log_4 (3 - x) + \log_{0.25} (3 + x) = \log_4 (1 - x) + \log_{0.25} (2x + 1)$
$\Rightarrow \log_4 (3 - x) - \log_4 (3 + x) = \log_4 (1 - x) - \log_4 (2x + 1)$
$\Rightarrow \log_4 (3 - x) + \log_4 (2x + 1) = \log_4 (1 - x) + \log_4 (3 + x)$
$\Rightarrow (3 - x)(2x + 1) = (1 - x)(3 + x)$
$\Rightarrow 3 + 5x - 2x^2 = 3 - 2x - x^2$
$\Rightarrow x^2 - 7x = 0$
$\Rightarrow x = 0, 7$
Only $x = 0$ is the solution and $x = 7$ is to be rejected.

32. d. $\dfrac{1 + 2\log_3 2}{(1 + \log_3 2)^2} + \dfrac{(\log_3 2)^2}{(1 + \log_3 2)^2} = \dfrac{(1 + \log_3 2)^2}{(1 + \log_3 2)^2} = 1$

33. d. Let $\log_8 x = y$, then the given equation reduces to $(1 - 2y)/y^2 = 3$.
$\Rightarrow 3y^2 + 2y - 1 = 0 \Rightarrow 3y^2 + 3y - y - 1 = 0$
$\Rightarrow 3y(y + 1) - 1(y + 1) = 0 \Rightarrow \log_8 x = y = 1/3, -1$
$\Rightarrow x = 2, 1/8$

34. d. Given equation can be written as $(a^{\log_2 x})^2 = 5 + 4\, a^{\log_2 x}$
Let $a^{\log_2 x} = t$, then the given equation is $t^2 - 4t - 5 = 0$. We get $(t - 5)(t + 1) = 0$
$\Rightarrow t = 5$ or $t = -1$ (rejected)
$\therefore\ a^{\log_2 x} = 5 \Rightarrow x^{\log_2 a} = 5 \Rightarrow x = 5^{\log_a 2}$

35. c. $(21.4)^a = 100 \Rightarrow a \log(21.4) = 2$

$\therefore \log(21.4) = 2/a$ (i)

Again $(0.00214)^b = 100$, we get $b(\log 0.00214) = 2$
$\Rightarrow b \log(21.4 \times 10^{-4}) = 2$

$$\Rightarrow b=\frac{2}{\log 21.4-4}=\frac{2}{\frac{2}{a}-4}=\frac{1}{\frac{1}{a}-2}$$

$$\Rightarrow b=\frac{a}{1-2a};\ \frac{1}{b}=\frac{1-2a}{a}$$

$$\Rightarrow \frac{1}{a}-\frac{1}{b}=2$$

36. b. $x^2-16\le 4x-11 \Rightarrow x^2-4x-5\le 0 \Rightarrow (x-5)(x+1)\le 0 \Rightarrow -1\le x\le 5$ (i)

Also $x^2-16>0 \Rightarrow x<-4$ or $x>4$ (ii)

And $4x-11>0 \Rightarrow x>11/4$ (iii)

From Eqs. (i), (ii) and (iii), we have $x\in(4,5]$.

37. b. $$\frac{\log x}{\log 3+(1/2)\log x}+\frac{(1/2)\log x}{\log 3+\log x}$$

$$\Rightarrow \frac{\log_3 x}{1+(1/2)\log_3 x}+\frac{1}{2}\frac{\log_3 x}{(1+\log_3 x)}=0$$

Let $\log_3 x=y$, we get $\dfrac{y}{1+(y/2)}+\dfrac{y}{2(1+y)}=0$

$$\Rightarrow y\left(\frac{2}{2+y}+\frac{1}{2(1+y)}\right)=0$$

$\Rightarrow y[4+4y+2+y]=0$

$\Rightarrow y=0$ or $y=-6/5$

$\Rightarrow \log_3 x=0$ or $\log_3 x=-6/5$

$\Rightarrow x=1$ or $x=3^{-6/5}$

38. b. Taking log of both the side with base 3, we have $(\log_3 x^2+(\log_3 x)^2-10)(\log_3 x)=-2\log_3 x$.

$\Rightarrow \log_3 x=0$ or $2\log_3 x+(\log_3 x)^2-8=0$

$\Rightarrow x=1, \log_3 x=-1\pm 3$ or $\log_3 x=2, \log_3 x=-4$.

Hence, $x=1, 3^2, 3^{-4}=1, 9, 1/81$.

39. d. $\log_3(\log_2 x)+\log_{1/3}(\log_{1/2} y)=1$

$\Rightarrow \log_3(\log_2 x)-\log_3(\log_{1/2} y)=1$

$\Rightarrow \log_3(\log_2(4/y^2))-\log_3(\log_{1/2} y)=1$

$\Rightarrow \log_2(4/y^2)=3(\log_{1/2} y)$

$\Rightarrow \log_2(4/y^2)=-3(\log_2 y)\Rightarrow \log_2(4/y^2)+(\log_2 y^3)=0\Rightarrow 4y=1\Rightarrow y=1/4\Rightarrow x=64$

40. c. Taking log, we have $(x+y)\log 2=y(\log 2+\log 3)$, therefore $x\log 2=y\log 3$.

or $\dfrac{x}{\log 3}=\dfrac{y}{\log 2}=\dfrac{x-y}{\log 3-\log 2}=\lambda$ say (i)

Also $(x-1)\log 3=(y+1)\log 2$.

or $x\log 3-y\log 2=\log 3+\log 2$

Using Eq. (i), we get $\lambda[(\log 3)^2-(\log 2)^2]=\log 3+\log 2$

$$\lambda=\frac{1}{\log 3-\log 2}\text{, therefore } \frac{1}{\lambda}=\log 3-\log 2=\log\frac{3}{2}$$

41. c. Given $\log_2 x + \log_2 y \geq 6 \Rightarrow \log_2 (xy) \geq 6$

$\Rightarrow xy \geq 64$

Also to define $\log_2 x$ and $\log_2 y$, we have $x > 0, y > 0$.

Since A.M. $\geq$ G.M.

$$\therefore \frac{x+y}{2} \geq \sqrt{xy} \Rightarrow x+y \geq 2\sqrt{xy} \geq 16$$

42. d. $$f(x_1)+f(x_2) = \log\left(\frac{1+x_1}{1-x_1}\cdot\frac{1+x_2}{1-x_2}\right)$$

$$= \log\left(\frac{1+x_1x_2+x_1+x_2}{1+x_1x_2-x_1-x_2}\right)$$

$$= \log\left(\frac{1+\frac{x_1+x_2}{1+x_1x_2}}{1-\frac{x_1+x_2}{1+x_1x_2}}\right) = f\left(\frac{x_1+x_2}{1+x_1x_2}\right)$$

43. c. Given inequality is defined if $x > 2/5$; $x \neq 1$

Case I: If $x > 1 \Rightarrow \frac{5}{2} - \frac{1}{x} > x \Rightarrow x + \frac{1}{x} < \frac{5}{2}$

$\Rightarrow 2(x^2+1) < 5x$

$\Rightarrow 2x^2 - 5x + 2 < 0$

$\Rightarrow 2x^2 - 4x - x + 2 < 0$

$\Rightarrow (x-2)(2x-1) < 0$

$\Rightarrow x \in (1, 2)$ (i)

Case II: $\frac{5}{2} < x < 1$, then $(x-2)(2x-1) > 0$.

$\Rightarrow x \in \left(\frac{2}{5}, \frac{1}{2}\right)$ (ii)

From Eqs. (i) and (ii), we get $x \in \left(\frac{2}{5}, \frac{1}{2}\right) \cup (1, 2)$.

44. $e^2 \cdot x^{\ln x} = x^3$

Taking log on both sides, we get $\ln(e^2 \cdot x^{\ln x}) = \ln(x^3)$

$\Rightarrow (\ln x)^2 - 3\ln x + 2 = 0$

$\Rightarrow (\ln x - 2)(\ln x - 1) = 0$

If $\ln x = 2 \Rightarrow x = e^2$

If $\ln x = 1 \Rightarrow x = e$

Since $x_1 > x_2$, we get $x_1 = e^2$ and $x_2 = e$

$\Rightarrow x_2^2 = x_1$

45. a. $\log_{16} x = \dfrac{1 \pm \sqrt{1 - 4\log_{16} k}}{2}$. For exactly one solution, $4\log_{16} k = 1$.

$\therefore k^4 = 16$, i.e., $k = 2, -2, 2i, -2i$.

Multiple Correct Answers Type

1. a, d.

$\log_{ax} a + \log_x a^2 + \log_{a^2x} a^3 = 0$

$$\Rightarrow \frac{1}{\log_a ax} + \frac{2}{\log_a x} + \frac{3}{(\log_a a^2x)} = 0$$

$$\Rightarrow \frac{1}{\log_a a + \log_a x} + \frac{2}{\log_a x} + \frac{3}{(2 + \log_a x)} = 0$$

Let $\log_a x = y$, we have $\frac{1}{y+1} + \frac{2}{y} + \frac{3}{2+y} = 0$

$\Rightarrow 6y^2 + 11y + 4 = 0$

$\Rightarrow y = \log_a x = -\frac{1}{2}, -\frac{4}{3}$

$\Rightarrow x = a^{-4/3}, a^{-1/2}$

2. b, c, d.

$2^{x+2}.5^{6-x} = 2^{x^2} \cdot 5^{x^2}$

$\Rightarrow 5^{6-x-x^2} = 2^{x^2-x-2}$

$\Rightarrow (6 - x - x^2) \log_{10} 5 = (x^2 - x - 2) \log_{10} 2$ (base 10)

$\Rightarrow (6 - x - x^2) [1 - \log_2 10] = (x^2 - x - 2) \log_{10} 2$

$\Rightarrow 6 - x - x^2 = (\log_{10} 2) [(x^2 - x - 2) - x^2 - x + 6]$

$\Rightarrow 6 - x - x^2 = (\log_{10} 2) [4 - 2x]$

$\Rightarrow x^2 + x - 6 = 2 (\log_{10} 2) (x - 2)$

$\Rightarrow (x + 3) (x - 2) = (\log_{10} 4) (x - 2)$

Therefore, either $x = 2$ or $x + 3 = \log_{10} 4$

$$\Rightarrow x = \log_{10} 4 - 3 = \log_{10}\left(\frac{4}{1000}\right); x = -\log_{10}(250)$$

3. a, b, c, d.

$$\text{Let } \frac{\log_k x}{b - c} = \frac{\log_k y}{c - a} = \frac{\log_k z}{a - b} = p$$

$\Rightarrow x = k^{p(b-c)}, y = k^{p(c-a)}, z = k^{p(a-b)}$

$\Rightarrow xyz = k^{p(b-c)} k^{p(c-a)} k^{p(a-b)} = k^{p(b-c)+p(c-a)+p(a-b)} = k^0 = 1$

$x^a y^b z^c = k^{pa(b-c)} k^{pb(c-a)} k^{pc(a-b)} = k^0 = 1$

$x^{b+c} y^{c+a} z^{a+b} = k^{p(b+c)(b-c)} k^{p(c+a)(c-a)} k^{p(a+b)(a-b)} = k^0 = 1$

4. b, c.

$\log_k x . \log_5 k = \log_x 5$

$$\Rightarrow \frac{\log x}{\log k}\frac{\log k}{\log 5} = \log_x 5$$

$\Rightarrow \dfrac{\log x}{\log 5} = \log_x 5$

$\Rightarrow \log_5 x = \dfrac{1}{\log_5 x}$

$\Rightarrow (\log_5 x)^2 = 1 \Rightarrow \log_5 x = \pm 1$

$\Rightarrow x = 5^{\pm 1} \Rightarrow x = \dfrac{1}{5}, 5$

5. a, c, d.

$x^{\sqrt{x}} = \left(\sqrt{x}\right)^x, p, q \in N$

$\Rightarrow \sqrt{x} \log x = x \log \sqrt{x}$

$\Rightarrow \log x\left[\sqrt{x} - \dfrac{x}{2}\right] = 0$

$\Rightarrow \log x = 0$ or $\left[\sqrt{x} - \dfrac{x}{2}\right] = 0$

$\Rightarrow x = 1$ or 4

6. a, b, c.

a. $\log_{10}\left(\dfrac{10}{2}\right) . \log_{10}(10 \times 2) + (\log_{10}2)^2 = (1 - \log_{10}2)(1 + \log_{10}2) + (\log_{10}2)^2 = 1$

b. $\dfrac{\log 2^2 \times 3}{\log(48/4)} = 1$

c. $-\log_5 \log_3 9^{1/10} = -\log_5 \log_3 3^{1/5} = -\log_5(1/5) = 1$

d. $\dfrac{1}{6} \log_{\frac{\sqrt{3}}{2}}\left(\dfrac{64}{27}\right) = \dfrac{1}{6} \log_{\frac{\sqrt{3}}{2}}\left(\dfrac{\sqrt{3}}{2}\right)^{-6} = -1$

7. a, b, c, d.

$\log_a x = b \Rightarrow x = a^b$

a. For $a = \sqrt{2}^{\sqrt{2}} \notin Q$ and $b = \sqrt{2} \notin Q; x = \left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}}$ which is rational.

b. For $a = 2 \in Q$ and $b = \log_2 3 \notin Q; x = 3$ which is rational.

c. For $a = \sqrt{2}$ and $b = 2; x = 2$

d. The option is obviously correct.

8. b, c, d.

$\dfrac{\log_2(x-0.5)}{\log_2(x+1)} = \dfrac{\log_2(x+1)}{\log_2(x-0.5)}$

$\Rightarrow [\log_2(x+1)]^2 = [\log_2(x-0.5)^2]$

$\Rightarrow \log_2(x+1) = \log_2(x-0.5)$ or $-\log_2(x-0.5)$

If $\log_2(x+1) = \log_2(x-0.5) \Rightarrow x + 1 = x - 0.5 \Rightarrow$ no solution

If $\log_2(x+1) = \log(x-0.5)^{-1}$

$$\Rightarrow x+1=\frac{1}{x-(1/2)}=\frac{2}{2x-1}$$

$\Rightarrow (x+1)(2x-1)=2$

$\Rightarrow 2x^2+x-3=0$

$\Rightarrow 2x^2+3x-2x-3=0$

$\Rightarrow (x-1)(2x+3)=0$

$\Rightarrow x=1 (x=-3/2$ rejected)

9. a, d.

$$\left(\sqrt{1+\frac{3}{2\log_3 x}}\right)\log_3 x+1=0$$

Let $\log_3 x=y$, we get

$$\left(\sqrt{1+\frac{3}{2y}}\right)y=-1 \Rightarrow \left(1+\frac{3}{2y}\right)=\frac{1}{y^2} \Rightarrow \frac{2y+3}{2y}=\frac{1}{y^2}$$

$\Rightarrow 2y^2+3y-2=0 \Rightarrow 2y^2+4y-y-2=0 \Rightarrow (y+2)(2y-1)=0$

$y=1/2$ or $y=-2 \Rightarrow x=3^{1/2}$ (rejected) or $x=1/9$

10. a, b.

$\log_{1/2}(4-x) \ge \log_{1/2}2-\log_{1/2}(x-1)$

$\Rightarrow \log_{1/2}(4-x)(x-1) \ge \log_{1/2}2$

$\Rightarrow (4-x)(x-1) \le 2$

$\Rightarrow x^2-5x+6 \ge 0$

$\Rightarrow (x-3)(x-2) \ge 0$

$\Rightarrow x \ge 3$ or $x \le 2$

But $x \in (1, 4)$

$\Rightarrow x \in (1, 2] \cup [3, 4)$

11. a, c.

$(\log_a x^2)\log_a x=(k-2)\log_a x-k$ (taking log on base a)

Let $\log_a x=t$, we get

$2t^2-(k-2)t+k=0$

Putting $D=0$ (has only one solution), we have

$(k-2)^2-8k=0$

$\Rightarrow k^2-12k+4=0$

$$\Rightarrow k=\frac{12\pm\sqrt{128}}{2}$$

$\Rightarrow k=6\pm4\sqrt{2}$

Matrix-Match Type

1. a → q; b → s; c → p; d → r

a. We have $\dfrac{1}{\log_3 \pi}+\dfrac{1}{\log_4 \pi}=\log_\pi 3+\log_\pi 4=\log_\pi 12$

But $\pi^2<12<\pi^3$, we have $2<\log_\pi 12<3$.

b. $3^a=4$; $a=\log_3 4$

Similarly, $b=\log_4 5$ etc.

Hence, $abcdef=\log_3 4\cdot\log_4 5\cdot\log_5 6\cdot\log_6 7\cdot\log_7 8\cdot\log_8 9=\log_3 9=2$

c. We have to find characteristic of $\log_2 2008$.

We know that $\log_2 1024=10$ and $\log_2 2048=11$, therefore

$10<\log_2 2008<11$

Hence, it has characteristic = 10.

d. $\log_2(\log_2(\log_3 x))=0\Rightarrow\log_2(\log_3 x)=1\Rightarrow\log_3 x=2\Rightarrow x=9$

Similarly, we have $\log_3(\log_2 y)=1$

$\Rightarrow\log_2 y=3\Rightarrow y=8$

Therefore, $x-y=1$.

2. a → s; b → p; c → q; d → r

a. $A=\log_2\log_2\log_4 256+2\log_{\sqrt{2}}2$

$=\log_2\log_2\log_4 4^4+2\log_{2^{1/2}}2$

$=\log_2\log_2 4+4\log_2 2+4=1+4=5$

b. $\log_3(5x-2)-2\log_3\sqrt{3x+1}=1-\log_3 4$

$\Rightarrow\log_3(5x-2)-\log_3(3x+1)+\log_3 4=1$

$\Rightarrow\log_3\left(\dfrac{(5x-2)(4)}{3x+1}\right)=1\Rightarrow\dfrac{(5x-2)(4)}{3x+1}=3\Rightarrow x=1$

c. $7^{\log_7(x^2-4x+5)}=(x-1)$

$\Rightarrow x^2-4x+5=x-1$

$\Rightarrow x^2-5x+6=0$

$\Rightarrow(x-2)(x-3)=0\Rightarrow x=2$ or $x=3$

Also we must have $x^2-4x+5>0$ and $x-1>0$

$\Rightarrow x>1$ (as $x^2-4x+5>0$ is true for all real numbers)

d. $x>0,\ \dfrac{1}{2}\log_2 x-2\left(\dfrac{\log_2 x}{2}\right)^2+1>0$

$\Rightarrow\log_2 x-(\log_2 x)^2+2>0$

$\Rightarrow(\log_2 x)^2-\log_2 x-2<0$

Let $\log_2 x=t$, we have $t^2-t-2<0$

$\Rightarrow(t-2)(t+1)<0\Rightarrow-1<t<2$

$\Rightarrow -1 < \log_2 x < 2 \Rightarrow \frac{1}{2} < x < 4$

Hence, the number of integers is 3, i.e., $\{1, 2, 3\}$.

3. **a → q; b → p, s; c → p; d → p, r**

a. $2^{\log_{(2\sqrt{2})}15} = 2^{\log_{2^{3/2}}15} = 2^{2/3\log_2 15} = 2^{\log_2 15^{2/3}} = 15^{2/3}$

b. $\sqrt[3]{\left(5^{1/\log_7 5} + \frac{1}{\sqrt{(-\log_{10} 0.1)}}\right)} = \sqrt[3]{\left(5^{\log_5 7} + \frac{1}{\sqrt{(\log_{10} 0.1^{-1})}}\right)} = \sqrt[3]{\left(7 + \frac{1}{\sqrt{\log_{10} 10}}\right)} = \sqrt[3]{(7+1)} = 2$

c. $\log_3 5 \cdot \log_{25} 27 = \frac{\log 5}{\log 3}\frac{\log 27}{\log 25} = \frac{\log 5}{\log 3} \times \frac{3}{2}\frac{\log 3}{\log 5} = \frac{3}{2}$

d. $(\log_{10} x)^2 = \log 100x = 2 + \log 10x$

Putting $\log x = t$, we get $t^2 = 2 + t$

$\Rightarrow t^2 - t - 2 = 0$

$\Rightarrow t = 2$ or $t = -1$

$\Rightarrow \log 10x = 2$ or $\log 10x = -1$

$\Rightarrow x = 100$ or $x = 1/10$

Hence, the product of roots is 10.

Integer Type

1.(3) $\log_3 c = 3 + \log_3 a \Rightarrow \log_3 \frac{c}{a} = 3 \Rightarrow c = 27a$ (i)

$\log_a b = 2;\ \log_b c = 2$

$\Rightarrow \log_a b \cdot \log_b c = 4 \Rightarrow \log_a c = 4 \Rightarrow c = a^4$ (ii)

From Eqs. (i) and (ii), we get $a = 3, c = 81$.

From relation (i), we have $b = a^2 = 9$.

Hence, $c/(ab) = 3$.

2.(1) Let $\log_2 10 = p$ and $\log_5 10 = q$

Hence, $p + q = 1$

$x = p^3 + 3pq + q^3$

$= (p+q)^3 - 3pq(p+q) + 3pq$

$= 1 - 3pq + 3pq$

$= 1$

3.(6) Let $\log_4 A = \log_6 B = \log_9(A+B) = x$

$\Rightarrow A = 4^x; B = 6^x$ and $A + B = 9^x$

$A + B = 9^x \Rightarrow 4^x + 6^x = 9^x$

$\Rightarrow 2^{2x} + 2^x \cdot 3^x = 3^{2x}$

$\Rightarrow \left(\frac{3}{2}\right)^{2x} - \left(\frac{3}{2}\right)^x - 1 = 0$

$\Rightarrow \left(\frac{3}{2}\right)^x = \frac{1+\sqrt{5}}{2}$

$$\Rightarrow \frac{B}{A} = \left(\frac{3}{2}\right)^x = \frac{1+\sqrt{5}}{2}$$

$$\Rightarrow 4\frac{B}{A} = 4\left(\frac{1+\sqrt{5}}{2}\right)$$

$$\Rightarrow \left[4\frac{B}{A}\right] = 6$$

4.(4) $\log_6 54 + \log_x 16 = \log_{\sqrt{2}} x - \log_{36}\frac{4}{9}$

$$\Rightarrow 1 + 2\log_6 3 + \log_x 16 = 2\log_2 x - \log_6 \frac{2}{3}$$

$$\Rightarrow 1 + 2\log_6 3 + \log_x 16 = 2\log_2 x - \log_6 2 + \log_6 3$$

$$\Rightarrow 1 + \log_x 16 = 2\log_2 x - (\log_6 2 + \log_6 3)$$

$$\Rightarrow 1 + \log_x 16 = 2\log_2 x - 1$$

$$\Rightarrow \frac{4}{\log_2 x} = 2\log_2 x - 2$$

Let $\log_2 x = t$, we have $(t-2)(t+1) = 0$

$\Rightarrow t = 2$ or $t = -1$

$\Rightarrow x = 4$ or $1/2$

5.(5) $a = \frac{\log_5 175}{\log_5 245} = \frac{2+\log_5 7}{1+2\log_5 7}$

$$\Rightarrow a + 2a\log_5 7 = 2 + \log_5 7$$

$$\Rightarrow \log_5 7 = \frac{a-2}{1-2a} \quad \text{(i)}$$

$$b = \frac{\log_5 875}{\log_5 1715} = \frac{3+\log_5 7}{1+3\log_5 7}$$

$$\Rightarrow b + 3b\log_5 7 = 3 + \log_5 7$$

$$\Rightarrow \log_5 7 = \frac{b-3}{1-3b} \quad \text{(ii)}$$

From Eqs. (i) and (ii); we get $\frac{a-2}{1-2a} = \frac{b-3}{1-3b} \Rightarrow \frac{1-ab}{a-b} = 5.$

6.(8) $(\log_{27} x^3)^2 = \log_{27} x^6$

$\Rightarrow (3\log_{27} x)^2 = 6\log_{27} x$

$\Rightarrow 3\log_{27} x(3\log_{27} x - 2) = 0$

$$\Rightarrow x = 1 \text{ or } \log_{27} x = \frac{2}{3}$$

$\Rightarrow x = (27)^{2/3} = 9$

Difference $= 9 - 1 = 8$

7.(3) We must have $12-3x>0$ and $x>0 \Rightarrow x \in (0,4)$
Therefore, the integral values are 1, 2, 3.

$$\text{For } x=1;\ \left(3^{\frac{5}{2}\log_3 9}\right)-\left(3^{\log_2 1}\right)=3^5-3^0>32$$

Similarly, $x=2$ satisfies but not $x=3$
Hence, the required sum = 3.

8.(3) $\log_2 15 \cdot \log_{1/6} 2 \cdot \log_3 1/6$.

$$=\frac{\log 15}{\log 2}\cdot\frac{\log 2}{-\log 6}\cdot\frac{-\log 6}{\log 3}$$

$=\log_3 15$

9.(6) Let $N=2\log_x 4+3\log_x 5$; where $x=(2000)^6$
$=\log_x 4^2+\log_x 5^3$

$$=\log_x 4^2\cdot 5^3=\log_{(2000)^6}(2000)=\frac{1}{6}$$

Hence reciprocal of given value is 6

10.(5) $\sqrt{\log_2 x-1}-\frac{3}{2}\log_2 x+2>0\,(x>0)$

$$\Rightarrow \sqrt{\log_2 x-1}-\frac{3}{2}(\log_2 x-1)+\frac{1}{2}>0 \quad \text{(i)}$$

Let $\sqrt{\log_2 x-1}=t\geq 0$, we have (ii)
$\log_2 x \geq 1 \Rightarrow x\geq 2$

Then from Eq. (i), we have $t-\frac{3}{2}t^2+\frac{1}{2}>0$

$\Rightarrow 3t^2-2t-1<0$
$\Rightarrow 1/3<t<1$ (iii)
From Eqs. (i) and (ii), we have $0\leq t<1$.

$0\leq\sqrt{\log_2 x-1}<1$
$0\leq\log_2 x-1<1$
$1\leq\log_2 x<2$
$2\leq x<4$
Hence, the integral values are 2 and 3, and their sum is 5.

11.(2) $\log_{1/2}|x-3|>-1$
$\Rightarrow |x-3|<2$
$\Rightarrow -2<x-3<2$
$\Rightarrow 1<x<5, x\neq 3$
$\therefore x\in\{2,4\}$

12.(9) We must be $x>1$

$$2\log_{1/2}(x-1)\leq\frac{1}{3}-\frac{1}{\log_{x^2-x}8}$$

$$\Rightarrow \frac{1}{3} - \frac{\log_2(x^2 - x)}{3} + 2\log_2(x-1) \geq 0$$

$$\Rightarrow \log_2 2 - \log_2(x^2 - x) + 6\log_2(x-1) \geq 0$$

$$\Rightarrow \log_2 \frac{2(x-1)^6}{x(x-1)} \geq 0$$

$$\Rightarrow \frac{2(x-1)^5}{x} \geq 1$$

Putting $x - 1 = y$, we have $y > 0$.

$$\Rightarrow \frac{2y^5}{y+1} - 1 \geq 0$$

$$\Rightarrow \frac{2y^5 - y - 1}{y+1} \geq 0$$

$$\Rightarrow \frac{2y^5 - 2y + y - 1}{y+1} \geq 0$$

$$\Rightarrow \frac{2y(y^4 - 1) + y - 1}{y+1} \geq 0$$

$$\Rightarrow \frac{(y-1)\left[2y(y+1)(y^2+1)+1\right]}{y+1} \geq 0$$

$$\Rightarrow \frac{y-1}{y+1} \geq 0 \Rightarrow y \geq 1$$

$$\Rightarrow x \geq 2$$

13.(6) $3 + 2\sqrt{2} = \left(\sqrt{2}+1\right)^2$ and $3 - 2\sqrt{2} = \left(\sqrt{2}-1\right)^2$

$$\Rightarrow \log_{\left(\sqrt{3+2\sqrt{2}}+\sqrt{3-2\sqrt{2}}\right)} 2^9 = \frac{1}{\log_{2^9}\left(\left(\sqrt{2}+1\right)+\left(\sqrt{2}-1\right)\right)}$$

$$= \frac{1}{\log_{2^9} 2^{3/2}}$$

$$= \frac{9}{3/2} = 6$$

CHAPTER

2

Trigonometric Ratios and Identities

- Measurement of Angles
- Trigonometric Functions
- Problems Based on Trigonometric Identities
- Trigonometric Ratios for Complementary and Supplementary Angles
- Trigonometric Ratios for Compound Angles
- Transformation Formulae
- Trigonometric Ratios of Multiples and Sub-Multiple Angles
- Values of Trigonometric Ratios of Standard Angles
- Sum of Sines or Cosines of n Angles in A.P.
- Conditional Identities
- Some Important Results and their Applications
- Important Inequalities

MEASUREMENT OF ANGLES

Angles in Trigonometry

In trigonometry, the idea of angle is more general; it may be positive or negative and has any magnitude (Fig. 2.1).

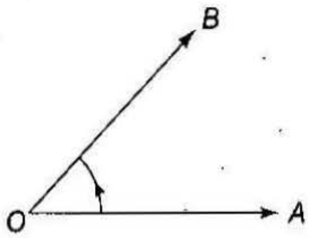


Fig. 2.1

In trigonometry, as in case of geometry, the measure of angle is the amount of rotation from the direction of one ray of the angle to the other. The initial and final positions of the revolving ray are respectively called the initial side (arm) and terminal side (arm), and the revolving line is called the generating line or the radius vector. For example, if OA and OB are the initial and final positions of the revolving ray, then the angle formed will be $\angle AOB$.

Angles Exceeding 360°

In geometry, we confine ourselves to angles from 0° to 360°. But there may be problems in which rotation involves more than one revolution, for example, the rotation of a flywheel. In trigonometry, we generalise the concept of angle to angles greater than 360°. This angle can be formed in the following way:

The revolving line (radius vector) starts from the initial position $\overrightarrow{OA}$ and makes n complete revolutions in anticlockwise direction and also a further angle α in the same direction. We then have a certain angle β_n given by $\beta_n = 360° \times n + \alpha$, where $0° < \alpha < 360°$ and n is a positive integer or zero.

Thus, there are infinitely many β_n angles with initial side $\overrightarrow{OA}$ and final side $\overrightarrow{OB}$.

For example, $\beta_0 = \alpha$, $\beta_1 = 360° + \alpha$, $\beta_2 = 720° + \alpha$, etc.

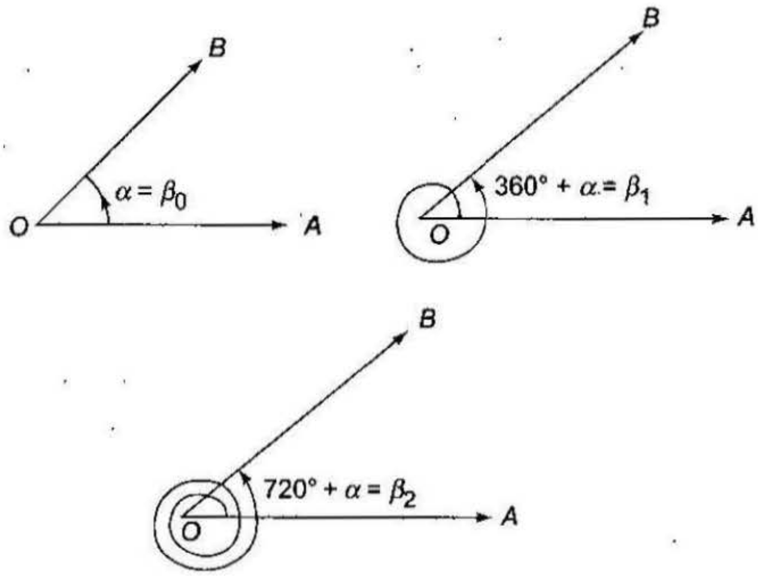


Fig. 2.2

Sign of Angles

Angles formed by anticlockwise rotation of the radius vector are taken as positive, whereas angles formed by clockwise rotation of the radius vector are taken as negative.

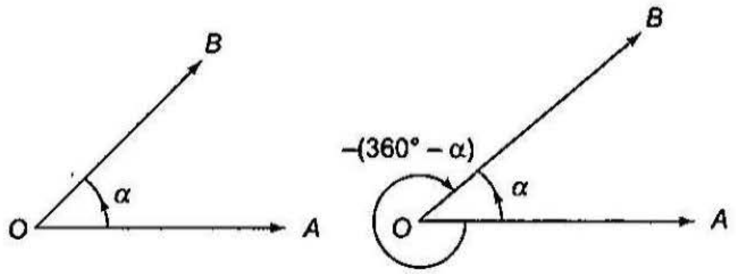


Fig. 2.3

Relation between Degree and Radian

Radian is a constant angle. One radian is the angle subtended by an arc of a circle at the centre. It is equal to arc/radius. It is expressed as rad.

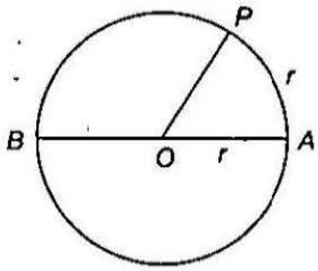


Fig. 2.4

Consider a circle with centre O and radius r. Let A be a point on the circle.
Join OA and cut off an arc of length equal to the radius of the circle.
Then, $\angle AOP = 1$ rad. Produce AO to meet the circle at B.
$\Rightarrow \quad \angle AOB$ = a straight angle = 2 right angles
We know that the angles at the centre of a circle are proportional to the arcs subtending them.

$$\Rightarrow \frac{\angle AOP}{\angle AOB} = \frac{\text{arc}AP}{\text{arc}APB}$$

$$\Rightarrow \frac{\angle AOP}{2 \text{ right angles}} = \frac{r}{\pi r} \qquad \left[\because \text{are } APB = \frac{1}{2}(\text{circumference})\right]$$

$$\Rightarrow \angle AOP = \frac{2 \text{ right angles}}{\pi} \qquad \Rightarrow 1^R = \frac{180°}{\pi}$$

Hence, $1 \text{ rad} = \dfrac{180°}{\pi} \Rightarrow \pi \text{ rad} = 180°$.

Note:

- *When an angle is expressed in radian, the word radian is generally omitted.*
- $1° = 60'$ (60 min) and $1' = 60''$ (60 sec)
- *Since* $180° = \pi$ *rad. Therefore,* $1° = \pi/180$ *rad.*

$$\text{Hence, } 30° = \frac{\pi}{180} \times 30 = \frac{\pi}{6} \text{ rad},$$

$$45° = \frac{\pi}{180} \times 45 = \frac{\pi}{4} \text{ rad},$$

$$60° = \frac{\pi}{180} \times 60 = \frac{\pi}{3} \text{ rad},$$

$$90° = \frac{\pi}{180} \times 90 = \frac{\pi}{2} \text{ rad, etc.}$$

- *We have π rad = 180°*

$$\Rightarrow \quad 1\,rad = \frac{180°}{\pi} = \left(\frac{180}{22} \times 7\right)^{\circ} = 57°16'22'' \ (approx)$$

- $180° = \pi\,rad \Rightarrow 1° = \frac{\pi}{180} rad = \left(\frac{22}{7 \times 180}\right) rad = 0.01746\,rad.$
- *Sum of interior angles on convex polygon of n sides is $(n-2)\pi$ rad.*

Example 2.1 **Express 45° 20′10″ in rad measure ($\pi = 3.1415$).**

Sol. $10'' = \frac{10}{60}$ min $= \frac{10}{60 \times 60}$ degrees $= \frac{1}{360}$ degrees

$20' = \frac{20}{60}$ degrees $= \frac{1}{3}$ degrees

$$\therefore \quad 45°20'10'' = \left(45 + \frac{1}{360} + \frac{1}{3}\right) \text{degrees} = \frac{16200 + 1 + 120}{360} = \frac{16321}{360}$$

$$\text{Now } \left(\frac{16321}{360}\right)^{\circ} = \frac{16321}{360} \times \frac{\pi}{180} \text{rad} = \frac{16321}{360} \times \frac{3.1416}{180} = \frac{51274.054}{64800} = 0.79 \text{ rad}$$

Example 2.2 **Express 1.2 rad in degree measure.**

Sol. $(1.2)^R = 1.2 \times \frac{180}{\pi}$ degrees $= 1.2 \times \frac{180 \times 7}{22}$ $\qquad \left[\because \pi = \frac{27}{7} (\text{approx})\right]$

$= 68.7272 = 68° (.7272 \times 60)' = 68° (43.63)' = 68° 43' (.63 \times 60)'' = 68° (43'37.8'')$

Example 2.3 **Find the length of an arc of a circle of radius 5 cm subtending a central angle measuring 15°.**

Sol. Let s be the length of the arc subtending an angle θ^R at the centre of a circle of radius r. Then, $\theta = s/r$.

Here, $r = 5$ cm and $\theta = 15° = \left(15 \times \frac{\pi}{180}\right)^R = \left(\frac{\pi}{12}\right)^R$

$$\therefore \quad \theta = \frac{s}{r} \Rightarrow \frac{\pi}{12} = \frac{s}{5} \Rightarrow s = \frac{5\pi}{12} \text{cm}$$

Example 2.4 **Find in degrees the angle subtended at the centre of a circle of diameter 50 cm by an arc of length 11 cm.**

Sol. Here, $r = 25$ cm and $s = 11$ cm.

$$\therefore \quad \theta = \left(\frac{s}{r}\right)^R \Rightarrow \theta = \left(\frac{11}{25}\right)^R = \left(\frac{11}{25} \times \frac{180}{\pi}\right)^{\circ}$$

$$= \left(\frac{11}{25} \times \frac{180}{22} \times 7\right)^{\circ}$$

$$= \left(\frac{126}{5}\right)^{\circ} = \left(25\frac{1}{5}\right)^{\circ} = 25°\left(\frac{1}{5} \times 60\right)' = 25°12'$$

Example 2.5 **If arcs of same length in two circles subtend angles of 60° and 75° at their centres, find the ratios of their radii.**

Sol. Let r_1 and r_2 be the radii of the given circles and let their arcs of same length s subtend angles of 60° and 75° at their centres.

Now, $60^R = \left(60\times\frac{\pi}{180}\right)^R = \left(\frac{\pi}{3}\right)^R$ and $75° = \left(75\times\frac{\pi}{180}\right)^R = \left(\frac{5\pi}{12}\right)^R$

$$\therefore \quad \frac{\pi}{3}=\frac{s}{r_1} \text{ and } \frac{5\pi}{12}=\frac{s}{r_2} \Rightarrow \frac{\pi}{3}r_1 = s \text{ and } \frac{5\pi}{12}r_2 = s \Rightarrow \frac{\pi}{3}r_1 = \frac{5\pi}{12}r_2 \Rightarrow 4r_1 = 5r_2 \Rightarrow r_1 : r_2 = 5:4$$

Hence, $r_1 : r_2 = 5:4$.

Example 2.6 **Assuming the distance of earth from the moon to be 38,400 km and the angle subtended by the moon at the eye of a person on earth to be 31′, find the diameter of the moon.**

Sol.

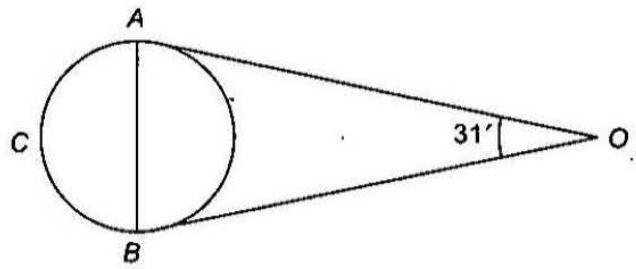


Fig. 2.5

Let AB be the diameter of the moon and O be the observer.

Given $\angle AOB = 31' = \frac{31}{60}\times\frac{\pi}{180}$ rad

Since the angle subtended by the moon is very small, its diameter will be approximately equal to the small arc of a circle whose centre is the eye of the observer and the radius is the distance of the earth from the moon. Also the moon subtends an angle of 31′ at the centre of this circle.

$$\Rightarrow \quad \theta = \frac{l}{r}, \quad \text{therefore } \frac{31}{60}\times\frac{\pi}{180} = \frac{AB}{38400}$$

$$\Rightarrow \quad AB = \frac{31}{60}\times\frac{22}{7\times180}\times 38,400 = 3464\frac{8}{63} \text{ km}$$

Example 2.7 **Find the angle between the minute hand and the hour hand of a clock when the time is 7:20 AM.**

Sol. We know that the hour hand completes one rotation in 12 hr, while the minute hand completes one rotation in 60 min.

Therefore, the angle traced by the hour hand in 12 hr = 360°

Angle traced by the hour hand in 7 hr 20 min, i.e., $\frac{22}{3}$ hr $= \left(\frac{360}{12}\times\frac{22}{3}\right)^\circ = 220°$

Also, the angle traced by the minute hand in 60 min = 360°

The angle traced by the minute hand in 20 min $= \left(\frac{360}{60}\times 20\right)^\circ = 120°$

Hence, the required angle between the two hands = 220° – 120° = 100°.

Example 2.8 **For each natural number, k, let C_k denote a circle with radius k centimeters and centre at origin O. On the circle C_k a particle moves k centimetres in the counter-clockwise direction. After completing its motion on C_k, the particle moves to C_{k+1} in the radial direction. The motion of the particle continues in this manner. The particle starts at (1, 0). If the particle crosses the positive direction of x-axis for the first time on the circle C_n, then find the value of n.**

Sol.

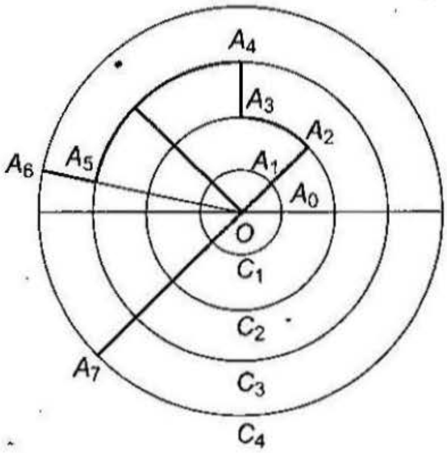


Fig. 2.6

The motion of the particle on the first four circles is shown with bold line in Fig. 2.6. Note that on every circle the particle travels just 1 rad. The particle crosses the positive direction of x-axis first time on C_n, where n is the least positive integer such that $n \geq 2\pi \Rightarrow n = 7$.

Concept Application Exercise 2.1

1. A horse is tied to a post by a rope. If the horse moves along a circular path always keeping the rope tight and describes 88 m when it has traced out 72° at the centre, find the length of the rope.
2. If the angular diameter of the moon is 30′, how far from the eye a coin of diameter 2.2 cm can be kept to hide the moon?
3. Find in degrees and radians the angle between the hour hand and the minute hand of a clock at half past three.
4. There is an equilateral triangle with side 4 and a circle with the centre on the one of the vertex of that triangle. The arc of that circle divides the triangle into two parts of equal area. How long is the radius of the circle?

TRIGONOMETRIC FUNCTIONS

Trigonometric Functions of Acute Angles

An angle whose measure is greater than 0° but less than 90° is called an acute angle. Consider a right-angled triangle ABC with right angle at B. The side opposite to the right angle is called the hypotenuse, side opposite to angle A is called the perpendicular for angle A and side opposite to the third angle is called the base for angle A.

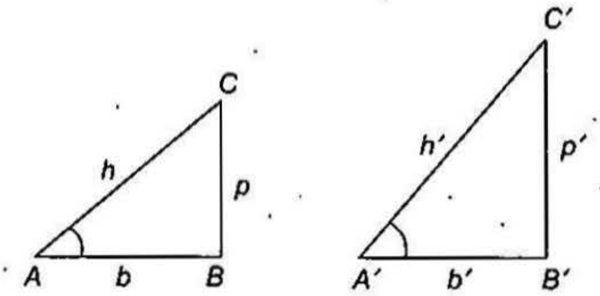


Fig. 2.7

The ratio of any two sides of the triangle depends only on measure of angle A, for if we take a larger and smaller right angle triangles as shown in Fig. 2.7, we have $\frac{h}{h'} = \frac{b}{b'} = \frac{p}{p'}$ (as these triangles are similar).

Thus, the ratio of the lengths of any two sides of a triangle is completely determined by angle A alone and is independent of the size of the triangle. There are six possible ratios that can be formed from the three sides of a right-angled triangle. Each of them has been given a name as follows.

Definitions

(i) $\sin A = \frac{p}{h}$ (ii) $\cos A = \frac{b}{h}$ (iii) $\tan A = \frac{p}{b}$

(iv) $\cot A = \frac{b}{p}$ (v) $\sec A = \frac{h}{b}$ (vi) $\operatorname{cosec} A = \frac{h}{p}$

The abbreviations stand for sine, cosine, tangent, cotangent, secant, and cosecant of A, respectively. These functions of angle A are called trigonometrical functions or trigonometrical ratios.

Example 2.9 **The circumference of a circle circumscribing an equilateral triangle is 24π units. Find the area of the circle inscribed in the equilateral triangle.**

Sol. $2\pi R = 24\pi$ (R is the radius of circumcircle)

$R = 12$

$\sin 30° = \frac{r}{R}$ (r is the radius of incircle)

$r = \frac{12}{2} = 6$

Therefore, area of incircle $= \pi r^2 = 36\pi$

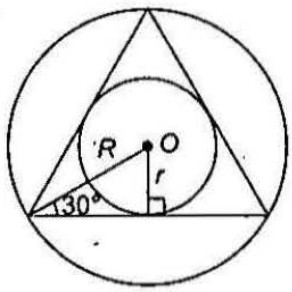


Fig. 2.8

Example 2.10 **In triangle ABC, $BC = 8$, $CA = 6$ and $AB = 10$. A line dividing the triangle ABC into two regions of equal area is perpendicular to AB at the point X. Then find the value of $BX/\sqrt{2}$.**

Sol. From the figure, $2\left(\frac{x \times y}{2}\right) = \frac{8 \times 6}{2} = 24$

$x \times x \tan B = 24$

$x^2 \times \frac{3}{4} = 24$

$x^2 = 32 \Rightarrow x = 4\sqrt{2}$

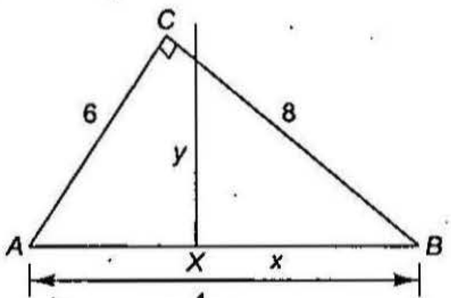


Fig. 2.9

Example 2.11 Let PQ and RS be tangents at the extremities of the diameter PR of a circle of radius r. If PS and RQ intersect at a point X on the circumference of the circle, then prove that $2r = \sqrt{PQ \times RS}$

Sol. From the figure, we have $\frac{PQ}{PR} = \tan(\pi/2 - \theta) = \cot\theta$.

and $\frac{RS}{PR} = \tan\theta$

$\Rightarrow \frac{PQ}{PR} \times \frac{RS}{PR} = 1$

$\Rightarrow (PR)^2 = PQ \times PS$

$\Rightarrow (2r)^2 = PQ \times PS$

$\Rightarrow 2r = \sqrt{PQ \times PS}$

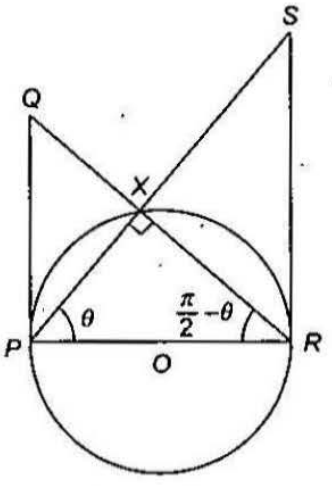


Fig. 2.10

Example 2.12 Two circles of radii 4 cm and 1 cm touch each other externally and θ is the angle contained by their direct common tangents. Then find $\sin\theta$.

Sol.

$\sin\frac{\theta}{2} = \frac{3}{5}$

$\cos\frac{\theta}{2} = \frac{4}{5}$

$\therefore \sin\theta = 2 \times \frac{3}{5} \times \frac{4}{5} = \frac{24}{25}$

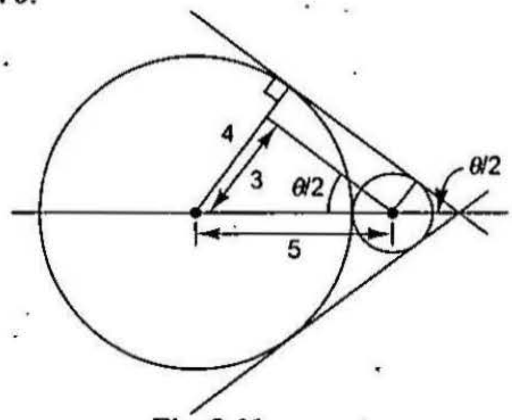


Fig. 2.11

Example 2.13 If angle C of triangle ABC is 90°, then prove that $\tan A + \tan B = \frac{c^2}{ab}$ (where, a, b, c are sides opposite to angles A, B, C respectively)

Sol. Draw ΔABC with $\angle C = 90°$

$\tan A + \tan B = \frac{a}{b} + \frac{b}{a}$

$= \frac{a^2 + b^2}{ab} = \frac{c^2}{ab}$

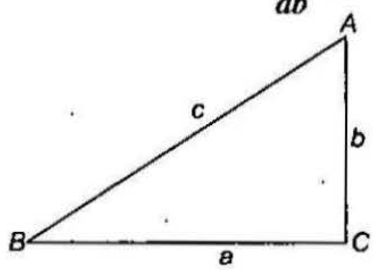


Fig. 2.12

Example 2.14 In the following diagram $\angle BAO = \tan^{-1}3$, then find the ratio $BC : CA$

Sol. $\therefore \tan\theta = 3$

$\therefore \frac{OC}{AC} = \tan\theta,\ \frac{OC}{BC} = \cot\theta$

$\Rightarrow \frac{BC}{AC} = \frac{\tan\theta}{\cot\theta} = \tan^2\theta = 9$

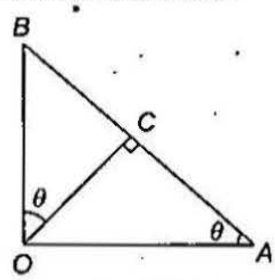


Fig. 2.13

Trigonometric Functions of Any Angle

Let A be a given angle with a specified initial ray. We introduce a rectangular coordinate system in the plane with the vertex of angle A as the origin and the initial ray of angle A as the positive ray of the x-axis (Fig. 2.14). We choose any point P on the terminal ray of angle A. Let the coordinates of P be (x, y) and its distance from the origin be r, then we define

(i) $\sin A = \dfrac{y}{r}$ (ii) $\cos A = \dfrac{x}{r}$ (iii) $\tan A = \dfrac{y}{x}$

(iv) $\cot A = \dfrac{x}{y}$ (v) $\sec A = \dfrac{r}{x}$ (vi) $\text{cosec}\, A = \dfrac{r}{y}$

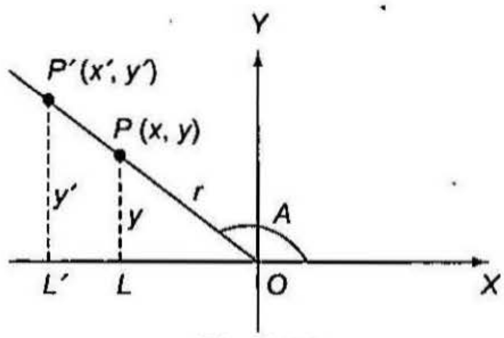


Fig. 2.14

These quantities are functions of angle A alone. They do not depend on the choice of point P and the terminal ray. If we choose a different point $P'(x', y')$ on the terminal ray of A at a distance r' from the origin, it is clear that x' and y' will have the same sign as that of x and y, respectively, because of similar triangles ΔOPL and $\Delta OP'L'$.

Also, any trigonometrical function of an angle A is equal to the same trigonometrical function of any angle $360n + A$, where n is any integer since all these angles will have the same terminal ray. For example, $\sin 60° = \sin 420° = \sin(-300°)$. After the coordinate system has been introduced, the plane is divided into four quadrants. An angle is said to be in that quadrant in which its terminal ray lies. For positive acute angles, this definition gives the same result as in case of a right-angled triangle since both x and y are positive for any point in the first quadrant. Consequently, they are the length of base and perpendicular of angle A.

Graphs and Other Useful Data of Trigonometric Functions

1. $y = f(x) = \sin x$

   Domain $\to R$, Range $\to [-1, 1]$

   Period $\to 2\pi$

   $\sin^2 x$, $|\sin x| \in [0, 1]$

   $\sin x = 0 \Rightarrow x = n\pi, n \in I$

   $\sin x = 1 \Rightarrow x = (4n+1)\pi/2, n \in I$

   $\sin x = -1 \Rightarrow x = (4n-1)\pi/2, n \in I$

   $\sin x = \sin\alpha \Rightarrow x = n\pi + (-1)^n\alpha, n \in I$

   $\sin x \geq 0 \Rightarrow x \in \bigcup_{n\in I}[2n\pi, \pi + 2n\pi]$

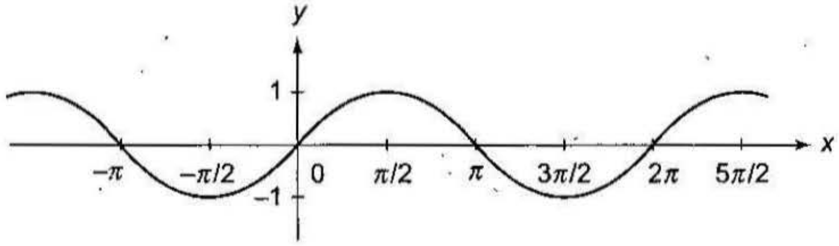


Fig. 2.15

2. $y = f(x) = \cos x$

Domain $\rightarrow R$, Range $\rightarrow [-1, 1]$

Period $\rightarrow 2\pi$

$\cos^2 x$, $|\cos x| \in [0, 1]$

$\cos x = 0 \Rightarrow x = (2n+1)\pi/2, n \in I$

$\cos x = 1 \Rightarrow x = 2n\pi, n \in I$

$\cos x = -1 \Rightarrow x = (2n+1)\pi, n \in I$

$\cos x = \cos\alpha \Rightarrow x = 2n\pi \pm \alpha, n \in I$

$$\cos x \geq 0 \Rightarrow x \in \bigcup_{n \in I} [2n\pi - \frac{\pi}{2}, 2n\pi + \pi/2]$$

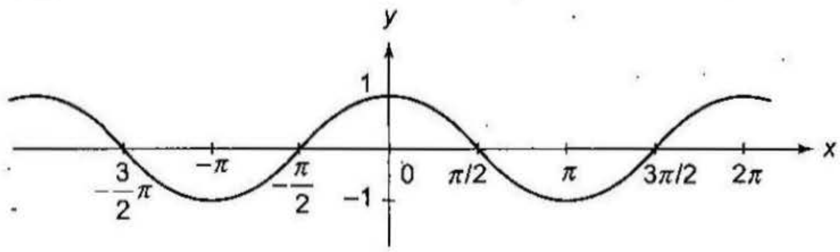


Fig. 2.16

3. $y = f(x) = \tan x$

Domain $\rightarrow R \sim (2n+1)\pi/2, n \in I$

Range $\rightarrow (-\infty, \infty)$

Period $\rightarrow \pi$

Discontinuous at $x = (2n+1)\,\pi/2, n \in I$

$\tan^2 x$, $|\tan x| \in [0, \infty)$

$\tan x = 0 \Rightarrow x = n\pi, n \in I$

$\tan x = \tan\alpha \Rightarrow x = n\pi + \alpha, n \in I$

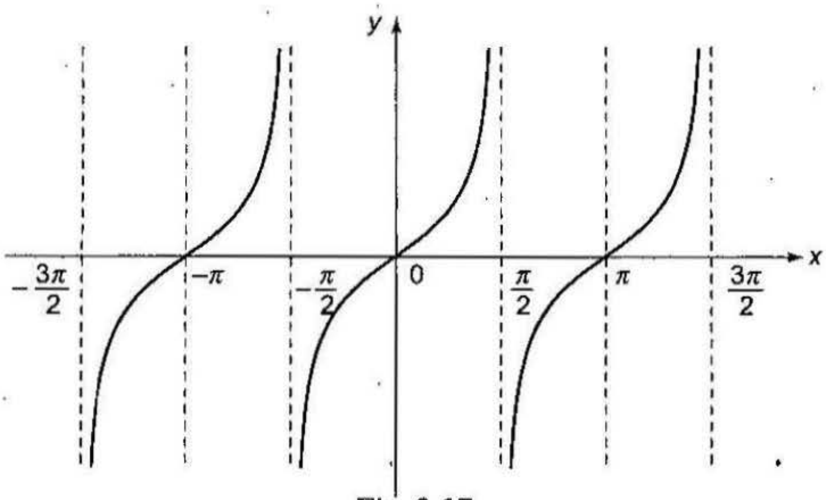


Fig. 2.17

4. $y = f(x) = \cot x$

Domain $\rightarrow R - n\pi, n \in I$; Range $\rightarrow (-\infty, \infty)$; Period $\rightarrow \pi$;

Discontinuous at $x = n\pi, n \in I$

$\cot^2 x$, $|\cot x| \in [0, \infty)$

$\cot x = 0 \Rightarrow x = (2n+1)\pi/2, n \in I$

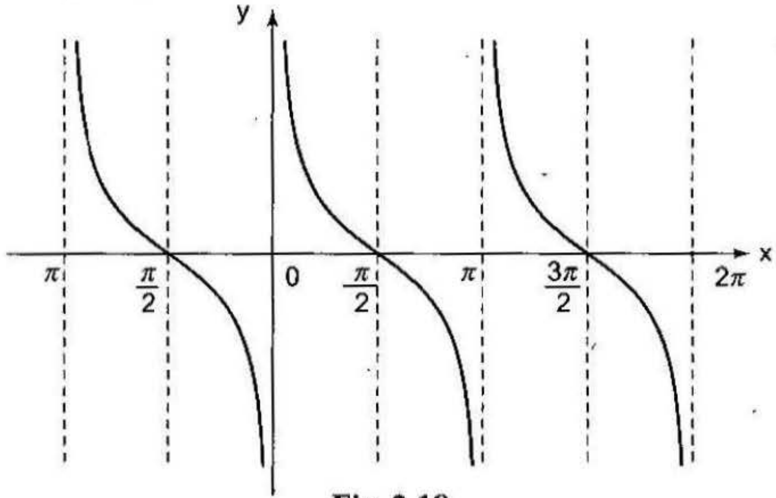


Fig. 2.18

5. $y = f(x) = \sec x$

Domain $\rightarrow R - (2n+1)\pi/2, n \in I$; Range $\rightarrow (-\infty, -1] \cup [1, \infty)$

Period $\rightarrow 2\pi$, $\sec^2 x$, $|\sec x| \in [1, \infty)$

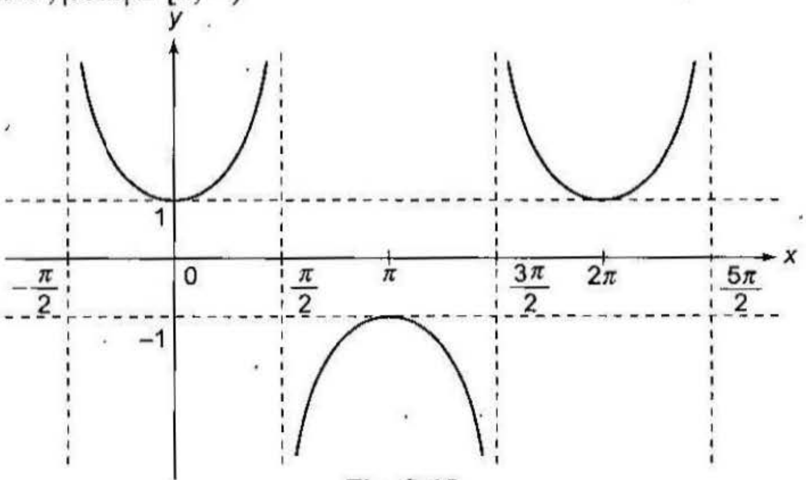


Fig. 2.19

6. $y = f(x) = \operatorname{cosec} x$

Domain $\rightarrow R - n\pi, n \in I$;

Range $\rightarrow (-\infty, -1] \cup [1, \infty)$

Period $\rightarrow 2\pi$, $\operatorname{cosec}^2 x$, $|\operatorname{cosec} x| \in [1, \infty)$

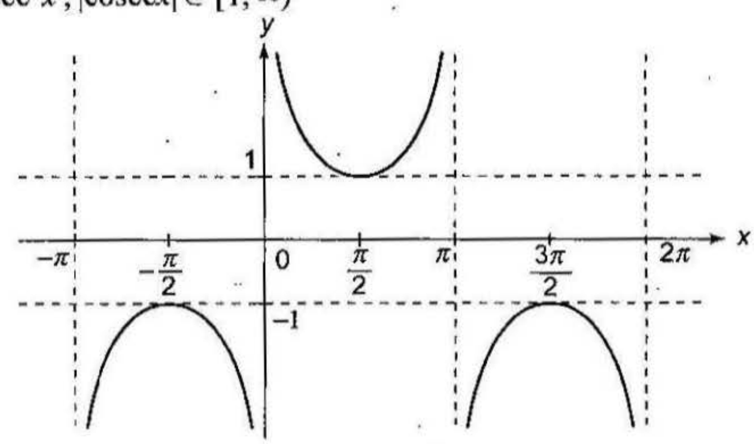


Fig. 2.20

Signs of the Trigonometric Ratios or Functions

The signs of trigonometric functions depend on the quadrant in which the terminal side of the angle lies. We always take the length $OP = r$ to be positive. Thus, $\sin\theta = y/r$ has the sign of y and $\cos\theta = x/r$ has the sign of x. The sign of $\tan\theta$ depends on the signs of x and y and similarly the signs of other trigonometric ratios are determined by the signs of x and/or y. Sign can also be determined by the graphs. Thus, we have the following:

Function	1st quadrant	2nd quadrant	3rd quadrant	4th quadrant
$\sin\theta$ $\operatorname{cosec}\theta$	+ve	+ve	–ve	–ve
$\cos\theta$ $\sec\theta$	+ve	–ve	–ve	+ve
$\tan\theta$ $\cot\theta$	+ve	–ve	+ve	–ve

Variations in the Values of Trigonometric Functions in Different Quadrants

	1st quadrant	2nd quadrant	3rd quadrant	4th quadrant
$\sin\theta$	$\uparrow$ from 0 to 1	$\downarrow$ from 1 to 0	$\downarrow$ from 0 to -1	$\uparrow$ from -1 to 0
$\cos\theta$	$\downarrow$ from 1 to 0	$\downarrow$ from 0 to -1	$\uparrow$ from -1 to 0	$\uparrow$ from 0 to 1
$\tan\theta$	$\uparrow$ from 0 to ∞	$\uparrow$ from $-\infty$ to 0	$\uparrow$ from 0 to ∞	$\uparrow$ from $-\infty$ to 0
$\cot\theta$	$\downarrow$ from ∞ to 0	$\downarrow$ from 0 to $-\infty$	$\downarrow$ from ∞ to 0	$\downarrow$ from 0 to $-\infty$
$\sec\theta$	$\uparrow$ from 1 to ∞	$\uparrow$ from $-\infty$ to -1	$\downarrow$ -1 to $-\infty$	$\downarrow$ from ∞ to 1
$\operatorname{cosec}\theta$	$\downarrow$ from ∞ to 1	$\uparrow$ from 1 to ∞	$\uparrow$ from $-\infty$ to -1	$\downarrow$ from -1 to $-\infty$

Note:

$+\infty$ and $-\infty$ are two symbols. These are not real numbers. When we say that $\tan\theta$ increases from 0 to ∞ as θ varies from 0 to $\pi/2$, it means that $\tan\theta$ increases in the interval $(0, \pi/2)$ and it attains arbitrarily large positive values as θ tends to $\pi/2$. Similarly, this happens for other trigonometrical functions as well.

Trigonometric Ratios of Standard Angles

Angle(θ) $\rightarrow$ T-Ratio $\downarrow$	30°	45°	60°
$\sin\theta$	$1/2$	$1/\sqrt{2}$	$\sqrt{3}/2$
$\cos\theta$	$\sqrt{3}/2$	$1/\sqrt{2}$	$1/2$
$\tan\theta$	$1/\sqrt{3}$	1	$\sqrt{3}$
$\operatorname{cosec}\theta$	2	$\sqrt{2}$	$2/\sqrt{3}$
$\sec\theta$	$2/\sqrt{3}$	$\sqrt{2}$	2
$\cot\theta$	$\sqrt{3}$	1	$1/\sqrt{3}$

Transformation of the Graphs of Trigonometric Functions

1. To draw the graph of $y=f(x+a)$; $(a>0)$ from the graph of $y=f(x)$, shift the graph of $y=f(x)$, a units left along the x-axis.
   Consider the following illustration.

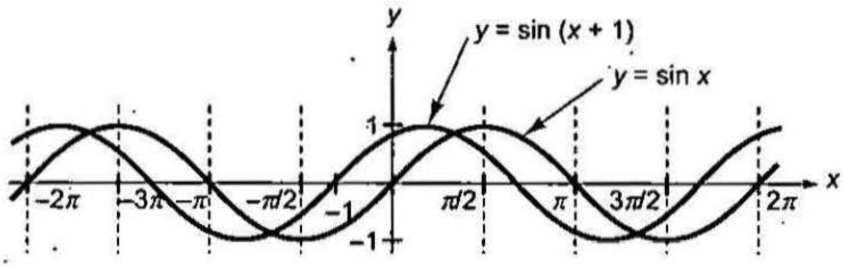


Fig. 2.21

To draw the graph of $y=f(x-a)$; $(a>0)$ from the graph of $y=f(x)$, shift the graph of $y=f(x)$, a units right along the x-axis.
Consider the following illustration.

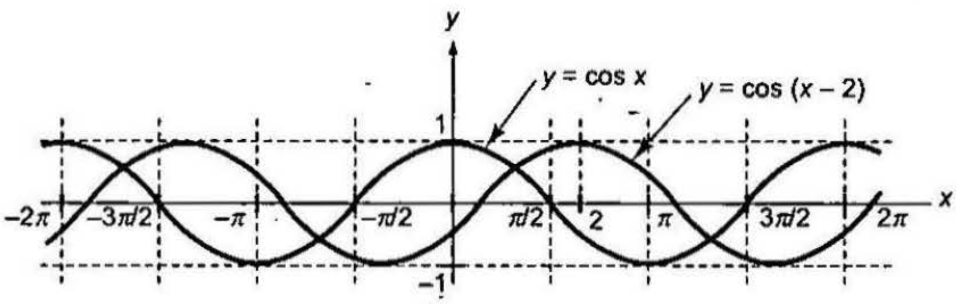


Fig. 2.22

2. To draw the graph of $y=f(x)+a$; $(a>0)$ from the graph of $y=f(x)$, shift the graph of $y=f(x)$, a units upward along the y-axis.
   To draw the graph of $y=f(x)-a$; $(a>0)$ from the graph of $y=f(x)$, shift the graph of $y=f(x)$, a units downward along the y-axis.

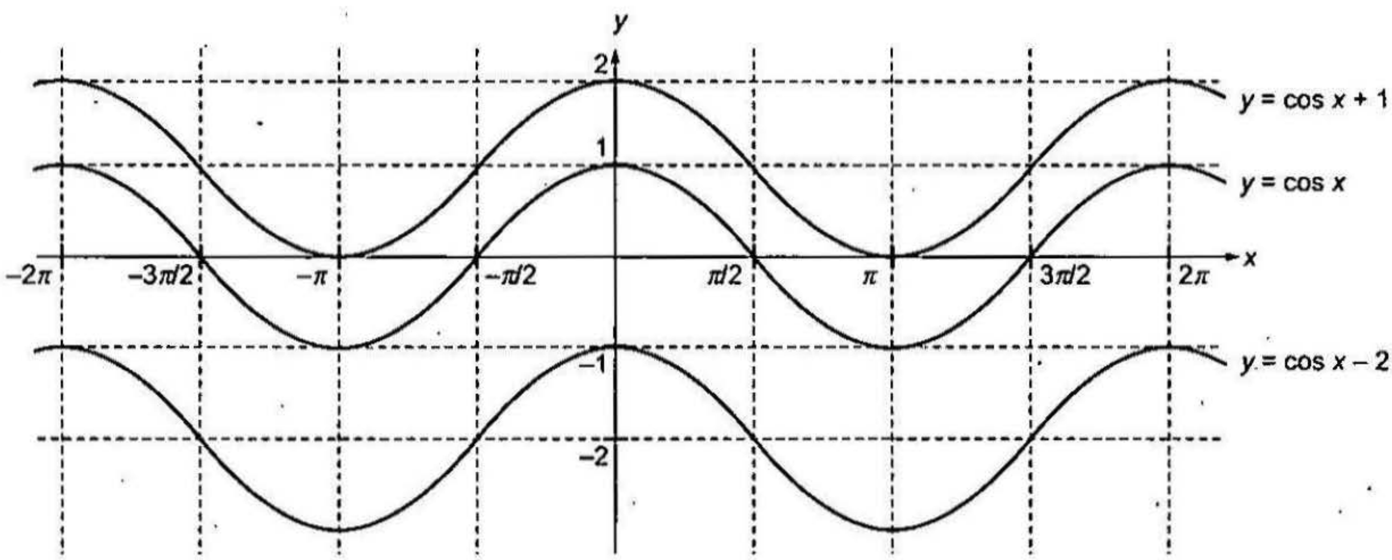


Fig. 2.23

3. If $y = f(x)$ has period T, then period of $y = f(ax)$ is $T/|a|$.

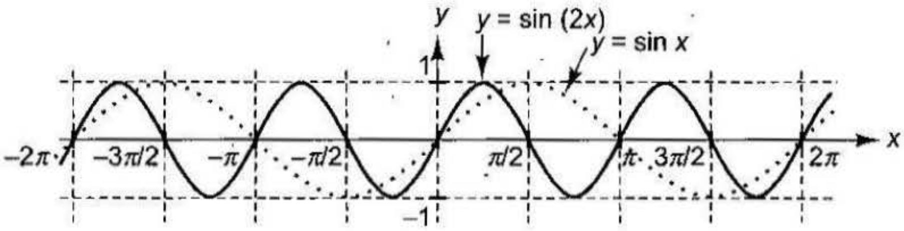


Fig. 2.24

Period of $y = \sin(2x)$ is $2\pi/2 = \pi$

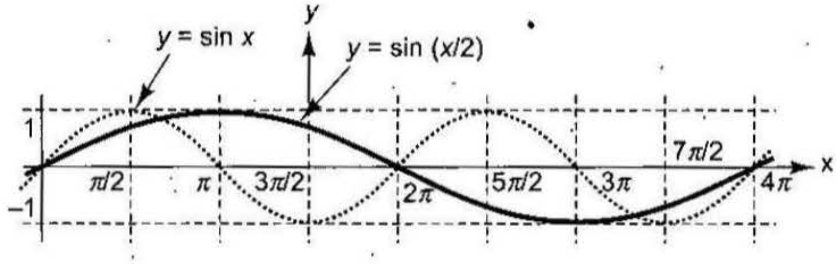


Fig. 2.25

Period of $y = \sin(x/2)$ is $2\pi/(1/2) = 4\pi$

4. Since $y = |f(x)| \geq 0$, to draw the graph of $y = |f(x)|$, take the mirror of the graph of $y = f(x)$ in x-axis for $f(x) < 0$, retaining the graph for $f(x) > 0$.
Consider the following illustrations.

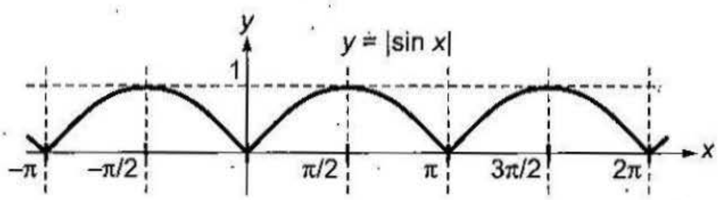


Fig. 2.26

Here period of $f(x) = |\sin x|$ is π.

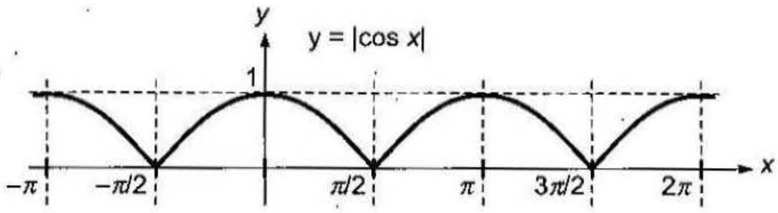


Fig. 2.27

Here period of $f(x) = |\cos x|$ is π

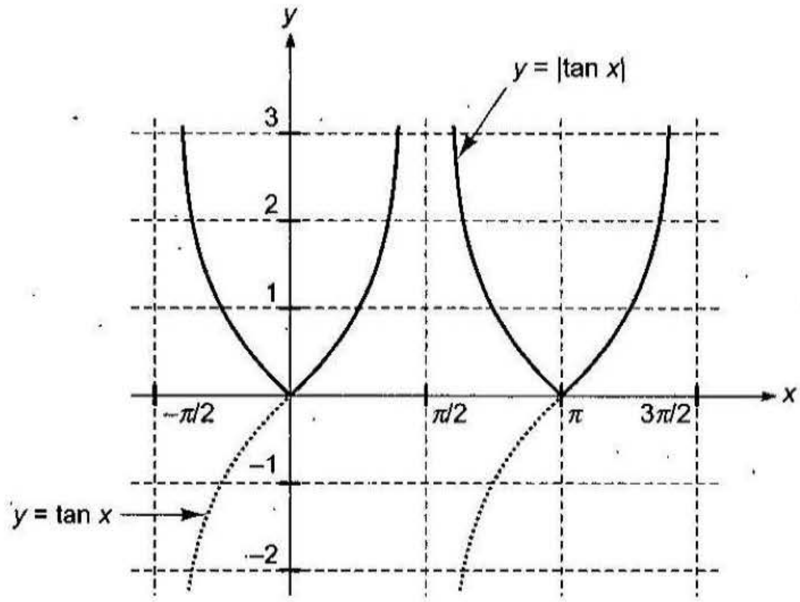
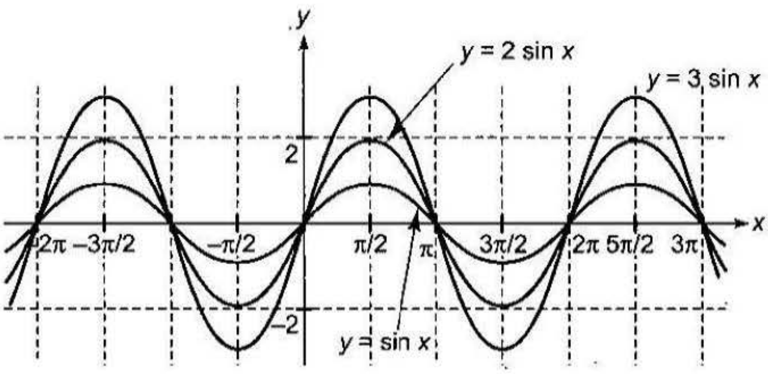


Fig. 2.28

5. Graph of $y = af(x)$ from the graph of $y\,f(x)$



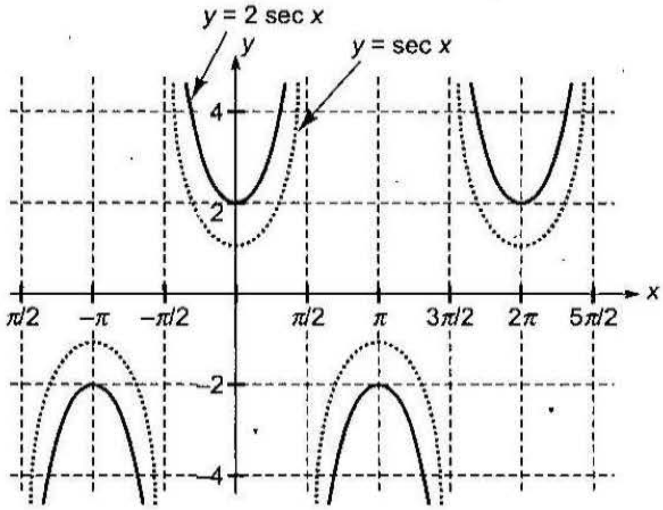


Fig. 2.29

Example 2.15 **Which of the following is possible?**

a. $\sin\theta = \dfrac{5}{3}$ **b.** $\tan\theta = 1002$

c. $\cos\theta = \dfrac{1+p^2}{1-p^2}, (p \neq \pm 1)$ **d.** $\sec\theta = \dfrac{1}{2}$

Sol. b. $\sin\theta = \dfrac{5}{3}$ is not possible as $-1 \le \sin\theta \le 1$.

$\cos\theta = \dfrac{1+p^2}{1-p^2}$ is not possible, as in $\dfrac{1+p^2}{1-p^2}$ numerator is always greater than the denominator for any value of p other than $p = 0$. Hence, $\dfrac{1+p^2}{1-p^2}$ does not lie in $[-1, 1]$.

$\sec\theta = \frac{1}{2}$ is not possible as $\sec\theta \in (-\infty, -1] \cup [1, \infty)$.

$\tan\theta = 1002$ is possible as $\tan\theta$ can take any real value.

Example 2.16 **Which of the following is greatest?**

a. $\tan 1$ **b.** $\tan 4$ **c.** $\tan 7$ **d.** $\tan 10$

Sol. a. $\tan 4 = \tan(\pi + (4-\pi)) = \tan(4-\pi) = \tan(0.86)$

$\tan 7 = \tan(2\pi + (7-2\pi)) = \tan(7-2\pi) = \tan(0.72)$

$\tan 10 = \tan(3\pi + (10-3\pi)) = \tan(10-3\pi) = \tan(0.58)$

Now $1 > 0.86 > 0.72 > 0.58$

$\Rightarrow$ $\tan 1 > \tan(0.86) > \tan(0.72) > \tan(0.58)$ [as 1, 0.86, 0.72, 0.58 lie in the first quadrant and tan functions increase in all the quadrant]

Hence, $\tan 1$ is greatest.

Example 2.17 **Which of the following is least?**

a. $\sin 3$ **b.** $\sin 2$ **c.** $\sin 1$ **d.** $\sin 7$

Sol. d. $\sin 3 = \sin[\pi - (\pi - 3)] = \sin(\pi - 3) = \sin(0.14)$

$\sin 2 = \sin[\pi - (\pi - 2)] = \sin(\pi - 2) = \sin(1.14)$

$\sin 7 = \sin[2\pi + (7-2\pi)] = \sin(7-2\pi) = \sin(0.72)$

Now $1.14 > 1 > 0.72 > 0.14$

$\Rightarrow$ $\sin(1.14) > \sin 1 > \sin(0.72) > \sin(0.14)$ [as 1.14, 1, 0.72, 0.14 lie in the first quadrant and sine functions increase in the first quadrant]

Hence, $\sin 3$ is least.

Alternative solution:

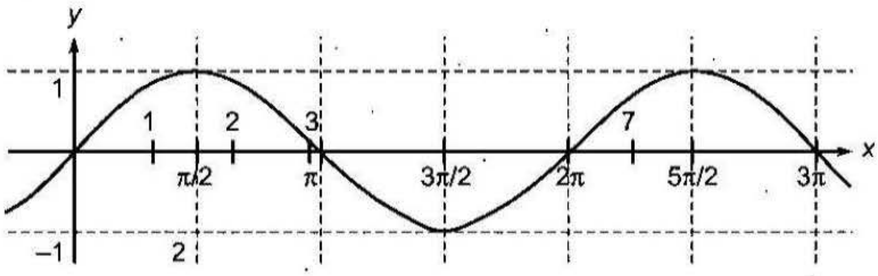


Fig. 2.30

From the graph, obviously $\sin 3$ is least.

Example 2.18 **If $A = 4\sin\theta + \cos^2\theta$, then which of the following is not true?**

a. **Maximum value of A is 5**

b. **Minimum value of A is -4**

c. **Maximum value of A occurs when $\sin\theta = 1/2$**

d. **Minimum value of A occurs when $\sin\theta = 1$**

Sol. a, c, d.

$$f(\theta) = 4\sin\theta + \cos^2\theta = 4\sin\theta + 1 - \sin^2\theta$$
$$= 5 - (4 - 4\sin\theta + \sin^2\theta) = 5 - (\sin\theta - 2)^2$$

Now maximum value of $f(\theta)$ occurs when $(\sin\theta - 2)^2$ is minimum.
Minimum value of $(\sin\theta - 2)^2$ occurs when $\sin\theta = 1$, then maximum value of $f(\theta)$ is $5 - (1-2)^2 = 4$.
Also minimum value of $f(\theta)$ occurs when $(\sin\theta - 2)^2$ is maximum.
Maximum value of $(\sin\theta - 2)^2$ occurs when $\sin\theta = -1$, then minimum value of $f(\theta)$ is $5 - (-1-2)^2 = -4$.

Example 2.19 **Is the equation $\sec^2\theta = \dfrac{4xy}{(x+y)^2}$ possible for real values of x and y?**

Sol. Given, $\sec^2\theta = \dfrac{4xy}{(x+y)^2}$

Since $\sec^2\theta \ge 1$, we get $\dfrac{4xy}{(x+y)^2} \ge 1$

$\Rightarrow \quad (x+y)^2 \le 4xy$

$\Rightarrow \quad (x+y)^2 - 4xy \le 0$ or $(x-y)^2 \le 0$

But for real values of x and y, $(x-y)^2 \ge 0$

Since $(x-y)^2 = 0$, $x = y$. Also $x + y \ne 0 \Rightarrow x \ne 0, y \ne 0$

Therefore, the given equation $\sec^2\theta = \dfrac{4xy}{(x+y)^2}$ is possible for real values of x and y only when $x = y\ (x \ne 0)$.

Example 2.20 **Show that the equation $\sin\theta = x + \dfrac{1}{x}$ is impossible if x is real.**

Sol. Given, $\sin\theta = x + \dfrac{1}{x}$

$\therefore \quad \sin^2\theta = x^2 + \dfrac{1}{x^2} + 2x\dfrac{1}{x} = x^2 + \dfrac{1}{x^2} + 2 = \left(x - \dfrac{1}{x}\right)^2 + 4 \ge 4$

which is not possible since $\sin\theta^2 \le 1$.

Example 2.21 **If $\sin^2\theta_1 + \sin^2\theta_2 + \sin^2\theta_3 = 0$, then which of the following is not the possible value of $\cos\theta_1 + \cos\theta_2 + \cos\theta_3$.**

a. 3 **b.** –3 **c.** –1 **d.** –2

Sol. **d.**

$\sin^2\theta_1 + \sin^2\theta_2 + \sin^2\theta_3 = 0$

$\Rightarrow \quad \sin^2\theta_1 = \sin^2\theta_2 = \sin^2\theta_3 = 0 \quad \Rightarrow \quad \cos^2\theta_1, \cos^2\theta_2, \cos^2\theta_3 = 1 \quad \Rightarrow \quad \cos\theta_1, \cos\theta_2, \cos\theta_3 = \pm 1$

$\cos\theta_1 + \cos\theta_2 + \cos\theta_3$ can be -3 (when all are -1)

or 3 (when all are $+1$)

or -1 (when any two are -1 and one $+1$)

or 1 (when any two are $+1$ and one -1)

but -2 is not a possible value.

Example 2.22 **For real values of θ, which of the following is/are positive?**

a. $\cos(\cos\theta)$ **b.** $\cos(\sin\theta)$ **c.** $\sin(\cos\theta)$ **d.** $\sin(\sin\theta)$

Sol. a, b. $\cos\theta, \sin\theta \in [-1, 1]$ or (value lies in 1st or 4th quadrant)

For which $\cos(\sin\theta)$ is always greater than 0.

$\sin(\cos\theta) < 0$, when $\cos\theta \in [-1, 0]$ and $\sin(\sin\theta) > 0$ when $\sin\theta \in [0, 1]$

Example 2.23 **Find the range of $f(x) = \dfrac{1}{4\cos x - 3}$.**

Sol. $-1 \le \cos x \le 1$

$\Rightarrow -4 \le 4\cos x \le 4$

$\Rightarrow -7 \le 4\cos x - 3 \le 1$

$\Rightarrow -7 \le 4\cos x - 3 < 0$ or $0 < 4\cos x - 3 \le 1$ $\quad (\because 4\cos x - 3 \ne 0)$

$\Rightarrow -\frac{1}{7} \ge \frac{1}{4\cos x - 3} > -\infty$ or $\infty > \frac{1}{4\cos x - 3} \ge 1$

$\Rightarrow \frac{1}{4\cos x - 3} \in \left(-\infty, -\frac{1}{7}\right] \cup [1, \infty)$

Example 2.24 **Find the range of $f(x) = \frac{1}{5\sin x - 6}$.**

Sol. $-1 \le \sin x \le 1$

$\Rightarrow -5 \le 5\sin x \le 5$

$\Rightarrow -11 \le 5\sin x - 6 \le -1$

$\Rightarrow -1 \le \frac{1}{5\sin x - 6} \le -1/11$

$\Rightarrow \frac{1}{5\sin x - 6} \in [-1, -1/11]$

Example 2.25 **Find the range of $f(x) = \cos^2 x + \sec^2 x$**

Sol. We have

$f(x) = \cos^2 x + \sec^2 x$

$= (\cos x - \sec x)^2 + 2\cos x \sec x$

$= 2 + (\cos x - \sec x)^2 \ge 2$

Example 2.26 **Find the range of $f(x) = \sin^2 x - 3\sin x + 2$**

Sol. $f(x) = \sin^2 x - 3\sin x + 2$

$= (\sin x - 3/2)^2 + 2 - 9/4$

$= (\sin x - 3/2)^2 - 1/4$

$-1 \le \sin x \le 1$

$\Rightarrow -5/2 \le \sin x - 3/2 \le -1/2$

$\Rightarrow 1/4 \le (\sin x - 3/2)^2 \le 25/4$

$\Rightarrow 0 \le (\sin x - 3/2)^2 - 1/4 \le 6$

Example 2.27 **Find the range of $f(x) = \sqrt{\sin^2 x - 6\sin x + 9} + 3$.**

Sol. $f(x) = \sqrt{\sin^2 x - 6\sin x + 9} + 3$

$= \sqrt{(\sin x - 3)^2} + 3$

$= |\sin x - 3| + 3$

Now $-1 \le \sin x \le 1$

$\Rightarrow -4 \le \sin x - 3 \le -2$

$\Rightarrow 2 \le |\sin x - 3| \le 4$

$\Rightarrow 5 \le |\sin x - 3| + 3 \le 7$

Example 2.28 **Find the range of $f(x) = \operatorname{cosec}^2 x + 25\sec^2 x$.**

Sol. $f(x) = (1 + \cot^2 x) + 25(1 + \tan^2 x)$

$= 26 + \cot^2 x + 25\tan^2 x$

$= 36 + 10 + (\cot^2 x + 25\tan^2 x - 2\cot x\, 5\tan x)$

$= 36 + (\cot x - 5\tan x)^2 \geq 36$

Example 2.29 **Find the value of x for which $f(x) = \sqrt{\sin x - \cos x}$ is defined, $x \in [0, 2\pi]$.**

Sol. $f(x) = \sqrt{\sin x - \cos x}$ is defined if $\sin x \geq \cos x$,

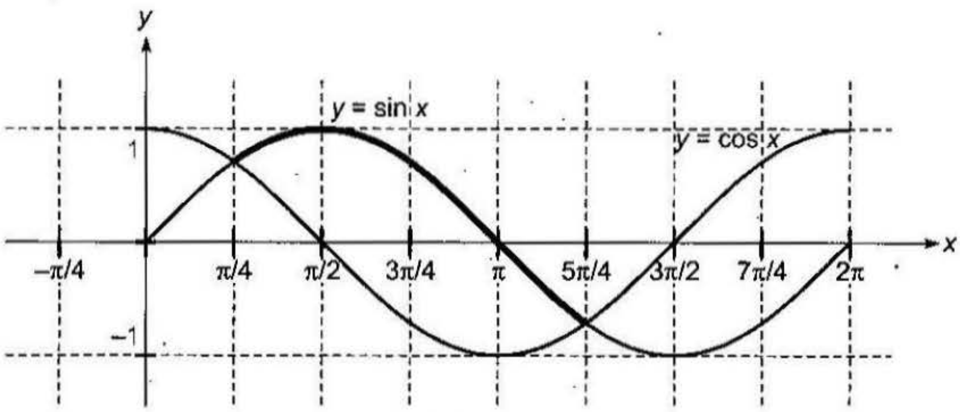


Fig. 2.31

From the graph, $\sin x \geq \cos x$, for $x \in \left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$.

Example 2.30 **Which of the following is highest?**

a. cosec 1 **b.** cosec 2 **c.** cosec 4 **d.** cosec (–6)

Sol. d. Consider sin 1, sin 2 and –sin 6 (sin 4 is negative; hence, cosec 4 cannot be maximum).

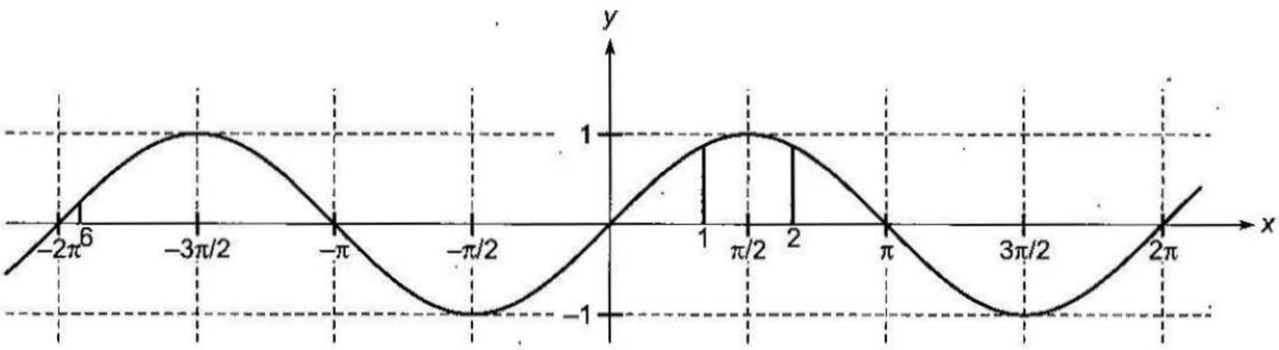


Fig. 2.32

From the graph, sin (–6) is least, hence cosec (–6) is maximum.

Example 2.31 **Solve $\tan x > \cot x$, where $x \in [0, 2\pi]$.**

Sol.

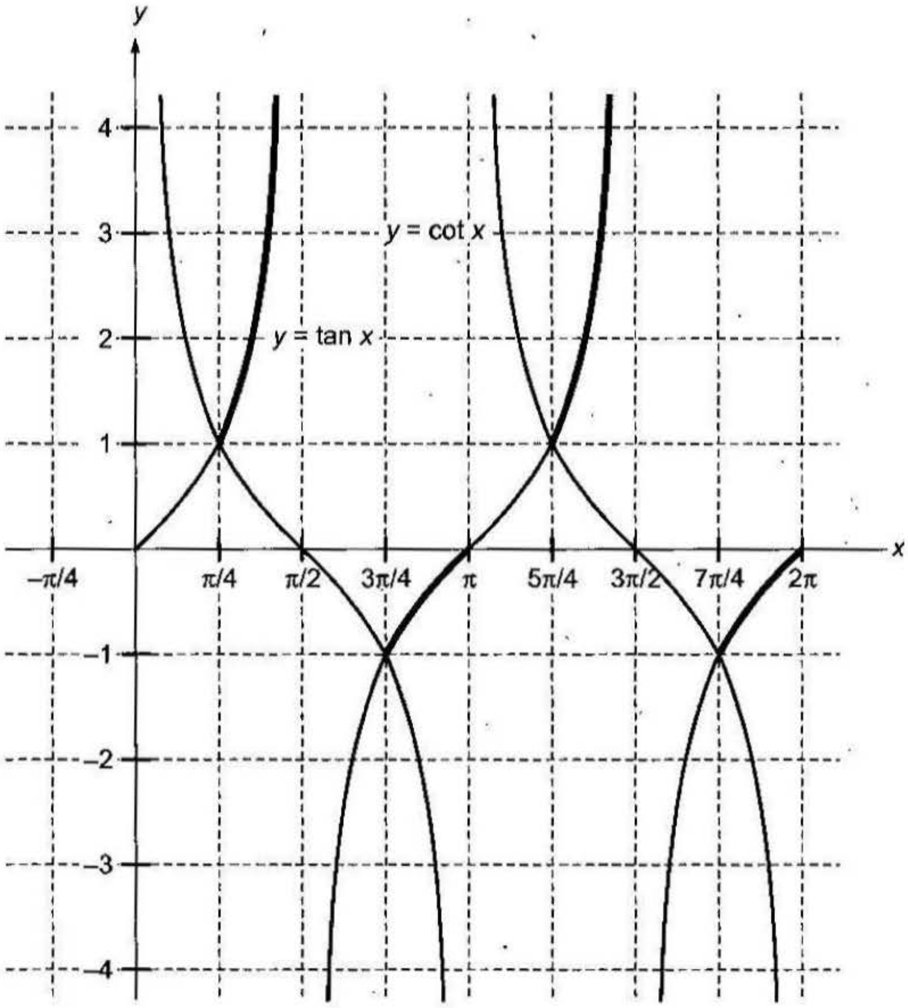


Fig. 2.33

We find that $\tan x \geq \cot x$
Therefore, values of $\tan x$ are more than the values of $\cot x$.
That is, values of x for which graph of $y = \tan x$ is above the graph of $y = \cot x$.
From the graph, it is clear that $x \in (\pi/4, \pi/2) \cup (3\pi/4, \pi) \cup (5\pi/4, 3\pi/2) \cup (7\pi/4, 2\pi)$.

Concept Application Exercise 2.2

1. Find the least value of $2 \sin^2 \theta + 3 \cos^2 \theta$.
2. Find the range of $f(x) = \sin(\cos x)$.
3. Find the range of $12 \sin \theta - 9 \sin^2 \theta$.
4. Find the minimum value of $9 \tan^2 \theta + 4 \cot^2 \theta$.
5. Which of following is correct (where $n \in N$)?

   a. $\sin \theta = \dfrac{n+1}{n}$ **b.** $\sin \theta = \dfrac{n^2+1}{n+1}$ **c.** $\sec \theta = \dfrac{n+2}{n-1}$ **d.** $\sec \theta = \dfrac{n}{\sqrt{n^2+1}}$

6. If $\sin^2 \theta_1 + \sin^2 \theta_2 + \cdots + \sin^2 \theta_n = 0$, then find the minimum value of $\cos \theta_1 + \cos \theta_2 + \cdots + \cos \theta_n$.
7. If $\sin^2 \theta = x^2 - 3x + 3$ is meaningful, then find the values of x.
8. Find the range of $f(x) = \sqrt{4 - \sqrt{1 + \tan^2 x}}$.

9. Find the range of $f(x) = \dfrac{1}{2|\cos x| - 3}$.
10. Find the range of $f(x) = \cos^4 x + \sin^2 x - 1$.
11. Find the minimum value of the function $f(x) = (1 + \sin x)(1 + \cos x)$, $\forall x \in R$.
12. Prove that $(\sin \theta + \operatorname{cosec} \theta)^2 + (\cos \theta + \sec \theta)^2 \geq 9$.

PROBLEMS BASED ON TRIGONOMETRIC IDENTITIES

Example 2.32 **Show that $2(\sin^6 x + \cos^6 x) - 3(\sin^4 x + \cos^4 x) + 1 = 0$.**

Sol. $2(\sin^6 x + \cos^6 x) - 3(\sin^4 x + \cos^4 x) + 1$

$= 2[(\sin^2 x)^3 + (\cos^2 x)^3] - 3(\sin^4 x + \cos^4 x) + 1 = 2[(\sin^2 x + \cos^2 x)^3 - 3\sin^2 x\cos^2 x(\sin^2 x + \cos^2 x)]$

$- 3[(\sin^2 x + \cos^2 x)^2 - 2\sin^2 x\cos^2 x] + 1 = 2[1 - 3\sin^2 x\cos^2 x] - 3[1 - 2\sin^2 x\cos^2 x] + 1 = 0$

Example 2.33 **Prove that $\sqrt{\dfrac{1+\sin\theta}{1-\sin\theta}} = \sec\theta + \tan\theta$, $-\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$.**

Sol. L.H.S. $= \sqrt{\dfrac{1+\sin\theta}{1-\sin\theta}} = \sqrt{\dfrac{1+\sin\theta}{1-\sin\theta}\,\dfrac{1+\sin\theta}{1+\sin\theta}}$

$= \sqrt{\dfrac{(1+\sin\theta)^2}{1-\sin^2\theta}} = \sqrt{\dfrac{(1+\sin\theta)^2}{\cos^2\theta}}$

$= \dfrac{1+\sin\theta}{\cos\theta} = \dfrac{1}{\cos\theta} + \dfrac{\sin\theta}{\cos\theta} = \sec\theta + \tan\theta$

= R.H.S.

Example 2.34 **Prove that $\dfrac{1}{\sec A - \tan A} - \dfrac{1}{\cos A} = \dfrac{1}{\cos A} - \dfrac{1}{\sec A + \tan A}$.**

Sol. To prove $\dfrac{1}{\sec A - \tan A} - \dfrac{1}{\cos A} = \dfrac{1}{\cos A} - \dfrac{1}{\sec A + \tan A}$

or, $\dfrac{1}{\sec A - \tan A} + \dfrac{1}{\sec A + \tan A} = \dfrac{1}{\cos A} + \dfrac{1}{\cos A} = \dfrac{2}{\cos A}$ (i)

Now L.H.S. $= \dfrac{1}{\sec A - \tan A} + \dfrac{1}{\sec A + \tan A} = \dfrac{\sec A + \tan A + \sec A - \tan A}{(\sec A - \tan A)(\sec A + \tan A)} = \dfrac{2}{\cos A}$

Example 2.35 **If $3\sin\theta + 5\cos\theta = 5$, then show that $5\sin\theta - 3\cos\theta = \pm 3$.**

Sol. Given, $3\sin\theta + 5\cos\theta = 5$ (i)

Let $5\sin\theta - 3\cos\theta = x$ (ii)

Squaring and adding, we get

$(9\sin^2\theta + 25\cos^2\theta + 30\sin\theta\cos\theta) + (25\sin^2\theta + 9\cos^2\theta - 30\sin\theta\cos\theta) = 25 + x^2$

$\Rightarrow$ $9(\sin^2\theta + \cos^2\theta) + 25(\sin^2\theta + \cos^2\theta) = 25 + x^2$

$\Rightarrow$ $34 = 25 + x^2$ or $x^2 = 9$

$\Rightarrow \quad x = \pm 3$

Example 2.36 **If $(\sec A + \tan A)(\sec B + \tan B)(\sec C + \tan C) = (\sec A - \tan A)(\sec B - \tan B)(\sec C - \tan C)$, prove that the value of each side is ± 1.**

Sol. Let $(\sec A + \tan A)(\sec B + \tan B)(\sec C + \tan C) = x$ (i)

$(\sec A - \tan A)(\sec B - \tan B)(\sec C - \tan C) = x.$ (ii)

Multiplying Eqs. (i) and (ii), we get

$(\sec^2 A - \tan^2 A)(\sec^2 B - \tan^2 B)(\sec^2 C - \tan^2 C) = x^2$

or $x^2 = 1$

$\therefore x = \pm 1$

Hence, each side is equal to ± 1.

Example 2.37 **If $\tan\theta + \sec\theta = 1.5$, find $\sin\theta$, $\tan\theta$, and $\sec\theta$.**

Sol. Given, $\sec\theta + \tan\theta = \dfrac{3}{2}$ (i)

Now, $\sec\theta - \tan\theta = \dfrac{1}{\sec\theta + \tan\theta} = \dfrac{2}{3}$ (see Fig. 2.34) (ii)

Adding Eqs. (i) and (ii), we get $2\sec\theta = \dfrac{3}{2} + \dfrac{2}{3} = \dfrac{13}{6}$

$\therefore \quad \sec\theta = \dfrac{13}{12}$

$\therefore \quad \tan\theta = \dfrac{5}{12}$

and $\sin\theta = \dfrac{5}{13}$

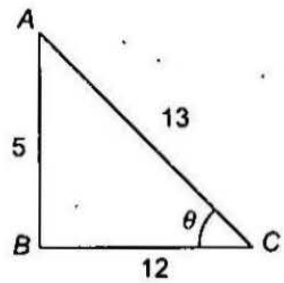


Fig. 2.34

Example 2.38 **If $\operatorname{cosec}\theta - \sin\theta = m$ and $\sec\theta - \cos\theta = n$, eliminate θ.**

Sol. Given $\operatorname{cosec}\theta - \sin\theta = m$, or $\dfrac{1}{\sin\theta} - \sin\theta = m$

or, $\dfrac{1 - \sin^2\theta}{\sin\theta} = m$, or $\dfrac{\cos^2\theta}{\sin\theta} = m$ (i)

Again $\sec\theta - \cos\theta = n$, or $\dfrac{1}{\cos\theta} - \cos\theta = n$

or, $\dfrac{1 - \cos^2\theta}{\cos\theta} = n$, or $\dfrac{\sin^2\theta}{\cos\theta} = n$ (ii)

From Eq. (i), $\sin\theta = \dfrac{\cos^2\theta}{m}$ (iii)

Putting in Eq. (ii), we get $\dfrac{\cos^4\theta}{m^2\cos\theta} = n$, or $\cos^3\theta = m^2 n$

$\therefore \quad \cos\theta = (m^2 n)^{\frac{1}{3}}$, or $\cos^2\theta = (m^2 n)^{\frac{2}{3}}$ (iv)

From Eq. (iii), $\sin\theta = \frac{\cos^2\theta}{m} = \frac{(m^2n)^{\frac{2}{3}}}{m} = \frac{m^{\frac{4}{3}}n^{\frac{2}{3}}}{m} = m^{\frac{1}{3}}n^{\frac{2}{3}} = (mn^2)^{\frac{1}{3}}$

$\therefore \quad \sin^2\theta = (mn^2)^{\frac{2}{3}}$ (v)

Adding Eqs. (iv) and (v), we get

$$(m^2n)^{\frac{2}{3}} + (mn^2)^{\frac{2}{3}} = \cos^2\theta + \sin^2\theta$$

or, $(m^2n)^{\frac{2}{3}} + (mn^2)^{\frac{2}{3}} = 1$

Example 2.39 If $\frac{\cos^4 A}{\cos^2 B} + \frac{\sin^4 A}{\sin^2 B} = 1$, then prove that

a. $\sin^4 A + \sin^4 B = 2\sin^2 A \sin^2 B$

b. $\frac{\cos^4 B}{\cos^2 A} + \frac{\sin^4 B}{\sin^2 A} = 1$

Sol. Given, $\frac{\cos^4 A}{\cos^2 B} + \frac{\sin^4 A}{\sin^2 B} = 1(\cos^2 A + \sin^2 A)$

$$\Rightarrow \frac{\cos^4 A}{\cos^2 B} - \cos^2 A = \sin^2 A - \frac{\sin^4 A}{\sin^2 B}$$

$$\Rightarrow \frac{\cos^2 A(\cos^2 A - \cos^2 B)}{\cos^2 B} = \sin^2 A\frac{(\sin^2 B - \sin^2 A)}{\sin^2 B}$$

$$\Rightarrow \frac{\cos^2 A}{\cos^2 B}(\cos^2 A - \cos^2 B) = \frac{\sin^2 A}{\sin^2 B}\left[(1 - \cos^2 B) - (1 - \cos^2 A)\right]$$

$$\Rightarrow \frac{\cos^2 A}{\cos^2 B}(\cos^2 A - \cos^2 B) = \frac{\sin^2 A}{\sin^2 B}(\cos^2 A - \cos^2 B)$$

$$\Rightarrow (\cos^2 A - \cos^2 B)\left(\frac{\cos^2 A}{\cos^2 B} - \frac{\sin^2 A}{\sin^2 B}\right) = 0$$

when $\cos^2 A - \cos^2 B = 0$, $\cos^2 A = \cos^2 B$ (i)

when $\frac{\cos^2 A}{\cos^2 B} - \frac{\sin^2 A}{\sin^2 B} = 0$, $\cos^2 A\sin^2 B = \sin^2 A\cos^2 B$

$$\Rightarrow \cos^2 A(1 - \cos^2 B) = (1 - \cos^2 A)\cos^2 B$$

$$\Rightarrow \cos^2 A - \cos^2 A\cos^2 B = \cos^2 B - \cos^2 A\cos^2 B$$

$\Rightarrow \cos^2 A = \cos^2 B$ (ii)

Thus, in both the cases, $\cos^2 A = \cos^2 B$.

$\therefore \quad 1-\sin^2 A = 1-\sin^2 B$, or $\sin^2 A = \sin^2 B$ (iii)

a. L.H.S. $= \sin^4 A + \sin^4 B$

$= (\sin^2 A - \sin^2 B)^2 + 2\sin^2 A \sin^2 B = 2\sin^2 A \sin^2 B =$ R.H.S. $[\because \sin^2 A = \sin^2 B]$

b. L.H.S. $= \dfrac{\cos^4 B}{\cos^2 A} + \dfrac{\sin^4 B}{\sin^2 A} = \dfrac{\cos^4 B}{\cos^2 B} + \dfrac{\sin^4 B}{\sin^2 B}$

$= \cos^2 B + \sin^2 B = 1$

Example 2.40 **If $x = \sec\theta - \tan\theta$ and $y = \operatorname{cosec}\theta + \cot\theta$, then prove that $xy + 1 = y - x$.**

Sol. $xy + 1 = \dfrac{1-\sin\theta}{\cos\theta}\dfrac{1+\cos\theta}{\sin\theta} + 1 = \dfrac{1-\sin\theta+\cos\theta}{\sin\theta\cos\theta}$

$= \dfrac{(\sin^2\theta + \cos^2\theta)}{\sin\theta\cos\theta} - \dfrac{(\sin\theta - \cos\theta)}{\sin\theta\cos\theta}$

$= (\tan\theta + \cot\theta) - (\sec\theta - \operatorname{cosec}\theta)$

$= (\operatorname{cosec}\theta + \cot\theta) - (\sec\theta - \tan\theta) = y - x$

Concept Application Exercise 2.3

1. Show that $3(\sin x - \cos x)^4 + 6(\sin x + \cos x)^2 + 4(\sin^6 x + \cos^6 x) = 13$.
2. If $\sec\theta + \tan\theta = p$, then find the value of $\tan\theta$.
3. If $(1 + \sin A)(1 + \sin B)(1 + \sin C) = (1 - \sin A)(1 - \sin B)(1 - \sin C)$, then find the value of $(1 + \sin A)(1 + \sin B)(1 + \sin C)$.
4. If $(\sec\theta + \tan\theta)(\sec\phi + \tan\phi)(\sec\psi + \tan\psi) = \tan\theta\tan\phi\tan\psi$, then $(\sec\theta - \tan\theta)(\sec\phi - \tan\phi)(\sec\psi - \tan\psi)$ is equal to

   a. $\cot\theta\cot\phi\cot\psi$ b. $\tan\theta\tan\phi\tan\psi$

   c. $\tan\theta + \tan\phi + \tan\psi$ d. $\cot\theta + \cot\phi + \cot\psi$

5. If $\dfrac{x}{a}\cos\theta + \dfrac{y}{b}\sin\theta = 1$, $\dfrac{x}{a}\sin\theta - \dfrac{y}{b}\cos\theta = 1$, then eliminate θ.
6. If $a + b\tan\theta = \sec\theta$ and $b - a\tan\theta = 3\sec\theta$, then find the value of $a^2 + b^2$.
7. If $a\sin^2 x + b\cos^2 x = c$, $b\sin^2 y + a\cos^2 y = d$ and $a\tan x = b\tan y$, then prove that $\dfrac{a^2}{b^2} = \dfrac{(d-a)(c-a)}{(b-c)(b-d)}$.

TRIGONOMETRIC RATIOS FOR COMPLEMENTARY AND SUPPLEMENTARY ANGLES

In each of the following figures, x and y are positive. Also triangles OPM, $OP'M'$, or $OP'M$ are congruent.

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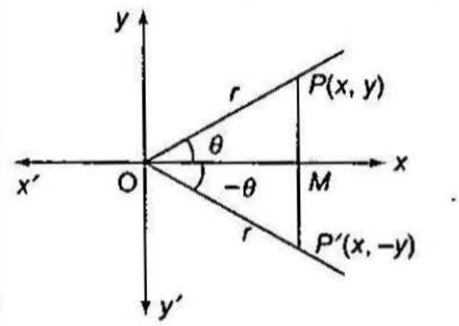
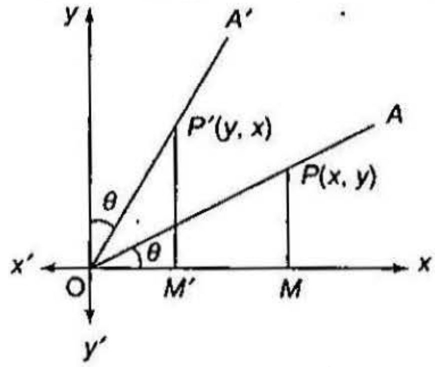
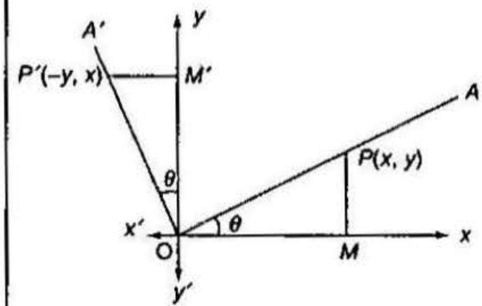
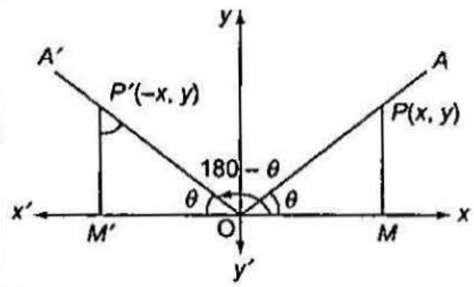
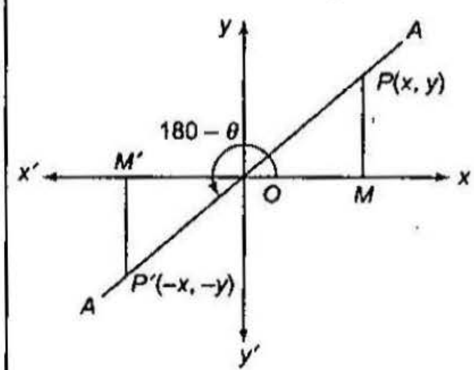
$\sin(-\theta) = -\sin\theta$ $\cos(-\theta) = \cos\theta$	$\sin(-\theta) = \frac{-y}{r} = -\frac{y}{r} = -\sin\theta$, $\cos(-\theta) = \frac{x}{r} = \cos\theta$, $\tan(-\theta) = -\frac{y}{x} = \tan\theta$ Taking the reciprocals of these trigonometric ratios, we have $\operatorname{cosec}(-\theta) = -\operatorname{cosec}\theta$, $\sec(-\theta) = \sec\theta$ and $\cot(-\theta) = -\cot\theta$	
$\sin(90° - \theta) = \cos\theta$ $\cos(90° - \theta) = \sin\theta$	$\sin(90 - \theta) = \frac{x}{r} = \cos\theta$ $\cos(90 - \theta) = \frac{y}{r} = \sin\theta$ $\tan(90 - \theta) = \frac{x}{y} = \cot\theta$	
$\sin(90° + \theta) = \cos\theta$ $\cos(90° + \theta) = -\sin\theta$	$\sin(90 + \theta) = \frac{x}{r} = \cos\theta$ $\cos(90 + \theta) = \frac{-y}{r} = -\sin\theta$ $\tan(90 + \theta) = \frac{x}{-y} = \frac{-x}{y} = -\cot\theta$	
$\sin(180° - \theta) = \sin\theta$ $\cos(180° - \theta)$ $= -\cos\theta$	Now, $\sin(180 - \theta) = \frac{y}{r} = \sin\theta$ $\cos(180 - \theta) = -\frac{x}{r} = -\cos\theta$ and, $\tan(180 - \theta) = \frac{y}{-x} = -\tan\theta$	
$\sin(180° + \theta)$ $= -\sin\theta$ $\cos(180° + \theta)$ $= -\cos\theta$	$= \sin(180 + \theta) = \frac{-y}{r} = -\sin\theta$ $\cos(180 + \theta) = \frac{-x}{r} = -\cos\theta$ $\tan(180 + \theta) = \frac{-y}{-x} = \frac{y}{x} = \tan\theta$	

Since the terminal sides of co-terminal angles coincide, hence their trigonometrical ratios are same.
Clearly, $360° - \theta$ and $-\theta$ are coterminal angles.

Therefore, $\sin(360° - \theta) = \sin(-\theta) = -\sin\theta$, $\cos(360° - \theta) = \cos(-\theta) = \cos\theta$, and $\tan(360° - \theta) = \tan(-\theta) = -\tan\theta$. Similarly, $\operatorname{cosec}(360° - \theta) = -\operatorname{cosec}\theta$, $\sec(360° - \theta) = \sec\theta$ and $\cot(360° - \theta) = -\cot\theta$.

Also θ and $360° + \theta$ are co-terminal angles. Therefore, $\sin(360° + \theta) = \sin\theta$, $\cos(360° + \theta) = \cos\theta$, $\tan(360° + \theta) = \tan\theta$, $\sec(360° + \theta) = \sec\theta$, $\operatorname{cosec}(360° + \theta) = \operatorname{cosec}\theta$ and $\cot(360° + \theta) = \cot\theta$.

In fact, for any positive integer n, $(360° \times n + \theta)$ is co-terminal to θ. Therefore, for any positive integer n, we have $\sin(360° \times n + \theta) = \sin\theta$, $\cos(360° \times n + \theta) = \cos\theta$, $\tan(360° \times n + \theta) = \tan\theta$, $\operatorname{cosec}(360° \times n + \theta) = \operatorname{cosec}\theta$, $\sec(360° \times n + \theta) = \sec\theta$ and $\cot(360° \times n + \theta) = \cot\theta$.

Example 2.41 **Prove that $\sin(-420°)(\cos 390°) + \cos(-660°)(\sin 330°) = -1$.**

Sol. L.H.S. $= \sin(-420°)(\cos 390°) + \cos(-660°)(\sin 330°)$

$= -\sin 420° \cos 390° + \cos 660° \sin 330°$ $\quad [\because \sin(-\theta) = -\sin\theta, \cos(-\theta) = \cos\theta]$

$= -\sin(90° \times 4 + 60°)\cos(90° \times 4 + 30°) + \cos(90° \times 7 + 30°)\sin(90° \times 3 + 60°)$

$= -(\sin 60°)(\cos 30°) + (\sin 30°)(-\cos 60°)$

$$= -\frac{\sqrt{3}}{2} \times \frac{\sqrt{3}}{2} + \frac{1}{2}\left(-\frac{1}{2}\right) = -1 = \text{R.H.S.}$$

Example 2.42 **Prove that $\dfrac{\cos(90° + \theta)\sec(-\theta)\tan(180° - \theta)}{\sec(360° - \theta)\sin(180° + \theta)\cot(90° - \theta)} = -1$.**

Sol. $$\text{L.H.S} = \frac{\cos(90° + \theta)\sec(-\theta)\tan(180° - \theta)}{\sec(360° - \theta)\sin(180° + \theta)\cot(90° - \theta)} = \frac{(-\sin\theta)(\sec\theta)(-\tan\theta)}{(\sec\theta)(-\sin\theta)(\tan\theta)} = -1 = \text{R.H.S.}$$

Example 2.43 **If A, B, C, D are angles of a cyclic quadrilateral, then prove that $\cos A + \cos B + \cos C + \cos D = 0$.**

Sol. We know that the opposite angles of a cyclic quadrilateral are supplementary, i.e., $A + C = \pi$ and $B + D = \pi$.

$\therefore \quad A = \pi - C$ and $B = \pi - D$

$\Rightarrow \quad \cos A = \cos(\pi - C) = -\cos C$

and $\cos B = \cos(\pi - D) = -\cos D$

$\therefore \quad \cos A + \cos B + \cos C + \cos D = -\cos C - \cos D + \cos C + \cos D = 0$

Example 2.44 **Show that $\tan 1° \tan 2° \ldots \tan 89° = 1$.**

Sol. L.H.S. $= (\tan 1° \tan 89°)(\tan 2° \tan 88°) \ldots (\tan 44° \tan 46°)\tan 45°$

$= [\tan 1° \tan(90° - 1°)][\tan 2° \tan(90° - 2°)] \ldots [\tan 44° \tan(90° - 44°)]\tan 45°$

$= (\tan 1° \cot 1°)(\tan 2° \cot 2°) \ldots (\tan 44° \cot 44°)\tan 45°$

$= 1 \quad [\because \tan\theta \cot\theta = 1 \text{ and } \tan 45° = 1]$

Example 2.45 **Show that $\sin^2 5° + \sin^2 10° + \sin^2 15° + \cdots + \sin^2 90° = 9\frac{1}{2}$.**

Sol. L.H.S. $= (\sin^2 5° + \sin^2 85°) + (\sin^2 10° + \sin^2 80°) + \cdots + (\sin^2 40° + \sin^2 50°) + \sin^2 45° + \sin^2 90°$

$= (\sin^2 5° + \cos^2 5°) + (\sin^2 10° + \cos^2 10°) + \cdots + (\sin^2 40° + \cos^2 40°) + \sin^2 45° + \sin^2 90°$

$$= (1 + 1 + 1 + 1 + 1 + 1 + 1 + 1) + \left(\frac{1}{\sqrt{2}}\right)^2 + 1 = 9\frac{1}{2}$$

Example 2.46 Find the value of $\cos^2\frac{\pi}{16}+\cos^2\frac{3\pi}{16}+\cos^2\frac{5\pi}{16}+\cos^2\frac{7\pi}{16}$.

Sol. L.H.S. $=\cos^2\frac{\pi}{16}+\cos^2\frac{3\pi}{16}+\cos^2\left(\frac{\pi}{2}-\frac{3\pi}{16}\right)+\cos^2\left(\frac{\pi}{2}-\frac{\pi}{16}\right)$

$=\cos^2\frac{\pi}{16}+\cos^2\frac{3\pi}{16}+\sin^2\frac{3\pi}{16}+\sin^2\frac{\pi}{16}$

$=\left(\cos^2\frac{\pi}{16}+\sin^2\frac{\pi}{16}\right)+\left(\cos^2\frac{3\pi}{16}+\sin^2\frac{3\pi}{16}\right)$

$=1+1=2$

Example 2.47 If $\sin(120°-\alpha)=\sin(120°-\beta)$, $0<\alpha,\beta<\pi$, then find the relation between α and β.

Sol. If $\sin A=\sin B$, where $A=120°-\alpha$ and $B=120°-\beta$

$\Rightarrow$ $A=B$ or $A=\pi-B$, i.e., $A+B=\pi$

$\Rightarrow$ $120°-\alpha=120°-\beta$, or $120°-\alpha+120°-\beta=180°$

$\Rightarrow$ $\alpha=\beta$ or $\alpha+\beta=60°$

Concept Application Exercise 2.4

1. In triangle ABC prove that
   a. $\sin A=\sin(B+C)$
   b. $\sin 2A=-\sin(2B+2C)$
   c. $\cos A=-\cos(A+B)$
   d. $\tan\left(\frac{A+B}{2}\right)=\cot\frac{C}{2}$
2. Prove that $\sin(-420°)(\cos 390°)+\cos(-660°)(\sin 330°)=-1$.
3. Prove that
   a. $\tan 720°-\cos 270°-\sin 150°\cos 120°=\frac{1}{4}$
   b. $\sin 780°\sin 480°+\cos 120°\sin 150°=\frac{1}{2}$
4. If $\alpha=\frac{\pi}{3}$, prove that $\cos\alpha\cos 2\alpha\cos 3\alpha\cos 4\alpha\cos 5\alpha\cos 6\alpha=-\frac{1}{16}$.
5. Find the value of $\tan\frac{\pi}{20}\tan\frac{3\pi}{20}\tan\frac{5\pi}{20}\tan\frac{7\pi}{20}\tan\frac{9\pi}{20}$.
6. Find the value of $\frac{\cot 54°}{\tan 36°}+\frac{\tan 20°}{\cot 70°}$.
7. Prove that $\sin^2\frac{\pi}{18}+\sin^2\frac{\pi}{9}+\sin^2\frac{7\pi}{18}+\sin^2\frac{4\pi}{9}=2$.
8. Prove that $\sec\left(\frac{3\pi}{2}-\theta\right)\sec\left(\theta-\frac{5\pi}{2}\right)+\tan\left(\frac{5\pi}{2}+\theta\right)\tan\left(\theta-\frac{3\pi}{2}\right)=-1$.
9. In any quadrilateral $ABCD$, prove that
   a. $\sin(A+B)+\sin(C+D)=0$
   b. $\cos(A+B)=\cos(C+D)$

TRIGONOMETRIC RATIOS FOR COMPOUND ANGLES

Cosine of the Difference and Sum of Two Angles

1. $\cos(A-B) = \cos A \cos B + \sin A \sin B$
2. $\cos(A+B) = \cos A \cos B - \sin A \sin B$

for all angles A and B.

Proof:

1. $\cos(A-B)$

Let $X'OX$ and YOY' be the coordinate axes. Consider a unit circle with O as the centre (Fig. 2.35).

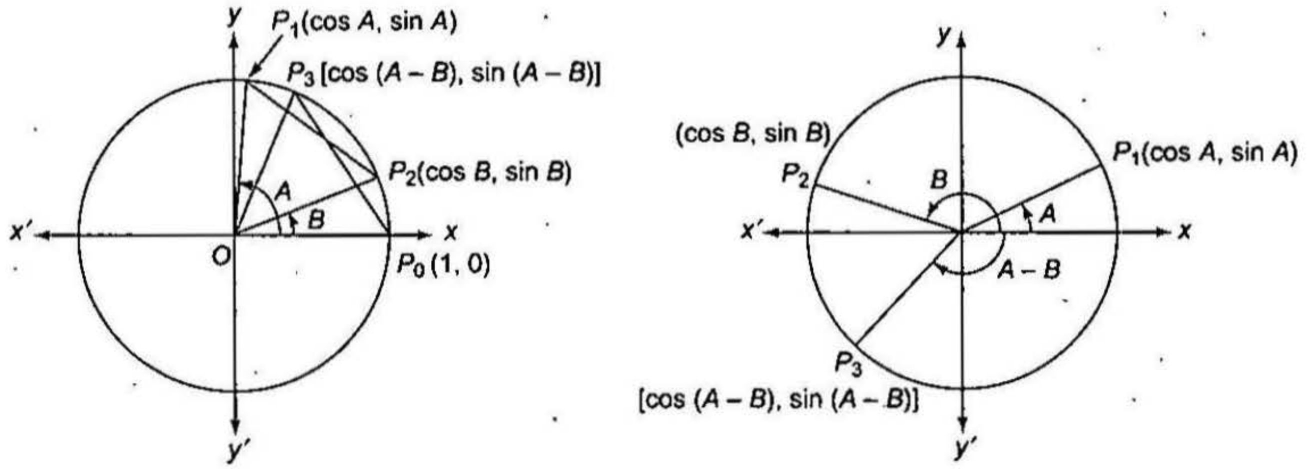


Fig. 2.35

Let P_1, P_2 and P_3 be the three points on the circle such that $\angle XOP_1 = A$, $\angle XOP_2 = B$ and $\angle XOP_3 = A - B$.

As we know that the terminal side of any angle intersects the circle with centre at O and unit radius at a point whose coordinates are the cosine and sine of the angle. Therefore, coordinates of P_1, P_2 and P_3 are $(\cos A, \sin A)$, $(\cos B, \sin B)$ and $(\cos(A-B), \sin(A-B))$, respectively.

We know that equal chords of a circle make equal angles at its centre and chords P_0P_3 and P_1P_2 subtend equal angles at O. Therefore,

Chord P_0P_3 = Chord P_1P_2

$$\Rightarrow \sqrt{\{\cos(A-B)-1\}^2 + \{\sin(A-B)-0\}^2} = \sqrt{(\cos B - \cos A)^2 + (\sin B - \sin A)^2}$$

$$\Rightarrow \{\cos(A-B)-1\}^2 + \sin^2(A-B) = (\cos B - \cos A)^2 + (\sin B - \sin A)^2$$

$$\Rightarrow \cos^2(A-B) - 2\cos(A-B) + 1 + \sin^2(A-B) = \cos^2 B + \cos^2 A - 2\cos A \cos B + \sin^2 B + \sin^2 A - 2\sin A \sin B$$

$$\Rightarrow 2 - 2\cos(A-B) = 2 - 2\cos A \cos B - 2\sin A \sin B$$

$$\Rightarrow \cos(A-B) = \cos A \cos B + \sin A \sin B \quad \text{(i)}$$

2. $\cos(A+B)$

$= \cos(A-(-B))$

$= \cos A \cos(-B) + \sin A \sin(-B)$ [Using Eq. (i)]

$= \cos A \cos B - \sin A \sin B$ [$\because \cos(-B) = \cos B$, $\sin(-B) = -\sin B$]

Hence, $\cos(A+B) = \cos A \cos B - \sin A \sin B$

> ***Note:***
> *This method of proof of the above formula is true for all values of angles A and B whether positive, zero or negative.*

Sine of the Difference and Sum of Two Angles

1. $\sin(A-B) = \sin A \cos B - \cos A \sin B$
2. $\sin(A+B) = \sin A \cos B + \cos A \sin B$

Proof:

1. We have

$$\sin(A-B) = \cos(90^\circ - (A-B)) \quad [\because \cos(90^\circ - \theta) = \sin\theta]$$
$$= \cos((90^\circ - A) + B)$$
$$= \cos(90^\circ - A)\cos B - \sin(90^\circ - A)\sin B$$
$$= \sin A \cos B - \cos A \sin B \quad \text{(i)}$$

2. $\sin(A+B) = \sin(A-(-B))$

$$= \sin A \cos(-B) - \cos A \sin(-B) \quad [\text{Using equation (i)}]$$
$$= \sin A \cos B + \cos A \sin B \quad [\because \sin(-B) = -\sin B]$$

Tangent of the Difference and Sum of Two Angles

1. $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$
2. $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

Proof:

1. We have

$$\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)} = \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}$$
$$= \frac{\tan A + \tan B}{1 - \tan A \tan B} \quad \text{(i) [On dividing the numerator and denominator by } \cos A \cos B]$$

2. $\tan(A-B) = \tan(A+(-B))$

$$= \frac{\tan A + \tan(-B)}{1 - \tan A \tan(-B)} \quad [\text{Using Eq. (i)}]$$
$$= \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

Similarly, it can be proved that

$$\cot(A+B) = \frac{\cot A \cot B - 1}{\cot B + \cot A}$$

and $\cot(A-B) = \dfrac{\cot A \cot B + 1}{\cot B - \cot A}$

Some More Results

1. $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A$
2. $\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A$
3. $\sin(A+B+C) = \sin A \cos B \cos C + \cos A \sin B \cos C + \cos A \cos B \sin C - \sin A \sin B \sin C$
4. $\cos(A+B+C) = \cos A \cos B \cos C - \cos A \sin B \sin C - \sin A \cos B \sin C - \sin A \sin B \cos C$
5. $\tan(A+B+C) = \dfrac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A}$

Proof:

1. $\sin(A+B)\sin(A-B) = (\sin A\cos B + \cos A\sin B)(\sin A\cos B - \cos A\sin B)$
$= \sin^2 A\cos^2 B - \cos^2 A\sin^2 B$
$= \sin^2 A(1-\sin^2 B) - (1-\sin^2 A)\sin^2 B$
$= \sin^2 A - \sin^2 A\sin^2 B - \sin^2 B + \sin^2 A\sin^2 B = \sin^2 A - \sin^2 B$
$= (1-\cos^2 A) - (1-\cos^2 B) = \cos^2 B - \cos^2 A$

2. $\cos(A+B)\cos(A-B) = (\cos A\cos B - \sin A\sin B)(\cos A\cos B + \sin A\sin B)$
$= \cos^2 A\cos^2 B - \sin^2 A\sin^2 B$
$= \cos^2 A(1-\sin^2 B) - (1-\cos^2 A)\sin^2 B = \cos^2 A - \sin^2 B$
$= (1-\sin^2 A) - (1-\cos^2 B) = \cos^2 B - \sin^2 A$

5. $\tan(A+B+C) = \tan((A+B)+C)$

$$= \frac{\tan(A+B)+\tan C}{1-\tan(A+B)\tan C} = \frac{\dfrac{\tan A+\tan B}{1-\tan A\tan B}+\tan C}{1-\left(\dfrac{\tan A+\tan B}{1-\tan A\tan B}\right)\tan C} = \frac{\tan A+\tan B+\tan C-\tan A\tan B\tan C}{1-\tan A\tan B-\tan B\tan C-\tan C\tan A}$$

Example 2.48 **Prove that $\dfrac{\sin(B-C)}{\cos B\cos C}+\dfrac{\sin(C-A)}{\cos C\cos A}+\dfrac{\sin(A-B)}{\cos A\cos B}=0.$**

Sol. First term of L.H.S. is

$$\frac{\sin(B-C)}{\cos B\cos C} = \frac{\sin B\cos C-\cos B\sin C}{\cos B\cos C} = \frac{\sin B\cos C}{\cos B\cos C}-\frac{\cos B\sin C}{\cos B\cos C} = \tan B-\tan C$$

Similarly, second term of L.H.S. $= \tan C - \tan A$
and, third term of L.H.S. $= \tan A - \tan B$
Now L.H.S. $= (\tan B - \tan C) + (\tan C - \tan A) + (\tan A - \tan B) = 0.$

Example 2.49 **If $\sin\alpha\sin\beta - \cos\alpha\cos\beta + 1 = 0$, then prove that $1 + \cot\alpha\tan\beta = 0$.**

Sol. Given, $\sin\alpha\sin\beta - \cos\alpha\cos\beta + 1 = 0$
or $\cos\alpha\cos\beta - \sin\alpha\sin\beta = 1$
or $\cos(\alpha+\beta) = 1$ (i)

Now $1+\cot\alpha\tan\beta = 1+\dfrac{\cos\alpha}{\sin\alpha}\times\dfrac{\sin\beta}{\cos\beta}$

$$= \frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\sin\alpha\cos\beta}$$

$$= \frac{\sin(\alpha+\beta)}{\sin\alpha\cos\beta}$$

$$= \frac{0}{\sin\alpha\cos\beta} = 0 \qquad [\because \sin^2(\alpha+\beta) = 1-\cos^2(\alpha+\beta) = 1-1 = 0]$$

Example 2.50 **Show that $\cos^2\theta+\cos^2(\alpha+\theta)-2\cos\alpha\cos\theta\cos(\alpha+\theta)$ is independent of θ.**

Sol. $\cos^2\theta+\cos^2(\alpha+\theta)-2\cos\alpha\cos\theta\cos(\alpha+\theta) = \cos^2\theta+\cos(\alpha+\theta)[\cos(\alpha+\theta)-2\cos\alpha\cos\theta]$
$= \cos^2\theta + \cos(\alpha+\theta)[\cos\alpha\cos\theta - \sin\alpha\sin\theta - 2\cos\alpha\cos\theta]$

$= \cos^2\theta - \cos(\alpha+\theta)[\cos\alpha\cos\theta + \sin\alpha\sin\theta]$
$= \cos^2\theta - \cos(\alpha+\theta)\cos(\alpha-\theta)$
$= \cos^2\theta - [\cos^2\alpha - \sin^2\theta] = \cos^2\theta + \sin^2\theta - \cos^2\alpha$
$= 1 - \cos^2\alpha$, which is independent of θ.

Example 2.51 **If $3\tan\theta\tan\varphi = 1$, then prove that $2\cos(\theta+\varphi) = \cos(\theta-\varphi)$.**

Sol. Given, $3\tan\theta\tan\varphi = 1$ or $\cot\theta\cot\varphi = 3$

or, $\dfrac{\cos\theta\cos\varphi}{\sin\theta\sin\varphi} = \dfrac{3}{1}$

By componendo and dividendo, we get

$$\frac{\cos\theta\cos\varphi + \sin\theta\sin\varphi}{\cos\theta\cos\varphi - \sin\theta\sin\varphi} = \frac{3+1}{3-1}$$

$$\Rightarrow \frac{\cos(\theta-\varphi)}{\cos(\theta+\varphi)} = 2$$

$$\Rightarrow 2\cos(\theta+\varphi) = \cos(\theta-\varphi)$$

Example 2.52 **If $\sin(A-B) = \dfrac{1}{\sqrt{10}}$, $\cos(A+B) = \dfrac{2}{\sqrt{29}}$, find the value of $\tan 2A$ where A and B lie between 0 and $\pi/4$.**

Sol. $\tan 2A = \tan[(A+B)+(A-B)] = \dfrac{\tan(A+B) + \tan(A-B)}{1 - \tan(A+B)\tan(A-B)}$ (i)

Given that, $0 < A < \dfrac{\pi}{4}$ and $0 < B < \dfrac{\pi}{4}$

$\therefore \quad 0 < A+B < \dfrac{\pi}{2}$

Also, $-\dfrac{\pi}{4} < A - B < \dfrac{\pi}{4}$ and $\sin(A-B) = \dfrac{1}{\sqrt{10}} = (+)$ ve

$\therefore \quad 0 < A - B < \dfrac{\pi}{4}$

Now, $\sin(A-B) = \dfrac{1}{\sqrt{10}}$

$\Rightarrow \quad \tan(A-B) = \dfrac{1}{3}$ (ii)

$\cos(A+B) = \dfrac{2}{\sqrt{29}}$

$\Rightarrow \quad \tan(A+B) = \dfrac{5}{2}$ (iii)

From Eqs. (i), (ii) and (iii), we get

$$\tan 2A = \frac{\frac{5}{2}+\frac{1}{3}}{1-\frac{5}{2}\times\frac{1}{3}} = \frac{17}{6}\times\frac{6}{1} = 17$$

Example 2.53 Prove that $\dfrac{\cos 10° + \sin 10°}{\cos 10° - \sin 10°} = \tan 55°$.

Sol. $\dfrac{\cos 10° + \sin 10°}{\cos 10° - \sin 10°} = \dfrac{1 + \tan 10°}{1 - \tan 10°} = \dfrac{\tan 45° + \tan 10°}{1 - \tan 45° \tan 10°} = \tan(45° + 10°) = \tan 55°$ (dividing by $\cos 10°$)

Example 2.54 Prove that $\tan 70° = 2\tan 50° + \tan 20°$.

Sol. $\tan 70° = \tan(50° + 20°) = \dfrac{\tan 50° + \tan 20°}{1 - \tan 50° \tan 20°}$ (i)

$\Rightarrow \tan 70° (1 - \tan 50° \tan 20°) = \tan 50° + \tan 20°$

$\Rightarrow \tan 70° - \tan 50° \tan 20° \tan 70° = \tan 50° + \tan 20°$

$\Rightarrow \tan 70° = \tan 70° \tan 50° \tan 20° + \tan 50° + \tan 20°$

$= \tan(90° - 20°) \tan 50° \tan 20° + \tan 50° + \tan 20°$

$= \cot 20° \tan 50° \tan 20° + \tan 50° + \tan 20°$

$= \tan 50° + \tan 50° + \tan 20° = 2\tan 50° + \tan 20°$

Example 2.55 Let A, B, C be three angles such that $A + B + C = \pi$. If $\tan A \cdot \tan B = 2$. Then find the value of $\dfrac{\cos A \cos B}{\cos C}$.

Sol. Given $\tan A \cdot \tan B = 2$

Let $y = \dfrac{\cos A \cos B}{\cos C} = -\dfrac{\cos A \cdot \cos B}{\cos(A + B)} = \dfrac{\cos A \cdot \cos B}{\sin A \sin B - \cos A \cos B}$

$= \dfrac{1}{\tan A \tan B - 1} = \dfrac{1}{2 - 1} = 1$

Range of $f(\theta) = a\cos\theta + b\sin\theta$

Let $a = r \sin\alpha$ and $b = r \cos\alpha$, then $r^2 = a^2 + b^2$ and $\tan\alpha = a/b$

Now $f(\theta) = a\cos\theta + b\sin\theta = r \sin\alpha \cos\theta + r \cos\alpha \sin\theta = r \sin(\theta + \alpha) = \sqrt{a^2 + b^2} \sin\left(\theta + \tan^{-1}\dfrac{a}{b}\right)$

Now $-1 \le \sin\left(\theta + \tan^{-1}\dfrac{a}{b}\right) \le 1$

$\Rightarrow -\sqrt{a^2 + b^2} \le \sqrt{a^2 + b^2} \sin\left(\theta + \tan^{-1}\dfrac{a}{b}\right) \le \sqrt{a^2 + b^2}$

Hence, range is $\left[-\sqrt{a^2 + b^2}, \sqrt{a^2 + b^2}\right]$.

Example 2.56 Find the maximum value of $\sqrt{3}\sin x + \cos x$ and x for which a maximum value occurs.

Sol. $\sqrt{3}\sin x + \cos x = 2\left(\frac{\sqrt{3}}{2}\sin x + \frac{1}{2}\sin x\right) = 2\sin(x + \pi/6)$

which is maximum when $x + \pi/6 = \pi/2$ or $x = 60°$ and has a maximum value 2.

Example 2.57 Find the maximum and minimum values of $\cos^2\theta - 6\sin\theta\cos\theta + 3\sin^2\theta + 2$.

Sol. $\cos^2\theta - 6\sin\theta\cos\theta + 3\sin^2\theta + 2$

$$= \frac{1+\cos 2\theta}{2} - 3\sin 2\theta + 3\frac{(1-\cos 2\theta)}{2} + 2$$

$$= 4 - \cos 2\theta - 3\sin 2\theta$$

Now, $-\cos 2\theta - 3\sin 2\theta \in \left[-\sqrt{10}, \sqrt{10}\right]$

$$\Rightarrow \quad 4 - \cos 2\theta - 3\sin 2\theta \in \left[4-\sqrt{10}, 4+\sqrt{10}\right]$$

Concept Application Exercise 2.5

1. If $A + B = 225°$, then find the value of $\frac{\cot A}{1+\cot A} \times \frac{\cot B}{1+\cot B}$.
2. If $\tan A - \tan B = x$ and $\cot B - \cot A = y$, then find the value of $\cot(A - B)$.
3. If x is A.M. of $\tan \pi/9$ and $\tan 5\pi/18$ and y is A.M. of $\tan \pi/9$ and $\tan 7\pi/18$, then relate x and y.
4. Prove that $\frac{\tan^2 2\theta - \tan^2\theta}{1-\tan^2 2\theta \tan^2\theta} = \tan 3\theta \tan\theta$.
5. If $A + B = 45°$, show that $(1 + \tan A)(1 + \tan B) = 2$.
6. If $\tan A = 1/2$, $\tan B = 1/3$, then prove that $\cos 2A = \sin 2B$.
7. Find the maximum value of $1 + \sin\left(\frac{\pi}{4}+\theta\right) + 2\sin\left(\frac{\pi}{4}-\theta\right)$ for all real values of θ.
8. Find the maximum and minimum values of $6\sin x\cos x + 4\cos 2x$.
9. If $p(x) = \sin x(\sin^3 x + 3) + \cos x(\cos^3 x + 4) + \frac{1}{2}\sin^2 2x + 5$, then find the range of $p(x)$.
10. Find the value of $\cos\frac{\pi}{12}\left(\sin\frac{5\pi}{12} + \cos\frac{\pi}{4}\right) + \sin\frac{\pi}{12}\left(\cos\frac{5\pi}{12} - \sin\frac{\pi}{4}\right)$.
11. If $\cos(\alpha+\beta) + \sin(\alpha-\beta) = 0$ and $\tan\beta \neq 1$, then find the value of $\tan\alpha$.
12. If $\sin A + \cos 2A = 1/2$ and $\cos A + \sin 2A = 1/3$, then find the value of $\sin 3A$.
13. If $\sin x + \sin y + \sin z = 0 = \cos x + \cos y + \cos z$, then find the value of expression $\cos(\theta - x) + \cos(\theta - y) + \cos(\theta - z)$.

TRANSFORMATION FORMULAE

Formulae to Transform the Product into Sum or Difference

We know that

$$\sin A\cos B+\cos A\sin B=\sin(A+B) \quad \text{(i)}$$

$$\sin A\cos B-\cos A\sin B=\sin(A-B) \quad \text{(ii)}$$

$$\cos A\cos B-\sin A\sin B=\cos(A+B) \quad \text{(iii)}$$

$$\cos A\cos B+\sin A\sin B=\cos(A-B) \quad \text{(iv)}$$

Adding Eqs. (i) and (ii), we obtain

$$2\sin A\cos B=\sin(A+B)+\sin(A-B) \quad \text{(v)}$$

Subtracting Eqs. (ii) from (i), we get

$$2\cos A\sin B=\sin(A+B)-\sin(A-B) \quad \text{(vi)}$$

Adding Eqs. (iii) and (iv), we get

$$2\cos A\cos B=\cos(A+B)+\cos(A-B) \quad \text{(vii)}$$

Subtracting Eqs. (iii) from (iv), we get

$$2\sin A\sin B=\cos(A-B)-\cos(A+B) \quad \text{(viii)}$$

Formulae to Transform the Sum or Difference into Product

Let $A+B=C$ and $A-B=D$. Then, $A=\dfrac{C+D}{2}$ and $B=\dfrac{C-D}{2}$

Substituting the values of A, B, C, and D in Eqs. (v), (vi), (vii), (viii), we get

$$\sin C+\sin D=2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right) \quad \text{(ix)}$$

$$\sin C-\sin D=2\sin\left(\frac{C-D}{2}\right)\cos\left(\frac{C+D}{2}\right) \quad \text{(x)}$$

$$\cos C+\cos D=2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right) \quad \text{(xi)}$$

$$\left.\begin{aligned}
&\cos D-\cos C=2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)\\
\text{or}\quad &\cos C-\cos D=-2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)\\
\text{or}\quad &\cos C-\cos D=2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{D-C}{2}\right)
\end{aligned}\right\} \quad \text{(xii)}$$

These four formulae are used to convert the sum or difference of two sines or two cosines into the product of sines and cosines.

Example 2.58 **If $\sin A=\sin B$ and $\cos A=\cos B$, then prove that $\sin\dfrac{A-B}{2}=0$.**

Sol. We have $\sin A=\sin B$ and $\cos A=\cos B$

$\Rightarrow\quad \sin A-\sin B=0$ and $\cos A-\cos B=0$

$$\Rightarrow \quad 2\sin\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right)=0 \text{ and } -2\sin\left(\frac{A-B}{2}\right)\sin\left(\frac{A+B}{2}\right)=0$$

$\Rightarrow \quad \sin\dfrac{A-B}{2}=0$, which is common for both the equations.

Example 2.59 **Prove that $\cos 55° + \cos 65° + \cos 175° = 0$.**

Sol. L.H.S. $= \cos 55° + \cos 65° + \cos 175°$

$$= 2\cos\frac{55°+65°}{2}\cos\frac{55°-65°}{2} + \cos 175°$$

$$= 2\cos 60°\cos(-5°) + \cos 175° = 2\times\frac{1}{2}\cos 5° + \cos(180°-5°) = \cos 5° - \cos 5° = 0$$

Example 2.60 **Prove that $\cos 18° - \sin 18° = \sqrt{2}\sin 27°$.**

Sol. L.H.S. $= \cos 18° - \sin 18° = \cos 18° - \sin(90° - 72°) = \cos 18° - \cos 72°$

$$= 2\sin\frac{18°+72°}{2}\sin\frac{72°-18°}{2}$$

$$= 2\sin 45°\sin 27° = 2\frac{1}{\sqrt{2}}\sin 27° = \sqrt{2}\sin 27°$$

Example 2.61 **Prove that**

a. $\dfrac{\sin 5A - \sin 3A}{\cos 5A + \cos 3A} = \tan A$

b. $\dfrac{\sin A + \sin 3A}{\cos A + \cos 3A} = \tan 2A$

Sol.

a. L.H.S. $= \dfrac{\sin 5A - \sin 3A}{\cos 5A + \cos 3A} = \dfrac{2\sin\left(\frac{5A-3A}{2}\right)\cos\left(\frac{5A+3A}{2}\right)}{2\cos\left(\frac{5A+3A}{2}\right)\cos\left(\frac{5A-3A}{2}\right)} = \dfrac{2\sin A\cos 4A}{2\cos 4A\cos A} = \tan A = $ R.H.S.

b. L.H.S. $= \dfrac{\sin 3A + \sin A}{\cos 3A + \cos A} = \dfrac{2\sin\left(\frac{3A+A}{2}\right)\cos\left(\frac{3A-A}{2}\right)}{2\cos\left(\frac{3A+A}{2}\right)\cos\left(\frac{3A-A}{2}\right)} = \dfrac{\sin 2A\cos A}{\cos 2A\cos A} = \tan 2A = $ R.H.S.

Example 2.62 **Prove that $\cos\alpha + \cos\beta + \cos\gamma + \cos(\alpha+\beta+\gamma) = 4\cos\dfrac{\alpha+\beta}{2}\cos\dfrac{\beta+\gamma}{2}\cos\dfrac{\gamma+\alpha}{2}$.**

Sol. L.H.S. $= \cos\alpha + \cos\beta + \cos\gamma + \cos(\alpha+\beta+\gamma)$

$$= (\cos\alpha + \cos\beta) + [\cos\gamma + \cos(\alpha+\beta+\gamma)]$$

$$= 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) + 2\cos\left(\frac{\alpha+\beta+\gamma+\gamma}{2}\right)\cos\left(\frac{\alpha+\beta+\gamma-\gamma}{2}\right)$$

$$= 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) + 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha+\beta+2\gamma}{2}\right)$$

$$= 2\cos\left(\frac{\alpha+\beta}{2}\right)\left\{\cos\left(\frac{\alpha-\beta}{2}\right) + \cos\left(\frac{\alpha+\beta+2\gamma}{2}\right)\right\}$$

$$= 2\cos\left(\frac{\alpha+\beta}{2}\right)\left\{2\cos\left(\frac{\frac{\alpha-\beta}{2}+\frac{\alpha+\beta+2\gamma}{2}}{2}\right)\cos\left(\frac{\frac{\alpha+\beta+2\gamma}{2}-\frac{\alpha-\beta}{2}}{2}\right)\right\}$$

$$= 2\cos\left(\frac{\alpha+\beta}{2}\right)\left\{2\cos\left(\frac{\alpha+\gamma}{2}\right)\cos\left(\frac{\beta+\gamma}{2}\right)\right\} = 4\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\beta+\gamma}{2}\right)\cos\left(\frac{\gamma+\alpha}{2}\right) = \text{R.H.S.}$$

Example 2.63 **Prove that** $\dfrac{\sin A+\sin 2A+\sin 4A+\sin 5A}{\cos A+\cos 2A+\cos 4A+\cos 5A}=\tan 3A$.

Sol. $\dfrac{\sin A+\sin 2A+\sin 4A+\sin 5A}{\cos A+\cos 2A+\cos 4A+\cos 5A}$

$$= \frac{(\sin 5A+\sin A)+(\sin 4A+\sin 2A)}{(\cos 5A+\cos A)+(\cos 4A+\cos 2A)}$$

$$= \frac{2\sin 3A\cos 2A+2\sin 3A\cos A}{2\cos 3A\cos 2A+2\cos 3A\cos A}$$

$$= \frac{2\sin 3A(\cos 2A+\cos A)}{2\cos 3A(\cos 2A+\cos A)} = \tan 3A$$

Example 2.64 **Prove that** $\left(\dfrac{\cos A+\cos B}{\sin A-\sin B}\right)^n+\left(\dfrac{\sin A+\sin B}{\cos A-\cos B}\right)^n = 2\cot^n\dfrac{A-B}{2}$ **or 0, accordingly as n is even or odd.**

Sol. L.H.S. $= \left(\dfrac{2\cos\frac{A+B}{2}\cos\frac{A-B}{2}}{2\cos\frac{A+B}{2}\sin\frac{A-B}{2}}\right)^n + \left(\dfrac{2\sin\frac{A+B}{2}\cos\frac{A-B}{2}}{2\sin\frac{A+B}{2}\sin\frac{B-A}{2}}\right)^n$

$$= \left(\cot\frac{A-B}{2}\right)^n + \left(-\cot\frac{A-B}{2}\right)^n \qquad [\because \sin(-\theta) = -\sin\theta]$$

$$= \cot^n\frac{A-B}{2} + (-1)^n\cot^n\frac{A-B}{2} = \cot^n\frac{A-B}{2}\left[1+(-1)^n\right]$$

$$= \begin{cases} 0, \text{ if } n \text{ is odd} \\ 2\cot^n\dfrac{A-B}{2}, & \text{if } n \text{ is even.}\end{cases}$$

Example 2.65 **Prove that** $(\cos\alpha+\cos\beta)^2+(\sin\alpha+\sin\beta)^2 = 4\cos^2\left(\dfrac{\alpha-\beta}{2}\right)$.

Sol. L.H.S. $= (\cos\alpha+\cos\beta)^2+(\sin\alpha+\sin\beta)^2$

$$= \left\{2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)\right\}^2 + \left\{2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)\right\}^2$$

$$= 4\cos^2\left(\frac{\alpha-\beta}{2}\right)\left\{\cos^2\frac{\alpha+\beta}{2}+\sin^2\frac{\alpha+\beta}{2}\right\}$$

$$= 4\cos^2\left(\frac{\alpha-\beta}{2}\right) \quad \left[\because \cos^2\frac{\alpha+\beta}{2}+\sin^2\frac{\alpha+\beta}{2}=1\right]$$

$= \text{R.H.S.}$

Example 2.66 **If $\sec(\theta+\alpha)+\sec(\theta-\alpha)=2\sec\theta$, then show that $\cos^2\theta = 1+\cos\alpha$.**

Sol. $\sec(\theta+\alpha)+\sec(\theta-\alpha)=2\sec\theta$

$$\Rightarrow \frac{1}{\cos(\theta+\alpha)}+\frac{1}{\cos(\theta-\alpha)}=\frac{2}{\cos\theta}$$

$$\Rightarrow \frac{\cos(\theta-\alpha)+\cos(\theta+\alpha)}{\cos(\theta+\alpha)\cos(\theta-\alpha)}=\frac{2}{\cos\theta}$$

$$\Rightarrow \frac{2\cos\theta\cos\alpha}{\cos^2\theta-\sin^2\alpha}=\frac{2}{\cos\theta}$$

$\Rightarrow \cos^2\theta\cos\alpha=\cos^2\theta-\sin^2\alpha$

$\Rightarrow \sin^2\alpha=\cos^2\theta(1-\cos\alpha)$

$\Rightarrow 1-\cos^2\alpha=\cos^2\theta(1-\cos\alpha) \quad \Rightarrow 1+\cos\alpha=\cos^2\theta$

Example 2.67 **In quadrilateral $ABCD$ if $\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)+\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)=2$, then find the value of $\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\sin\frac{D}{2}$.**

Sol. $\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)+\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)=2$

$\Rightarrow \frac{1}{2}[\sin A+\sin B+\sin C+\sin D]=2$

$\Rightarrow \sin A+\sin B+\sin C+\sin D=4$

$\Rightarrow A=B=C=D=90°$

$\Rightarrow \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\sin\frac{D}{2}=1/4$

Concept Application Exercise 2.6

1. a. Prove that $\sin 65°+\cos 65°=\sqrt{2}\cos 20°$.
   b. Prove that $\sin 47°+\cos 77°=\cos 17°$.
2. Prove that $\cos 80°+\cos 40°-\cos 20°=0$.
3. Prove that $\sin 10°+\sin 20°+\sin 40°+\sin 50°=\sin 70°+\sin 80°$.
4. Prove that $\cos\frac{\pi}{5}+\cos\frac{2\pi}{5}+\cos\frac{6\pi}{5}+\cos\frac{7\pi}{5}=0$.
5. If $\sin\alpha-\sin\beta=\frac{1}{3}$ and $\cos\beta-\cos\alpha=\frac{1}{2}$, show that $\cot\frac{\alpha+\beta}{2}=\frac{2}{3}$.
6. If $\operatorname{cosec} A+\sec A=\operatorname{cosec} B+\sec B$, prove that $\tan A\tan B=\cot\frac{A+B}{2}$.
7. Prove that $\sin 25°\cos 115°=\frac{1}{2}(\sin 40°-1)$.

8. If $\cos A = \frac{3}{4}$, then find the value of $32 \sin\left(\frac{A}{2}\right)\sin\left(\frac{5A}{2}\right)$.

9. If $x \cos \theta = y \cos\left(\theta + \frac{2\pi}{3}\right) = z \cos\left(\theta + \frac{4\pi}{3}\right)$, prove that $xy + yz + zx = 0$.

10. If $y \sin \varphi = x \sin(2\theta + \varphi)$, show that $(x + y) \cot(\theta + \varphi) = (y - x) \cot \theta$.

11. If $\cos(A + B) \sin(C + D) = \cos(A - B) \sin(C - D)$, prove that $\cot A \cot B \cot C = \cot D$.

12. If $\tan(A + B) = 3 \tan A$, prove that

    a. $\sin(2A + B) = 2 \sin B$

    b. $\sin 2(A + B) + \sin 2A = 2 \sin 2B$

TRIGONOMETRIC RATIOS OF MULTIPLES AND SUB-MULTIPLE ANGLES

Formulae for Multiple Angles

1. $\cos 2A = \cos(A + A) = \cos^2 A - \sin^2 A = 1 - 2\sin^2 A = 2\cos^2 A - 1$

   Also $\sin^2 A = \frac{1}{2}(1 - \cos 2A)$, $\cos^2 A = \frac{1}{2}(1 + \cos 2A)$

2. $\sin 2A = \sin(A + A) = \sin A \cos A + \sin A \cos A = 2\sin A \cos A$

3. $\tan 2A = \tan(A + A) = \frac{\tan A + \tan A}{1 - \tan A \tan A} = \frac{2 \tan A}{1 - \tan^2 A}$

4. $\sin 3A = \sin(2A + A)$

   $= \sin 2A \cos A + \cos 2A \sin A = 2 \sin A \cos A \cos A + (1 - 2 \sin^2 A) \sin A$

   $= 2 \sin A \cos^2 A + \sin A - 2 \sin^3 A$

   $= 2 \sin A (1 - \sin^2 A) + \sin A - 2 \sin^3 A$

   $= 2 \sin A - 2 \sin^3 A + \sin A - 2 \sin^3 A$

   $= 3 \sin A - 4 \sin^3 A$

5. $\cos 3A = \cos(2A + A)$

   $= \cos 2A \cos A - \sin 2A \sin A = (2 \cos^2 A - 1) \cos A - 2 \sin A \cos A \sin A$

   $= 2 \cos^3 A - \cos A - 2 \cos A (1 - \cos^2 A)$

   $= 2 \cos^3 A - \cos A - 2 \cos A + 2 \cos^3 A$

   $= 4 \cos^3 A - 3 \cos A$

6. $\sin 2A$ and $\cos 2A$ in terms of $\tan A$

   $$\sin 2A = 2 \sin A \cos A = \frac{2\sin A \cos A}{\cos^2 A + \sin^2 A} = \frac{2 \tan A}{1 + \tan^2 A}$$

   [dividing numerator and denominator by $\cos^2 A$]

   $$\cos 2A = \cos^2 A - \sin^2 A = \frac{\cos^2 A - \sin^2 A}{\cos^2 A + \sin^2 A} = \frac{1 - \tan^2 A}{1 + \tan^2 A}$$

   [dividing numerator and denominator by $\cos^2 A$]

   Also $\tan^2 A = \frac{1 - \cos 2A}{1 + \cos 2A}$

7. In the formula of $\tan(A+B+C)$, putting $B=A$ and $C=A$, we get

$$\tan 3A = \frac{3\tan A - \tan^3 A}{1-3\tan^2 A}$$

Similarly, we can prove that $\cot 3A = \dfrac{\cot^3 A - 3\cot A}{3\cot^2 A - 1}$

8. $\tan(A_1 + A_2 + \cdots A_n) = \dfrac{S_1 - S_3 + S_5 - S_7 + \cdots}{1 - S_2 + S_4 - S_6 + \cdots}$,

where

$S_1 = \tan A_1 + \tan A_2 + \cdots + \tan A_n$ = Sum of the tangents of the separate angles,

$S_2 = \tan A_1 \tan A_2 + \tan A_1 \tan A_3 + \cdots$ = Sum of the product of tangents taken two at a time,

$S_3 = \tan A_1 \tan A_2 \tan A_3 + \tan A_2 \tan A_3 \tan A_4 + \cdots$ = Sum of the product of tangents taken three at a time, and so on.

If $A_1 = A_2 = \cdots = A_n = A$, then we have

$S_1 = n\tan A,\ S_2 = {}^nC_2 \tan^2 A,\ S_3 = {}^nC_3 \tan^3 A, \cdots$

Example 2.68 Prove that

a. $\dfrac{\sin 2\theta}{1+\cos 2\theta} = \tan\theta$

b. $\dfrac{\sin 2\theta}{1-\cos 2\theta} = \cot\theta$

c. $\dfrac{1+\sin 2\theta+\cos 2\theta}{1+\sin 2\theta-\cos 2\theta} = \cot\theta$

d. $\dfrac{1+\sin\theta-\cos\theta}{1+\sin\theta+\cos\theta} = \tan\theta/2$

e. $\dfrac{\cos 2\theta}{1+\sin 2\theta} = \tan(\pi/4-\theta)$

f. $\dfrac{\cos\theta}{1+\sin\theta} = \tan\left(\dfrac{\pi}{4}-\dfrac{\theta}{2}\right)$

Sol.

a. L.H.S. $= \dfrac{\sin 2\theta}{1+\cos 2\theta} = \dfrac{2\sin\theta\cos\theta}{2\cos^2\theta} = \tan\theta =$ R.H.S.

b. L.H.S. $= \dfrac{\sin 2\theta}{1-\cos 2\theta} = \dfrac{2\sin\theta\cos\theta}{2\sin^2\theta} = \cot\theta =$ R.H.S.

c. L.H.S. $= \dfrac{1+\sin 2\theta+\cos 2\theta}{1+\sin 2\theta-\cos 2\theta} = \dfrac{(1+\cos 2\theta)+\sin 2\theta}{(1-\cos 2\theta)+\sin 2\theta}$

$$= \frac{2\cos^2\theta + 2\sin\theta\cos\theta}{2\sin^2\theta + 2\sin\theta\cos\theta}$$

$$= \frac{2\cos\theta(\cos\theta+\sin\theta)}{2\sin\theta(\cos\theta+\sin\theta)} = \frac{\cos\theta}{\sin\theta} = \cot\theta = \text{R.H.S.}$$

d. L.H.S. $= \dfrac{1+\sin\theta-\cos\theta}{1+\sin\theta+\cos\theta} = \dfrac{(1-\cos\theta)+\sin\theta}{(1+\cos\theta)+\sin\theta}$

$$= \frac{2\sin^2\frac{\theta}{2} + 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2} + 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} = \frac{2\sin\frac{\theta}{2}\left(\sin\frac{\theta}{2}+\cos\frac{\theta}{2}\right)}{2\cos\frac{\theta}{2}\left(\sin\frac{\theta}{2}+\cos\frac{\theta}{2}\right)}$$

$$= \tan\frac{\theta}{2} = \text{R.H.S.}$$

e. L.H.S. $= \dfrac{\cos 2\theta}{1+\sin 2\theta}$

$$= \frac{\sin\left(\frac{\pi}{2}-2\theta\right)}{1+\cos\left(\frac{\pi}{2}-2\theta\right)}$$

$$= \frac{2\sin\left(\frac{\pi}{4}-\theta\right)\cos\left(\frac{\pi}{4}-\theta\right)}{2\cos^2\left(\frac{\pi}{4}-\theta\right)}$$

$$= \tan\left(\frac{\pi}{4}-\theta\right) = \text{R.H.S.}$$

f. L.H.S. $= \dfrac{\cos\theta}{1+\sin\theta} = \dfrac{\sin\left(\frac{\pi}{2}-\theta\right)}{1+\cos\left(\frac{\pi}{2}-\theta\right)} = \dfrac{2\sin\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\cos\left(\frac{\pi}{4}-\frac{\theta}{2}\right)}{2\cos^2\left(\frac{\pi}{4}-\frac{\theta}{2}\right)} = \tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right) =$ R.H.S.

Example 2.69 **Prove that $\dfrac{1+\sin 2\theta}{1-\sin 2\theta} = \left(\dfrac{1+\tan\theta}{1-\tan\theta}\right)^2$.**

Sol. L.H.S. $= \dfrac{1+\sin 2\theta}{1-\sin 2\theta}$

$$= \frac{\sin^2\theta+\cos^2\theta+2\sin\theta\cos\theta}{\sin^2\theta+\cos^2\theta-2\sin\theta\cos\theta}$$

$$= \left(\frac{\sin\theta+\cos\theta}{\sin\theta-\cos\theta}\right)^2 = \left(\frac{1+\tan\theta}{1-\tan\theta}\right)^2$$ (dividing numerator and denominator by $\cos\theta$)

Example 2.70 **If $\alpha+\beta=90°$, find the maximum value of $\sin\alpha\sin\beta$.**

Sol. Let $y = \sin\alpha\sin\beta = \sin\alpha\sin(90°-\alpha) = \sin\alpha\cos\alpha = \dfrac{1}{2}\sin 2\alpha$

which has the maximum value 1/2 when $\sin 2\alpha = 1$.

Example 2.71 **Prove that $\dfrac{1-\tan^2\left(\frac{\pi}{4}-A\right)}{1+\tan^2\left(\frac{\pi}{4}-A\right)} = \sin 2A$.**

Sol. $\dfrac{1-\tan^2\left(\frac{\pi}{4}-A\right)}{1+\tan^2\left(\frac{\pi}{4}-A\right)} = \dfrac{1-\tan^2\theta}{1+\tan^2\theta}$ $\left(\text{where } \frac{\pi}{4}-A=\theta\right) = \cos 2\theta = \cos\left(\frac{\pi}{2}-2A\right) = \sin 2A$

Example 2.72 **Prove that $(\cos A-\cos B)^2+(\sin A-\sin B)^2 = 4\sin^2\dfrac{A-B}{2}$.**

Sol. L.H.S. $= (\cos A-\cos B)^2+(\sin A-\sin B)^2$

$$= \cos^2 A+\cos^2 B-2\cos A\cos B+\sin^2 A+\sin^2 B-2\sin A\sin B$$

$$= (\cos^2 A + \sin^2 A) + (\cos^2 B + \sin^2 B) - 2(\cos A \cos B + \sin A \sin B)$$

$$= 2 - 2\cos(A - B) = 2(1 - \cos(A - B)) = 4 \sin^2 \frac{A - B}{2}$$

Example 2.73 **If $\sin A = \frac{3}{5}$ and $0° < A < 90°$, find the values of $\sin 2A$, $\cos 2A$, $\tan 2A$, and $\sin 4A$.**

Sol. Given $\sin A = \frac{3}{5}$ and A is an acute angle.

$\therefore \quad \cos A = \frac{4}{5}$ [$\because A$ is acute]

and $\tan A = \frac{3}{4}$

Now, $\sin 2A = 2 \sin A \cos A = 2 \times \frac{3}{5} \times \frac{4}{5} = \frac{24}{25}$

$$\cos 2A = 1 - 2\sin^2 A = 1 - 2 \times \frac{9}{25} = \frac{7}{25}$$

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A} = \frac{2 \times \frac{3}{4}}{1 - \left(\frac{3}{4}\right)^2} = \frac{\frac{6}{4}}{1 - \frac{9}{16}} = \frac{\frac{6}{4}}{\frac{7}{16}} = \frac{6}{4} \times \frac{16}{7} = \frac{24}{7}$$

$$\sin 4A = 2 \sin 2A \cos 2A = 2 \times \frac{24}{25} \times \frac{7}{25} = \frac{336}{625}$$

Example 2.74 **If $\tan \alpha = \frac{1}{7}$, $\sin \beta = \frac{1}{\sqrt{10}}$, prove that $\alpha + 2\beta = \frac{\pi}{4}$, where $0 < \alpha < \frac{\pi}{2}$ and $0 < \beta < \frac{\pi}{2}$.**

Sol. $\tan(\alpha + 2\beta) = \dfrac{\tan \alpha + \tan 2\beta}{1 - \tan \alpha \tan 2\beta} = \dfrac{\frac{1}{7} + \tan 2\beta}{1 - \frac{1}{7} \tan 2\beta}$ (i)

Now, $\tan 2\beta = \dfrac{2 \tan \beta}{1 - \tan^2 \beta} = \dfrac{2 \times \frac{1}{3}}{1 - \frac{1}{9}} = \dfrac{3}{4}$ [$\tan \beta > 0$ as $0 < \beta < \pi/2$]

Substituting the value of $\tan 2\beta$ in Eq. (i), we get

$$\tan(\alpha + 2\beta) = \frac{\frac{1}{7} + \frac{3}{4}}{1 - \frac{1}{7} \times \frac{3}{4}} = \frac{25}{25} = 1$$

Now, $0 < \alpha < \frac{\pi}{2}$ and $0 < \beta < \frac{\pi}{2}$

$\therefore \quad 0 < 2\beta < \pi$, but $\tan 2\beta = \frac{3}{4} > 0$

$\Rightarrow \quad 0 < 2\beta < \frac{\pi}{2}$

Hence, $0 < \alpha + 2\beta < \pi$.

In the interval $(0, \pi)$, $\tan\theta$ takes value 1 at $\pi/4$ only

$$\therefore \quad \alpha + 2\beta = \frac{\pi}{4}$$

Example 2.75 **Show that $\sqrt{2+\sqrt{2+\sqrt{2+2\cos 8\theta}}} = 2\cos\theta$, $0 < \theta < \pi/16$.**

Sol. L.H.S. $= \sqrt{2+\sqrt{2+\sqrt{2(1+\cos 8\theta)}}}$

$$= \sqrt{2+\sqrt{2+\sqrt{2(2\cos^2 4\theta)}}} \qquad \left[\because 1+\cos 8\theta = 2\cos^2\frac{8\theta}{2}\right]$$

$$= \sqrt{2+\sqrt{2+\sqrt{(4\cos^2 4\theta)}}}$$

$$= \sqrt{2+\sqrt{2+2\cos 4\theta}}$$

$$= \sqrt{2+\sqrt{2(1+\cos 4\theta)}}$$

$$= \sqrt{2+\sqrt{2(2\cos^2 2\theta)}} \qquad [\because 1+\cos 4\theta = 2\cos^2 2\theta]$$

$$= \sqrt{2+2\cos 2\theta} = \sqrt{2(1+\cos 2\theta)} = \sqrt{2(2\cos^2\theta)} = 2\cos\theta = \text{R.H.S.}$$

Example 2.76 **Prove that $\dfrac{\sec 8\theta - 1}{\sec 4\theta - 1} = \dfrac{\tan 8\theta}{\tan 2\theta}$.**

Sol. L.H.S. $= \dfrac{\sec 8\theta - 1}{\sec 4\theta - 1} = \dfrac{\dfrac{1}{\cos 8\theta} - 1}{\dfrac{1}{\cos 4\theta} - 1} = \dfrac{1-\cos 8\theta}{\cos 8\theta} \times \dfrac{\cos 4\theta}{1-\cos 4\theta}$

$$= \frac{2\sin^2 4\theta}{\cos 8\theta} \times \frac{\cos 4\theta}{2\sin^2 2\theta} \qquad \left[\begin{array}{l}\because 1-\cos 8\theta = 2\sin^2\dfrac{8\theta}{2} = 2\sin^2 4\theta \\ \text{and } 1-\cos 4\theta = 2\sin^2\dfrac{4\theta}{2} = 2\sin^2 2\theta\end{array}\right]$$

$$= \frac{(2\sin 4\theta\cos 4\theta)}{\cos 8\theta} \times \frac{\sin 4\theta}{2\sin^2 2\theta}$$

$$= \left(\frac{2\sin 4\theta\cos 4\theta}{\cos 8\theta}\right) \times \left(\frac{2\sin 2\theta\cos 2\theta}{2\sin^2 2\theta}\right)$$

$$= \left(\frac{\sin 2(4\theta)}{\cos 8\theta}\right) \times \left(\frac{\cos 2\theta}{\sin 2\theta}\right) = \left(\frac{\sin 8\theta}{\cos 8\theta}\right) \times \left(\frac{\cos 2\theta}{\sin 2\theta}\right)$$

$$= \tan 8\theta \cot 2\theta = \frac{\tan 8\theta}{\tan 2\theta} = \text{R.H.S.}$$

Example 2.77 **Show that $\sqrt{3}$ cosec $20° - \sec 20° = 4$.**

Sol. L.H.S. $= \dfrac{\sqrt{3}}{\sin 20°} - \dfrac{1}{\cos 20°} = \dfrac{\sqrt{3}\cos 20° - \sin 20°}{\sin 20° \cos 20°}$

$$= \frac{2\left(\frac{\sqrt{3}}{2}\cos 20^\circ - \frac{1}{2}\sin 20^\circ\right)}{\sin 20^\circ \cos 20^\circ}$$

$$= \frac{2\left(\sin 60^\circ \cos 20^\circ - \cos 60^\circ \sin 20^\circ\right)}{\sin 20^\circ \cos 20^\circ}$$

$$= \frac{2\sin\left(60^\circ - 20^\circ\right)}{\sin 20^\circ \cos 20^\circ} = \frac{2\sin 40^\circ}{\sin 20^\circ \cos 20^\circ}$$

$$= \frac{4\sin 40^\circ}{2\sin 20^\circ \cos 20^\circ} = \frac{4\sin 40^\circ}{\sin 40^\circ} = 4$$

Example 2.78 **Prove that $\left(1+\cos\frac{\pi}{8}\right)\left(1+\cos\frac{3\pi}{8}\right)\left(1+\cos\frac{5\pi}{8}\right)\left(1+\cos\frac{7\pi}{8}\right)=\frac{1}{8}$.**

Sol. We have $\cos\frac{7\pi}{8} = \cos\left(\pi - \frac{\pi}{8}\right) = -\cos\frac{\pi}{8}$

and $\cos\frac{5\pi}{8} = \cos\left(\pi - \frac{3\pi}{8}\right) = -\cos\frac{3\pi}{8}$

$\therefore$ L.H.S. $= \left(1+\cos\frac{\pi}{8}\right)\left(1+\cos\frac{3\pi}{8}\right)\left(1-\cos\frac{3\pi}{8}\right)\left(1-\cos\frac{\pi}{8}\right)$

$$= \left(1-\cos^2\frac{\pi}{8}\right)\left(1-\cos^2\frac{3\pi}{8}\right)$$

$$= \sin^2\frac{\pi}{8}\sin^2\frac{3\pi}{8}$$

$$= \frac{1}{4}\left(2\sin^2\frac{\pi}{8}\right)\left(2\sin^2\frac{3\pi}{8}\right)$$

$$= \frac{1}{4}\left[\left(1-\cos\frac{\pi}{4}\right)\left(1-\cos\frac{3\pi}{4}\right)\right] \qquad \left[\because 1-\cos\theta = 2\sin^2\frac{\theta}{2}\right]$$

$$= \frac{1}{4}\left[\left(1-\frac{1}{\sqrt{2}}\right)\left(1+\frac{1}{\sqrt{2}}\right)\right] = \frac{1}{4}\left(1-\frac{1}{2}\right) = \frac{1}{8} = \text{R.H.S.}$$

Example 2.79 **Prove that $\cos^4\frac{\pi}{8}+\cos^4\frac{3\pi}{8}+\cos^4\frac{5\pi}{8}+\cos^4\frac{7\pi}{8}=\frac{3}{2}$.**

Sol. We have $\frac{7\pi}{8} = \pi - \frac{\pi}{8}$ and $\frac{5\pi}{8} = \pi - \frac{3\pi}{8}$

$\Rightarrow \quad \cos\frac{7\pi}{8} = -\cos\frac{\pi}{8}$ and $\cos\frac{5\pi}{8} = -\cos\frac{3\pi}{8}$

$\Rightarrow \quad \cos^4\frac{7\pi}{8} = \cos^4\frac{\pi}{8}$ and $\cos^4\frac{5\pi}{8} = \cos^4\frac{3\pi}{8}$

$\therefore$ L.H.S. $= 2\cos^4\frac{\pi}{8} + 2\cos^4\frac{3\pi}{8}$

$$= 2\left[\left(\cos^2\frac{\pi}{8}\right)^2 + \left(\cos^2\frac{3\pi}{8}\right)^2\right]$$

$$= 2\left\{\frac{1+\cos\frac{\pi}{4}}{2}\right\}^2 + \left\{\frac{1+\cos\frac{3\pi}{4}}{2}\right\}^2 \qquad \left[\because \frac{1+\cos 2\theta}{2} = \cos^2\theta\right]$$

$$= \frac{1}{2}\left\{\left(1+\cos\frac{\pi}{4}\right)^2 + \left(1+\cos\frac{3\pi}{4}\right)^2\right\}^2$$

$$= \frac{1}{2}\left\{\left(1+\frac{1}{\sqrt{2}}\right)^2 + \left(1-\frac{1}{\sqrt{2}}\right)^2\right\}$$

$$= \frac{1}{2}\left\{\left(1+\frac{1}{2}+\sqrt{2}\right) + \left(1+\frac{1}{2}-\sqrt{2}\right)\right\} = \frac{3}{2} = \text{R.H.S.}$$

Example 2.80 If $\pi < x < 2\pi$, prove that $\dfrac{\sqrt{1+\cos x}+\sqrt{1-\cos x}}{\sqrt{1+\cos x}-\sqrt{1-\cos x}} = \cot\left(\dfrac{x}{2}+\dfrac{\pi}{4}\right)$.

Sol. L.H.S. $= \dfrac{\sqrt{1+\cos x}+\sqrt{1-\cos x}}{\sqrt{1+\cos x}-\sqrt{1-\cos x}} = \dfrac{\sqrt{2\cos^2\frac{x}{2}}+\sqrt{2\sin^2\frac{x}{2}}}{\sqrt{2\cos^2\frac{x}{2}}-\sqrt{2\sin^2\frac{x}{2}}}$

$$= \frac{\sqrt{2}\left|\cos\frac{x}{2}\right| + \sqrt{2}\left|\sin\frac{x}{2}\right|}{\sqrt{2}\left|\cos\frac{x}{2}\right| - \sqrt{2}\left|\sin\frac{x}{2}\right|}$$

$$= \frac{\left|\cos\frac{x}{2}\right| + \left|\sin\frac{x}{2}\right|}{\left|\cos\frac{x}{2}\right| - \left|\sin\frac{x}{2}\right|}$$

$$= \frac{-\cos\frac{x}{2}+\sin\frac{x}{2}}{-\cos\frac{x}{2}-\sin\frac{x}{2}} \qquad \left[\because \pi < x < 2\pi, \therefore \frac{\pi}{2} < \frac{x}{2} < \pi\right]$$

$\Rightarrow$ $\cos x/2$ is negative and $\sin x/2$ is positive.

$$= \frac{\cos\frac{x}{2}-\sin\frac{x}{2}}{\cos\frac{x}{2}+\sin\frac{x}{2}} = \frac{\cot\frac{x}{2}-1}{\cot\frac{x}{2}+1} \qquad \text{[dividing numerator and denominator by } \sin x/2]$$

$$= \cot\left(\frac{x}{2}+\frac{\pi}{4}\right) = \text{R.H.S.}$$

Example 2.81 If $\sin\alpha + \sin\beta = a$ and $\cos\alpha + \cos\beta = b$, prove that $\tan\dfrac{\alpha-\beta}{2} = \pm\sqrt{\dfrac{4-a^2-b^2}{a^2+b^2}}$.

Sol. Given, $\sin\alpha + \sin\beta = a$ (i)

and $\cos\alpha + \cos\beta = b$ (ii)

Now $(\cos\alpha+\cos\beta)^2+(\sin\alpha+\sin\beta)^2=b^2+a^2$

$\Rightarrow\ \cos^2\alpha+\cos^2\beta+2\cos\alpha\cos\beta+\sin^2\alpha+\sin^2\beta+2\sin\alpha\sin\beta=b^2+a^2$

$\Rightarrow\ (\cos^2\alpha+\sin^2\alpha)+(\cos^2\beta+\sin^2\beta)+2(\cos\alpha\cos\beta+\sin\alpha\sin\beta)=a^2+b^2$

$\Rightarrow\ 2+2\cos(\alpha-\beta)=a^2+b^2$

or $\cos(\alpha-\beta)=\dfrac{a^2+b^2-2}{2}$

Now, $\tan\dfrac{\alpha-\beta}{2}=\pm\sqrt{\dfrac{1-\cos(\alpha-\beta)}{1+\cos(\alpha-\beta)}}$

$$=4\pm\sqrt{\dfrac{1-\dfrac{a^2+b^2-2}{2}}{1+\dfrac{a^2+b^2-2}{2}}}=\pm\sqrt{\dfrac{4-a^2-b^2}{a^2+b^2}}$$

Example 2.82 If $\tan\dfrac{\theta}{2}=\sqrt{\dfrac{a-b}{a+b}}\tan\dfrac{\varphi}{2}$, prove that $\cos\alpha=\dfrac{a\cos\varphi+b}{a+b\cos\varphi}$.

Sol. Given, $\tan\dfrac{\theta}{2}=\sqrt{\dfrac{a-b}{a+b}}\tan\dfrac{\varphi}{2}$ (i)

Now, $\cos\theta=\dfrac{1-\tan^2\dfrac{\theta}{2}}{1+\tan^2\dfrac{\theta}{2}}=\dfrac{1-\dfrac{a-b}{a+b}\tan^2\dfrac{\varphi}{2}}{1+\dfrac{a-b}{a+b}\tan^2\dfrac{\varphi}{2}}$

$$=\dfrac{1-\dfrac{a-b}{a+b}\dfrac{\sin^2\dfrac{\varphi}{2}}{\cos^2\dfrac{\varphi}{2}}}{1+\dfrac{a-b}{a+b}\dfrac{\sin^2\dfrac{\varphi}{2}}{\cos^2\dfrac{\varphi}{2}}}$$

$$=\dfrac{(a+b)\cos^2\dfrac{\varphi}{2}-(a-b)\sin^2\dfrac{\varphi}{2}}{(a+b)\cos^2\dfrac{\varphi}{2}+(a-b)\sin^2\dfrac{\varphi}{2}}$$

$$=\dfrac{a\left(\cos^2\dfrac{\varphi}{2}-\sin^2\dfrac{\varphi}{2}\right)+b\left(\cos^2\dfrac{\varphi}{2}+\sin^2\dfrac{\varphi}{2}\right)}{a\left(\cos^2\dfrac{\varphi}{2}+\sin^2\dfrac{\varphi}{2}\right)+b\left(\cos^2\dfrac{\varphi}{2}-\sin^2\dfrac{\varphi}{2}\right)}$$

$$=\dfrac{a\cos\varphi+b}{a+b\cos\varphi}$$

Example 2.83 If $\cos\theta = \cos\alpha\cos\beta$, prove that $\tan\dfrac{\theta+\alpha}{2}\tan\dfrac{\theta-\alpha}{2} = \tan^2\dfrac{\beta}{2}$.

Sol. Given, $\cos\theta = \cos\alpha\cos\beta$, we have $\cos\beta = \dfrac{\cos\theta}{\cos\alpha}$ (i)

$$\text{Now, } \tan^2\frac{\beta}{2} = \frac{1-\cos\beta}{1+\cos\beta} = \frac{1-\dfrac{\cos\theta}{\cos\alpha}}{1+\dfrac{\cos\theta}{\cos\alpha}}$$

$$= \frac{\cos\alpha - \cos\theta}{\cos\alpha + \cos\theta}$$

$$= \frac{2\sin\dfrac{\alpha+\theta}{2}\sin\dfrac{\theta-\alpha}{2}}{2\cos\dfrac{\theta+\alpha}{2}\cos\dfrac{\theta-\alpha}{2}} = \tan\frac{\theta+\alpha}{2}\tan\frac{\theta-\alpha}{2}$$

Concept Application Exercise 2.7

1. Prove that $\cot\theta - \tan\theta = 2\cot 2\theta$.
2. Prove that $\dfrac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta} = \sec 2\theta - \tan 2\theta$.
3. Prove that $\tan\left(\dfrac{\pi}{4}+\theta\right) - \tan\left(\dfrac{\pi}{4}-\theta\right) = 2\tan 2\theta$.
4. Prove that $1 + \tan\theta\tan 2\theta = \sec 2\theta$.
5. Prove that $\dfrac{1+\sin 2A - \cos 2A}{1+\sin 2A + \cos 2A} = \tan A$.
6. Show that $\dfrac{1}{\sin 10°} - \dfrac{\sqrt{3}}{\cos 10°} = 4$.
7. Prove that $\operatorname{cosec} A - 2\cot 2A\cos A = 2\sin A$.
8. Prove that $\dfrac{1+\sin 2A}{\cos 2A} = \dfrac{\cos A + \sin A}{\cos A - \sin A} = \tan\left(\dfrac{\pi}{4}+A\right)$.
9. Prove that $\cos^3\theta\sin 3\theta + \sin^3\theta\cos 3\theta = \dfrac{3}{4}\sin 4\theta$.
10. Prove that $\tan\theta + \tan(60° + \theta) + \tan(120° + \theta) = 3\tan 3\theta$.
11. If α and β are the two different roots of equation $a\cos\theta + b\sin\theta = c$, prove that

    a. $\tan(\alpha+\beta) = \dfrac{2ab}{a^2-b^2}$ b. $\cos(\alpha+\beta) = \dfrac{a^2-b^2}{a^2+b^2}$
12. If $\cos\theta = \dfrac{\cos\alpha - \cos\beta}{1 - \cos\alpha\cos\beta}$, prove that one of the values of $\tan\dfrac{\theta}{2}$ is $\tan\dfrac{\alpha}{2}\cot\dfrac{\beta}{2}$.

13. If $\tan\theta\tan\phi = \sqrt{\frac{(a-b)}{(a+b)}}$, prove that $(a - b\cos 2\theta)(a - b\cos 2\phi)$ is independent of θ and ϕ.

14. If θ is an acute angle and $\sin\frac{\theta}{2} = \sqrt{\frac{x-1}{2x}}$, find $\tan\theta$ in terms of x.

15. Prove that $(1+\sec 2\theta)(1+\sec 4\theta)(1+\sec 8\theta) = \frac{\tan 8\theta}{\tan\theta}$.

16. Prove that $\frac{\sin^2 3A}{\sin^2 A} - \frac{\cos^2 3A}{\cos^2 A} = 8\cos 2A$.

17. If $A = 110°$, then prove that $\frac{1+\sqrt{1+\tan^2 2A}}{\tan 2A} = -\tan A$.

18. In triangle ABC, $a = 3$, $b = 4$ and $c = 5$. Then find the value of $\sin A + \sin 2B + \sin 3C$.

VALUES OF TRIGONOMETRIC RATIOS OF STANDARD ANGLES

1. **Value of $\sin 15°$, $\cos 15°$, $\sin 75°$, $\cos 75°$, $\tan 15°$, $\tan 75°$:**

$$\sin 15° = \sin(45° - 30°) = \sin 45°\cos 30° - \sin 30°\cos 45° = \frac{1}{\sqrt{2}}\frac{\sqrt{3}}{2} - \frac{1}{2}\frac{1}{\sqrt{2}} = \frac{\sqrt{3}-1}{2\sqrt{2}}$$

Also, $\sin 15° = \cos 75° = -\cos 105°$

Similarly, we can prove that $\cos 15° = \frac{\sqrt{3}+1}{2\sqrt{2}}$

Also, $\cos 15° = \sin 75° = \sin 105°$

$$\tan 15° = \tan(60° - 45°) = \frac{\tan 60° - \tan 45°}{1 + \tan 60°\ \tan 45°} = \frac{\sqrt{3}-1}{\sqrt{3}+1} = 2 - \sqrt{3}$$

$$\tan 75° = \tan(60° + 45°) = \frac{\tan 60° + \tan 45°}{1 - \tan 60°\ \tan 45°} = \frac{\sqrt{3}+1}{\sqrt{3}-1} = 2 + \sqrt{3}$$

2. **Value of $\sin 18°$, $\cos 18°$:**

Let $\theta = 18°$, then $5\theta = 90°$

$\Rightarrow\ 2\theta + 3\theta = 90°$

$\Rightarrow\ 2\theta = 90° - 3\theta$

$\Rightarrow\ \sin 2\theta = \sin(90° - 3\theta)$

$\Rightarrow\ \sin 2\theta = \cos 3\theta$

$\Rightarrow\ 2\sin\theta\cos\theta = 4\cos^3\theta - 3\cos\theta$

$\Rightarrow\ 2\sin\theta = 4\cos^2\theta - 3$ [dividing by $\cos\theta$]

$\Rightarrow\ 2\sin\theta = 4(1 - \sin^2\theta) - 3 = 1 - 4\sin^2\theta$

$\Rightarrow\ 4\sin^2\theta + 2\sin\theta - 1 = 0$

$\Rightarrow\ \sin\theta = \frac{-2 \pm \sqrt{4+16}}{8} = \frac{-2 \pm 2\sqrt{5}}{8} = \frac{-1 \pm \sqrt{5}}{4}$

$\because\ \theta = 18°$

$\therefore$ $\sin\theta = \sin 18° > 0$, for 18° lies in the first quadrant.

$\therefore$ $\sin\theta$, i.e., $\sin 18° = \dfrac{\sqrt{5}-1}{4}$

Value of cos 18°:

$$\cos^2 18° = 1 - \sin^2 18° = 1 - \left(\frac{\sqrt{5}-1}{4}\right)^2 = 1 - \frac{5+1-2\sqrt{5}}{16} = \frac{10+2\sqrt{5}}{16}$$

$\Rightarrow$ $\cos 18° = \dfrac{1}{4}\sqrt{10+2\sqrt{5}}$ [$\because \cos 18° > 0$]

3. Value of cos 36°, sin36°:

$$\cos 36° = 1 - 2\sin^2 18° = 1 - 2\left(\frac{\sqrt{5}-1}{4}\right)^2 = \frac{\sqrt{5}+1}{4}$$

Value of sin 36°:

$$\sin^2 36° = 1 - \cos^2 36° = 1 - \left(\frac{\sqrt{5}+1}{4}\right)^2 = 1 - \frac{6+2\sqrt{5}}{16} = \frac{16-6-2\sqrt{5}}{16} = \frac{10-2\sqrt{5}}{16}$$

$\therefore$ $\sin 36° = \dfrac{1}{4}\sqrt{10-2\sqrt{5}}$ [$\because \sin 36° > 0$]

Note:

- $\sin 54° = \sin(90° - 36°) = \cos 36° = \dfrac{\sqrt{5}+1}{4}$
- $\cos 54° = \cos(90° - 36°) = \sin 36° = \dfrac{1}{4}(\sqrt{10-2\sqrt{5}})$

4. Value of tan $7\frac{1°}{2}$, cot $7\frac{1°}{2}$:

Let $\theta = 7\frac{1}{2}°$, then $2\theta = 15°$

$\tan\theta = \dfrac{1-\cos 2\theta}{\sin 2\theta}$ [$\because 1 - \cos 2\theta = 2\sin^2\theta$ and $\sin 2\theta = 2\sin\theta\cos\theta$]

$$= \frac{1-\cos 15°}{\sin 15°} = \frac{1 - \frac{\sqrt{3}+1}{2\sqrt{2}}}{\frac{\sqrt{3}-1}{2\sqrt{2}}} = \frac{2\sqrt{2}-\sqrt{3}-1}{\sqrt{3}-1} = (\sqrt{3}-\sqrt{2})(\sqrt{2}-1)$$

Value of cot $82\frac{1°}{2}$:

$$\cot 82\frac{1}{2}° = \cot\left(90° - 7\frac{1°}{2}\right) = \tan 7\frac{1°}{2} = (\sqrt{3}-\sqrt{2})(\sqrt{2}-1)$$

Value of cot $7\frac{1°}{2}$:

Let $\theta = 7\frac{1°}{2}$, then $2\theta = 15°$

Now, $\cot\theta = \dfrac{1+\cos 2\theta}{\sin 2\theta} = \dfrac{1+\cos 15^\circ}{\sin 15^\circ} = \dfrac{1+\dfrac{\sqrt{3}+1}{2\sqrt{2}}}{\dfrac{\sqrt{3}-1}{2\sqrt{2}}} = \dfrac{2\sqrt{2}+\sqrt{3}+1}{\sqrt{3}-1} = (\sqrt{3}+\sqrt{2})(\sqrt{2}+1)$

Value of $\tan 82\frac{1^\circ}{2}$:

$\tan 82\frac{1^\circ}{2} = \tan(90^\circ - 7\frac{1^\circ}{2}) = \cot 7\frac{1^\circ}{2} = (\sqrt{3}+\sqrt{2})(\sqrt{2}+1)$

All these values are tabulated as follows:

	7.5°	15°	18°	22.5°	36°	67.5°	75°
sin	$\dfrac{\sqrt{8-2\sqrt{6}-2\sqrt{2}}}{4}$	$\dfrac{\sqrt{3}-1}{2\sqrt{2}}$	$\dfrac{\sqrt{5}-1}{4}$	$\dfrac{\sqrt{2-\sqrt{2}}}{2}$	$\dfrac{\sqrt{10-2\sqrt{5}}}{4}$	$\dfrac{\sqrt{2+\sqrt{2}}}{2}$	$\dfrac{\sqrt{3}+1}{2\sqrt{2}}$
cos	$\dfrac{\sqrt{8+2\sqrt{6}+2\sqrt{2}}}{4}$	$\dfrac{\sqrt{3}+1}{2\sqrt{2}}$	$\dfrac{\sqrt{10+2\sqrt{5}}}{4}$	$\dfrac{\sqrt{2+\sqrt{2}}}{2}$	$\dfrac{\sqrt{5}+1}{4}$	$\dfrac{\sqrt{2-\sqrt{2}}}{2}$	$\dfrac{\sqrt{3}-1}{2\sqrt{2}}$
tan	$(\sqrt{3}-\sqrt{2})(\sqrt{2}-1)$	$2-\sqrt{3}$	$\dfrac{\sqrt{10+2\sqrt{5}}}{4}$	$\sqrt{2}-1$	$\sqrt{5-2\sqrt{5}}$	$\sqrt{2}+1$	$2+\sqrt{3}$
cot	$(\sqrt{3}+\sqrt{2})(\sqrt{2}+1)$	$2+\sqrt{3}$	$\sqrt{(5+2\sqrt{5})}$	$\sqrt{2}+1$	$\sqrt{\left(1+\dfrac{2}{\sqrt{5}}\right)}$	$\sqrt{2}-1$	$2-\sqrt{3}$

Example 2.84 **Find the angle θ whose cosine is equal to its tangent.**

Sol. Given, $\cos\theta = \tan\theta \quad \Rightarrow \quad \cos^2\theta = \sin\theta$

$\Rightarrow \quad 1-\sin^2\theta = \sin\theta$ or $\sin^2\theta + \sin\theta - 1 = 0$

$\Rightarrow \quad \sin\theta = \dfrac{-1\pm\sqrt{5}}{2} = 2\;\dfrac{\sqrt{5}-1}{4} = 2\sin 18^\circ$

$\Rightarrow \quad \theta = \sin^{-1}(2\sin 18^\circ)$

Example 2.85 **Find the value of $\cos 12^\circ + \cos 84^\circ + \cos 156^\circ + \cos 132^\circ$.**

Sol. $\cos 12^\circ + \cos 84^\circ + \cos 156^\circ + \cos 132^\circ$

$= (\cos 12^\circ + \cos 132^\circ) + (\cos 84^\circ + \cos 156^\circ)$

$= 2\cos\left(\dfrac{12^\circ+132^\circ}{2}\right)\cos\left(\dfrac{132^\circ-12^\circ}{2}\right) + 2\cos\left(\dfrac{84^\circ+156^\circ}{2}\right)\cos\left(\dfrac{156^\circ-84^\circ}{2}\right)$

$= 2\cos 72^\circ\cos 60^\circ + 2\cos 120^\circ\cos 36^\circ$

$= 2\sin 18^\circ\cos 60^\circ + 2\cos 120^\circ\cos 36^\circ$

$= 2\left(\dfrac{\sqrt{5}-1}{4}\right)\dfrac{1}{2} + 2\left(-\dfrac{1}{2}\right)\left(\dfrac{\sqrt{5}+1}{4}\right) = -\dfrac{1}{2}$

Example 2.86 **Prove that $\cos 36° \cos 72° \cos 108° \cos 144° = \dfrac{1}{16}$.**

Sol. $\cos 36° \cos 72° \cos 108° \cos 144°$

$$= \cos 36° \sin 18° (-\sin 18°)(-\cos 36°)$$

$$= \cos^2 36° \sin^2 18° = \left(\frac{\sqrt{5}+1}{4}\right)^2 \left(\frac{\sqrt{5}-1}{4}\right)^2$$

$$= \left[\left(\frac{\sqrt{5}+1}{4}\right)\left(\frac{\sqrt{5}-1}{4}\right)\right]^2 = \frac{1}{16}$$

Concept Application Exercise 2.8

1. Prove that $\sin^2 48° - \cos^2 12° = \dfrac{\sqrt{5}+1}{8}$.
2. Prove that $4(\sin 24° + \cos 6°) = \sqrt{3} + \sqrt{15}$.
3. Find the value of $\sin 47° + \sin 61° - \sin 11° - \sin 25°$.

SUM OF SINES OR COSINES OF *N* ANGLES IN A.P.

$$\sin\alpha + \sin(\alpha + \alpha\beta) + \sin(\alpha + 2\beta) + \cdots + \sin(\alpha + \overline{n-1}\beta) = \frac{\sin\dfrac{n\beta}{2}}{\sin\dfrac{\beta}{2}} \times \sin\left[\alpha + (n-1)\frac{\beta}{2}\right]$$

Proof:

Let $S = \sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \cdots + \sin(\alpha + \overline{n-1}\beta)$

Here angle are in A.P. and common difference of angles $= \beta$

Multiplying both sides by $2\sin\dfrac{\beta}{2}$, we get

$$2S\sin\frac{\beta}{2} = 2\sin\alpha\sin\frac{\beta}{2} + 2\sin(\alpha+\beta)\sin\frac{\beta}{2} + \cdots + 2\sin(\alpha + \overline{n-1}\beta)\sin\frac{\beta}{2} \quad \text{(i)}$$

Now, $2\sin\alpha\sin\dfrac{\beta}{2} = \cos\left(\alpha - \dfrac{\beta}{2}\right) - \cos\left(\alpha + \dfrac{\beta}{2}\right)$

$$2\sin(\alpha+\beta)\sin\frac{\beta}{2} = \cos\left(\alpha + \frac{\beta}{2}\right) - \cos\left(\alpha + \frac{3\beta}{2}\right)$$

$$2\sin(\alpha+2\beta)\sin\frac{\beta}{2} = \cos\left(\alpha + \frac{3\beta}{2}\right) - \cos\left(\alpha + \frac{5\beta}{2}\right)$$

$\vdots$

$$2\sin(\alpha + \overline{n-1}\beta)\sin\frac{\beta}{2} = \cos\left[\alpha + (2n-3)\frac{\beta}{2}\right] - \cos\left[\alpha(2n-1)\frac{\beta}{2}\right]$$

Adding, we get R.H.S. of Eq. (*i*) $= \cos\left(\alpha - \dfrac{\beta}{2}\right) - \cos\left[\alpha + (2n-1)\dfrac{\beta}{2}\right]$

$$\text{or } 2\sin\frac{\beta}{2}S = 2\sin\left[\alpha+(n-1)\frac{\beta}{2}\right]\sin\frac{n\beta}{2}$$

$$\Rightarrow\quad S = \frac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}\sin\left[\alpha+(n-1)\frac{\beta}{2}\right]$$

In the above result replacing α by $\pi/2+\alpha$, we get

$$\cos\alpha+\cos(\alpha+\beta)+\cos(\alpha+2\beta)+\cdots+\cos(\alpha+\overline{n-1}\beta) = \frac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}\cos\left[\alpha+(n-1)\frac{\beta}{2}\right].$$

Example 2.87 Find the value of $\cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{6\pi}{7}$.

Sol. $S = \cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{6\pi}{7}$

$$= \frac{\sin\left(3\frac{\pi}{7}\right)}{\sin\left(\frac{\pi}{7}\right)}\cos\left(\frac{\pi}{7}+\frac{3\pi}{7}\right)$$

$$= \frac{2\sin\left(\frac{3\pi}{7}\right)\cos\left(\frac{4\pi}{7}\right)}{2\sin\left(\frac{2\pi}{7}\right)} = \frac{\sin\left(\frac{7\pi}{7}\right)-\sin\left(\frac{\pi}{7}\right)}{2\sin\left(\frac{2\pi}{7}\right)} = -\frac{1}{2}$$

Example 2.88 Prove that $\sin\theta+\sin3\theta+\sin5\theta+\cdots+\sin(2n-1)\theta = \frac{\sin^2 n\theta}{\sin\theta}$.

Sol. $$\sin\theta+\sin3\theta+\sin5\theta+\cdots+\sin(2n-1)\theta = \frac{\sin\left[n\left(\frac{2\theta}{2}\right)\right]}{\sin\left(\frac{2\theta}{2}\right)}\sin\left(\frac{\theta+(2n-1)\theta}{2}\right)$$

$$= \frac{\sin^2 n\theta}{\sin\theta}$$

Concept Application Exercise 2.9

1. Find the value of $\cos\frac{\pi}{11}+\cos\frac{3\pi}{11}+\cos\frac{5\pi}{11}+\cos\frac{7\pi}{11}+\cos\frac{9\pi}{11}$.
2. Find the average value of $\sin 2°, \sin 4°, \sin 6°, \ldots, \sin 180°$.
3. Find the value of $\sum_{r=1}^{n-1}\sin^2\frac{r\pi}{n}$.

CONDITIONAL IDENTITIES

Some Standard Identities in Triangle

1. $\tan A+\tan B+\tan C=\tan A\tan B\tan C$

Proof:

In ΔABC, we have $A+B+C=\pi$

$\Rightarrow \quad A+B=\pi-C$

$\Rightarrow \quad \tan(A+B)=\tan(\pi-C)$

$$\Rightarrow \quad \frac{\tan A+\tan B}{1-\tan A\tan B}=-\tan C$$

$\Rightarrow \quad \tan A+\tan B=-\tan C+\tan A\tan B\tan C$

$\Rightarrow \quad \tan A+\tan B+\tan C=\tan A\tan B\tan C$

2. $\tan\frac{A}{2}\tan\frac{B}{2}+\tan\frac{C}{2}\tan\frac{B}{2}+\tan\frac{C}{2}\tan\frac{A}{2}=1$

Proof:

Since $A+B+C=\pi$, we have $\frac{A}{2}+\frac{B}{2}=\frac{\pi}{2}-\frac{C}{2}$

$$\Rightarrow \tan\left(\frac{A}{2}+\frac{B}{2}\right)=\tan\left(\frac{\pi}{2}-\frac{C}{2}\right)=\cot\frac{C}{2}$$

$$\Rightarrow \frac{\tan\frac{A}{2}+\tan\frac{B}{2}}{1-\tan\frac{A}{2}\tan\frac{B}{2}}=\frac{1}{\tan\frac{C}{2}}$$

$$\Rightarrow \tan\frac{A}{2}\tan\frac{C}{2}+\tan\frac{B}{2}\tan\frac{C}{2}=1-\tan\frac{A}{2}\tan\frac{B}{2}$$

$$\Rightarrow \tan\frac{A}{2}\tan\frac{B}{2}+\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{C}{2}\tan\frac{A}{2}=1$$

3. $\sin 2A+\sin 2B+\sin 2C=4\sin A\sin B\sin C$

Proof:

$$\begin{aligned}(\sin 2A+\sin 2B)+\sin 2C &= 2\sin(A+B)\cos(A-B)+\sin 2C\\ &= 2\sin(\pi-C)\cos(A-B)+\sin 2C\\ &= 2\sin C\cos(A-B)+2\sin C\cos C\\ &= 2\sin C[\cos(A-B)+\cos C]\\ &= 2\sin C[\cos(A-B)+\cos\{\pi-(A+B)\}]\\ &= 2\sin C[\cos(A-B)-\cos(A+B)]\\ &= 2\sin C\times 2\sin A\sin B=4\sin A\sin B\sin C\end{aligned}$$

4. $\cos 2A + \cos 2B + \cos 2C = -1 - 4\cos A \cos B \cos C$

Proof:

$$
\begin{aligned}
&(\cos 2A + \cos 2B) + \cos 2C \\
&= 2\cos(A+B)\cos(A-B) + 2\cos^2 C - 1 \\
&= 2\cos(\pi - C)\cos(A-B) + 2\cos^2 C - 1 \\
&= -2\cos C\cos(A-B) + 2\cos^2 C - 1 \\
&= -2\cos C[\cos(A-B) - \cos C] - 1 \\
&= -2\cos C[\cos(A-B) - \cos\{\pi - (A+B)\}] - 1 \\
&= -2\cos C[\cos(A-B) + \cos(A+B)] - 1 \\
&= -1 - 4\cos A\cos B\cos C
\end{aligned}
$$

5. $\cos A + \cos B + \cos C = 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

Proof:

$$
\begin{aligned}
&(\cos A + \cos B) + \cos C - 1 \\
&= 2\cos\frac{A+B}{2}\cos\frac{A-B}{2} + \cos C - 1 \\
&= 2\cos\left(\frac{\pi}{2} - \frac{C}{2}\right)\cos\frac{A-B}{2} + \cos C - 1 \\
&= 2\sin\frac{C}{2}\cos\frac{A-B}{2} + 1 - 2\sin^2\frac{C}{2} - 1 \\
&= 2\sin\frac{C}{2}\cos\frac{A-B}{2} - 2\sin^2\frac{C}{2} \\
&= 2\sin\frac{C}{2}\left[\cos\frac{A-B}{2} - \sin\frac{C}{2}\right] \\
&= 2\cos\frac{C}{2}\left[\cos\frac{A-B}{2} - \sin\left(\frac{\pi}{2} - \frac{A+B}{2}\right)\right] \\
&= 2\sin\frac{C}{2}\left[\cos\frac{A-B}{2} - \cos\frac{A+B}{2}\right] \\
&= 2\sin\frac{C}{2}\left(2\sin\frac{A}{2}\sin\frac{B}{2}\right) = 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}
\end{aligned}
$$

6. $\sin A + \sin B + \sin C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

Proof:

$$
\begin{aligned}
&(\sin A + \sin B) + \sin C \\
&= 2\sin\frac{A+B}{2}\cos\frac{A-B}{2} + \sin C \\
&= 2\sin\left(\frac{\pi}{2} - \frac{C}{2}\right)\cos\frac{A-B}{2} + \sin C \\
&= 2\cos\frac{C}{2}\cos\frac{A-B}{2} + 2\sin\frac{C}{2}\cos\frac{C}{2}
\end{aligned}
$$

$$= 2\cos\frac{C}{2}\left[\cos\frac{A-B}{2} + \sin\frac{C}{2}\right]$$

$$= 2\cos\frac{C}{2}\left[\cos\frac{A-B}{2} + \sin\left(\frac{\pi}{2} - \frac{A+B}{2}\right)\right]$$

$$= 2\cos\frac{C}{2}\left[\cos\frac{A-B}{2} + \cos\frac{A+B}{2}\right]$$

$$= 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$$

Note:

$\tan A + \tan B + \tan C = \tan A \tan B \tan C$ *is true for* $A + B + C = n\pi$*, where* $n \in N$.

Example 2.89 **If $A + B + C = 180°$, prove that $\cos^2 A + \cos^2 B + \cos^2 C = 1 - 2\cos A \cos B \cos C$.**

Sol. $\cos^2 A + \cos^2 B + \cos^2 C = \frac{1+\cos 2A}{2} + \frac{1+\cos 2B}{2} + \frac{1+\cos 2C}{2}$

$$= \frac{1}{2}(\cos 2A + \cos 2B + \cos 2C) + \frac{3}{2}$$

$$= \frac{1}{2}(-1 - 4\cos A\cos B\cos C) + \frac{3}{2}$$

$$= 1 - 2\cos A \cos B \cos C$$

Example 2.90 **Prove that in triangle ABC, $\cos^2 A + \cos^2 B - \cos^2 C = 1 - 2\sin A \sin B \cos C$.**

Sol. $\cos^2 A + \cos^2 B - \cos^2 C = \cos^2 A + \sin^2 C - \sin^2 B$

$$= \cos^2 A + \sin(C+B)\sin(C-B)$$

$$= 1 - \sin^2 A + \sin A \sin(C-B)$$

$$= 1 - \sin A[\sin A - \sin(C-B)]$$

$$= 1 - \sin A[\sin(B+C) - \sin(C-B)]$$

$$= 1 - 2\sin A \sin B \cos C$$

Example 2.91 **In triangle ABC, prove that**

$\sin(B+C-A) + \sin(C+A-B) + \sin(A+B-C) = 4\sin A \sin B \sin C$.

Sol. $\sin(B+C-A) + \sin(C+A-B) + \sin(A+B-C) = \sin(\pi-2A) + \sin(\pi-2B) + \sin(\pi-2C)$

$$= \sin 2A + \sin 2B + \sin 2C = 4\sin A \sin B \sin C$$

Example 2.92 If $x+y+z=xyz$, prove that $\frac{2x}{1-x^2}+\frac{2y}{1-y^2}+\frac{2z}{1-z^2}=\frac{2x}{1-x^2}\frac{2y}{1-y^2}\frac{2z}{1-z^2}$.

Sol. Let $x=\tan A, y=\tan B, z=\tan C$

Now $x+y+z=xyz$

$\Rightarrow$ $\tan A+\tan B+\tan C=\tan A\tan B\tan C$

$\Rightarrow$ $A+B+C=n\pi$

$\Rightarrow$ $2A+2B+2C=2n\pi$

$\Rightarrow$ $\tan 2A+\tan 2B+\tan 2C=\tan 2A\tan 2B\tan 2C$

$$\Rightarrow \frac{2\tan A}{1-\tan^2 A}+\frac{2\tan B}{1-\tan^2 B}+\frac{2\tan C}{1-\tan^2 C}=\frac{2\tan A}{1-\tan^2 A}\frac{2\tan B}{1-\tan^2 B}\frac{2\tan C}{1-\tan^2 C}$$

$$\Rightarrow \frac{2x}{1-x^2}+\frac{2y}{1-y^2}+\frac{2z}{1-z^2}=\frac{2x}{1-x^2}\frac{2y}{1-y^2}\frac{2z}{1-z^2}$$

Example 2.93 If $A+B+C=\pi$, prove that $\sin^2\frac{A}{2}+\sin^2\frac{B}{2}-\sin^2\frac{C}{2}=1-2\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$.

Sol.
$$\sin^2\frac{A}{2}-\sin^2\frac{C}{2}+\sin^2\frac{B}{2}=\sin\left(\frac{A+C}{2}\right)\sin\left(\frac{A-C}{2}\right)+1-\cos^2\frac{B}{2}$$

$$=\cos\left(\frac{B}{2}\right)\sin\left(\frac{A-C}{2}\right)-\cos^2\frac{B}{2}+1$$

$$=\cos\left(\frac{B}{2}\right)\left[\sin\left(\frac{A-C}{2}\right)-\cos\frac{B}{2}\right]+1$$

$$=\cos\left(\frac{B}{2}\right)\left[\sin\left(\frac{A-C}{2}\right)-\sin\left(\frac{A+C}{2}\right)\right]+1$$

$$=1-2\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$$

Example 2.94 The product of the sines of the angles of a triangle is p and the product of their cosines is q. Show that the tangents of the angles are the roots of the equation $qx^2-px^2+(1+q)x-p=0$.

Sol. From the question, $\sin A\sin B\sin C=p$ and $\cos A\cos B\cos C=q$

$$\therefore \tan A\tan B\tan C=\frac{p}{q} \quad \text{(i)}$$

$$\text{Also, } \tan A+\tan B+\tan C=\tan A\tan B\tan C=\frac{p}{q} \quad \text{(ii)}$$

Now, $\tan A\tan B+\tan B\tan C+\tan C\tan A$

$$=\frac{\sin A\sin B\cos C+\sin B\sin C\cos A+\sin C\sin A\cos B}{\cos A\cos B\cos C}$$

$$=\frac{1}{2q}[(\sin^2A+\sin^2B-\sin^2C)+(\sin^2B+\sin^2C\ \sin^2A)+\sin^2C+\sin^2A-\sin^2B)]$$

$[\because A+B+C=\pi \text{ and } 2\sin A\sin B\cos C=\sin^2A+\sin^2B-\sin^2C]$

$$= \frac{1}{2q}[\sin^2 A + \sin^2 B + \sin^2 C] = \frac{1}{4q}[3 - (\cos 2A + \cos 2B + \cos 2C)] = \frac{1}{q}[1 + \cos A \cos B \cos C] = \frac{1}{q}(1+q)$$

The equation whose roots are $\tan A$, $\tan B$ $\tan C$ will be given by

$x^3 - (\tan A + \tan B + \tan C)x^2 + (\tan A \tan B + \tan B \tan C + \tan C \tan A)x - \tan A \tan B \tan C = 0$

or $x^3 - \frac{p}{q}x^2 + \frac{1+q}{q}x - \frac{p}{q} = 0$, or $qx^3 - px^2 + (1+q)x - p = 0$

Concept Application Exercise 2.10

1. In triangle ABC, prove that
   a. $\cos^2\frac{A}{2} + \cos^2\frac{B}{2} - \cos^2\frac{C}{2} = 2\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$
   b. $\cos^2\frac{A}{2} + \cos^2\frac{B}{2} + \cos^2\frac{C}{2} = 2 + 2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$
2. If $A + B + C = \pi/2$, show that
   a. $\sin^2 A + \sin^2 B + \sin^2 C = 1 - 2\sin A \sin B \sin C$
   b. $\cos^2 A + \cos^2 B + \cos^2 C = 2 + 2\sin A \sin B \sin C$
3. a. If $A + B = C$, prove that $\cos^2 A + \cos^2 B + \cos^2 C = 1 + 2\cos A \cos B \cos C$.
   b. If $\alpha + \beta = 60°$, prove that $\cos^2\alpha + \cos^2\beta - \cos\alpha\cos\beta = 3/4$.
4. Prove that $\cos^2(\beta - \gamma) + \cos^2(\gamma - \alpha) + \cos^2(\alpha - \beta) = 1 + 2\cos(\beta - \alpha)\cos(\gamma - \alpha)\cos(\alpha - \beta)$.
5. If $A + B + C = \pi/2$, show that
   a. $\cot A + \cot B + \cot C = \cot A \cot B \cot C$
   b. $\tan A \tan B + \tan B \tan C + \tan C \tan A = 1$
6. If $A + B + C = \pi$, prove that
   a. $\tan 3A + \tan 3B + \tan 3C = \tan 3A \tan 3B \tan 3C$
   b. $\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2} = \cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$
7. If $A + B + C = \pi$, prove that $\frac{\cos A}{\sin B \sin C} + \frac{\cos B}{\sin C \sin A} + \frac{\cos C}{\sin A \sin B} = 2$.

SOME IMPORTANT RESULTS AND THEIR APPLICATIONS

Result 1. $\cos A \cos(60 - A)\cos(60 + A) = \frac{1}{4}\cos 3A$

Proof:

We have

L.H.S. $= \cos A \cos(60 - A)\cos(60 + A)$

$= \cos A(\cos^2 60° - \sin^2 A)$ $\quad [\because \cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B]$

$$= \cos A\left(\frac{1}{4} - \sin^2 A\right) = \cos A\left(\frac{1}{4} - (1 - \cos^2 A)\right) = \cos A\left(-\frac{3}{4} + \cos^2 A\right)$$

$$= \frac{1}{4}\cos A(-3 + 4\cos^2 A) = \frac{1}{4}(4\cos^3 A - 3\cos A)$$

$$= \frac{1}{4}\cos 3A = \text{R.H.S.}$$

Result 2. $\sin A \sin(60-A)\sin(60+A) = \frac{1}{4}\sin 3A$

Proof:

We have

$$\begin{aligned} \text{L.H.S.} &= \sin A \sin(60-A)\sin(60+A) \\ &= \sin A(\sin^2 60° - \sin^2 A) \quad [\because \sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B] \\ &= \sin A\left(\frac{3}{4} - \sin^2 A\right) = \frac{1}{4}\sin A(3-4\sin^2 A) \\ &= \frac{1}{4}(3\sin A - 4\sin^3 A) \\ &= \frac{1}{4}\sin 3A = \text{R.H.S.} \end{aligned}$$

Result 3. $\tan\alpha \tan(60° - \alpha)\tan(60° + \alpha) = \tan 3\alpha$

Using the above two results, we can prove this result

Example 2.95 **Prove that $\cos 20° \cos 40° \cos 60° \cos 80° = \frac{1}{16}$.**

Sol. $\cos 20° \cos 40° \cos 80° \cos 60° = \cos 20° \cos(60° - 20°)\cos(60° + 20°)\cos 60°$

$$= \frac{1}{4}\cos(3\times 20°)\ \cos 60° = \frac{1}{4}\cos^2 60° = \frac{1}{4}\times\frac{1}{4} = \frac{1}{16}$$

Example 2.96 **Prove that $\sin 10° \sin 30° \sin 50° \sin 70° = \frac{1}{16}$.**

Sol. $\sin 10° \sin 30° \sin 50° \sin 70° = \sin 10° \sin(60° - 10°)\sin(60° + 10°)\sin 30°$

$$= \frac{1}{4}\sin(3\times 10°)\ \sin 30° = \frac{1}{4}\sin^2 30° = \frac{1}{16}$$

Example 2.97 **Prove that $\tan 20° \tan 40° \tan 80° = \tan 60°$.**

Sol. $\tan 20° \tan 40° \tan 80°$

$= \tan 20° \tan(60° - 20°)\tan(60° + 20°) = \tan(3\times 20°) = \tan 60°$

Result 4. $\cos A \cos 2A \cos 2^2A \cos 2^3A \cdots \cos 2^{n-1}A = \frac{\sin 2^n A}{2^n \sin A}$

Proof:

$$\begin{aligned} \text{L.H.S.} &= \cos A \cos 2A \cos 2^2A \cos 2^3A \cdots \cos 2^{n-1}A \\ &= \frac{1}{2\sin A}[(2\sin A\cos A)\cos 2A \cos 2^2A \cos 2^3A \cdots \cos 2^{n-1}A] \\ &= \frac{1}{2\sin A}[(\sin 2A \cos 2A \cos 2^2A \cos 2^3A \cdots \cos 2^{n-1}A] \\ &= \frac{1}{2^2\sin A}[(2\sin 2A\cos 2A)\cos 2^2A \cos 2^3A \cdots \cos 2^{n-1}A] \\ &= \frac{1}{2^2\sin A}[\sin 2(2A).\cos 2^2A \cos 2^3A \cdots \cos 2^{n-1}A] \\ &= \frac{1}{2^3\sin A}[(2\sin 2^2A\cos 2^2A)\cos 2^3A \cdots \cos 2^{n-1}A] \end{aligned}$$

$$= \frac{1}{2^3 \sin A}[\sin(2\times 2^2 A)\cos 2^3 A \cdots \cos 2^{n-1}A]$$

$$= \frac{1}{2^3 \sin A}[(\sin 2^3 A \cos 2^3 A \cos 2^4 A \cdots \cos 2^{n-1}A]$$

...

...

$$= \frac{1}{2^{n-1} \sin A}[\sin 2^{n-1} A \cos 2^{n-1} A]$$

$$= \frac{1}{2^n \sin A}[2 \sin 2^{n-1} A \cos 2^{n-1} A]$$

$$= \frac{1}{2^n \sin A}\sin(2\times 2^{n-1} A)$$

$$= \frac{1}{2^n \sin A}\sin 2^n A = \text{R.H.S.}$$

Example 2.98 If $\theta = \dfrac{\pi}{2^n+1}$, show that $\cos\theta \cos 2\theta\, 2\cos 2^2\theta \cdots \cos 2^{n-1}\theta = \dfrac{1}{2^n}$.

Sol. In the above result, put $\theta = \dfrac{\pi}{2^n+1}$

$$\Rightarrow \text{R.H.S.} = \frac{\sin 2^n\theta}{2^n \sin\theta} = \frac{\sin\left(\frac{\pi}{2^n+1}\right)2^n}{2^n \sin\left(\frac{\pi}{2^n+1}\right)} = \frac{\sin\left(\frac{2^n+1-1}{2^n+1}\right)\pi}{2^n \sin\left(\frac{\pi}{2^n+1}\right)}$$

$$= \frac{\sin\left(\pi - \frac{\pi}{2^n+1}\right)}{2^n \sin\left(\frac{\pi}{2^n+1}\right)}$$

$$= \frac{\sin\left(\frac{\pi}{2^n+1}\right)}{2^n \sin\left(\frac{\pi}{2^n+1}\right)}.$$

$$= \frac{1}{2^n}.$$

Example 2.99 **Prove that $\cos\dfrac{2\pi}{15}\cos\dfrac{4\pi}{15}\cos\dfrac{8\pi}{15}\cos\dfrac{14\pi}{15} = \dfrac{1}{16}$.**

Sol. We have

$$\text{L.H.S.} = \cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{8\pi}{15}\cos\left(\pi - \frac{\pi}{15}\right)$$

$$= \left(\cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{8\pi}{15}\right)\left(-\cos\frac{\pi}{15}\right)$$

$$= -\cos\frac{\pi}{15}\cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{8\pi}{15}$$

$$= -\cos A\cos 2A\cos 2^2 A\cos 2^3 A,\text{ where } A = \pi/15$$

$$= -\left[\frac{\sin 2^4 A}{2^4 \sin A}\right] = -\frac{\sin 16A}{2^4 \sin A}$$

$$= -\frac{\sin(15A + A)}{16\sin A} = \frac{-\sin(\pi + A)}{16\sin A} \qquad [\because 15A = \pi]$$

$$= \frac{\sin A}{16\sin A} = \frac{1}{16} = \frac{1}{16} = \text{R.H.S.}$$

Example 2.100 **Prove that $\sin 6^\circ \sin 42^\circ \sin 66^\circ \sin 78^\circ = \frac{1}{16}$.**

Sol. $\sin 6^\circ \sin 42^\circ \sin 66^\circ \sin 78^\circ$

$= \sin 6^\circ \cos 48^\circ \cos 24^\circ \cos 12^\circ$

$$= \sin 6^\circ \frac{2^3 \sin 12^\circ \cos 12^\circ \cos 24^\circ \cos 48^\circ}{2^3 \sin 12^\circ}$$

$$= \sin 6^\circ \frac{\sin 96^\circ}{2^3 \sin 12^\circ}$$

$$= \frac{2\sin 6^\circ \cos 6^\circ}{2^4 \sin 12^\circ} = \frac{\sin 12^\circ}{2^4 \sin 12^\circ} = \frac{1}{16}$$

Concept Application Exercise 2.11

1. If $\alpha = \frac{\pi}{15}$, prove that $\cos 2\alpha \cos 4\alpha \cos 8\alpha \cos 14\alpha = \frac{1}{16}$.
2. Prove that $\sin 20^\circ \sin 40^\circ \sin 60^\circ \sin 80^\circ = \frac{3}{16}$.
3. Prove that $\cos 10^\circ \cos 30^\circ \cos 50^\circ \cos 70^\circ = \frac{3}{16}$.

IMPORTANT INEQUALITIES

Example 2.101 **In ΔABC, $\tan A + \tan B + \tan C \geq 3\sqrt{3}$, where A, B, C are acute angles.**

Sol. In ΔABC,

$\tan A + \tan B + \tan C = \tan A \tan B \tan C$

$$\text{Also, } \frac{\tan A + \tan B + \tan C}{3} \geq \sqrt[3]{\tan A \tan B \tan C} \qquad [\text{since A.M.} \geq \text{G.M.}]$$

$\Rightarrow \quad \tan A \tan B \tan C \geq \sqrt[3]{\tan A \tan B \tan C}$

$\Rightarrow \quad \tan^2 A \tan^2 B \tan^2 C \geq 27$

$\Rightarrow \quad \tan A \tan B \tan C \geq 3\sqrt{3}$ [cubing both sides]

$\Rightarrow \quad \tan A + \tan B + \tan C \geq 3\sqrt{3}$

Example 2.102 **In ΔABC, prove that $\cos A + \cos B + \cos C \leq 3/2$.**

Sol. Let $\cos A + \cos B + \cos C = x$

$$\Rightarrow \quad 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) + 1 - 2\sin^2\frac{C}{2} = x$$

$$\Rightarrow \quad 2\sin\frac{C}{2}\cos\left(\frac{A-B}{2}\right) + 1 - 2\sin^2\frac{C}{2} = x$$

$$\Rightarrow \quad 2\sin^2\frac{C}{2} - 2\sin\frac{C}{2}\cos\left(\frac{A-B}{2}\right) + x - 1 = 0$$

This is quadratic in $\sin C/2$ which is real. So, discriminant $D \geq 0$.

$$4\cos^2\left(\frac{A-B}{2}\right) - 4 \times 2(x-1) \geq 0$$

$$\Rightarrow \quad 2(x-1) \leq \cos^2\left(\frac{A-B}{2}\right)$$

$\Rightarrow \quad 2(x-1) \leq 1$

$\Rightarrow \quad x \leq 3/2$

Thus, $\cos A + \cos B + \cos C \leq 3/2$

Note: Since $\cos A + \cos B + \cos C = 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

We have $\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \leq \frac{1}{8}$.

Students are advised to remember this as a standard result.

Example 2.103 **Find the least value of $\sec A + \sec B + \sec C$ in an acute angle triangle.**

Sol. In an acute angle triangle, $\sec A$, $\sec B$ and $\sec C$ are positive.

Now A.M. $\geq$ H.M.

$$\Rightarrow \quad \frac{\sec A + \sec B + \sec C}{3} \geq \frac{3}{\cos A + \cos B + \cos C}$$

But in ΔABC, $\cos A + \cos B + \cos C \leq 3/2$

$$\Rightarrow \quad \frac{\sec A + \sec B + \sec C}{3} \geq 2$$

$\Rightarrow \quad \sec A + \sec B + \sec C \geq 6$

Example 2.104 **In ΔABC, prove that $\operatorname{cosec}\frac{A}{2}+\operatorname{cosec}\frac{B}{2}+\operatorname{cosec}\frac{C}{2}\geq 6$.**

Sol. In ΔABC, we know that $\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\leq\frac{1}{8}$

Now A.M. $\geq$ G.M.

$$\Rightarrow \frac{\operatorname{cosec}\frac{A}{2}+\operatorname{cosec}\frac{B}{2}+\operatorname{cosec}\frac{C}{2}}{3}\geq\left(\operatorname{cosec}\frac{A}{2}\operatorname{cosec}\frac{B}{2}\operatorname{cosec}\frac{C}{2}\right)^{1/3}$$

$$\Rightarrow \frac{\operatorname{cosec}\frac{A}{2}+\operatorname{cosec}\frac{B}{2}+\operatorname{cosec}\frac{C}{2}}{3}\geq\left(\frac{1}{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}\right)^{1/3}$$

$$\Rightarrow \frac{\operatorname{cosec}\frac{A}{2}+\operatorname{cosec}\frac{B}{2}+\operatorname{cosec}\frac{C}{2}}{3}\geq(8)^{1/3}$$

$$\Rightarrow \operatorname{cosec}\frac{A}{2}+\operatorname{cosec}\frac{B}{2}+\operatorname{cosec}\frac{C}{2}\geq 6$$

EXERCISES

Subjective Type

Solutions on page 2.85

1. Are the set of angles α and β given by $\alpha=\left(2n+\frac{1}{2}\right)\pi\pm A$ and $\beta=m\pi+(-1)^m\left(\frac{\pi}{2}-A\right)$ same, where $n, m\in I$?
2. If ABC is a triangle and $\tan\frac{A}{2}$, $\tan\frac{B}{2}$, $\tan\frac{C}{2}$ are in H.P., then find the minimum value of $\cot B/2$.
3. Find the sum of the series $\operatorname{cosec}\theta+\operatorname{cosec}2\theta+\operatorname{cosec}4\theta+\cdots$ to n terms.
4. In ΔABC, if $\sin^3\theta=\sin(A-\theta)\sin(B-\theta)\sin(C-\theta)$, prove that $\cot\theta=\cot A+\cot B+\cot C$.
5. In triangle ABC, prove that $\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}\leq\frac{3}{2}$. Hence, deduce that $\cos\frac{\pi+A}{4}\cos\frac{\pi+B}{4}\cos\frac{\pi+C}{4}\leq\frac{1}{8}$.
6. If $\frac{x}{\tan(\theta+\alpha)}=\frac{y}{\tan(\theta+\beta)}=\frac{z}{\tan(\theta+\gamma)}$, then show that $\sum\frac{x+y}{x-y}\sin^2(\alpha-\beta)=0$.
7. If $\tan 6\theta=p/q$, find the value of $\frac{1}{2}(p\operatorname{cosec}2\theta-q\sec 2\theta)$ in terms of p and q.
8. If $0<\alpha<\pi/2$ and $\sin\alpha+\cos\alpha+\tan\alpha+\cot\alpha+\sec\alpha+\operatorname{cosec}\alpha=7$, then prove that $\sin 2\alpha$ is a root of the equation $x^2-44x-36=0$.
9. Prove that $1+\cot\theta\leq\cot\frac{\theta}{2}$ for $0<\theta<\pi$. Find θ when equality sign holds.
10. Show that $2^{\sin x}+2^{\cos x}\geq 2^{1-1/\sqrt{2}}$.
11. If A, B and C are the angles of a triangle, show that $\tan^2\frac{A}{2}+\tan^2\frac{B}{2}+\tan^2\frac{C}{2}\geq 1$.

12. Let A, B, C be three angles such that $A = \pi/4$ and $\tan B \tan C = p$. Find all possible values of p such that A, B, C are the angles of a triangle.
13. Eliminate x from the equations, $\sin(a + x) = 2b$ and $\sin(a - x) = 2c$.
14. If $\tan\beta = \dfrac{n\sin\alpha\cos\alpha}{1 - n\sin^2\alpha}$, prove that $\tan(\alpha - \beta) = (1 - n)\tan\alpha$.
15. Show that $\dfrac{\sin x}{\cos 3x} + \dfrac{\sin 3x}{\cos 9x} + \dfrac{\cos 9x}{\cos 27x} = \dfrac{1}{2}[\tan 27x - \tan x]$.
16. Prove that $\dfrac{2\cos 2^n\theta + 1}{2\cos\theta + 1} = (2\cos\theta - 1)(2\cos 2\theta - 1)(2\cos 2^2\theta - 1)\cdots(2\cos 2^{n-1}\theta - 1)$.
17. Prove that $\dfrac{\tan 2^n\theta}{\tan\theta} = (1 + \sec 2\theta)(1 + \sec 2^2\theta)(1 + \sec 2^3\theta)\cdots(1 + \sec 2^n\theta)$.

Objective Type

Solutions on page 2.92

Each question has four choices a, b, c, and d, out of which *only one* answer is correct. Find the correct answer.

1. Which of the following is correct?

   a. $\sin 1° > \sin 1$ b. $\sin 1° < \sin 1$ c. $\sin 1° = \sin 1$ d. $\sin 1° = \dfrac{\pi}{180}\sin 1$

2. The equation $\sin^2\theta = \dfrac{x^2 + y^2}{2xy}$ is possible if

   a. $x = y$ b. $x = -y$ c. $2x = y$ d. none of these

3. If $1 + \sin x + \sin^2 x + \sin^3 x + \cdots \infty$ is equal to $4 + 2\sqrt{3}$, $0 < x < \pi$, then x is equal to

   a. $\dfrac{\pi}{6}$ b. $\dfrac{\pi}{4}$ c. $\dfrac{\pi}{3}$ or $\dfrac{\pi}{6}$ d. $\dfrac{\pi}{3}$ or $\dfrac{2\pi}{3}$

4. If $\dfrac{\sin x}{a} = \dfrac{\cos x}{b} = \dfrac{\tan x}{c} = k$, then $bc + \dfrac{1}{ck} + \dfrac{ak}{1 + bk}$ is equal to

   a. $k\left(a + \dfrac{1}{a}\right)$ b. $\dfrac{1}{k}\left(a + \dfrac{1}{a}\right)$ c. $\dfrac{1}{k^2}$ d. $\dfrac{a}{k}$

5. If A, B, C are angles of a triangle, then $2\sin\dfrac{A}{2}\operatorname{cosec}\dfrac{B}{2}\sin\dfrac{C}{2} - \sin A\cot\dfrac{B}{2} - \cos A$ is

   a. independent of A, B, C b. function of A, B
   c. function of C d. none of these

6. The least value of $6\tan^2\phi + 54\cot^2\phi + 18$ is

   I: 54 when A.M. ≥ G.M. is applicable for $6\tan^2\phi$, $54\cot^2\phi$, 18.
   II: 54 when A.M. ≥ G.M is applicable for $6\tan^2\phi$, $54\cot^2\phi$ and 18 added further.
   III: 78 when $\tan^2\phi = \cot^2\phi$.

   a. I is correct b. I and II are correct
   c. III is correct d. none of the above is correct

7. If $5\tan\theta = 4$, then $\dfrac{5\sin\theta - 3\cos\theta}{5\sin\theta + 2\cos\theta}$ is equal to
   a. 0 b. 1 c. 1/6 d. 6
8. If $2\sec 2\theta = \tan\phi + \cot\phi$, then one of the values of $\theta + \phi$ is
   a. $\pi/2$ b. $\pi/4$ c. $\pi/3$ d. none of these
9. If $\sin x + \operatorname{cosec} x = 2$, then $\sin^n x + \operatorname{cosec}^n x$ is equal to
   a. 2 b. 2^n c. 2^{n-1} d. 2^{n-2}
10. A quadratic equation whose roots are $\operatorname{cosec}^2\theta$ and $\sec^2\theta$ can be
   a. $x^2 - 5x + 2 = 0$ b. $x^2 - 3x + 6 = 0$ c. $x^2 - 5x + 5 = 0$ d. none of these
11. If $\pi < \alpha < \dfrac{3\pi}{2}$, then $\sqrt{\dfrac{1-\cos\alpha}{1+\cos\alpha}} + \sqrt{\dfrac{1+\cos\alpha}{1-\cos\alpha}}$ is equal to
   a. $\dfrac{2}{\sin\alpha}$ b. $-\dfrac{2}{\sin\alpha}$ c. $\dfrac{1}{\sin\alpha}$ d. $-\dfrac{1}{\sin\alpha}$
12. The value of $\cos\dfrac{\pi}{7} + \cos\dfrac{2\pi}{7} + \cos\dfrac{3\pi}{7} + \cos\dfrac{4\pi}{7} + \cos\dfrac{5\pi}{7} + \cos\dfrac{6\pi}{7} + \cos\dfrac{7\pi}{7}$ is
   a. 1 b. -1 c. 0 d. none of these
13. The least value of $2\sin^2\theta + 3\cos^2\theta$ is
   a. 1 b. 2 c. 3 d. 5
14. The greatest value of $\sin^4\theta + \cos^4\theta$ is
   a. 1/2 b. 1 c. 2 d. 3
15. If $f(x) = \cos^2\theta + \sec^2\theta$, then
   a. $f(x) < 1$ b. $f(x) = 1$ c. $2 > f(x) > 1$ d. $f(x) \ge 2$
16. If $f(x) = \sin^6 x + \cos^6 x$, then range of $f(x)$ is
   a. $\left[\dfrac{1}{4}, 1\right]$ b. $\left[\dfrac{1}{4}, \dfrac{3}{4}\right]$ c. $\left[\dfrac{3}{4}, 1\right]$ d. none of these
17. If $a \le 3\cos x + 5\sin(x - \pi/6) \le b$ for all x, then (a, b) is
   a. $(-\sqrt{19}, \sqrt{19})$ b. $(-17, 17)$ c. $(-\sqrt{21}, \sqrt{21})$ d. none of these
18. The equation $\sin x(\sin x + \cos x) = k$ has real solutions if and only if k is a real number such that
   a. $0 \le k \le \dfrac{1+\sqrt{2}}{2}$ b. $2-\sqrt{3} \le k \le 2+\sqrt{3}$ c. $0 \le k \le 2-\sqrt{3}$ d. $\dfrac{1-\sqrt{2}}{2} \le k \le \dfrac{1+\sqrt{2}}{2}$
19. If $\cot(\alpha + \beta) = 0$, then $\sin(\alpha + 2\beta)$ can be
   a. $-\sin\alpha$ b. $\sin\beta$ c. $\cos\alpha$ d. $\cos\beta$
20. If $\dfrac{x}{\cos\theta} = \dfrac{y}{\cos\left(\theta - \dfrac{2\pi}{3}\right)} = \dfrac{z}{\cos\left(\theta + \dfrac{2\pi}{3}\right)}$, then $x + y + z$ is equal to
   a. 1 b. 0 c. -1 d. none of these
21. $\sin^{2n} x + \cos^{2n} x$ lies between
   a. -1 and 1 b. 0 and 1 c. 1 and 2 d. none of these
22. The roots of the equation $4x^2 - 2\sqrt{5}\,x + 1 = 0$ are
   a. $\sin 36°, \sin 18°$ b. $\sin 18°, \cos 36°$ c. $\sin 36°, \cos 18°$ d. $\cos 18°, \cos 36°$
23. If $\dfrac{3\pi}{4} < \alpha < \pi$, then $\sqrt{2\cot\alpha + \dfrac{1}{\sin^2\alpha}}$ is equal to
   a. $1 + \cot\alpha$ b. $-1 - \cot\alpha$ c. $1 - \cot\alpha$ d. $-1 + \cot\alpha$

24. If $f(\theta) = 5\cos\theta + 3\cos\left(\theta + \frac{\pi}{3}\right) + 3$, then range of $f(\theta)$ is

a. $[-5, 11]$ b. $[-3, 9]$ c. $[-2, 10]$ d. $[-4, 10]$

25. If $\alpha, \beta, \gamma, \delta$ are the smallest positive angles in ascending order of magnitude which have their sines equal to the positive quantity k, then the value of $4\sin\frac{\alpha}{2} + 3\sin\frac{\beta}{2} + 2\sin\frac{\gamma}{2} + \sin\frac{\delta}{2}$ is equal to

a. $2\sqrt{1-k}$ b. $2\sqrt{1+k}$ c. $\frac{\sqrt{1+k}}{2}$ d. none of these

26. Let $A_0A_1A_2A_3A_4A_5$ be a regular hexagon inscribed in a circle of unit radius. Then the product of the lengths of the line segments A_0A_1, A_0A_2 and A_0A_4 is

a. $3/4$ b. $3\sqrt{3}$ c. 3 d. $3\sqrt{3}/2$

27. If $\sin\theta_1 + \sin\theta_2 + \sin\theta_3 = 3$, then $\cos\theta_1 + \cos\theta_2 + \cos\theta_3$ is equal to

a. 3 b. 2 c. 1 d. 0

28. If $\sin^2\theta = \frac{x^2 + y^2 + 1}{2x}$, then x must be

a. -3 b. -2 c. 1 d. none of these

29. If $\sin(\alpha + \beta) = 1$, $\sin(\alpha - \beta) = \frac{1}{2}$, then $\tan(\alpha + 2\beta)\tan(2\alpha + \beta)$ is equal to, $\alpha, \beta \in (0, \pi/2)$

a. 1 b. -1 c. 0 d. none of these

30. Which of the following is not the value of $\sin 27° - \cos 27°$?

a. $-\frac{\sqrt{3-\sqrt{5}}}{2}$ b. $-\frac{\sqrt{5-\sqrt{5}}}{2}$ c. $-\frac{\sqrt{5}-1}{2\sqrt{2}}$ d. none of these

31. If $\operatorname{cosec}\theta - \cot\theta = q$, then the value of $\operatorname{cosec}\theta$ is

a. $q + \frac{1}{q}$ b. $q - \frac{1}{q}$ c. $\frac{1}{2}\left(q + \frac{1}{q}\right)$ d. none of these

32. If $\sin\theta + \cos\theta = \frac{1}{5}$ and $0 \le \theta < \pi$, then $\tan\theta$ is

a. $-4/3$ b. $-3/4$ c. $3/4$ d. $4/3$

33. If $x = \frac{2\sin\theta}{1 + \cos\theta + \sin\theta}$, then $\frac{1 - \cos\theta + \sin\theta}{1 + \sin\theta}$ is equal to

a. $1 + x$ b. $1 - x$ c. x d. $1/x$

34. If $\theta = \pi/4n$, then the value of $\tan\theta\tan 2\theta \cdots \tan(2n-2)\theta\tan(2n-1)\theta$ is

a. -1 b. 1 c. 0 d. 2

35. The value of the expression $\frac{2(\sin 1° + \sin 2° + \sin 3° + \cdots + \sin 89°)}{2(\cos 1° + \cos 2° + \cdots + \cos 44°) + 1}$ equals

a. $\sqrt{2}$ b. $1/\sqrt{2}$ c. $1/2$ d. 1

36. If $\sec\alpha$ and $\operatorname{cosec}\alpha$ are the roots of $x^2 - px + q + 0$, then

a. $p^2 = q(q-2)$ b. $p^2 = q(q+2)$ c. $p^2 + q^2 = 2q$ d. none of these

37. If $\sin x + \sin^2 x = 1$, then the value of $\cos^{12} x + 3\cos^{10} x + 3\cos^8 x + \cos^6 x - 2$ is equal to

a. 0 b. 1 c. -1 d. 2

38. If $\cos(A - B) = 3/5$ and $\tan A \tan B = 2$, then

a. $\cos A \cos B = 1/5$ b. $\sin A \sin B = -2/5$ c. $\cos A \cos B = -1/5$ d. $\sin A \sin B = -1/5$

39. If $(1 + \tan\alpha)(1 + \tan 4\alpha) = 2$, $\alpha \in (0, \pi/16)$ then α is equal to

a. $\frac{\pi}{20}$ b. $\frac{\pi}{30}$ c. $\frac{\pi}{40}$ d. $\frac{\pi}{60}$

40. If $A = \sin 45° + \cos 45°$ and $B = \sin 44° + \cos 44°$, then

a. $A > B$ b. $A < B$ c. $A = B$ d. none of these

41. $\frac{1}{4}\left[\sqrt{3}\cos 23° - \sin 23°\right]$ is equal to

a. $\cos 43°$ b. $\cos 7°$ c. $\cos 53°$ d. none of these

42. If $\cos\theta_1 = 2\cos\theta_2$, then $\tan\frac{\theta_1 - \theta_2}{2}\tan\frac{\theta_1 + \theta_2}{2}$ is equal to

a. $\frac{1}{3}$ b. $-\frac{1}{3}$ c. 1 d. -1

43. Value of $\frac{3 + \cot 80° \cot 20°}{\cot 80° + \cot 20°}$ is equal to

a. $\cot 20°$ b. $\tan 50°$ c. $\cot 50°$ d. $\cot\sqrt{20°}$

44. If $\tan\beta = 2\sin\alpha\sin\gamma\,\mathrm{cosec}(\alpha + \gamma)$, then $\cot\alpha$, $\cot\beta$, $\cot\gamma$ are in

a. A.P. b. G.P. c. H.P. d. none of these

45. In triangle ABC, if $\sin A\cos B = \frac{1}{4}$ and $3\tan A = \tan B$, then $\cot^2 A$ is equal to

a. 2 b. 3 c. 4 d. 5

46. $\tan 100° + \tan 125° + \tan 100° \tan 125°$ is equal to

a. 0 b. 1/2 c. -1 d. 1

47. $\tan 20° + \tan 40° + \sqrt{3}\tan 20° \tan 40°$ is equal to

a. $\frac{1}{\sqrt{3}}$ b. $\sqrt{3}$ c. $-\frac{1}{\sqrt{3}}$ d. $-\sqrt{3}$

48. $\frac{\sqrt{2} - \sin\alpha - \cos\alpha}{\sin\alpha - \cos\alpha}$ is equal to

a. $\sec\left(\frac{\alpha}{2} - \frac{\pi}{8}\right)$ b. $\cos\left(\frac{\pi}{8} - \frac{\alpha}{2}\right)$ c. $\tan\left(\frac{\alpha}{2} - \frac{\pi}{8}\right)$ d. $\cot\left(\frac{\alpha}{2} - \frac{\pi}{2}\right)$

49. If $\sin\theta_1 - \sin\theta_2 = a$ and $\cos\theta_1 + \cos\theta_2 = b$, then

a. $a^2 + b^2 \ge 4$ b. $a^2 + b^2 \le 4$ c. $a^2 + b^2 \ge 3$ d. $a^2 + b^2 \le 2$

50. If $\frac{1 + \sin 2x}{1 - \sin 2x} = \cot^2(a + x)\ \forall x \in R \sim \left(n\pi + \frac{\pi}{4}\right)$, $n \in N$, then a can be

a. $\frac{\pi}{4}$ b. $\frac{\pi}{2}$ c. $\frac{3\pi}{4}$ d. none of these

51. If $\tan\alpha$ is equal to the integral solution of the inequality $4x^2 - 16x + 15 < 0$ and $\cos\beta$ is equal to the slope of the bisector of the first quadrant, then $\sin(\alpha+\beta)\sin(\alpha-\beta)$ is equal to

a. $\frac{3}{5}$ b. $\frac{3}{5}$ c. $\frac{2}{\sqrt{5}}$ d. $\frac{4}{5}$

52. If $\frac{\cos(x-y)}{\cos(x+y)} + \frac{\cos(z+t)}{\cos(z-t)} = 0$, then the value of $\tan x \tan y \tan z \tan t$ is equal to

a. 1 b. -1 c. 2 d. -2

53. Let $f(n) = 2\cos nx\ \forall\ n \in N$, then $f(1)f(n+1) - f(n)$ is equal to

a. $f(n+3)$ b. $f(n+2)$ c. $f(n+1)f(2)$ d. $f(n+2)f(2)$

54. If in triangle ABC, $\sin A \cos B = 1/4$ and $3\tan A = \tan B$, then the triangle is

a. right angled b. equilateral c. isosceles d. none of these

55. If A and B are acute positive angles satisfying the equations $3\sin^2 A + 2\sin^2 B = 1$ and $3\sin 2A - 2\sin 2B = 0$, then $A + 2B$ is equal to

a. π b. $\frac{\pi}{2}$ c. $\frac{\pi}{4}$ d. $\frac{\pi}{6}$

56. Let $f(\theta) = \frac{\cot\theta}{1+\cot\theta}$ and $\alpha+\beta = \frac{5\pi}{4}$, then the value $f(\alpha)f(\beta)$ is

a. $\frac{1}{2}$ b. $-\frac{1}{2}$ c. 2 d. none of these

57. If $y = (1+\tan A)(1-\tan B)$ where $A - B = \frac{\pi}{4}$, then $(y+1)^{y+1}$ is equal to

a. 9 b. 4 c. 27 d. 81

58. If $\sin(y+z-x)$, $\sin(z+x-y)$, $\sin(x+y-z)$ are in A.P., then $\tan x$, $\tan y$, $\tan z$ are in

a. A.P. b. GP. c. H.P. d. none of these

59. If $\cos\alpha + \cos\beta = 0 = \sin\alpha + \sin\beta$, then $\cos 2\alpha + \cos 2\beta$ is equal to

a. $-2\sin(\alpha+\beta)$ b. $-2\cos(\alpha+\beta)$ c. $2\sin(\alpha+\beta)$ d. $2\cos(\alpha+\beta)$

60. If $x_1, x_2, x_3, \ldots, x_n$ are in A.P. whose common difference is α, then the value of $\sin\alpha(\sec x_1 \sec x_2 + \sec x_2 \sec x_3 + \cdots + \sec x_{n-1}\sec x_n)$ is

a. $\frac{\sin(n-1)\alpha}{\cos x_1 \cos x_n}$ b. $\frac{\sin n\alpha}{\cos x_1 \cos x_n}$ c. $\sin(n-1)\alpha \cos x_1 \cos x_n$ d. $\sin n\alpha \cos x_1 \cos x_n$

61. If $\tan\frac{\pi}{9}$, x and $\tan\frac{5\pi}{18}$ are in A.P. and $\tan\frac{\pi}{9}$, y and $\tan\frac{7\pi}{18}$ are also in A.P., then

a. $2x = y$ b. $x > 2$ c. $x = y$ d. none of these

62. Let $x = \sin 1°$, then the value of the expression

$$\frac{1}{\cos 0°\cdot\cos 1°} + \frac{1}{\cos 1°\cdot\cos 2°} + \frac{1}{\cos 2°\cdot\cos 3°} + \cdots + \frac{1}{\cos 44°\cdot\cos 45°}$$ is equal to

a. x b. $1/x$ c. $\sqrt{2}/x$ d. $x/\sqrt{2}$

63. Let α and, β be such that $\pi < \alpha - \beta < 3\pi$. If $\sin\alpha + \sin\beta = -\frac{21}{65}$ and $\cos\alpha + \cos\beta = -\frac{17}{65}$, then the value of $\cos\frac{\alpha-\beta}{2}$ is

a. $-\dfrac{3}{\sqrt{130}}$ b. $\dfrac{3}{\sqrt{130}}$ c. $\dfrac{6}{65}$ d. $-\dfrac{6}{65}$

64. If $\dfrac{\sin(x+y)}{\sin(x-y)}=\dfrac{a+b}{a-b}$, then $\dfrac{\tan x}{\tan y}$ is equal to

a. $\dfrac{b}{a}$ b. $\dfrac{a}{b}$ c. ab d. none of these

65. $\dfrac{\sin 3\theta+\sin 5\theta+\sin 7\theta+\sin 9\theta}{\cos 3\theta+\cos 5\theta+\cos 7\theta+\cos 9\theta}$ is equal to

a. $\tan 3\theta$ b. $\cot 3\theta$ c. $\tan 6\theta$ d. $\cot 6\theta$

66. If x, y, z are in A.P, then $\dfrac{\sin x-\sin z}{\cos z-\cos x}$ is equal to

a. $\tan y$ b. $\cot y$ c. $\sin y$ d. $\cos y$

67. If $\cos 25°+\sin 25°=p$, then $\cos 50°$ is

a. $\sqrt{2-p^2}$ b. $-\sqrt{2-p^2}$ c. $p\sqrt{2-p^2}$ d. $-p\sqrt{2-p^2}$

68. $\dfrac{\sin^2 A-\sin^2 B}{\sin A\cos A-\sin B\cos B}$ is equal to

a. $\tan(A-B)$ b. $\tan(A+B)$ c. $\cot(A-B)$ d. $\cot(A+B)$

69. If $\tan A=\dfrac{1-\cos B}{\sin B}$, then $\tan 2A$ is

a. $\tan 2A=\tan B$ b. $\tan 2A=\tan^2 B$

c. $\tan 2A=\tan^2 B+2\tan B$ d. none of these

70. If $a+b=3-\cos 4\theta$ and $a-b=4\sin 2\theta$, then ab is always less than or equal to

a. $\dfrac{1}{2}$ b. 1 c. $\dfrac{2}{3}$ d. $\dfrac{3}{4}$

71. The value of $\cos^2 10°-\cos 10°\cos 50°+\cos^2 50°$ is equal to

a. $\dfrac{4}{3}$ b. $\dfrac{1}{3}$ c. $\dfrac{3}{4}$ d. 3

72. The numerical value of $\tan 20°\tan 80°\cot 50°$ is equal to

a. $\sqrt{3}$ b. $\dfrac{1}{\sqrt{3}}$ c. $2\sqrt{3}$ d. $\dfrac{1}{2\sqrt{3}}$

73. If $\tan^2\theta=2\tan^2\phi+1$, then $\cos 2\theta+\sin^2\phi$ equals

a. -1 b. 0 c. 1 d. none of these

74. The value of $\cot 70°+4\cos 70°$ is

a. $\dfrac{1}{\sqrt{3}}$ b. $\sqrt{3}$ c. $2\sqrt{3}$ d. $\dfrac{1}{2}$

75. If x_1 and x_2 are two distinct roots of the equation $a\cos x+b\sin x=c$, then $\tan\dfrac{x_1+x_2}{2}$ is equal to

a. $\dfrac{a}{b}$ b. $\dfrac{b}{a}$ c. $\dfrac{c}{a}$ d. $\dfrac{a}{c}$

76. Given that $(1+\sqrt{1+x})\tan y = 1+\sqrt{1-x}$. Then $\sin 4y$ is equal to

a. $4x$ b. $2x$ c. x d. none of these

77. If $\cos x = \tan y$, $\cos y = \tan z$, $\cos z = \tan x$, then the value of $\sin x$ is

a. $2\cos 18°$ b. $\cos 18°$ c. $\sin 18°$ d. $2\sin 18°$

78. If $\sin 2\theta = \cos 3\theta$ and θ is an acute angle, then $\sin\theta$ equals

a. $\frac{\sqrt{5}-1}{4}$ b. $-\left(\frac{\sqrt{5}-1}{4}\right)$ c. $\frac{\sqrt{5}+1}{4}$ d. $\frac{-\sqrt{5}-1}{4}$

79. If θ_1 and θ_2 are two values lying in $[0, 2\pi]$ for which $\tan\theta = \lambda$, then $\tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2}$ is equal to

a. 0 b. -1 c. 2 d. 1

80. If $\tan\theta = \sqrt{n}$ where $n \in N, \geq 2$, then $\sec 2\theta$ is always

a. a rational number b. an irrational number c. a positive integer d. a negative integer

81. If $\sin x + \cos x = \frac{\sqrt{7}}{2}$ where $x \in A$, then $\tan\frac{x}{2}$ is equal to

a. $\frac{3-\sqrt{7}}{3}$ b. $\frac{\sqrt{7}-2}{3}$ c. $\frac{4-\sqrt{7}}{4}$ d. none of these

82. The value of $\sin^2\frac{\pi}{8}+\sin^2\frac{3\pi}{8}+\sin^2\frac{5\pi}{8}+\sin^2\frac{7\pi}{8}$ is

a. 1 b. 2 c. $1\frac{1}{8}$ d. $2\frac{1}{8}$

83. If $x \in \left(\pi, \frac{3\pi}{2}\right)$, then $4\cos^2\left(\frac{\pi}{4}-\frac{x}{2}\right)+\sqrt{4\sin^4 x+\sin^2 2x}$ is always equal to

a. 1 b. 2 c. -2 d. none of these

84. $\cos^3 x \sin 2x = \sum_{x=0}^{n} a_r \sin(rx)\ \forall x \in R$, then

a. $n=5, a_1=1/2$ b. $n=5, a_1=1/4$ c. $n=5, a_2=1/8$ d. $n=5, a_2=1/4$

85. The value of $\cos 2(\theta+\phi)+4\cos(\theta+\phi)\sin\theta\sin\phi+2\sin^2\phi$ is

a. independent of θ only b. independent of ϕ only
c. independent of both θ and ϕ d. dependent on θ and ϕ

86. If $\cos 2B = \frac{\cos(A+C)}{\cos(A-C)}$, then $\tan A, \tan B, \tan C$ are in

a. A.P. b. G.P. c. H.P. d. none of these

87. If $\cos x = \frac{2\cos y-1}{2-\cos y}$, where $x, y \in (0, \pi)$, then $\tan\frac{x}{2}\cot\frac{y}{2}$ is equal to

a. $\sqrt{2}$ b. $\sqrt{3}$ c. $\frac{1}{\sqrt{2}}$ d. $\frac{1}{\sqrt{3}}$

88. If $\tan x = b/a$, then $\sqrt{(a+b)/(a-b)}+\sqrt{(a-b)/(a+b)}$ is equal to

a. $2\sin x/\sqrt{\sin 2x}$ b. $2\cos x/\sqrt{\cos 2x}$ c. $2\cos x/\sqrt{\sin 2x}$ d. $2\sin x/\sqrt{\cos 2x}$

89. If α is a root of $25\cos^2\theta + 5\cos\theta - 12 = 0$, $\frac{\pi}{2} < \alpha < \pi$, then $\sin 2\alpha$ is equal to

a. $\frac{24}{25}$ b. $-\frac{24}{25}$ c. $\frac{13}{18}$ d. $-\frac{13}{18}$

90. The value of $\tan 9° - \tan 27° - \tan 63° + \tan 81°$ is

a. 2 b. 3 c. 4 d. none of these

91. If $\sin^{-1} a + \sin^{-1} b + \sin^{-1} c = \pi$, then $a\sqrt{1-a^2} + b\sqrt{1-b^2} + c\sqrt{1-c^2}$ is equal to

a. $a+b+c$ b. $a^2b^2c^2$ c. $2abc$ d. $4abc$

92. If $A + B + C = 3\pi/2$, then $\cos 2A + \cos 2B + \cos 2C$ is equal to

a. $1 - 4\cos A\cos B\cos C$ b. $4\sin A\sin B\sin C$

c. $1 + 2\cos A\cos B\cos C$ d. $1 - 4\sin A\sin B\sin C$

93. In triangle ABC, $\tan\frac{A}{2}$, $\tan\frac{B}{2}$, $\tan\frac{C}{2}$ are in H.P., then the value of $\cot\frac{A}{2} \times \cot\frac{C}{2}$ is equal to

a. 1 b. 2 c. 3 d. 4

94. In any triangle ABC, $\sin^2 A - \sin^2 B + \sin^2 C$ is always equal to

a. $2\sin A\sin B\cos C$ b. $2\sin A\cos B\sin C$

c. $2\sin A\cos B\cos C$ d. $2\sin A\sin B\sin C$

95. If $\tan^2\alpha\tan^2\beta + \tan^2\beta\tan^2\gamma + \tan^2\gamma\tan^2\alpha + 2\tan^2\alpha\tan^2\beta\tan^2\gamma = 1$, then the value of $\sin^2\alpha + \sin^2\beta + \sin^2\gamma$ is

a. 3 b. 2 c. 1 d. none of these

96. In triangle ABC, $\frac{\sin A + \sin B + \sin C}{\sin A + \sin B - \sin C}$ is equal to

a. $\tan\frac{A}{2}\cot\frac{B}{2}$ b. $\cot\frac{A}{2}\tan\frac{B}{2}$ c. $\cot\frac{A}{2}\cot\frac{B}{2}$ d. $\tan\frac{A}{2}\tan\frac{B}{2}$

97. $\frac{\sin 2A + \sin 2B + \sin 2C}{\sin A + \sin B + \sin C}$ is equal to

a. $8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$ b. $8\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

c. $8\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}$ d. $8\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$

98. If $\cos^2 A + \cos^2 B + \cos^2 C = 1$, then ΔABC is

a. equilateral b. isosceles c. right angled d. none of these

99. In triangle ABC, $\tan A + \tan B + \tan C = 6$ and $\tan A\tan B = 2$, then the values of $\tan A$, $\tan B$, $\tan C$ are

a. 1, 2, 3 b. 3, 2/3, 7/3 c. 4, 1/2, 3/2 d. none of these

100. The value of $\tan 6° \tan 42° \tan 66° \tan 78°$ is

a. 1 b. 1/2 c. 1/4 d. 1/8

101. If $0 < \alpha < \frac{\pi}{6}$, then $\alpha(\operatorname{cosec}\alpha)$ is

a. less than $\pi/6$ b. greater than $\pi/6$ c. less than $\pi/3$ d. greater than $\pi/3$

102. If θ is eliminated from the equations $x = a\cos(\theta - \alpha)$ and $y = b\cos(\theta - \beta)$, then $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2xy}{ab}\cos(\alpha - \beta)$ is equal to

a. $\sec^2(\alpha - \beta)$ b. $\operatorname{cosec}^2(\alpha - \beta)$ c. $\cos^2(-\beta)$ d. $\sin^2(\alpha - \beta)$

103. If $\left|\cos\theta\left\{\sin\theta + \sqrt{\sin^2\theta + \sin^2\alpha}\right\}\right| \le k$, then the value of k is

a. $\sqrt{1+\cos^2\alpha}$ b. $\sqrt{1+\sin^2\alpha}$ c. $\sqrt{2+\sin^2\alpha}$ d. $\sqrt{2+\cos^2\alpha}$

104. If $\sin\theta_1 \sin\theta_2 - \cos\theta_1 \cos\theta_2 + 1 = 0$, then the value of $\tan(\theta_1/2)\cot(\theta_2/2)$ is always equal to

a. -1 b. 1 c. 2 d. -2

105. The numerical value of $\tan\frac{\pi}{3} + 2\tan\frac{2\pi}{3} + 4\tan\frac{4\pi}{3} + 8\tan\frac{8\pi}{3}$ is equal to

a. $-5\sqrt{3}$ b. $-5/\sqrt{3}$ c. $5\sqrt{3}$ d. $5/\sqrt{3}$

106. $\tan^6\frac{\pi}{9} - 33\tan^4\frac{\pi}{9} + 27\tan^2\frac{\pi}{9}$ is equal to

a. 0 b. $\sqrt{3}$ c. 3 d. 9

107. If $\cos x + \cos y - \cos(x+y) = \frac{3}{2}$, then

a. $x+y=0$ b. $x=2y$ c. $x=y$ d. $2x=y$

108. If $a\sin x + b\cos(x+\theta) + b\cos(x-\theta) = d$, then the minimum value of $|\cos\theta|$ is equal to

a. $\frac{1}{2|b|}\sqrt{d^2-a^2}$ b. $\frac{1}{2|a|}\sqrt{d^2-a^2}$ c. $\frac{1}{2|d|}\sqrt{d^2-a^2}$ d. none of these

109. If $\frac{\sin x}{\sin y} = \frac{1}{2}$, $\frac{\cos x}{\cos y} = \frac{3}{2}$ where $x, y \in \left(0, \frac{\pi}{2}\right)$, then the value of $\tan(x+y)$ is equal to

a. $\sqrt{13}$ b. $\sqrt{14}$ c. $\sqrt{17}$ d. $\sqrt{15}$

110. If $x \in (\pi, 2\pi)$ and $\frac{\sqrt{1+\cos x}+\sqrt{1-\cos x}}{\sqrt{1+\cos x}-\sqrt{1-\cos x}} = \cot\left(a + \frac{x}{2}\right)$, then a is equal to

a. $\frac{\pi}{4}$ b. $\frac{\pi}{2}$ c. $\frac{\pi}{3}$ d. none of these

111. If $\tan x = n\tan y$, $n \in R^+$, then the maximum value of $\sec^2(x-y)$ is equal to

a. $\frac{(n+1)^2}{2n}$ b. $\frac{(n+1)^2}{n}$ c. $\frac{(n+1)^2}{2}$ d. $\frac{(n+1)^2}{4n}$

112. If $\cot^2 x = \cot(x-y)\cot(x-z)$, then $\cot 2x$ is equal to (where $x \ne \pm\pi/4$)

a. $\frac{1}{2}(\tan y + \tan z)$ b. $\frac{1}{2}(\cot y + \cot z)$ c. $\frac{1}{2}(\sin y + \sin z)$ d. none of these

113. If A, B, C are acute positive angles such that $A + B + C = \pi$ and $\cot A\cot B\cot C = k$, then

a. $K \le \frac{1}{3\sqrt{3}}$ b. $K \ge \frac{1}{3\sqrt{3}}$ c. $K < \frac{1}{9}$ d. $K > \frac{1}{3}$

114. If $u=\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}+\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}$, then the difference between the maximum and minimum values of u^2 is given by

a. $2(a^2+b^2)$ b. $2\sqrt{a^2+b^2}$ c. $(a+b)^2$ d. $(a-b)^2$

115. If $(\sin x+\cos x)^2+k\sin x\cos x=1$ holds $\forall x\in R$, then the value of k equals

a. 2 b. 2 c. –2 d. 3

116. The range of k for which the inequality $k\cos^2x-k\cos x+1\geq 0\ \forall x\in(-\infty,\infty)$, is

a. $k<\frac{-1}{2}$ b. $\frac{-1}{2}\leq k\leq 4$ c. $k>4$ d. $\frac{1}{2}\leq k\leq 5$

117. The minimum vertical distance between the graphs of $y=2+\sin x$ and $y=\cos x$ is

a. 2 b. 1 c. $\sqrt{2}$ d. $2-\sqrt{2}$

118. If $\theta=3\alpha$ and $\sin\theta=\frac{a}{\sqrt{a^2+b^2}}$ The value of the expression $a\operatorname{cosec}\alpha-b\sec\alpha$ is

a. $\frac{a}{\sqrt{a^2+b^2}}$ b. $2\sqrt{a^2+b^2}$ c. $a+b$ d. none of these

119. If the equation $\cot^4x-2\operatorname{cosec}^2x+a^2=0$ has at least one solution, then the sum of all possible integral values of 'a' is equal to

a. 4 b. 3 c. 2 d. 0

120. If the inequality $\sin^2x+a\cos x+a^2>1+\cos x$ holds for any $x\in R$ then the largest negative integral value of 'a' is

a. –4 b. –3 c. –2 d. –1

121. In triangle ABC if angle C is 90° and area of triangle is 30 sq. units, then the minimum possible value of the hypotenuse c is equal to

a. $30\sqrt{2}$ b. $60\sqrt{2}$ c. $120\sqrt{2}$ d. $\sqrt{30}$

122. The distance between the two parallel lines is 1 unit. A point 'A' is chosen to lie between the lines at a distance 'd' from one of them. Triangle ABC is equilateral with B on one line and C on the other parallel line. The length of the side of the equilateral triangle is

a. $\frac{2}{3}\sqrt{d^2+d+1}$ b. $2\sqrt{\frac{d^2-d+1}{3}}$ c. $2\sqrt{d^2-d+1}$ d. $\sqrt{d^2-d+1}$

123. Given that a, b, c are the sides of a triangle ABC which is right angled at C, then the minimum value of $\left(\frac{c}{a}+\frac{c}{b}\right)^2$ is

a. 0 b. 4 c. 6 d. 8

124. Let $y=(\sin x+\operatorname{cosec}x)^2+(\cos x+\sec x)^2+(\cos x+\sec x)^2$, then the minimum value of y, $\forall x\in R$, is

a. 7 b. 3 c. 9 d. 0

Multiple Correct Answers Type

Solutions on page 2.116

Each question has four choices a, b, c, and d, out of which one or more answers are correct.

1. If $\cos\beta$ is the geometric mean between $\sin\alpha$ and $\cos\alpha$, where $0<\alpha,\beta<\pi/2$, then $\cos 2\beta$ is equal to

a. $-2\sin^2\left(\frac{\pi}{4}-\alpha\right)$ b. $-2\cos^2\left(\frac{\pi}{4}+\alpha\right)$ c. $2\sin^2\left(\frac{\pi}{4}+\alpha\right)$ d. $2\cos^2\left(\frac{\pi}{4}-\alpha\right)$

2. If $0\leq\theta\leq\pi$ and $81^{\sin^2\theta}+81^{\cos^2\theta}=30$, then θ is

a. 30° b. 60° c. 120° d. 150°

3. Suppose $ABCD$ (in order) is a quadrilateral inscribed in a circle. Which of the following is/are always true?
 a. $\sec B = \sec D$ b. $\cot A + \cot C = 0$ c. $\text{cosec } A = \text{cosec } C$ d. $\tan B + \tan D = 0$
4. Which of the following statements are always correct (where Q denotes the set of rationals)?
 a. $\cos 2\theta \in Q$ and $\sin 2\theta \in Q \Rightarrow \tan \theta \in Q$ (if defined)
 b. $\tan \theta \in Q \Rightarrow \sin 2\theta, \cos 2\theta$ and $\tan 2\theta \in Q$ (if defined)
 c. if $\sin \theta \in Q$ and $\cos \theta \in Q \Rightarrow \tan 3\theta \in Q$ (if defined)
 d. if $\sin \theta \in Q \Rightarrow \cos 3\theta \in Q$
5. Which of the following quantities are rational?
 a. $\sin\left(\frac{11\pi}{12}\right) \sin\left(\frac{5\pi}{12}\right)$ b. $\text{cosec}\left(\frac{9\pi}{10}\right) \sec\left(\frac{4\pi}{5}\right)$
 c. $\sin^4\left(\frac{\pi}{8}\right) + \cos^4\left(\frac{\pi}{8}\right)$ d. $\left(1+\cos\frac{2\pi}{9}\right)\left(1+\cos\frac{4\pi}{9}\right)\left(1+\cos\frac{8\pi}{9}\right)$
6. In which of the following sets the inequality $\sin^6 x + \cos^6 x > 5/8$ holds good?
 a. $(-\pi/8, \pi/8)$ b. $(3\pi/8, 5\pi/8)$ c. $(\pi/4, 3\pi/4)$ d. $(7\pi/8, 9\pi/8)$
7. Which of the following inequalities hold true in any triangle ABC?
 a. $\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \le \frac{1}{8}$ b. $\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} \le \frac{3\sqrt{3}}{8}$
 c. $\sin^2\frac{A}{2} + \sin^2\frac{B}{2} + \sin^2\frac{C}{2} < \frac{3}{4}$ d. $\cos^2\frac{A}{2} + \cos^2\frac{B}{2} + \cos^2\frac{C}{2} \le \frac{9}{4}$
8. For $\alpha = \pi/7$ which of the following hold(s) good?
 a. $\tan\alpha \tan 2\alpha \tan 3\alpha = \tan 3\alpha - \tan 2\alpha - \tan\alpha$
 b. $\text{cosec }\alpha = \text{cosec } 2\alpha + \text{cosec } 4\alpha$
 c. $\cos\alpha - \cos 2\alpha + \cos 3\alpha = 1/2$
 d. $8\cos\alpha\cos 2\alpha\cos 4\alpha = 1$
9. Which of the following is/are correct?
 a. $(\tan x)^{\ln(\sin x)} > (\cot x)^{\ln(\sin x)}, \forall x \in (0, \pi/4)$ b. $4^{\ln \text{cosec } x} < 5^{\ln \text{cosec } x}, \forall x \in (0, \pi/2)$
 c. $(1/2)^{\ln(\cos x)} < (1/3)^{\ln(\cos x)}, \forall x \in (0, \pi/2)$ d. $2^{\ln(\tan x)} > 2^{\ln(\sin x)}, \forall x \in (0, \pi/2)$
10. Which of the following do/does not reduce to unity?
 a. $\frac{\sin(180°+A)}{\tan(180°+A)}\frac{\cot(90°+A)}{\tan(90°+A)}\frac{\cos(360°-A)\text{ cosec } A}{\sin(-A)}$
 b. $\frac{\sin(-A)}{\sin(180°+A)} - \frac{\tan(90°+A)}{\cot A} + \frac{\cos A}{\sin(90°+A)}$
 c. $\frac{\sin 24°\cos 6° - \sin 6°\cos 24°}{\sin 21°\cos 39° - \cos 51°\sin 69°}$
 d. $\frac{\cos(90°+A)\sec(-A)\tan(180°-A)}{\sec(360°+A)\sin(180°+A)\cot(90°-A)}$

11. Which of the following identities, wherever defined, hold(s) good?

a. $\cot\alpha - \tan\alpha = 2\cot 2\alpha$ **b.** $\tan(45^\circ + \alpha) - \tan(45^\circ - \alpha) = 2\operatorname{cosec} 2\alpha$

c. $\tan(45^\circ + \alpha) + \tan(45^\circ - \alpha) = 2\sec 2\alpha$ **d.** $\tan\alpha + \cot\alpha = 2\tan 2\alpha$

12. A circle centred at O has radius 1 and contains the point A. Segment AB is tangent to the circle at A and $\angle AOB = \theta$. If point C lies on OA and BC bisects the angle ABO, then OC equals

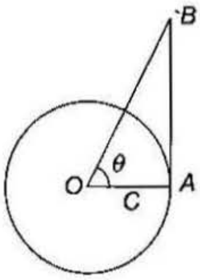


Fig. 2.36

a. $\sec\theta(\sec\theta - \tan\theta)$ **b.** $\dfrac{\cos^2\theta}{1+\sin\theta}$ **c.** $\dfrac{1}{1+\sin\theta}$ **d.** $\dfrac{1-\sin\theta}{\cos^2\theta}$

13. The expression $(\tan^4 x + 2\tan^2 x + 1)\cos^2 x$ when $x = \pi/12$ can be equal to

a. $4(2-\sqrt{3})$ **b.** $4(\sqrt{2}+1)$ **c.** $16\cos^2 \pi/12$ **d.** $16\sin^2 \pi/12$

14. Let α, β and γ be some angles in the first quadrant satisfying $\tan(\alpha+\beta) = 15/8$ and $\operatorname{cosec}\gamma = 17/8$, then which of the following hold(s) good?

a. $\alpha+\beta+\gamma = \pi$ **b.** $\cot\alpha\cot\beta\cot\gamma = \cot\alpha + \cot\beta + \cot\gamma$

c. $\tan\alpha + \tan\beta + \tan\gamma = \tan\alpha\tan\beta\tan\gamma$ **d.** $\tan\alpha\tan\beta + \tan\beta\tan\gamma + \tan\gamma\tan\alpha = 1$

15. $(a+2)\sin\alpha + (2a-1)\cos\alpha = (2a+1)$ if $\tan\alpha$ is

a. 3/4 **b.** 4/3 **c.** $2a/(a^2+1)$ **d.** $2a/(a^2-1)$

16. Let $f(x) = \log(\log_{1/3}(\log_7(\sin x + a)))$ be defined for every real value of x, then the possible value of a is

a. 3 **b.** 4 **c.** 5 **d.** 6

17. If $b > 1$, $\sin t > 0$, $\cos t > 0$ and $\log_b(\sin t) = x$, then $\log_b(\cos t)$ is equal to

a. $\dfrac{1}{2}\log_b(1-b^{2x})$ **b.** $2\log(1-b^{x/2})$

c. $\log_b\sqrt{1-b^{2x}}\ \log_b(1-b^{2x})$ **d.** $\sqrt{1-x^2}$

18. The equation $x^3 - \dfrac{3}{4}x = -\dfrac{\sqrt{3}}{8}$ is satisfied by

a. $x = \cos\left(\dfrac{5\pi}{18}\right)$ **b.** $x = \cos\left(\dfrac{7\pi}{18}\right)$ **c.** $x = \cos\left(\dfrac{23\pi}{18}\right)$ **d.** $x = \cos\left(\dfrac{17\pi}{18}\right)$

19. If $\sin(x + 20°) = 2 \sin x \cos 40°$ where $x \in (0, \pi/2)$ then which of the following hold(s) good?

a. $\cos 2x = 1/2$ b. $\text{cosec } 4x = 2$ c. $\sec \frac{x}{2} = \sqrt{6} - \sqrt{2}$ d. $\tan \frac{x}{2} = (2 - \sqrt{3})$

Reasoning Type

Solutions on page 2.122

Each question has four choices a, b, c, and d, out of which *only one* is correct. Each question contains STATEMENT 1 and STATEMENT 2.

a. Both the statements are TRUE and STATEMENT 2 is the correct explanation of STATEMENT 1

b. Both the statements are TRUE but STATEMENT 2 is NOT the correct explanation of STATEMENT 1

c. STATEMENT 1 is TRUE and STATEMENT 2 is FALSE

d. STATEMENT 1 is FALSE and STATEMENT 2 is TRUE

1. **Statement 1:** If $x + y + z = xyz$, then at most one of the numbers can be negative.
   Statement 2: In a triangle ABC, $\tan A + \tan B + \tan C = \tan A \tan B \tan C$ and there can be at most one obtuse angle in a triangle.
2. **Statement 1:** $\cos 1 < \cos 7$.
   Statement 2: $1 < 7$.
3. **Statement 1:** $\tan 4 < \tan 7.5$.
   Statement 2: $\tan x$ is always an increasing function.
4. **Statement 1:** $\cos 1 < \sin 1$.
   Statement 2: In the first quadrant, cosine decreases but sine increases.
5. **Statement 1:** If $f(\theta) = (\sin \theta + \text{cosec } \theta)^2 + (\cos \theta + \sec \theta)^2$, then the minimum value of $f(\theta)$ is 9.
   Statement 2: Maximum value of $\sin 2\theta$ is 1.
6. **Statement 1:** If $\sin^2 \theta_1 + \sin^2 \theta_2 + \cdots + \sin^2 \theta_n = 0$, then the different sets of values of $(\theta_1, \theta_2, \cdots, \theta_n)$ for which $\cos \theta_1 + \cos \theta_2 + \cdots + \cos \theta_n = n - 4$ is $n(n-1)$.
   Statement 2: If $\sin^2 \theta_1 + \sin^2 \theta_2 + \cdots + \sin^2 \theta_n = 0$, then $\cos \theta_1, \cos \theta_2, \cdots, \cos \theta_n = \pm 1$.
7. **Statement 1:** The minimum value of $27^{\cos 2x} 81^{\sin 2x}$ is $\frac{1}{243}$.
   Statement 2: The minimum value of $a \cos \theta + b \sin \theta$ is $-\sqrt{a^2 + b^2}$.
8. **Statement 1:** If A, B, C are the angles of a triangle such that angle A is obtuse, then $\tan B \tan C > 1$.
   Statement 2: In any triangle, $\tan A = \frac{\tan B + \tan C}{\tan B \tan C - 1}$.
9. **Statement 1:** $\tan 5°$ is an irrational number.
   Statement 2: $\tan 15°$ is an irrational number.
10. **Statement 1:** $\sin \pi/18$ is a root of $8x^3 - 6x + 1 = 0$.
    Statement 2: For any $\theta \in R$, $\sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta$.
11. Let f be any one of the six trigonometric functions. Let $A, B \in R$ satisfying $f(2A) = f(2B)$.
    Statement 1: $A = n\pi + B$, for some $n \in Z$.
    Statement 2: 2π is one of the period of f.

12. **Statement 1:** $\sin 3 < \sin 1 < \sin 2$.
Statement 2: $\sin x$ is positive in first and second quadrants.

13. **Statement 1:** The maximum and minimum values of the function $f(x) = \dfrac{1}{3\sin x + 4\cos x - 2}$ do not exist.
Statement 2: The given function is an unbounded function.

14. **Statement 1:** The minimum value of the expression $\sin\alpha + \sin\beta + \sin\gamma$ is negative, where α, β, γ are real numbers such that $\alpha + \beta + \gamma = \pi$.
Statement 2: If α, β, γ are the angles of a triangle, then $\sin\alpha + \sin\beta + \sin\gamma = 4\cos\alpha/2 \cos\beta/2 \cos\gamma/2$.

15. **Statement 1:** If in a triangle, $\sin^2 A + \sin^2 B + \sin^2 C = 2$ then one of the angles must be $90°$.
Statement 2: In any triangle $\sin^2 A + \sin^2 B + \sin^2 C = 2 + 2\cos A\cos B\cos C$.

16. **Statement 1:** In a triangle, the least value of the sum of cosines of its angles is unity.
Statement 2: $\cos A + \cos B + \cos C = 1 + 4\sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2}$, if A, B, C are the angles of a triangle.

17. Let α, β, and γ satisfy $0 < \alpha < \beta < \gamma < 2\pi$ and $\cos(x+\alpha) + \cos(x+\beta) + \cos(x+\gamma) = 0\ \forall x \in R$.
Statement 1: $\gamma - \alpha = \dfrac{2\pi}{3}$.
Statement 2: $\cos\alpha + \cos\beta + \cos\gamma = 0$ and $\sin\alpha + \sin\beta + \sin\gamma = 0$.

18. If $A + B + C = \pi$, then
Statement 1: $\cos^2 A + \cos^2 B + \cos^2 C$ has its minimum value $\dfrac{3}{4}$.
Statement 2: Maximum value of $\cos A\cos B\cos C$ is $\dfrac{1}{8}$.

19. **Statement 1:** If $xy + yz + zx = 1$ where $x, y, z \in R^+$,
then $\dfrac{x}{1+x^2} + \dfrac{y}{1+y^2} + \dfrac{z}{1+z^2} = \dfrac{2}{\sqrt{(1+x^2)(1+y^2)(1+z^2)}}$.
Statement 2: In a triangle ABC, $\sin 2A + \sin 2B + \sin 2C = 4\sin A\sin B\sin C$.

20. **Statement 1:** In any triangle ABC,
$\ln\left(\cot\dfrac{A}{2} + \cot\dfrac{B}{2} + \cot\dfrac{C}{2}\right) = \ln\cot\dfrac{A}{2} + \ln\cot\dfrac{B}{2} + \ln\cot\dfrac{C}{2}$.
Statement 2: $\ln\left(1 + \sqrt{3} + (2+\sqrt{3})\right) = \ln 1 + \ln\sqrt{3} + \ln(2+\sqrt{3})$.

Linked Comprehension Type

Solutions on page 2.126

Based upon each paragraph, three multiple choice questions have to be answered. Each question has 4 choices a, b, c, and d, out of which only one is correct.

For Problems 1 – 3

If $\sin\alpha = A\sin(\alpha+\beta)$, $A \neq 0$, then

1. The value of $\tan\alpha$ is

a. $\dfrac{A\sin\beta}{1 - A\cos\beta}$ b. $\dfrac{A\sin\beta}{1 + A\cos\beta}$ c. $\dfrac{A\cos\beta}{1 - A\sin\beta}$ d. $\dfrac{A\sin\beta}{1 + A\cos\beta}$

2. The value of $\tan\beta$ is

a. $\dfrac{\sin\alpha(1+A\cos\beta)}{A\cos\alpha\cos\beta}$ b. $\dfrac{\sin\alpha(1-A\cos\beta)}{A\cos\alpha\cos\beta}$ c. $\dfrac{\cos\alpha(1-A\sin\beta)}{A\cos\alpha\cos\beta}$ d. $\dfrac{\cos\alpha(1+A\sin\beta)}{A\cos\alpha\cos\beta}$

3. Which of the following is not the value of $\tan(\alpha+\beta)$?

a. $\dfrac{\sin\beta}{\cos\beta-A}$ b. $\dfrac{\sin\alpha\cos\alpha}{A\cos\beta-\sin^2\alpha}$ c. $\dfrac{\sin\alpha\cos\alpha}{A\cos\beta+\sin^2\alpha}$ d. none of these

For Problems 4 – 6

If $\alpha, \beta, \gamma, \delta$ are the solutions of the equation $\tan\left(\theta+\dfrac{\pi}{4}\right)=3\tan 3\theta$, no two of which have equal tangents.

4. The value of $\tan\alpha+\tan\beta+\tan\gamma+\tan\delta$ is

a. 1/3 b. 8/3 c. –8/3 d. 0

5. The value of $\tan\alpha\tan\beta\tan\gamma\tan\delta$ is

a. –1/3 b. –2 c. 0 d. none of these

6. The value of $\dfrac{1}{\tan\alpha}+\dfrac{1}{\tan\beta}+\dfrac{1}{\tan\gamma}+\dfrac{1}{\tan\delta}$ is

a. –8 b. 8 c. 2/3 d. 1/3

For Problems 7 – 9

$\sin\alpha+\sin\beta=\dfrac{1}{4}$ and $\cos\alpha+\cos\beta=\dfrac{1}{3}$

7. The value of $\sin(\alpha+\beta)$ is

a. $\dfrac{24}{25}$ b. $\dfrac{13}{25}$ c. $\dfrac{12}{13}$ d. none of these

8. The value of $\cos(\alpha+\beta)$ is

a. $\dfrac{12}{25}$ b. $\dfrac{7}{25}$ c. $\dfrac{12}{13}$ d. none of these

9. The value of $\tan(\alpha+\beta)$ is

a. $\dfrac{25}{7}$ b. $\dfrac{25}{12}$ c. $\dfrac{25}{13}$ d. $\dfrac{24}{7}$

For Problems 10 – 12

To find the sum $\sin^2\dfrac{2\pi}{7}+\sin^2\dfrac{4\pi}{7}+\sin^2\dfrac{8\pi}{7}$ we follow the following method.

Put $7\theta=2n\pi$, where n is any integer.

Then $\sin 4\theta=\sin(2n\pi-3\theta)=-\sin 3\theta$. (i)

This means that $\sin\theta$ takes the values 0, $\pm\sin(2\pi/7)$, $\pm\sin(4\pi/7)$ and $\pm\sin(8\pi/7)$.

Since $\sin(6\pi/7)=\sin(8\pi/7)$, from equation (1), we now get

$2\sin 2\theta\cos 2\theta=4\sin^3\theta-3\sin\theta$

$\Rightarrow 4\sin\theta\cos\theta(1-2\sin^2\theta)=\sin\theta(4\sin^2\theta-3)$

Rejecting the value $\sin\theta=0$, we get

$4\cos\theta(1-2\sin^2\theta)=4\sin^2\theta-3$

$\Rightarrow \quad 16\cos^2\theta(1-2\sin^2\theta)^2 = (4\sin^2\theta - 3)^2$

$\Rightarrow \quad 16(1-\sin^2\theta)(1-4\sin^2\theta + 4\sin^4\theta) = 16\sin^4\theta - 24\sin^2\theta + 9$

$\Rightarrow \quad 64\sin^6\theta - 112\sin^4\theta - 56\sin^2\theta - 7 = 0$

This is cubic in $\sin^2\theta$ with the roots $\sin^2(2\pi/7)$, $\sin^2(4\pi/7)$ and $\sin^2(8\pi/7)$.

The sum of these roots is $\sin^2\frac{2\pi}{7} + \sin^2\frac{4\pi}{7} + \sin^2\frac{8\pi}{7} = \frac{112}{64} = \frac{7}{4}$.

Now answer the following questions.

10. The value of $\left(\tan^2\frac{\pi}{7} + \tan^2\frac{2\pi}{7} + \tan^2\frac{3\pi}{7}\right)\left(\cot^2\frac{\pi}{7} + \cot^2\frac{2\pi}{7} + \cot^2\frac{3\pi}{7}\right)$ is

    a. 105 b. 35 c. 210 d. none of these

11. The value of $\dfrac{\tan^2\frac{\pi}{7} + \tan^2\frac{2\pi}{7} + \tan^2\frac{3\pi}{7}}{\cot^2\frac{\pi}{7} + \cot^2\frac{2\pi}{7} + \cot^2\frac{3\pi}{7}}$ is

    a. 7 b. 35/3 c. 21/5 d. none of these

12. The value of $\tan^2\frac{\pi}{7}\tan^2\frac{2\pi}{7}\tan^2\frac{3\pi}{7}$ is

    a. -3 b. -7 c. -5 d. none of these

For Problems 13 – 15

An altitude BD and a bisector BE are drawn in the triangle ABC from the vertex B. It is known that the length of side $AC = 1$, and the magnitudes of the angles BEC, ABD, ABE, BAC form an arithmetic progression.

13. The area of circle circumscribing ΔABC is

    a. $\frac{\pi}{8}$ b. $\frac{\pi}{4}$ c. $\frac{\pi}{2}$ d. π

14. Let 'O' be the circumcentre of ΔABC, the radius of the circle inscribed in ΔBOC is

    a. $\frac{1}{8\sqrt{3}}$ b. $\frac{1}{4\sqrt{3}}$ c. $\frac{1}{2\sqrt{3}}$ d. $\frac{1}{2}$

15. Let B' be the image of point B with respect to side AC of ΔABC, then the length BB' is equal to

    a. $\frac{\sqrt{3}}{4}$ b. $\frac{\sqrt{2}}{4}$ c. $\frac{1}{\sqrt{2}}$ d. $\frac{\sqrt{3}}{2}$

Matrix-Match Type

Solutions on page 2.129

Each question contains statements given in two columns which have to be matched.

Statements (a, b, c, d) in column I have to be matched with statements (p, q, r, s) in column II. If the correct matches are a-p, a-s, b-q, b-r, c-p, c-q and d-s, then the correctly bubbled 4 × 4 matrix should be as follows:

	p	q	r	s
a	p	q	r	s
b	p	q	r	s
c	p	q	r	s
d	p	q	r	s

1. If $\cos\theta - \sin\theta = \frac{1}{5}$ where $0 < \theta < \frac{\pi}{2}$

Column I	Column II
a. $(\cos\theta + \sin\theta)/2$	p. $\frac{4}{5}$
b. $\sin 2\theta$	q. $\frac{7}{10}$
c. $\cos 2\theta$	r. $\frac{24}{25}$
d. $\cos\theta$	s. $\frac{7}{25}$

2. For all real values of θ

Column I	Column II
a. $A = \sin^2\theta + \cos^4\theta$	p. $A \in [-1, 1]$
b. $A = 3\cos^2\theta + \sin^4\theta$	q. $A \in \left[\frac{3}{4}, 1\right]$
c. $A = \sin^2\theta - \cos^4\theta$	r. $A \in [2\sqrt{2}, \infty)$
d. $A = \tan^2\theta + 2\cot^2\theta$	s. $A \in [1, 3]$

3. If $\cos\alpha + \cos\beta = 1/2$ and $\sin\alpha + \sin\beta = 1/3$

Column I	Column II
a. $\cos\left(\frac{\alpha+\beta}{2}\right)$	p. $\pm\frac{\sqrt{13}}{12}$
b. $\cos\left(\frac{\alpha-\beta}{2}\right)$	q. $\frac{2}{3}$
c. $\tan\left(\frac{\alpha+\beta}{2}\right)$	r. $\pm\frac{3}{\sqrt{13}}$
d. $\tan\left(\frac{\alpha-\beta}{2}\right)$	s. $\pm\sqrt{\frac{131}{13}}$

4.

Column I	Column II
a. $\sin(410°-A)\cos(400°+A)+\cos(410°-A)\sin(400°+A)$ has the value equal to	p. -1
b. $\dfrac{\cos^2 1° - \cos^2 2°}{2\sin 3° \sin 1°}$ is equal to	q. 0
c. $\sin(-870°)+\text{cosec}(-660°)+\tan(-855°)+2\cot(840°)+\cos(480°)+\sec(900°)$	r. $\dfrac{1}{2}$
d. If $\cos\theta=\dfrac{4}{5}$ where $\theta\in\left(\dfrac{3\pi}{2}, 2\pi\right)$ and $\cos\phi=\dfrac{3}{5}$ where $\phi\in\left(0, \dfrac{\pi}{2}\right)$, then $\cos(\theta-\phi)$ has the value equal to	s. 1

5.

Column I	Column II
a. The maximum value of $\{\cos(2A+\theta)+\cos(2B+\theta)\}$, where A, B are constants, is	p. $2\sin(A+B)$
b. The maximum value of $\{\cos 2A+\cos 2B\}$, where $(A+B)$ is constant and $A, B\in(0, \pi/2)$, is	q. $2\sec(A+B)$
c. The minimum value of $\{\sec 2A+\sec 2B\}$, where $(A+B)$ is constant and $A, B\in(0, \pi/4)$, is	r. $2\cos(A+B)$
d. The minimum value of $\sqrt{\{\tan\theta+\cot\theta-2\cos 2(A+B)\}}$, where A, B are constants and $\theta\in(0, \pi/2)$, is	s. $2\cos(A-B)$

6.

Column I	Column II
a. $\cos 20°+\cos 80°-\sqrt{3}\cos 50°$	p. -1
b. $\cos 0°+\cos\dfrac{\pi}{7}+\cos\dfrac{2\pi}{7}+\cos\dfrac{3\pi}{7}+\cos\dfrac{4\pi}{7}+\cos\dfrac{5\pi}{7}+\cos\dfrac{6\pi}{7}$	q. $-\dfrac{3}{4}$
c. $\cos 20°+\cos 40°+\cos 60°-4\cos 10°\cos 20°\cos 30°$	r. 1
d. $\cos 20°\cos 100°+\cos 100°\cos 140°-\cos 140°\cos 200°$	s. 0

7.

Column I	Column II
a. Suppose ABC is a triangle with three acute angles A, B and C. The point whose coordinates are $(\cos B - \sin A, \sin B - \cos A)$ can be in the	p. 1^{st} quadrant
b. If $2^{\sin\theta} > 1$ and $3^{\cos\theta} < 1$, then $\theta \in$	q. 2^{nd} quadrant
c. $\lvert\cos x + \sin x\rvert = \lvert\sin x\rvert + \lvert\cos x\rvert$	r. 3^{rd} quadrant
d. If $\sqrt{\dfrac{1-\sin A}{1+\sin A}} + \dfrac{\sin A}{\cos A} = \dfrac{1}{\cos A}$, for all permissible values of A, then A can belong to	s. 4^{th} quadrant

8.

Column I	Column II
a. If $x^2 + y^2 = 1$ and $P = (3x - 4x^3)^2 + (3y - 4y^3)^2$, then P is equal to	p. 1
b. If $a + b = 3 - \cos 4\theta$ and $a - b = 4 \sin 2\theta$, then the maximum value of (ab) is	q. 4
c. The least positive integral value of x for which $3 \cos \theta = x^2 - 8x + 19$ holds good is	r. 5
d. If $x = \dfrac{4\lambda}{1+\lambda^2}$ and $y = \dfrac{2-2\lambda^2}{1+\lambda^2}$, where λ is a real parameter, then $x^2 - xy + y^2$ lies between $[a, b]$ then $(a + b)$ is	s. 8

9.

Column I	Column II
a. In triangle ABC, $3 \sin A + 4 \cos B = 6$ and $3 \cos A + 4 \sin B = 1$, then $\angle C$ can be	p. $60°$
b. In any triangle, if $(\sin A + \sin B + \sin C)(\sin A + \sin B - \sin C) = 3 \sin A \sin B$, then the angle C	q. $30°$
c. If $8 \sin x \cos^5 x - 8 \sin^5 x \cos x = 1$, then $x =$	r. $165°$
d. 'O' is the centre of the inscribed circle in a $30° - 60° - 90°$ triangle ABC with right angled at C. If the circle is tangent to AB at D, then the angle $\angle COD$ is	s. $7.5°$

Integer Type

Solutions on page 2.135

1. If $f(\theta) = \dfrac{1 - \sin 2\theta + \cos 2\theta}{2 \cos 2\theta}$ then value of $8f(11°) \cdot f(34°)$ is ________.
2. If $f(x) = 2(7 \cos x + 24 \sin x)(7 \sin x - 24 \cos x)$, for every $x \in R$, then maximum value of $(f(x))^{1/4}$ is ________.
3. In a triangle ABC, $\angle C = \dfrac{\pi}{2}$. If $\tan\left(\dfrac{A}{2}\right)$ and $\tan\left(\dfrac{B}{2}\right)$ are the roots of the equation $ax^2 + bx + c = 0$ $(a \neq 0)$, then the value of $\dfrac{a+b}{c}$ (where, a, b, c are sides of Δ opposite to angles A, B, C resp.) is ________.

4. If $(1 + \tan 5°)(1 + \tan 10°)(1 + \tan 15°)...(1 + \tan 45°) = 2^k$, then the value of '$k$' is ______.

5. The value of $\sqrt{3}\left|\dfrac{\dfrac{2\sin(140°)\sec(280°)}{\sec(220°)}+\dfrac{\sec(340°)}{\operatorname{cosec}(20°)}}{\dfrac{\cot(200°)-\tan(280°)}{\cot(200°)}}\right|$ is ______.

6. If $x, y \in R$ satisfy $(x+5)^2 + (y-12)^2 = (14)^2$, then the minimum value of is ______.

7. Suppose x and y are real numbers such that $\tan x + \tan y = 42$ and $\cot x + \cot y = 49$. Then the prime number by which the value of $\tan(x + y)$ is not divisible by 5 is ______.

8. Let $0 \le a, b, c, d \le \pi$ where b and c are not complementary, such that
$2 \cos a + 6 \cos b + 7 \cos c + 9 \cos d = 0$
and $2 \sin a - 6 \sin b + 7 \sin c - 9 \sin d = 0$, then the value of $3\dfrac{\cos(a+d)}{\cos(b+c)}$ is ______.

9. Suppose A and B are two angles such that $A, B \in (0, \pi)$, and satisfy $\sin A + \sin B = 1$ and $\cos A + \cos B = 0$. Then the value of $12 \cos 2A + 4 \cos 2B$ is ______.

10. α and β are the positive acute angles and satisfying equations $5 \sin 2\beta = 3 \sin 2\alpha$ and $\tan \beta = 3 \tan \alpha$ simultaneously. Then the value of $\tan \alpha + \tan \beta$ is ______.

11. The absolute value of the expression $\tan\dfrac{\pi}{16} + \tan\dfrac{5\pi}{16} + \tan\dfrac{9\pi}{16} + \tan\dfrac{13\pi}{16}$ is ______.

12. The greatest integer less than or equal to $\dfrac{1}{\cos 290°} + \dfrac{1}{\sqrt{3}\sin 250°}$ is ______.

13. The maximum value of $y = \dfrac{1}{\sin^6 x + \cos^6 x}$ is ______.

14. The maximum value of $\cos^2(45° + x) + (\sin x - \cos x)^2$ is ______.

15. The value of $9\dfrac{\sin^4 t + \cos^4 t - 1}{\sin^6 t + \cos^6 t - 1}$ is ______.

16. The value of $\operatorname{cosec}10° + \operatorname{cosec}50° - \operatorname{cosec}70°$ is ______.

17. The minimum value of $\sqrt{(3\sin x - 4\cos x - 10)(3\sin x + 4\cos x - 10)}$ is ______.

18. Number of triangles ABC if $\tan A = x$, $\tan B = x + 1$ and $\tan C = 1 - x$ is ______.

19. If $\log_{10}\sin x + \log_{10}\cos x = -1$ and $\log_{10}(\sin x + \cos x) = \dfrac{(\log_{10} n) - 1}{2}$, then the value of '$n/3$' is ______.

20. The value of $\dfrac{\sin 1° + \sin 3° + \sin 5° + \sin 7°}{\cos 1° \cdot \cos 2° \cdot \sin 4°}$ is ______.

21. In a triangle ABC, if $A - B = 120°$ and $\sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2} = \dfrac{1}{32}$ then, the value of $8\cos C$ is ______.

22. In a triangle ABC if $\tan A = \dfrac{1}{2}$, $\tan B = k + \dfrac{1}{2}$ and $\tan C = 2k + \dfrac{1}{2}$, then the possible value of $[k]$, where $[\cdot]$ represents greatest integer function is ______.

23. If $\sin^3 x \cos 3x + \cos^3 x \sin 3x = 3/8$, then the value of $8\sin 4x$ is ______.

Archives

Solutions on page 2.141

Subjective

1. If $\tan\alpha = \frac{m}{m+1}$ and $\tan\beta = \frac{1}{2m+1}$, find the possible values of $(\alpha+\beta)$. **(IIT-JEE, 1978)**

2. a. Draw the graph of $y = \frac{1}{\sqrt{2}}(\sin x + \cos x)$ from $x = -\frac{\pi}{2}$ to $x = \frac{\pi}{2}$. **(IIT-JEE, 1979)**

   b. If $\cos(\alpha+\beta) = \frac{4}{5}$, $\sin(\alpha-\beta) = \frac{5}{13}$, and α, β lie between 0 and $\pi/4$, find $\tan 2\alpha$.

3. Prove that $5\cos\theta + 3\cos\left(\theta+\frac{\pi}{3}\right) + 3$ lies between -4 and 10.

4. Given $\alpha+\beta-\gamma=\pi$, prove that $\sin^2\alpha + \sin^2\beta - \sin^2\gamma = 2\sin\alpha\sin\beta\cos\gamma$. **(IIT-JEE, 1980)**
5. For all θ in $[0, \pi/2]$ show that $\cos(\sin\theta) \geq \sin(\cos\theta)$. **(IIT-JEE, 1981)**
6. Without using tables, prove that $(\sin 12°)(\sin 48°)(\sin 54°) = 1/8$. **(IIT-JEE, 1980)**
7. Show that $16\cos\left(\frac{2\pi}{15}\right)\cos\left(\frac{4\pi}{15}\right)\cos\left(\frac{8\pi}{15}\right)\cos\left(\frac{16\pi}{15}\right) = 1$. **(IIT-JEE, 1983)**
8. Prove that $\tan\alpha + 2\tan 2\alpha + 4\tan 4\alpha + 8\cot 8\alpha = \cot\alpha$. **(IIT-JEE, 1988)**
9. ABC is a triangle such that $\sin(2A+B) = \sin(C-A) = -\sin(B+2C) = 1/2$. If A, B and C are in A.P. determine the values of A, B, and C. **(IIT-JEE, 1990)**
10. Show that the value of $\frac{\tan x}{\tan 3x}$, wherever defined, never lies between $\frac{1}{3}$ and 3. **(IIT-JEE, 1992)**
11. Prove that $\sum_{k=1}^{n-1}(n-k)\cos\frac{2k\pi}{n} = -\frac{n}{2}$, where $n \geq 3$ is an integer. **(IIT-JEE, 1997)**
12. Find the range of values of t for which $2\sin t = \frac{1-2x+5x^2}{3x^2-2x-1}$, $t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. **(IIT-JEE, 2005)**
13. Find the maximum value of the expression $\frac{1}{\sin^2\theta + 3\sin\theta\cos\theta + 5\cos^2\theta}$. **(IIT-JEE, 2010)**

Objective

Fill in the blanks

1. Suppose $\sin^3 x \sin 3x = \sum_{m=0}^{n} C_m \cos mx$ is an identity in x, where $C_0, C_1, \ldots, C_n$ are constants, and $C_n \neq 0$, then the value of n is______. **(IIT-JEE, 1981)**
2. The side of a triangle inscribed in a given circle subtends angles α, β and γ at the centre. The minimum value of the arithmetic mean of $\cos\left(\alpha+\frac{\pi}{2}\right)$, $\cos\left(\beta+\frac{\pi}{2}\right)$ and $\cos\left(\gamma+\frac{\pi}{2}\right)$ is equal to______. **(IIT-JEE, 1987)**

3. The value of $\sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14}\sin\frac{7\pi}{14}\sin\frac{9\pi}{14}\sin\frac{11\pi}{14}\sin\frac{13\pi}{14}$ is equal to______. **(IIT-JEE, 1991)**
4. If $K = \sin(\pi/18)\sin(5\pi/18)\sin(7\pi/18)$, then the numerical value of K is______. **(IIT-JEE, 1993)**
5. If $A > 0$, $B > 0$ and $A + B = \pi/3$, then the maximum value of $\tan A \tan B$ is______. **(IIT-JEE, 1993)**
6. If $\cos(x-y)$, $\cos x$ and $\cos(x+y)$ are in H.P., then $\cos x \sec\left(\frac{y}{2}\right) =$ ______. **(IIT-JEE, 1997)**

True or false

1. If $\tan A = \frac{1-\cos B}{\sin B}$, then $\tan 2A = \tan B$. **(IIT-JEE, 1983)**

Multiple choice questions with one correct answer

1. If $\tan\theta + \sin\theta = m$ and $\tan\theta - \sin\theta = n$, then
   a. $m^2 - n^2 = 4mn$
   b. $m^2 + n^2 = 4mn$
   c. $m^2 - n^2 = m^2 + n^2$
   d. $m^2 - n^2 = 4\sqrt{mn}$ **(IIT-JEE, 1970)**
2. If $\tan\theta = -\frac{4}{3}$, then $\sin\theta$ is **(IIT-JEE, 1979)**
   a. $-\frac{4}{5}$ but not $\frac{4}{5}$
   b. $-\frac{4}{5}$ or $\frac{4}{5}$
   c. $\frac{4}{5}$ but not $-\frac{4}{5}$
   d. none of these
3. If $\alpha + \beta + \gamma = 2\pi$, then
   a. $\tan\frac{\alpha}{2} + \tan\frac{\beta}{2} + \tan\frac{\gamma}{2} = \tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}$
   b. $\tan\frac{\alpha}{2}\tan\frac{\beta}{2} + \tan\frac{\beta}{2}\tan\frac{\gamma}{2} + \tan\frac{\gamma}{2}\tan\frac{\alpha}{2} = 1$
   c. $\tan\frac{\alpha}{2} + \tan\frac{\beta}{2} + \tan\frac{\gamma}{2} = -\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}$
   d. none of these **(IIT-JEE, 1979)**
4. Given $A = \sin^2\theta + \cos^4\theta$, then for all real θ,
   a. $1 \le A \le 2$
   b. $3/4 \le A \le 1$
   c. $13/16 \le A \le 1$
   d. $3/4 \le A \le 13/16$ **(IIT-JEE, 1980)**
5. The value of $\left(1+\cos\frac{\pi}{8}\right)\left(1+\cos\frac{3\pi}{8}\right)\left(1+\cos\frac{5\pi}{8}\right)\left(1+\cos\frac{7\pi}{8}\right)$ is
   a. 1/4
   b. 3/4
   c. 1/8
   d. 3/8 **(IIT-JEE, 1984)**
6. The value of the expression $\sqrt{3}$ cosec 20° − sec 20° is equal to
   a. 2
   b. 2 sin 20°/sin 40°
   c. 4
   d. 4 sin 20°/sin 40° **(IIT-JEE, 1988)**
7. $3(\sin x - \cos x)^4 + 6(\sin x + \cos x)^2 + 4(\sin^6 x + \cos^6 x)$ is equal to
   a. 11
   b. 12
   c. 13
   d. 14 **(IIT-JEE, 1995)**
8. $\sec^2\theta = \frac{4xy}{(x+y)^2}$ is true if and only if
   a. $x + y \ne 0$
   b. $x = y, x \ne 0$
   c. $x = y$
   d. $x \ne 0, y \ne 0$ **(IIT-JEE, 1996)**
9. Let $f(\theta) = \sin\theta(\sin\theta + \sin 3\theta)$. Then $f(\theta)$ is
   a. ≥ 0 only when $\theta \ge 0$
   b. ≤ 0 for all real θ
   c. ≥ 0 for all real θ
   d. ≤ 0 only when $\theta \le 0$ **(IIT-JEE, 2000)**

10. The maximum value of $(\cos \alpha_1)(\cos \alpha_2) \cdots (\cos \alpha_n)$, under the restrictions $0 \le \alpha_1, \alpha_2, \ldots, \alpha_n \le \pi/2$ and $(\cot \alpha_1)(\cot \alpha_2) \cdots (\cot \alpha_n) = 1$ is

a. $1/2^{n/2}$ b. $1/2^n$ c. $1/2n$ d. 1

(IIT-JEE, 2001)

11. If $\alpha + \beta = \pi/2$ and $\beta + \gamma = \alpha$, then $\tan \alpha$ equals

a. $2(\tan \beta + \tan \gamma)$ b. $\tan \beta + \tan \gamma$ c. $\tan \beta + 2 \tan \gamma$ d. $2 \tan \beta + \tan \gamma$

(IIT-JEE, 2001)

12. Given both θ and ϕ are acute angles and $\sin \theta = 1/2$, $\cos \phi = 1/3$, then the value of $\theta + \phi$ belongs to

a. $\left(\frac{\pi}{3}, \frac{\pi}{2}\right]$ b. $\left(\frac{\pi}{2}, \frac{2\pi}{3}\right]$ c. $\left(\frac{2\pi}{3}, \frac{5\pi}{6}\right]$ d. $\left(\frac{5\pi}{6}, \pi\right]$

(IIT-JEE, 2004)

13. Let $0 < x < \pi/4$, then $(\sec 2x - \tan 2x)$ equals

a. $\tan\left(x - \frac{\pi}{4}\right)$ b. $\tan\left(\frac{\pi}{4} - x\right)$ c. $\tan\left(x + \frac{\pi}{4}\right)$ d. $\tan^2\left(x + \frac{\pi}{4}\right)$

(IIT-JEE, 1994)

14. Let n be a positive integer such that $\sin \frac{\pi}{2n} + \cos \frac{\pi}{2n} = \frac{\sqrt{x}}{2}$. Then

a. $6 \le n \le 8$ b. $4 < n \le 8$ c. $4 \le n \le 8$ d. $4 < n < 8$

(IIT-JEE, 1994)

15. Let $\theta \in (0, \pi/4)$ and $t_1 = (\tan \theta)^{\tan \theta}$, $t_2 = (\tan \theta)^{\cot \theta}$, $t_3 = (\cot \theta)^{\tan \theta}$ and $t_4 = (\cot \theta)^{\cot \theta}$, then

a. $t_1 > t_2 > t_3 > t_4$ b. $t_4 > t_3 > t_1 > t_2$ c. $t_3 > t_1 > t_2 > t_4$ d. $t_2 > t_3 > t_1 > t_2$

(IIT-JEE, 2006)

Multiple choice questions with one or more than one correct answers

1. The expression $3\left[\sin^4\left(\frac{3}{2}\pi - \alpha\right) + \sin^4(3\pi + \alpha)\right] - 2\left[\sin^6\left(\frac{1}{2}\pi + \alpha\right) + \sin^6(5\pi - \alpha)\right]$ is equal to

a. 0 b. 1 c. 3 d. none of these

(IIT-JEE, 1984)

2. For $0 < \phi \le \pi/2$, if $x = \sum_{n=0}^{\infty} \cos^{2n} \phi$, $y = \sum_{n=0}^{\infty} \sin^{2n} \phi$, $z = \sum_{n=0}^{\infty} \cos^{2n} \phi \sin^{2n} \phi$, then

a. $xyz = xz + y$ b. $xyz = xy + z$ c. $xyz = x + y + z$ d. $xyz = yz + x$

(IIT-JEE, 1992)

3. Which of the following number(s) is/are rational?

a. $\sin 15°$ b. $\cos 15°$ c. $\sin 15° \cos 15°$ d. $\sin 15° \cos 75°$

(IIT-JEE, 1998)

4. For a positive integer n, let $f_n(\theta) = (\tan \theta/2)(1 + \sec \theta)(1 + \sec 2\theta)(1 + \sec 4\theta) \cdots (1 + \sec 2^n \theta)$. Then

a. $f_2(\pi/16) = 1$ b. $f_3(\pi/32) = 1$ c. $f_4(\pi/64) = 1$ d. $f_5(\pi/128) = 1$

(IIT-JEE, 1999)

5. The minimum value of the expression $\sin\alpha+\sin\beta+\sin\gamma$, where α, β, γ are real numbers satisfying $\alpha+\beta+\gamma=\pi$ is
   a. positive b. zero c. negative d. -3 **(IIT-JEE, 1995)**

ANSWERS AND SOLUTIONS

Subjective Type

1. Let $m=2k$, i.e., m is even where $k\in I$

Now, $\beta=2k\pi+\dfrac{\pi}{2}-A=\left(2k+\dfrac{1}{2}\right)\pi-A$ (i)

If $m=2k+1$, i.e., m is odd, then

$\beta=(2k+1)\pi-\left(\dfrac{\pi}{2}-A\right)=\left(2k+\dfrac{1}{2}\right)\pi+A$ (ii)

From Eqs. (i) and (ii), β can be expressed as

$\beta=\left(2k+\dfrac{1}{2}\right)\pi\pm A, k\in I$

which is same as α.

2. $A+B+C=\pi$

$$\Rightarrow \frac{A}{2}+\frac{B}{2}=\frac{\pi}{2}-\frac{C}{2}$$

$$\Rightarrow \cot\left(\frac{A}{2}+\frac{B}{2}\right)=\cot\left(\frac{\pi}{2}-\frac{C}{2}\right)$$

$$\Rightarrow \frac{\cot\frac{A}{2}\cot\frac{B}{2}-1}{\cot\frac{A}{2}+\cot\frac{B}{2}}=\tan\frac{C}{2}=\frac{1}{\cot\frac{C}{2}}$$

$$\Rightarrow \cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}=\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}$$

But $\tan\dfrac{A}{2}, \tan\dfrac{B}{2}, \tan\dfrac{C}{2}$ are in H.P. (i)

$\therefore \cot\dfrac{A}{2}, \cot\dfrac{B}{2}, \cot\dfrac{C}{2}$ are in A.P.

So, $\cot\dfrac{A}{2}+\cot\dfrac{C}{2}=2\cot\dfrac{B}{2}$.

Hence, Eq. (i) becomes $\cot\dfrac{A}{2}\cot\dfrac{B}{2}\cot\dfrac{C}{2}=3\cot\dfrac{B}{2} \Rightarrow \cot\dfrac{A}{2}\cot\dfrac{C}{2}=3$

$$\Rightarrow \text{G.M. of } \cot\frac{A}{2} \text{ and } \cot\frac{C}{2}=\sqrt{\cot\frac{A}{2}\cot\frac{C}{2}}=\sqrt{3}$$

$$\text{and A.M. of } \cot\frac{A}{2} \text{ and } \cot\frac{C}{2}=\frac{\cot\frac{A}{2}+\cot\frac{C}{2}}{2}=\cot\frac{B}{2}$$

But A.M. ≥ G.M.

$\Rightarrow \quad \cot \dfrac{B}{2} \geq \sqrt{3}$

Therefore, the minimum value of cot $B/2$ is $\sqrt{3}$.

3. $\operatorname{cosec} \theta = \dfrac{1}{\sin\theta}$

$$= \frac{1}{\sin\theta}\,\frac{\sin\theta/2}{\sin\theta/2} = \frac{\sin\left(\theta - \dfrac{\theta}{2}\right)}{\sin\theta\sin\left(\dfrac{\theta}{2}\right)} = \frac{\sin\theta\cos\dfrac{\theta}{2} - \cos\theta\sin\dfrac{\theta}{2}}{\sin\theta\sin\dfrac{\theta}{2}}$$

$\therefore \operatorname{cosec} \theta = \cot\dfrac{\theta}{2} - \cot\theta$

Similarly, $\operatorname{cosec} 2\theta = \cot\theta - \cot 2\theta$

$\operatorname{cosec} 4\theta = \cot 2\theta - \cot 4\theta$

$\vdots$

$\operatorname{cosec} 2^{n-1}\theta = \cot 2^{n-2}\theta - \cot 2^{n-1}\theta$

Therefore, sum $= \cot\dfrac{\theta}{2} - \cot 2^{n-1}\theta$

4. Let $\cot\theta = \cot A + \cot B + \cot C$

$\Rightarrow \cot\theta - \cot A = \cot B + \cot C$

$\Rightarrow \quad \dfrac{\sin(A-\theta)}{\sin A\sin\theta} = \dfrac{\sin(B+C)}{\sin B\sin C}$

$\Rightarrow \quad \sin(A-\theta) = \dfrac{\sin^2 A\sin\theta}{\sin B\sin C}$ (i)

Similarly, $\sin(B-\theta) = \dfrac{\sin^2 B\sin\theta}{\sin A\sin C}$ (ii)

and $\sin(C-\theta) = \dfrac{\sin^2 C\sin\theta}{\sin A\sin B}$ (iii)

By multiplying corresponding sides of Eqs. (i), (ii), and (iii), we have

$\sin^3\theta = \sin(A-\theta)\sin(B-\theta)\sin(C-\theta)$

5. Let $\sin\dfrac{A}{2} + \sin\dfrac{B}{2} + \sin\dfrac{C}{2} = k$

or $2\sin\dfrac{A+B}{4}\cos\dfrac{A-B}{4} + \cos\dfrac{A+B}{2} = k$

$\Rightarrow \quad 2\sin^2\dfrac{A+B}{4} - 2\cos\dfrac{A-B}{4}\sin\dfrac{A+B}{4} + k - 1 = 0$

Since $\sin\dfrac{A+B}{4}$ is real, $4\cos^2\dfrac{A-B}{4} - 8(k-1) \geq 0$

$\Rightarrow \quad 2(k-1) \leq \cos^2\dfrac{A-B}{4} \leq 1 \Rightarrow k \leq 3/2$

Hence, $2\sin\frac{A+B}{4}\left[\cos\frac{A-B}{4}-\sin\frac{\pi-C}{4}\right]\le\frac{1}{2}$

$\Rightarrow\quad 2\sin\frac{A+B}{4}\left[\cos\frac{A-B}{4}-\cos\frac{\pi+C}{4}\right]\le\frac{1}{2}$

$\Rightarrow\quad 4\sin\frac{A+B}{4}\sin\frac{\pi+C+A-B}{8}\sin\frac{\pi+C-A+B}{8}\le\frac{1}{2}$

$\Rightarrow\quad 4\sin\frac{\pi-C}{4}\sin\frac{\pi-B}{4}\sin\frac{\pi-A}{4}\le\frac{1}{2}\qquad\Rightarrow\cos\frac{\pi+C}{4}\cos\frac{\pi+B}{4}\cos\frac{\pi+A}{4}\le\frac{1}{8}$

6. Here $\frac{x}{y}=\frac{\tan(\theta+\alpha)}{\tan(\theta+\beta)}$. By componendo and dividendo, we get

$$\frac{x+y}{x-y}=\frac{\tan(\theta+\alpha)+\tan(\theta+\beta)}{\tan(\theta+\alpha)-\tan(\theta+\beta)}=\frac{\sin(2\theta+\alpha+\beta)}{\sin(\alpha-\beta)}$$

$$\therefore\ \frac{x+y}{x-y}\sin^2(\alpha-\beta)=\sin(2\theta+\alpha+\beta)\ \sin(\alpha-\beta)$$

$$=\frac{1}{2}[\cos 2(\theta+\beta)-\cos 2(\theta+\alpha)] \qquad \text{(i)}$$

Similarly, $\frac{y+z}{y-z}\sin^2(\beta-\gamma)=\frac{1}{2}[\cos 2(\theta+\gamma)-\cos 2(\theta+\beta)]$ (ii)

and $\frac{z+x}{z-x}\sin^2(\gamma-\alpha)=\frac{1}{2}[\cos 2(\theta+\alpha)-\cos 2(\theta+\gamma)]$ (iii)

Adding Eqs. (i), (ii), and (iii), we get L.H.S. = 0.

7. Here, we have $\tan 6\theta=p/q$

$\Rightarrow\quad \frac{\sin 6\theta}{\cos 6\theta}=\frac{p}{q}\qquad\Rightarrow\frac{p}{\sin 6\theta}=\frac{q}{\cos 6\theta}=\frac{\sqrt{p^2+q^2}}{\sqrt{1}}=\sqrt{p^2+q^2}=k$ (say)

Now $y=\frac{1}{2}(p\operatorname{cosec}\theta-q\sec 2\theta)=\frac{1}{2}\left(\frac{p}{\sin 2\theta}-\frac{q}{\cos 2\theta}\right)$

$\Rightarrow\quad y=\frac{1}{2}\left[\frac{p\cos 2\theta-q\sin 2\theta}{\sin 2\theta\cos 2\theta}\right]$

$$=\left[\frac{2k\sin 6\theta\cos 2\theta-2k\cos 6\theta\sin 2\theta}{4\sin 2\theta\cos 2\theta}\right]=k\frac{\sin(6\theta-2\theta)}{\sin 4\theta}=k=\sqrt{p^2+q^2}$$

8. $\sin\alpha+\cos\alpha+(\tan\alpha+\cot\alpha)+(\sec\alpha+\operatorname{cosec}\alpha)=7$

$\Rightarrow\quad (\sin\alpha+\cos\alpha)+\frac{1}{\sin\alpha\cos\alpha}+\frac{\sin\alpha+\cos\alpha}{\sin\alpha\cos\alpha}=7$

$\Rightarrow\quad (\sin\alpha+\cos\alpha)\left(1+\frac{1}{\sin\alpha\cos\alpha}\right)=7-\frac{1}{\sin\alpha\cos\alpha}$

$\Rightarrow\quad (1+\sin 2\alpha)\left(1+\frac{4}{\sin 2\alpha}+\frac{4}{\sin^2 2\alpha}\right)=49-\frac{28}{\sin 2\alpha}+\frac{4}{\sin^2 2\alpha}$

Let $\sin 2\alpha = x$

$$\Rightarrow \quad (1+x)\left(1+\frac{4}{x}+\frac{4}{x^2}\right)=49-\frac{28}{x}+\frac{4}{x^2} \quad \Rightarrow (1+x)\left(x^2+4x+4\right)=49x^2-28x+4$$

$$\Rightarrow \quad x^3-44x^2-36x=0$$

$$\Rightarrow \quad x^2-44x-36=0. \text{ (as } x=\sin 2a \neq 0)$$

9. We have

$$1+\cot\theta-\cot\frac{\theta}{2}=1+\frac{\cot^2\frac{\theta}{2}-1}{2\cot\frac{\theta}{2}}-\cot\frac{\theta}{2}$$

$$=\frac{2\cot\frac{\theta}{2}+\cot^2\frac{\theta}{2}-1-2\cot^2\frac{\theta}{2}}{2\cot\frac{\theta}{2}}$$

$$=\frac{-\left(\cot\frac{\theta}{2}-1\right)^2}{2\cot\frac{\theta}{2}}\leq 0 \text{ for } 0<\theta<\pi$$

$$\Rightarrow \quad 1+\cot\theta \leq \cot\frac{\theta}{2}$$

Equality holds when $\cot\frac{\theta}{2}-1=0 \Rightarrow \theta=\frac{\pi}{2}$.

10. Since A.M. of two positive quantities $\geq$ their G.M., we have

$$\frac{2^{\sin x}+2^{\cos x}}{2}\geq\sqrt{2^{\sin x}\,2^{\cos x}}=\sqrt{2^{\sin x+\cos x}}=\sqrt{2^{\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)}}\geq\sqrt{2^{-\sqrt{2}}}$$

$$\Rightarrow \quad 2^{\sin x}+2^{\cos x}\geq 2\times 2^{\frac{-1}{\sqrt{2}}}=2^{1-\frac{1}{\sqrt{2}}}$$

11. We have $\frac{A}{2}+\frac{B}{2}+\frac{C}{2}=\frac{\pi}{2}$, so that

$$\tan\left(\frac{A}{2}+\frac{B}{2}\right)=\tan\left(\frac{\pi}{2}-\frac{C}{2}\right)$$

$$\Rightarrow \quad \frac{\tan\frac{A}{2}+\tan\frac{B}{2}}{1-\tan\frac{A}{2}\tan\frac{B}{2}}=\frac{1}{\tan\frac{C}{2}}$$

$$\Rightarrow \quad \tan\frac{A}{2}\tan\frac{B}{2}+\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{C}{2}\tan\frac{A}{2}=1$$

Now $\tan^2\frac{A}{2}+\tan^2\frac{B}{2}+\tan^2\frac{C}{2}-1$

$$=\frac{1}{2}\left[\sum 2\tan^2\frac{A}{2}-\sum 2\tan\frac{A}{2}\tan\frac{B}{2}\right]$$

$$=\frac{1}{2}\left[\left(\tan\frac{A}{2}-\tan\frac{B}{2}\right)^2+\left(\tan\frac{B}{2}-\tan\frac{C}{2}\right)^2+\left(\tan\frac{C}{2}-\tan\frac{A}{2}\right)^2\right]\geq 0$$

12. $A+B+C=\pi$

$\Rightarrow \quad B+C=\frac{3\pi}{4}\Rightarrow 0<B,\ C<\frac{3\pi}{4}$ Also $\tan B\tan C=p$

$$\Rightarrow \quad \frac{\sin B\sin C}{\cos B\cos C}=\frac{p}{1}$$

$$\Rightarrow \quad \frac{\cos B\cos C-\sin B\sin C}{\cos B\cos C+\sin B\sin C}=\frac{1-p}{1+p}$$

$$\Rightarrow \quad \frac{\cos(B+C)}{\cos(B-C)}=\frac{1-p}{1+p}$$

$$\Rightarrow \quad \frac{1+p}{\sqrt{2}(p-1)}=\cos(B-C) \qquad \text{(i)}$$

Since B or C can vary from 0 to $3\pi/4$, we get

$$0\leq B-C<\frac{3\pi}{4}\Rightarrow -\frac{1}{\sqrt{2}}<\cos(B-C)\leq 1.$$

Equation (i) will now lead to $-\frac{1}{\sqrt{2}}<\frac{p+1}{\sqrt{2}(p-1)}\leq 1$

For $0<1+\frac{p+1}{p-1} \quad \Rightarrow \frac{2p}{(p-1)}>0 \quad \Rightarrow p<0$ or $p>1$ (ii)

Also $\frac{p+1-\sqrt{2}(p-1)}{\sqrt{2}(p-1)}\leq 0$

$$\Rightarrow \quad \frac{\left(p-\left(\sqrt{2}+1\right)^2\right)}{(p-1)}\geq 0$$

$\Rightarrow \quad p<1$ or $p\geq\left(\sqrt{2}+1\right)^2$ (iii)

Combining Eqs. (ii) and (iii), we get $p<0$ or $p\geq\left(\sqrt{2}+1\right)^2$.

13. Adding $\sin(a+x)+\sin(a-x)=2(b+c)$

$\Rightarrow 2\sin a\cos x=2(b+c)$

$$\Rightarrow \cos x=\frac{b+c}{\sin a} \qquad \text{(i)}$$

Subtracting, we get

$\sin(a+x)-\sin(a-x)=2(b-c)$

$\Rightarrow 2\cos a \sin x = 2(b-c)$

$$\Rightarrow \sin x = \frac{b-c}{\cos a} \quad \text{(ii)}$$

Squaring and adding Eq. (i) and Eq. (ii), we get

$$\frac{(b+c)^2}{\sin^2 a}+\frac{(b-c)^2}{\cos^2 a}=1$$

14. $\tan\beta = \dfrac{n\sin\alpha\cos\alpha}{1-n\sin^2\alpha}$

$$=\frac{\dfrac{n\sin\alpha\cos\alpha}{\cos^2\alpha}}{\dfrac{1}{\cos^2\alpha}-\dfrac{n\sin^2\alpha}{\cos^2\alpha}} \quad \text{[dividing numerator and denominator by } \cos^2\alpha]$$

$$=\frac{n\tan\alpha}{\sec^2\alpha-n\tan^2\alpha}=\frac{n\tan\alpha}{1+\tan^2\alpha-n\tan^2\alpha}=\frac{n\tan\alpha}{1+(1-n)\tan^2\alpha} \quad \text{(i)}$$

Now, L.H.S. $=\tan(\alpha-\beta)=\dfrac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$

$$=\frac{\tan\alpha-\dfrac{n\tan\alpha}{1+(1-n)\tan^2\alpha}}{1+\tan\alpha\dfrac{n\tan\alpha}{1+(1-n)\tan^2\alpha}} \quad \text{[From Eq. (i)]}$$

$$=\frac{\tan\alpha+(1-n)\tan^3\alpha-n\tan\alpha}{1+(1-n)\tan^2\alpha+n\tan^2\alpha}=\frac{(1-n)\tan\alpha+(1-n)\tan^3\alpha}{1+\tan^2\alpha}=(1-n)\tan\alpha$$

15. L.H.S. contains x, $3x$, $9x$ and $27x$, whereas R.H.S contains $27x$ and x only. So, we will manipulate terms as shown below

R.H.S.$=\dfrac{1}{2}[\tan 27x-\tan x]$

$$=\frac{1}{2}[(\tan 27x-\tan 9x)+(\tan 9x-\tan 3x)+(\tan 3x-\tan x)]$$

$$=\frac{1}{2}\left[\left(\frac{\sin 27x}{\cos 27x}-\frac{\sin 9x}{\cos 9x}\right)+\left(\frac{\sin 9x}{\cos 9x}-\frac{\sin 3x}{\cos 3x}\right)+\left(\frac{\sin 3x}{\cos 3x}-\frac{\sin x}{\cos x}\right)\right]$$

$$=\frac{1}{2}\left[\frac{\sin(27x-9x)}{\cos 27x\cos 9x}+\frac{\sin(9x-3x)}{\cos 9x\cos 3x}+\frac{\sin(3x-x)}{\cos 3x\cos x}\right]$$

$$=\frac{1}{2}\left[\frac{\sin 18x}{\cos 27x\cos 9x}+\frac{\sin 6x}{\cos 9x\cos 3x}+\frac{\sin 2x}{\cos 3x\cos x}\right]$$

$$= \frac{1}{2}\left[\frac{2\sin 9x\cos 9x}{\cos 27x\cos 9x} + \frac{2\sin 3x\cos 3x}{\cos 9x\cos 3x} + \frac{2\sin x\cos x}{\cos 3x\cos x}\right]$$

$$= \frac{\sin 9x}{\cos 27x} + \frac{\sin 3x}{\cos 9x} + \frac{\sin x}{\cos 3x} = \frac{\sin x}{\cos 3x} + \frac{\sin 3x}{\cos 9x} + \frac{\sin 9x}{\cos 27x} = \text{L.H.S.}$$

16. We have to prove $\frac{2\cos 2^n\theta+1}{2\cos\theta+1} = (2\cos\theta-1)(2\cos 2\theta-1)(2\cos 2^2\theta-1)\ldots(2\cos 2^{n-1}\theta-1)$

or $2\cos 2^n\theta+1 = [(2\cos\theta+1)(2\cos\theta-1)](2\cos 2\theta-1)(2\cos 2^2\theta-1)\ldots(2\cos 2^{n-1}\theta-1)$

Now $[(2\cos\theta+1)(2\cos\theta-1)](2\cos 2\theta-1)(2\cos 2^2\theta-1)\ldots(2\cos 2^{n-1}\theta-1)$

$= (4\cos^2\theta-1)(2\cos 2\theta-1)(2\cos 2^2\theta-1\ldots(2\cos 2^{n-1}\theta-1)$

$= (2\cos 2\theta+1)(2\cos 2\theta-1)(2\cos 2^2\theta-1)\ldots(2\cos 2^{n-1}\theta-1)$ [using $\cos 2\theta = 2\cos^2\theta-1$]

$= (4\cos^2 2\theta-1)(2\cos 2^2\theta-1)\ldots(2\cos 2^{n-1}\theta-1)$

$= (2\cos 2^2\theta+1)(2\cos 2^2\theta-1)\ldots(2\cos 2^{n-1}\theta-1)$

$= (4\cos^2 2^2\theta-1)(2\cos 2^3\theta-1)\ldots(2\cos 2^{n-1}\theta-1)$

$\vdots$

$= (2\cos 2^{n-1}\theta+1)(2\cos 2^{n-1}\theta-1)$

$= 4\cos^2 2^{n-1}\theta-1$

$= 2\cos 2^n\theta+1$

17. We have to prove that $\frac{\tan 2^n\theta}{\tan\theta} = (1+\sec 2\theta)(1+\sec 2^2\theta)(1+\sec 2^3\theta)\ldots(1+\sec 2^n\theta)$

or $\tan 2^n\theta = \tan\theta(1+\sec 2\theta)(1+\sec 2^2\theta)(1+\sec 2^3\theta)\ldots(1+\sec 2^n\theta)$

Now $\tan\theta(1+\sec 2\theta)(1+\sec 2^2\theta)(1+\sec 2^3\theta)\ldots(1+\sec 2^n\theta)$

$$= \tan\theta\left(\frac{1+\cos 2\theta}{\cos 2\theta}\right)(1+\sec 2^2\theta)(1+\sec 2^3\theta)\ldots(1+\sec 2^n\theta)$$

$$= \frac{\sin\theta}{\cos\theta}\left(\frac{2\cos^2\theta}{\cos 2\theta}\right)(1+\sec 2^2\theta)(1+\sec 2^3\theta)\ldots(1+\sec 2^n\theta)$$

$= (\tan 2\theta)(1+\sec 2^2\theta)(1+\sec 2^3\theta)\ldots(1+\sec 2^n\theta)$

$$= (\tan 2\theta)\left(\frac{1+\cos 2^2\theta}{\cos 2^2\theta}\right)(1+\sec 2^3\theta)\ldots(1+\sec 2^n\theta)$$

$$= (\tan 2\theta)\left(\frac{2\cos^2 2\theta}{\cos 2^2\theta}\right)(1+\sec 2^3\theta)\ldots(1+\sec 2^n\theta)$$

$= (\tan 2^2\theta)(1+\sec 2^3\theta)\ldots(1+\sec 2^n\theta)$

$\vdots$

$= \tan 2^{n-1}\theta(1+\sec 2^n\theta)$

$$= \tan 2^{n-1}\theta\left(\frac{1+\cos 2^n\theta}{\cos 2^n\theta}\right)$$

$$= \tan 2^{n-1}\theta\left(\frac{2\cos^2 2^{n-1}\theta}{\cos 2^n\theta}\right)$$

$= \tan 2^n\theta$

Objective Type

1. b. Since $f(x) = \sin x$ is an increasing function for $0 < x < \pi/2$ and 1 radian is approximately 57°. Therefore, $1° < 1^R \Rightarrow \sin 1° < \sin 1$.

2. a. We have $\dfrac{x^2+y^2}{2xy} = \dfrac{1}{2}\left(\dfrac{x}{y}+\dfrac{y}{x}\right)$

Now, $\sin^2\theta = \dfrac{x^2+y^2}{2xy} \quad \Rightarrow \dfrac{x^2+y^2}{2xy} \geq 0$ $\quad [\because \sin^2\theta \geq 0]$

Therefore, x and y have the same sign.

Now, $\dfrac{x^2+y^2}{2xy} = \dfrac{1}{2}\left(\dfrac{x}{y}+\dfrac{y}{x}\right) \quad \Rightarrow \dfrac{x^2+y^2}{2xy} \geq 1$ ($\because$ A.M. $\geq$ G.M.)

But $\sin^2\theta \leq 1$. Therefore, $\dfrac{x^2+y^2}{2xy} = 1 \Rightarrow x = y$.

3. d. Since $0 < x < \pi$. Therefore, $\sin x > 0$

We have $1 + \sin x + \sin^2 x + \cdots \infty = 4 + 2\sqrt{3}$

$$\Rightarrow \quad \frac{1}{1-\sin x} = 4+2\sqrt{3}$$

$$\Rightarrow \quad \sin x = 1 - \frac{1}{4+2\sqrt{3}}$$

$$= \frac{3+2\sqrt{3}}{4+2\sqrt{3}} = \frac{\sqrt{3}}{2}$$

$$\Rightarrow \quad x = \frac{\pi}{3} \text{ or } \frac{2\pi}{3}$$

4. b. $\dfrac{\cos x \tan x}{k^2} + \dfrac{1}{\tan x} + \dfrac{\sin x}{1+\cos x}$

$$= \frac{\sin x}{k^2} + \frac{\cos x\,(1+\cos x)+\sin^2 x}{\sin x\,(1+\cos x)} = \frac{\cos x(1+\cos x)+(1-\cos^2 x)}{\sin x(1+\cos x)} = \frac{a}{k} + \frac{1}{\sin x} = \frac{a}{k} + \frac{1}{ak}$$

5. a. $2\sin\dfrac{A}{2}\operatorname{cosec}\dfrac{B}{2}\left(\sin\dfrac{C}{2} - \cos\dfrac{A}{2}\cos\dfrac{B}{2}\right) - \cos A$

$$= 2\sin\frac{A}{2}\operatorname{cosec}\frac{B}{2}\left(\cos\frac{A+B}{2} - \cos\frac{A}{2}\cos\frac{B}{2}\right) - \cos A$$

$$= 2\sin\frac{A}{2}\operatorname{cosec}\frac{B}{2}\left(-\sin\frac{A}{2}\sin\frac{B}{2}\right) - \cos A = -2\sin^2\frac{A}{2} - \cos A = -1$$

6. b. Applying A.M. ≥ G.M. in $6\tan^2\phi$, $54\cot^2\phi$, 18, we get

$$\frac{6\tan^2\phi+54\cot^2\phi+18}{3}\geq(6\times54\times18)^{1/3}\geq18$$

This is true if $6\tan^2\phi=54\cot^2\phi=18$

$\Rightarrow\quad \tan^2\phi=3$ and $\cot^2\phi=1/3$

Therefore, I and II are correct.

7. c. $5\tan\theta=4\Rightarrow\tan\theta=\dfrac{4}{5}$

Now $\dfrac{5\sin\theta-3\cos\theta}{5\sin\theta+2\cos\theta}=\dfrac{5\dfrac{\sin\theta}{\cos\theta}-3}{5\dfrac{\sin\theta}{\cos\theta}+2}=\dfrac{5\tan\theta-3}{5\tan\theta+2}=\dfrac{5\times\dfrac{4}{5}-3}{5\times\dfrac{4}{5}+2}=\dfrac{1}{6}$

8. b. $2\sec2\theta=\tan\phi+\cot\phi$

$$\Rightarrow\quad\frac{2}{\cos2\theta}=\frac{\sin^2\phi+\cos^2\phi}{\sin\phi\cos\phi}$$

$$\Rightarrow\quad\frac{2}{\cos2\theta}=\frac{1}{\sin\phi\cos\phi}$$

$$\Rightarrow\quad\cos2\theta=\sin2\phi$$

$$\Rightarrow\quad2\theta=90^\circ-2\phi$$

$$\Rightarrow\quad\theta+\phi=\frac{\pi}{4}$$

9. a. $\sin x+\operatorname{cosec}x=2$

$\Rightarrow\quad(\sin x-1)^2=0$

$\Rightarrow\quad\sin x=1$

$\Rightarrow\quad\sin^n x+\operatorname{cosec}^n x=1+1=2$

10. c. $\sec^2\theta+\operatorname{cosec}^2\theta=\dfrac{1}{\cos^2\theta}+\dfrac{1}{\sin^2\theta}=\dfrac{4}{\sin^2 2\theta}\geq4$

Also, $\sec^2\theta\operatorname{cosec}^2\theta=\dfrac{4}{\sin^2 2\theta}\geq4$

Hence, the only equation which can have roots $\operatorname{cosec}^2\theta$ and $\sec^2\theta$ is $x^2-5x+5=0$.

11. b. $\sqrt{\dfrac{1-\cos\alpha}{1+\cos\alpha}}+\sqrt{\dfrac{1+\cos\alpha}{1-\cos\alpha}}=\dfrac{1-\cos\alpha+1+\cos\alpha}{\sqrt{1-\cos^2\alpha}}$

$$=\frac{2}{|\sin x|}=\frac{2}{-\sin\alpha}\ (\text{since }\pi<\alpha<3\pi/2)$$

12. b. We have $\cos\frac{\pi}{7}+\cos\frac{2\pi}{7}+\cos\frac{3\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{5\pi}{7}+\cos\frac{6\pi}{7}+\cos\frac{7\pi}{7}$

$$=\left(\cos\frac{\pi}{7}+\cos\frac{6\pi}{7}\right)+\left(\cos\frac{2\pi}{7}+\cos\frac{5\pi}{7}\right)+\left(\cos\frac{3\pi}{7}+\cos\frac{4\pi}{7}\right)+\cos\pi$$

$$=\left(\cos\frac{\pi}{7}-\cos\frac{\pi}{7}\right)+\left(\cos\frac{2\pi}{7}-\cos\frac{2\pi}{7}\right)+\left(\cos\frac{3\pi}{7}-\cos\frac{3\pi}{7}\right)+\cos\pi$$

$=\cos\pi=-1$

13. b. $2\sin^2\theta+3\cos^2\theta=2(\sin^2\theta+\cos^2\theta)+\cos^2\theta=2+\cos^2\theta\geq 2$
$[\because \cos^2\theta>0]$

14. b. $\sin^4\theta+\cos^4\theta=(\sin^2\theta+\cos^2\theta)^2-2\sin^2\theta\cos^2\theta=1-2\sin^2\theta\cos^2\theta\leq 1$

15. d. We have

$f(x)=\cos^2\theta+\sec^2\theta=(\cos\theta-\sec\theta)^2+2\cos\theta\sec\theta=2+(\cos\theta-\sec\theta)^2\geq 2$

16. a. $f(x)=\cos^6 x+\sin^6 x$

$=(\cos^2 x+\sin^2 x)(\sin^4 x+\cos^4 x-\cos^2 x\sin^2 x)$

$=((\sin^2 x+\cos^2 x)^2-3\sin^2 x\cos^2 x)$

$=1-\frac{3}{4}\sin^2 2x$

$\Rightarrow\ f(x)\in\left[\frac{1}{4},1\right]$

17. a. $f(x)=3\cos x+5\sin(x-\pi/6)$

$=\frac{1}{2}\cos x+5\times\frac{\sqrt{3}}{2}\sin x$

Then, $-\sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{5\sqrt{3}}{2}\right)^2}\leq f(x)\leq\sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{5\sqrt{3}}{2}\right)^2}$

$\Rightarrow\ -\sqrt{19}\leq f(x)\leq\sqrt{19}$

18. d. $1-\cos 2x+\sin 2x=2k$

$\Rightarrow\ \sin 2x-\cos 2x=2k-1$

$\Rightarrow\ \sin(2x-\alpha)=\frac{2k-1}{\sqrt{2}}$

$\Rightarrow\ -1\leq\frac{2k-1}{\sqrt{2}}\leq 1$

$\Rightarrow\ \frac{1-\sqrt{2}}{2}\leq k\leq\frac{1+\sqrt{2}}{2}$

19. d. $\cot(\alpha+\beta)=0\Rightarrow\alpha+\beta=\pi/2+n\pi, n\in I$

$\Rightarrow\ \sin(\alpha+2\beta)=\sin(90°+\beta)=\cos\beta$ (for $n=0$).

20. b. We have

$$\frac{x}{\cos\theta}=\frac{y}{\cos\left(\theta-\frac{2\pi}{3}\right)}=\frac{z}{\cos\left(\theta+\frac{2\pi}{3}\right)}$$

Therefore, each ratio is equal to

$$\frac{x+y+z}{\cos\theta+\cos\left(\theta-\frac{2\pi}{3}\right)+\cos\left(\theta+\frac{2\pi}{3}\right)}=\frac{x+y+z}{0}$$

$\Rightarrow \quad x+y+z=0.$

21. b. Since $0 \le \sin^{2n} x \le \sin^2 x$

$0 \le \cos^{2n} x \le \cos^2 x$ [as $\sin^4 x = \sin^2 x \sin^2 x \le \sin^2 x$, $\sin^4 x \le \sin^2 x$ and so on]

$\Rightarrow \quad 0 \le \sin^{2n} x + \cos^{2n} x \le \sin^2 x + \cos^2 x = 1$

$\Rightarrow \quad 0 \le \sin^{2n} x + \cos^{2n} x \le 1$

22. b. $4x^2 - 2\sqrt{5}\,x + 1 = 0$

Let α and β be the roots, we have

$$\alpha+\beta=\frac{2\sqrt{5}}{4}=\frac{\sqrt{5}}{2},\ \alpha\beta=\frac{1}{4}$$

Since $\sin 18° = \dfrac{\sqrt{5}-1}{4}$, $\cos 36° = \dfrac{\sqrt{5}+1}{4}$

$$\therefore \quad \sin 18° + \cos 36° = \frac{2\sqrt{5}}{4}=\frac{\sqrt{5}}{2} \quad \sin 18° \cos 36° = \frac{5-1}{16}=\frac{4}{16}=\frac{1}{4}$$

Here the required roots are $\sin 18°$, $\cos 36°$.

23. b. $$\sqrt{2\cot\alpha+\frac{1}{\sin^2\alpha}}=\sqrt{2\cot\alpha+\operatorname{cosec}^2\alpha}$$

$$=\sqrt{2\cot\alpha+1+\cot^2\alpha}=|1+\cot\alpha|=-1-\cot\alpha$$

[since $\cot\alpha < -1$ when $3\pi/4 < \alpha < \pi$, $|1+\cot\alpha| = -1-\cot\alpha$]

24. d. $$f(\theta)=5\cos\theta+3\cos\left(\theta+\frac{\pi}{3}\right)+3=5\cos\theta+\frac{3}{2}\cos\theta-\frac{3\sqrt{3}}{2}\sin\theta+3=\frac{13}{2}\cos\theta-\frac{3\sqrt{3}}{2}\sin\theta+3$$

$$=\sqrt{\left(\frac{169}{4}+\frac{27}{4}\right)}\sin(\theta-\alpha)+3.$$ Thus, the range of $f(\theta)$ is $[-4, 10]$.

25. b. Since $\alpha<\beta<\gamma<\delta$ and $\sin\alpha=\sin\beta=\sin\gamma=\sin\delta=K$, therefore $\beta=\pi-\alpha$, $\gamma=2\pi+\alpha$, $\delta=3\pi-\alpha$

$$\Rightarrow \quad 4\sin\frac{\alpha}{2}+3\sin\frac{\beta}{2}+2\sin\frac{\gamma}{2}+\sin\frac{\delta}{2}=4\sin\frac{\alpha}{2}+3\cos\frac{\alpha}{2}-2\sin\frac{\alpha}{2}-\cos\frac{\alpha}{2}$$

$$=2\sin\frac{\alpha}{2}+2\cos\frac{\alpha}{2}=2\sqrt{1+\sin\alpha}=2\sqrt{1+K}$$

26. c. Let O be the centre of the circle

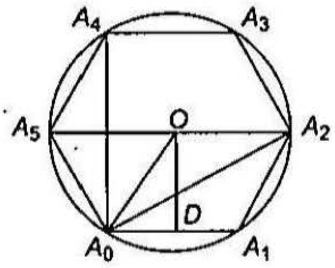


Fig. 2.37

Since $\angle A_0OA_1 = \dfrac{360°}{6} = 60°$

A_0OA_1 is an equilateral triangle, we get $A_0A_1 = 1$ [radius of circle = 1]

Also $A_0A_2 = A_0A_4 = 2OD = 2[OA_0] \sin 60° = 2(1)\dfrac{\sqrt{3}}{2} = \sqrt{3}$

$\therefore \quad (A_0A_1)(A_0A_2)(A_0A_4) = (1)(\sqrt{3})(\sqrt{3}) = 3$

27. d. The given relation is satisfied only when $\sin\theta_1 = \sin\theta_2 = \sin\theta_3 = 1$

$\Rightarrow \quad \cos\theta_1 = \cos\theta_2 = \cos\theta_3 = 0$

$\Rightarrow \quad \cos\theta_1 + \cos\theta_2 + \cos\theta_3 = 0$

28. c. $\sin^2\theta \le 1$

$\Rightarrow \quad \dfrac{x^2+y^2+1}{2x} \le 1 \qquad \Rightarrow x^2+y^2-2x+1 \le 0$ [as $x > 0$]

$\Rightarrow \quad (x-1)^2 + y^2 \le 0$

It is possible, iff $x - 1$ and $y = 0$.

29. a. $\sin(\alpha+\beta) = 1 \Rightarrow \alpha+\beta = \dfrac{\pi}{2}$ $\sin(\alpha-\beta) = \dfrac{1}{2} \Rightarrow \alpha-\beta = \dfrac{\pi}{6}$

Solving, we get $\alpha = \pi/3$ and $\beta = \pi/6$

Now $\tan(\alpha+2\beta)\tan(2\alpha+\beta) = \tan\left(\dfrac{\pi}{3}+\dfrac{\pi}{3}\right)\tan\left(\dfrac{2\pi}{3}+\dfrac{\pi}{6}\right) = \tan\dfrac{2\pi}{3}\tan\dfrac{5\pi}{6} = \left(-\cot\dfrac{\pi}{3}\right)\left(-\cot\dfrac{\pi}{6}\right)$

$$= \left(-\frac{1}{\sqrt{3}}\right)(-\sqrt{3}) = 1$$

30. a. $\sin 27° - \sin 63° = -2\cos 45° \sin 18°$

$$= -\sqrt{2}\left(\frac{\sqrt{5}-1}{4}\right) = -\frac{\sqrt{5}-1}{2\sqrt{2}} = -\frac{\sqrt{3-\sqrt{5}}}{2}$$

31. c. $\operatorname{cosec}\theta - \cot\theta = q$ (i)

$\therefore \quad \operatorname{cosec}\theta + \cot\theta = \dfrac{1}{q}$

$\therefore \quad \operatorname{cosec}\theta = \dfrac{1}{2}[q + (1/q)]$ (on addition).

32. a. Squaring both the sides, we get

$1 + \sin 2\theta = \dfrac{1}{25}$

$\Rightarrow \sin 2\theta = -\dfrac{24}{25}$

Let $t = \tan\theta$, we get $\dfrac{2t}{1+t^2} = -\dfrac{24}{25}$

$\Rightarrow 50t + 24 + 24t^2 = 0$
$\Rightarrow 12t^2 + 25t + 12 = 0$
$\Rightarrow (4t+3)(3t+4) = 0$
$\Rightarrow t = -4/3$ (as for $t = -3/4$ (rejected) as if $\tan\theta = -3/4$, then $\theta \in [\pi/2, \pi)$ and $\sin\theta + \cos\theta = -1/5$)

33. c. Multiplying x above and below by $1 - \cos\theta + \sin\theta$, we get

$$x = \frac{2\sin\theta(1-\cos\theta+\sin\theta)}{(1+\sin\theta)^2 - \cos^2\theta} = \frac{2\sin\theta(1-\cos\theta+\sin\theta)}{(1+\sin\theta)^2 - (1-\sin^2\theta)}$$

Putting $1 - \sin^2\theta = (1+\sin\theta)(1-\sin\theta)$, we get $\dfrac{2\sin\theta}{2\sin\theta}\dfrac{1-\cos\theta+\sin\theta}{1+\sin\theta} = x.$

34. b. $2n\theta = \pi/2$

$\therefore \quad \theta, (2n-1)\theta = (\pi/2) - \theta; 2\theta, (2n-2)\theta = (\pi/2) - 2\theta, \ldots$

They form complementary angles A and B so that $\tan A \tan B = \tan A \cot A = 1$ for each pair.

35. a. $N^r = 2[(\sin 1° + \sin 89°) + (\sin 2° + \sin 88°) + \cdots + (\sin 44° + \sin 46°) + \sin 45°]$

$$\Rightarrow \frac{N^r}{D^r} = 2\{\sin 45°[2(\cos 44° + \cos 43° + \cdots + \cos 1°)] + 1\}$$

$$= 2\sin 45°$$

$$= \sqrt{2}$$

36. b. $\sec\alpha + \operatorname{cosec}\alpha = p$, $\sec\alpha \operatorname{cosec}\alpha = q$

$$\Rightarrow \frac{\sin\alpha + \cos\alpha}{\sin\alpha\cos\alpha} = p \text{ and } \frac{1}{\sin\alpha\cos\alpha} = q$$

$$\Rightarrow \frac{1 + 2\sin\alpha\cos\alpha}{\sin^2\alpha\cos^2\alpha} = p^2$$

$$\Rightarrow \frac{1 + \frac{2}{q}}{\frac{1}{q^2}} = p^2$$

$$\Rightarrow q^2\left(1 + \frac{2}{q}\right) = p^2 \Rightarrow q(q+2) = p^2$$

37. c. We have $\sin x + \sin^2 x = 1$

$\Rightarrow \sin x = 1 - \sin^2 x \Rightarrow \sin x = \cos^2 x$

Now $\cos^{12} x + 3\cos^{10} x + 3\cos^8 x + \cos^6 x - 2 = \sin^6 x + 3\sin^5 x + 3\sin^4 x + \sin^3 x - 2$

$= (\sin^2 x)^3 + 3(\sin^2 x)^2 \sin x + 3(\sin^2 x)(\sin x)^2 + (\sin x)^3 - 2$

$= (\sin^2 x + \sin x)^3 - 2 = (1)^3 - 2 = -1$

38. a. $\cos(A - B) = \dfrac{3}{5}$

$\Rightarrow 5\cos A\cos B + 5\sin A\sin B = 3$ (i)

From 2nd relation, we have

$\sin A\sin B = 2\cos A\cos B$ (ii)

$\Rightarrow \cos A\cos B = \dfrac{1}{5}$ and $\sin A\sin B = \dfrac{2}{5}$

39. a. $(1 + \tan A)(1 + \tan B) = 2$

$\Rightarrow \tan A + \tan B = 1 - \tan A\tan B$

$\Rightarrow \tan(A + B) = 1$, i.e., $A + B = \dfrac{\pi}{4}$

or $\alpha + 4\alpha = \dfrac{\pi}{4}$, i.e., $\alpha = \dfrac{\pi}{20}$

40. a. $A = \dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{2}} = \sqrt{2}\sin 90°$

$B = \sqrt{2}\left[\dfrac{1}{\sqrt{2}}\sin 44° + \dfrac{1}{\sqrt{2}}\cos 44°\right] = \sqrt{2}\sin(45° + 44°)$

$= \sqrt{2}\sin 89° < \sqrt{2}\sin 90° = \sqrt{2}$ $\quad \therefore A > B \Rightarrow$ (a)

41. d. $\dfrac{1}{4}(\sqrt{3}\cos 23° - \sin 23°) = \dfrac{1}{2}(\cos 30°\cos 23° - \sin 30°\sin 23°) = \dfrac{1}{2}\cos(30° + 23°) = \dfrac{1}{2}\cos 53°$

42. b. $\tan\left(\dfrac{\theta_1 - \theta_2}{2}\right)\tan\left(\dfrac{\theta_1 + \theta_2}{2}\right) = \dfrac{\sin\left(\dfrac{\theta_1 + \theta_2}{2}\right)\sin\left(\dfrac{\theta_1 - \theta_2}{2}\right)}{\cos\left(\dfrac{\theta_1 + \theta_2}{2}\right)\cos\left(\dfrac{\theta_1 - \theta_2}{2}\right)}$

$$= \frac{\cos\theta_2 - \cos\theta_1}{\cos\theta_1 + \cos\theta_2} = \frac{-1}{3}$$

43. b. $\dfrac{3 + \dfrac{\cos 80°\cos 20°}{\sin 80°\sin 20°}}{\dfrac{\cos 80°}{\sin 80°} + \dfrac{\cos 20°}{\sin 20°}} = \dfrac{2\sin 80°\sin 20° + (\cos 80°\cos 20° + \sin 80°\sin 20°)}{\sin 20°\cos 80° + \cos 20°\sin 80°}$

$$= \frac{-\cos 100° + \cos 60° + \cos 60°}{\sin 100°} = \frac{1 - \cos 100°}{\sin 100} = \tan 50°$$

44. a. $\tan\beta = 2\sin\alpha\sin\gamma\,\mathrm{cosec}\,(\alpha + \gamma) = \dfrac{2\sin\alpha\sin\gamma}{\sin(\alpha + \gamma)}$

$\Rightarrow \cot\beta = \dfrac{\sin(\alpha + \gamma)}{2\sin\alpha\sin\gamma}$

$$\Rightarrow \quad 2\cot\beta = \frac{\sin\alpha\cos\gamma + \cos\alpha\sin\gamma}{\sin\alpha\sin\gamma} = \cot\alpha + \cot\gamma$$

$\Rightarrow$ $\cot\alpha, \cot\beta, \cot\gamma$ are in A.P.

45. b. $3\sin A\cos B = \sin B\cos A$

$$\Rightarrow \quad \cos A\sin B = \frac{3}{4}$$

$$\Rightarrow \quad \sin(A+B) = 1$$

$$\Rightarrow \quad C = \frac{\pi}{2},\ B = \frac{\pi}{2} - A$$

$$\Rightarrow \quad 3\tan A = \tan\left(\frac{\pi}{2} - A\right)$$

$$\Rightarrow \quad 3 = \cot^2 A$$

46. d. $\tan(100° + 125°) = \dfrac{\tan 100° + \tan 125°}{1 - \tan 100° \tan 125°}$

$$\therefore \quad \tan 225° = \frac{\tan 100° + \tan 125°}{1 - \tan 100° \tan 125°}, \text{ i.e., } 1 = \frac{\tan 100° + \tan 125°}{1 - \tan 100° \tan 125°}$$

i.e., $\tan 100° + \tan 125° + \tan 100° \tan 125° = 1$

47. b. We know that $\tan(20° + 40°) = \dfrac{\tan 20° + \tan 40°}{1 - \tan 20° \tan 40°}$

$$\Rightarrow \quad \sqrt{3} = \frac{\tan 20° + \tan 40°}{1 - \tan 20° \tan 40°}$$

$$\Rightarrow \quad \sqrt{3} - \sqrt{3}\tan 20° \tan 40° = \tan 20° + \tan 40°$$

$$\Rightarrow \quad \tan 20° + \tan 40° + \sqrt{3}\tan 20° \tan 40° = \sqrt{3}$$

48. c. $\dfrac{\sqrt{2} - \sin\alpha - \cos\alpha}{\sin\alpha - \cos\alpha}$

$$= \frac{\sqrt{2} - \sqrt{2}\left(\frac{1}{\sqrt{2}}\sin\alpha + \frac{1}{\sqrt{2}}\cos\alpha\right)}{\sqrt{2}\left(\frac{1}{\sqrt{2}}\sin\alpha - \frac{1}{\sqrt{2}}\cos\alpha\right)}$$

$$= \frac{\sqrt{2} - \sqrt{2}\cos\left(\alpha - \frac{\pi}{4}\right)}{\sqrt{2}\sin\left(\alpha - \frac{\pi}{4}\right)}$$

$$= \frac{\sqrt{2}(1 - \cos\theta)}{\sqrt{2}\sin\theta}, \text{ where } \theta = \alpha - \frac{\pi}{4} = \frac{2\sin^2(\theta/2)}{2\sin(\theta/2)\cos(\theta/2)} = \tan\frac{\theta}{2} = \tan\left(\frac{\alpha}{2} - \frac{\pi}{8}\right)$$

49. b. $\sin\theta_1 - \sin\theta_2 = a$, $\cos\theta_1 + \cos\theta_2 = b$

$$\Rightarrow \quad a^2 + b^2 = 2 + 2\cos(\theta_1 + \theta_2)$$

$$\Rightarrow \quad 0 \le a^2 + b^2 \le 4$$

50. c. $\dfrac{1+\sin 2x}{1-\sin 2x}=\dfrac{(\sin x+\cos x)^2}{(\sin x-\cos x)^2}=\left(\dfrac{1+\tan x}{1-\tan x}\right)^2=\left(\tan\left(\dfrac{\pi}{4}+x\right)\right)^2=\tan^2\left(\dfrac{\pi}{4}+x\right)$

$$=\cot^2\left(\frac{\pi}{2}+\frac{\pi}{4}+x\right)=\cot^2\left(\frac{3\pi}{4}+x\right)$$

$$\Rightarrow \quad a=\frac{3\pi}{4}$$

51. d. We have $4x^2-16x+15<0 \Rightarrow \dfrac{3}{2}<x<\dfrac{5}{2}$

Therefore, the integral solution of $4x^2-16x+15<0$ is $x=2$

Thus, $\tan\alpha=2$. It is given that $\cos\beta=\tan 45°=1$.

$$\therefore \quad \sin(\alpha+\beta)\sin(\alpha-\beta)=\sin^2\alpha-\sin^2\beta=\frac{1}{1+\cot^2\alpha}-(1-\cos^2\beta)=\frac{1}{1+\frac{1}{4}}-0=\frac{4}{5}$$

52. b. $\dfrac{\cos(x-y)}{\cos(x+y)}+\dfrac{\cos(z+t)}{\cos(z-t)}=0$

$$\Rightarrow \quad \frac{1+\tan x\tan y}{1-\tan x\tan y}+\frac{1-\tan z\tan t}{1+\tan z\tan t}=0$$

$$\Rightarrow \quad 1+\tan z\tan t+\tan x\tan y+\tan x\tan y\tan z\tan t+1-\tan z\tan t-\tan x\tan y$$
$$+\tan x\tan y\tan z\tan t=0$$

$$\Rightarrow \quad \tan x\tan y\tan z\tan t=-1.$$

53. b. $f(n)=2\cos nx$

$$\Rightarrow \quad f(1)f(n+1)-f(n)=4\cos x\cos(n+1)x-2\cos nx=2[2\cos(n+1)x\cos x-\cos nx]$$
$$=2[\cos(n+2)x+\cos nx-\cos nx]=2\cos(n+2)x=f(n+2).$$

54. a. $\dfrac{\tan A}{\tan B}=\dfrac{1}{3} \Rightarrow \dfrac{\sin A\cos B}{\cos A\sin B}=\dfrac{1}{3}$

Put $\sin A\cos B=\dfrac{1}{4}$

$$\Rightarrow \quad \cos A\sin B=\frac{3}{4}$$

$$\Rightarrow \quad \sin(A+B)=\frac{1}{4}+\frac{3}{4}=1$$

$$\Rightarrow \quad \sin C=1=\sin\pi/2$$

$\Rightarrow \quad C=\pi/2$. Hence, the triangle is right angled.

55. b. $3\sin^2 A+2\sin^2 B=1$

$$\Rightarrow \quad 3\sin^2 A=\cos 2B$$

Also $3\sin 2A-2\sin 2B=0$

$$\Rightarrow \quad \sin 2B=\frac{3}{2}\sin 2A$$

Now, $\cos(A+2B)=\cos A\cos 2B-\sin A\sin 2B=\cos A\, 3\sin^2 A-\sin A\,\frac{3}{2}\sin 2A$

$$=3\sin^2 A\cos A-3\sin^2 A\cos A=0$$

$\therefore A+2B=\pi/2$

56. a. $f(\beta)=f\left(\frac{5\pi}{4}-\alpha\right)=\dfrac{\cot\left(\frac{5\pi}{4}-\alpha\right)}{1+\cot\left(\frac{5\pi}{4}-\alpha\right)}$

$$=\frac{1}{1+\tan\left(\frac{5\pi}{4}-\alpha\right)}$$

$$=\frac{1}{1+\dfrac{1-\tan\alpha}{1+\tan\alpha}}=\frac{1+\tan\alpha}{2}$$

As $f(\alpha)=\dfrac{\cot\alpha}{1+\cot\alpha}=\dfrac{1}{1+\tan\alpha}$, we have $f(\alpha)f(\beta)=\dfrac{1}{2}$

57. c. $A-B=\frac{\pi}{4}\Rightarrow\tan(A-B)=\tan\frac{\pi}{4}$

$\Rightarrow\ \dfrac{\tan A-\tan B}{1+\tan A\tan B}$

$\Rightarrow\ \tan A-\tan B-\tan A\tan B=1$

$\Rightarrow\ \tan A-\tan B-\tan A\tan B+1=2$

$\Rightarrow\ (1+\tan A)(1-\tan B)=2\Rightarrow y=2$

Hence, $(y+1)^{y+1}=(2+1)^{2+1}=(3)^3=27$.

58. a. Applying $b-a=c-b$ for A.P., we get $2\cos z\sin(x-y)=2\cos x\sin(y-z)$

Dividing by $2\cos x\cos y\cos z$, etc., we get $\tan x-\tan y=\tan y-\tan z$.

59. b. $(\cos\alpha+\cos\beta)^2-(\sin\alpha+\sin\beta)^2=0$

$\Rightarrow\ (\cos^2\alpha+\cos^2\beta+2\cos\alpha\cos\beta)-(\sin^2\alpha+\sin^2\beta+2\sin\alpha\sin\beta)=0$

$\Rightarrow\ \cos 2\alpha+\cos 2\beta=-2(\cos\alpha\cos\beta-\sin\alpha\sin\beta)$

$$=-2\cos(\alpha+\beta)$$

60. a. We have

$\sin\alpha\sec x_1\sec x_2+\sin\alpha\sec x_2\sec x_3+\cdots+\sin\alpha\sec x_{n-1}\sec x_n$

$$=\frac{\sin(x_2-x_1)}{\cos x_1\cos x_2}+\frac{\sin(x_3-x_2)}{\cos x_2\cos x_3}+\cdots+\frac{\sin(x_n-x_{n-1})}{\cos x_{n-1}\cos x_n}$$

$$=(\tan x_2-\tan x_1)+(\tan x_3-\tan x_2)+\cdots+(\tan x_n-\tan x_{n-1})$$

$$=\tan x_n-\tan x_1=\frac{\sin(x_n-x_1)}{\cos x_n\cos x_1}=\frac{\sin(n-1)\alpha}{\cos x_n\cos x_1}.\qquad [\because x_n=x_1+(n-1)\alpha]$$

61. a. By the given conditions $\tan\frac{\pi}{9}+\tan\frac{5\pi}{18}=2x$

$\tan\frac{\pi}{9}+\tan\frac{7\pi}{18}=2y$

$$\Rightarrow 2x=\tan 20°+\tan 50°$$

$$=\frac{\sin 20°}{\cos 20°}+\frac{\sin 50°}{\cos 50°}$$

$$=\frac{\sin 20°\cos 50°+\cos 20°\sin 50°}{\cos 20°\cos 50°}$$

$$=\frac{\sin 70°}{\cos 20°\cos 50°}$$

$$=\frac{\cos 20°}{\cos 20°\cos 50°}=\frac{1}{\cos 50°}=\frac{1}{\sin 40°}=\operatorname{cosec} 40°$$

$$2y=\tan 20°+\tan 70°$$

$$=\frac{\sin 20°}{\cos 20°}+\frac{\sin 70°}{\cos 70°}$$

$$=\frac{\sin 90°}{\cos 20°\cos 70°}$$

$$=\frac{1}{\cos 20°\cos 70°}=\frac{1}{\cos 20°\sin 20°}$$

$$=\frac{2}{2\sin 20°\cos 20°}=\frac{2}{\sin 40°}=2\operatorname{cosec} 40°$$

$\therefore\ 2y=2(2x)\Rightarrow y=2x$

62. b. $\frac{1}{\sin 1°}\left[\frac{\sin(1°-0°)}{\cos 0°\cos 1°}+\frac{\sin(2°-1°)}{\cos 1°\cos 2°}+\frac{\sin(3°-2°)}{\cos 2°\cos 3°}+\cdots+\frac{\sin(45°-44°)}{\cos 44°\cos 45°}\right]$

$$=\frac{1}{\sin 1°}[\tan 1°+(\tan 2°-\tan 1°)+(\tan 3°-\tan 2°)+(\tan 4°-\tan 3°)+\cdots+(\tan 45°-\tan 44°)]$$

$$=\frac{1}{\sin 1°}=\frac{1}{x}$$

63. a. We have $\sin\alpha+\sin\beta=-\frac{21}{65}$ (i)

$\cos\alpha+\cos\beta=-\frac{17}{65}$ (ii)

Squaring Eq. (i), we get $\sin^2\alpha+\sin^2\beta+2\sin\alpha\sin\beta=\left(\frac{21}{65}\right)^2$ (iii)

Squaring Eq. (ii), we get $\cos^2\alpha+\cos^2\beta+2\cos\alpha\cos\beta=\left(\frac{27}{65}\right)^2$ (iv)

Adding Eqs. (iii) and (iv), we get $2+2\cos(\alpha-\beta)=\dfrac{1}{(65)^2}\left[(27)^2+(21)^2\right]=\dfrac{1}{(65)^2}(729+441)$

$$\Rightarrow \quad 2+2\cos(\alpha-\beta)=\frac{1}{(65)^2}(1170)=\frac{18}{65}$$

$$\Rightarrow \quad 1+\cos(\alpha-\beta)=\frac{9}{65}$$

$$\Rightarrow \quad 2\cos^2\frac{\alpha-\beta}{2}=\frac{9}{65}$$

$$\Rightarrow \quad \cos\frac{\alpha-\beta}{2}=-\frac{3}{\sqrt{130}} \qquad \left[\because \pi<\alpha-\beta<3\pi \Rightarrow \frac{\pi}{2}<\frac{\alpha-\beta}{2}<\frac{3\pi}{2}\Rightarrow \cos\left(\frac{\alpha-\beta}{2}\right)<0\right]$$

64. b. $\dfrac{\sin(x+y)}{\sin(x-y)}=\dfrac{a+b}{a-b}$

$$\Rightarrow \quad \frac{\sin(x+y)+\sin(x-y)}{\sin(x+y)-\sin(x-y)}=\frac{(a+b)+(a-b)}{(a+b)-(a-b)}$$

$$\Rightarrow \quad \frac{2\sin x\cos y}{2\cos x\sin y}=\frac{2a}{2b}$$

$$\Rightarrow \quad \frac{\tan x}{\tan y}=\frac{a}{b}$$

65. c. $\dfrac{\sin 3\theta+\sin 5\theta+\sin 7\theta+\sin 9\theta}{\cos 3\theta+\cos 5\theta+\cos 7\theta+\cos 9\theta}=\dfrac{(\sin 3\theta+\sin 9\theta)+(\sin 5\theta+\sin 7\theta)}{(\cos 3\theta+\cos 9\theta)+(\cos 5\theta+\cos 7\theta)}$

$$=\frac{2\sin 6\theta\cos 3\theta+2\sin 6\theta\cos\theta}{2\cos 6\theta\cos 3\theta+2\cos 6\theta\cos\theta}=\frac{2\sin 6\theta(\cos 3\theta+\cos\theta)}{2\cos 6\theta(\cos 3\theta+\cos\theta)}=\tan 6\theta$$

66. b. $\dfrac{\sin x-\sin z}{\cos z-\cos x}=\dfrac{2\cos\left(\dfrac{x+z}{2}\right)\sin\left(\dfrac{x-z}{z}\right)}{2\sin\left(\dfrac{x+z}{z}\right)\sin\left(\dfrac{x-z}{z}\right)}=\cot\left(\dfrac{x+z}{2}\right)=\cot(y)$

67. c. $\cos 50°=\cos^2 25°-\sin^2 25°=(\cos 25°+\sin 25°)(\cos 25°-\sin 25°)=p(\cos 25°-\sin 25°)$ (i)

Now $(\cos 25°-\sin 25°)^2+(\cos 25°+\sin 25°)^2=1+1$

$\therefore \quad \cos 25°-\sin 25°=\sqrt{2-p^2}$ (ii)

We have taken +ve sign as $\cos 25°>\sin 25°$, therefore $\cos 50°=p\sqrt{2-p^2}$, by Eqs. (i) and (ii).

68. b. $\dfrac{\sin^2 A-\sin^2 B}{\sin A\cos A-\sin B\cos B}=\dfrac{2\sin(A+B)\sin(A-B)}{\sin 2A-\sin 2B}=\dfrac{2\sin(A+B)\sin(A-B)}{2\sin(A-B)\cos(A+B)}$

$$=\tan(A+B)$$

69. a. $\tan A=\dfrac{1-\cos B}{\sin B}=\dfrac{2\sin^2(B/2)}{2\sin(B/2)\cos(B/2)}=\tan\dfrac{B}{2} \quad \Rightarrow \tan 2A=\tan B$

70. b. On adding, we get $a=\dfrac{3-\cos 4\theta+4\sin 2\theta}{2}=(1+\sin 2\theta)^2$

On subtracting, we get $b=(1-\sin 2\theta)^2 \Rightarrow ab=\cos^4 2\theta\le 1$

71. c. $\cos^2 10° - \cos 10° \cos 50° + \cos^2 50°$

$$= \frac{1}{2}\left[1 + \cos 20° - (\cos 60° + \cos 40°) + (1 + \cos 100°)\right]$$

$$= \frac{1}{2}[1 + \cos 20° - \frac{1}{2} - \cos 40° + 1 - \cos 80°]$$

$$= \frac{1}{2}\left[\frac{3}{2} + \cos 20° - (2 \cos 60° \cos 20°)\right] = \frac{3}{4}$$

72. a. $\tan 20° \tan 80° \cot 50° = \tan 20° \tan 80° \tan 40°$

$$= \tan 20° \tan(60° - 20°) \tan(60° + 20°) = \tan 60° = \sqrt{3}$$

73. b. $\tan^2 \theta = 2 \tan^2 \phi + 1$

$\Rightarrow \quad 1 + \tan^2 \theta = 2(1 + \tan^2 \phi)$

$\Rightarrow \quad \sec^2 \theta = 2 \sec^2 \phi$

$\Rightarrow \quad \cos^2 \phi = 2 \cos^2 \theta$

$\qquad\qquad = 1 + \cos 2\theta$

$\Rightarrow \quad \cos 2\theta = \cos^2 \phi - 1$

$\qquad\qquad = -\sin^2 \phi$

$\Rightarrow \quad \sin^2 \phi + \cos 2\theta = 0$

74. b. $\cot 70° + 4 \cos 70° = \dfrac{\cos 70° + 4 \sin 70° \cos 70°}{\sin 70°}$

$$= \frac{\cos 70° + 2 \sin 140°}{\sin 70°}$$

$$= \frac{\cos 70° + 2 \sin(180° - 40°)}{\sin 70°}$$

$$= \frac{\sin 20° + \sin 40° + \sin 40°}{\sin 70°}$$

$$= \frac{2 \sin 30° \cos 10° + \sin 40°}{\sin 70°}$$

$$= \frac{\sin 80° + \sin 40°}{\sin 70°}$$

$$= \frac{2 \sin 60° \cos 20°}{\sin 70°} = \sqrt{3}$$

75. b. $a \cos x + b \sin x = c$

$$\Rightarrow \quad \frac{a\left(1 - \tan^2 \frac{x}{2}\right)}{1 + \tan^2 \frac{x}{2}} + \frac{2b \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} = c$$

$$\Rightarrow \quad (c+a)\tan^2\frac{x}{2}-2b\tan\frac{x}{2}+c-a=0$$

$$\Rightarrow \quad \tan\frac{x_1}{2}+\tan\frac{x_2}{2}=\frac{2b}{c+a} \text{ and } \tan\frac{x_1}{2}\tan\frac{x_2}{2}=\frac{c-a}{c+a}$$

$$\Rightarrow \quad \tan\left(\frac{x_1+x_2}{2}\right)=\frac{\frac{2b}{c+a}}{1-\frac{c-a}{c+a}}=\frac{2b}{2a}=\frac{b}{a}$$

76. c. $\tan y=\dfrac{1+\sqrt{1-x}}{1+\sqrt{1+x}}$

If $x=\cos\theta$, then $\sqrt{1-x}=\sqrt{2}\sin(\theta/2)$, $\sqrt{1+x}=\sqrt{2}\cos(\theta/2)$

$$\Rightarrow \quad \tan y=\frac{\sqrt{2}\left[\frac{1}{\sqrt{2}}+\sin\frac{\theta}{2}\right]}{\sqrt{2}\left[\frac{1}{\sqrt{2}}+\cos\frac{\theta}{2}\right]}=\frac{\sin\frac{\pi}{4}+\sin\frac{\theta}{2}}{\cos\frac{\pi}{4}+\cos\frac{\theta}{2}}$$

$$=\frac{2\sin\left(\frac{\pi}{8}+\frac{\theta}{4}\right)\cos\left(\frac{\pi}{8}-\frac{\theta}{4}\right)}{2\cos\left(\frac{\pi}{8}+\frac{\theta}{4}\right)\cos\left(\frac{\pi}{8}-\frac{\theta}{4}\right)}$$

$$=\tan\left(\frac{\pi}{8}+\frac{\theta}{4}\right)$$

$$\Rightarrow \quad 4y=\frac{\pi}{2}+\theta$$

$$\Rightarrow \quad \sin 4y=\cos\theta=x$$

77. d. We have $\cos x=\tan y$

$$\Rightarrow \quad \cos^2 x=\tan^2 y$$

$$=\sec^2 y-1$$

$$=\cot^2 z-1 \qquad [\because \cos y=\tan z, \sec y=\cot z]$$

$$\Rightarrow \quad 1+\cos^2 x=\cot^2 z$$

$$=\frac{\tan^2 x}{1-\tan^2 x} \qquad [\because \cos z=\tan x]$$

$$=\frac{\sin^2 x}{\cos^2 x-\sin^2 x}$$

$$\Rightarrow \quad 2\sin^4 x-6\sin^2 x+2=0$$

$$\Rightarrow \quad \sin^2 x=\frac{3-\sqrt{5}}{2}$$

$$\Rightarrow \quad \sin^2 x = \left(\frac{\sqrt{5}-1}{2}\right)^2$$

$$\Rightarrow \quad \sin x = \frac{\sqrt{5}-1}{2} = 2\sin 18^\circ$$

78. a. $\sin 2\theta = \cos 3\theta \quad \Rightarrow 2\sin\theta\cos\theta = 4\cos^3\theta - 3\cos\theta$

$\Rightarrow \quad 2\sin\theta = 4(1-\sin^2\theta) - 3 \Rightarrow 4\sin^2\theta + 2\sin\theta - 1 = 0$

$$\Rightarrow \quad \sin\theta = \frac{\sqrt{5}-1}{4}$$

79. b. $\tan\theta = \lambda$, we get $\dfrac{2\tan\theta/2}{1-\tan^2\theta/2} = \lambda$

$$\Rightarrow \quad \lambda\tan^2\frac{\theta}{2} + 2\tan\frac{\theta}{2} - \lambda = 0$$

$$\Rightarrow \quad \tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2} = -1$$

80. a. $\tan\theta = \sqrt{n} \quad \Rightarrow \cos 2\theta = \dfrac{1-\tan^2\theta}{1+\tan^2\theta} = \dfrac{1-n}{1+n}$ = rational.

81. b. $\sin x + \cos x = \dfrac{\sqrt{7}}{2}$

$$\Rightarrow \quad \frac{2\tan\frac{x}{2}}{\left(1+\tan^2\frac{x}{2}\right)} + \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}} = \frac{\sqrt{7}}{2}$$

$$\Rightarrow \quad (\sqrt{7}+2)\tan^2\frac{x}{2} - 4\tan\frac{x}{2} + (\sqrt{7}-2) = 0$$

$$\Rightarrow \quad \tan\frac{x}{2} = \frac{4 \pm \sqrt{16-4(7-4)}}{2(\sqrt{7}+2)} = \frac{1}{(\sqrt{7}+2)} \text{ as } \frac{x}{2} < \frac{\pi}{8}$$

$$= \frac{\sqrt{7}-2}{3}$$

82. b. $\sin\dfrac{7\pi}{8} = \sin\left(\pi - \dfrac{\pi}{8}\right) = \sin\dfrac{\pi}{8}$; $\sin\dfrac{5\pi}{8} = \sin\left(\pi - \dfrac{3\pi}{8}\right) = \sin\dfrac{3\pi}{8}$

Therefore, the given value $= 2\left[\sin^2\dfrac{\pi}{8} + \sin^2\dfrac{3\pi}{8}\right] = 2\left[\sin^2\dfrac{\pi}{8} + \cos^2\dfrac{\pi}{8}\right]$

$$= 2(1) = 2 \quad \left[\because \sin\frac{3\pi}{8} = \sin\left(\frac{\pi}{2} - \frac{\pi}{8}\right) = \cos\frac{\pi}{8}\right]$$

83. b. $4\cos^2\left(\dfrac{\pi}{4} - \dfrac{x}{2}\right) + \sqrt{4\sin^4 x + \sin^2 2x} = 4\cos^2\left(\dfrac{\pi}{4} - \dfrac{x}{2}\right) + \sqrt{4\sin^2 x(\cos^2 x + \sin^2 x)}$

$$= 2\left(1 + \cos\left(\frac{\pi}{2} - x\right)\right) + 2|\sin x| = 2 + 2\sin x - 2\sin x \text{ as } x \in \left(\pi, \frac{3\pi}{2}\right) = 2.$$

84. b. $\cos^3 x \sin 2x = \cos^2 x \cos x \sin 2x$

$$= \left(\frac{1-\cos 2x}{2}\right)\left(\frac{2\sin 2x \cos x}{2}\right)$$

$$= \frac{1}{4}(1-\cos 2x)(\sin 3x + \sin x)$$

$$= \frac{1}{4}\left[\sin 3x + \sin x - \frac{1}{2}(2\sin 3x.\cos 2x) - \frac{1}{2}(2\cos 2x \sin x)\right]$$

$$= \frac{1}{4}\left[\sin 3x + \sin x - \frac{1}{2}(\sin 5x + \sin x) - \frac{1}{2}(\sin 3x - \sin x)\right]$$

$$= \frac{1}{4}\left[\sin x + \frac{1}{2}\sin 3x - \frac{1}{2}\sin 5x\right]$$

$\Rightarrow \quad a_1 = 1/4, a_3 = 1/8, n = 5$

85. b. Given expression is $2\sin^2\phi + 4\cos(\theta+\phi)\sin\theta\sin\phi + \cos 2(\theta+\phi)$

$= (1-\cos 2\phi) + 4\cos(\theta+\phi)\sin\theta\sin\phi + 2\cos^2(\theta+\phi) - 1$

$= -\cos 2\phi + 4\cos(\theta+\phi)\sin\theta\sin\phi + 2\cos^2(\theta+\phi)$

$= -\cos 2\phi + 2\cos(\theta+\phi)[\cos(\theta+\phi) + 2\sin\theta\sin\phi]$

$= -\cos 2\phi + 2\cos(\theta+\phi)[\cos\theta\cos\phi + \sin\theta\sin\phi]$

$= -\cos 2\phi + 2\cos(\theta+\phi)\cos(\theta-\phi)$

$= -\cos 2\phi + \cos 2\theta + \cos 2\phi = \cos 2\theta$

86. b. $\dfrac{\cos 2B}{1} = \dfrac{\cos(A+C)}{\cos(A-C)}$

Applying componendo and dividendo, we get

$$\Rightarrow \quad \frac{1-\cos 2B}{1+\cos 2B} = \frac{\cos(A-C)-\cos(A+C)}{\cos(A-C)+\cos(A+C)}$$

$$\Rightarrow \quad \frac{2\sin^2 B}{2\cos^2 B} = \frac{2\sin A \sin C}{2\cos A \cos C}$$

$\Rightarrow \quad \tan^2 B = \tan A \tan C$

$\Rightarrow \quad \tan A, \tan B, \tan C$ are in G.P.

87. b. $\cos x = \dfrac{2\cos y - 1}{2 - \cos y}$

$$\Rightarrow \quad \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}} = \frac{\frac{2(1-\tan^2 y/2)}{1+\tan^2 y/2} - 1}{2 - \frac{1-\tan^2 y/2}{1+\tan^2 y/2}}$$

$\Rightarrow \quad 6\tan^2 y/2 = 2\tan^2\frac{x}{2}$

$\Rightarrow \quad \tan\frac{x}{2}\cot\frac{y}{2} = \sqrt{3}$

88. b. We have $\sqrt{\left(\frac{a+b}{a-b}\right)}+\sqrt{\left(\frac{a-b}{a+b}\right)} = \frac{a+b+a-b}{\sqrt{(a^2-b^2)}}$

$$= \frac{2a}{\sqrt{(a^2-b^2)}} = \frac{2}{\sqrt{[1-(b/a)^2]}}$$

$$= \frac{2}{\sqrt{(1-\tan^2 x)}} = \frac{2\cos x}{\sqrt{(\cos^2 x-\sin^2 x)}}$$

$$= \frac{2\cos x}{\sqrt{(\cos 2x)}}$$

89. b. Since α is a root of $25\cos^2\theta + 5\cos\theta - 12 = 0$

$\therefore \quad 25\cos^2\alpha + 5\cos\alpha - 12 = 0 \qquad \Rightarrow \cos\alpha = \frac{-5\pm\sqrt{25+1200}}{50} = -\frac{4}{5}\ \left[\because \frac{\pi}{2}<\alpha<\pi\right]$

and $\sin\alpha = \sqrt{1-\frac{16}{25}} = \frac{3}{5}$; therefore, $\sin 2\alpha = 2\sin\alpha\cos\alpha = 2\left(\frac{3}{5}\right)\left(\frac{-4}{5}\right) = \frac{-24}{25}$

90. c. We have $\tan 9° - \tan 27° - \tan 63° + \tan 81° = (\tan 9° + \tan 81°) - (\tan 27° + \tan 63°)$

$$= \frac{1}{\sin 9°\cos 9°} - \frac{1}{\sin 27°\cos 27°}$$

$$= \frac{2}{\sin 18°} - \frac{2}{\sin 54°}$$

$$= 2\left[\frac{\sin 54° - \sin 18°}{\sin 54°\sin 18°}\right]$$

$$= 2\left[\frac{2\cos 36°\sin 18°}{\sin 18°\cos 36°}\right] = 4$$

91. c. Let $A = \sin^{-1}a$, $B = \sin^{-1}b$ and $C = \sin^{-1}c$, we have $A + B + C = \pi$.

$$a\sqrt{1-a^2}+b\sqrt{1-b^2}+c\sqrt{1-c^2} = \frac{1}{2}(\sin 2A+\sin 2B+\sin 2C) = \frac{1}{2}[4\sin A\sin B\sin C] = 2abc$$

92. d. $\cos 2A + \cos 2B + \cos 2C = 2\cos(A+B)\cos(A-B) + \cos 2C$

$$= 2\cos\left(\frac{3\pi}{2}-C\right)\cos(A-B)+\cos 2C$$

$$= -2\sin C\cos(A-B) + 1 - 2\sin^2 C$$

$$= 1 - 2\sin C(\cos(A-B) + \sin C)$$

$$= 1 - 2\sin C\{\cos(A-B) + \sin[3\pi/2 - (A+B)]\}$$

$$= 1 - 2\sin C[\cos(A-B) - \cos(A+B)]$$

$= 1 - 4\sin A \sin B \sin C$

93. c. $\dfrac{2}{\tan\frac{B}{2}} = \dfrac{1}{\tan\frac{A}{2}} + \dfrac{1}{\tan\frac{C}{2}}$

$\Rightarrow \quad 2\tan\frac{A}{2}\tan\frac{C}{2} = \tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{B}{2}\tan\frac{A}{2} = 1 - \tan\frac{A}{2}\tan\frac{C}{2}$

$\Rightarrow \quad \tan\frac{A}{2}\tan\frac{C}{2} = \frac{1}{3}$

$\Rightarrow \quad \cot\frac{A}{2}\cot\frac{C}{2} = 3$

94. b. $\sin^2 A - \sin^2 B + \sin^2 C = \sin(A+B)\sin(A-B) + \sin^2 C = \sin C(\sin(A-B) + \sin C)$

$= \sin C(\sin(A-B) + \sin(A+B)) = 2\sin A\cos B\sin C$

95. c. $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = \dfrac{\tan^2\alpha}{1+\tan^2\alpha} + \dfrac{\tan^2\beta}{1+\tan^2\beta} + \dfrac{\tan^2\gamma}{1+\tan^2\gamma}$

$= \dfrac{x}{1+x} + \dfrac{y}{1+y} + \dfrac{z}{1+z}$ [where $x = \tan^2\alpha, y = \tan^2\beta, z = \tan^2\gamma$]

$= \dfrac{(x+y+z) + (xy+yz+zx+2xyz) + xy+yz+zx+xyz}{(1+x)(1+y)(1+z)}$

$= \dfrac{1+x+y+z+xy+yz+zx+xyz}{(1+x)(1+y)(1+z)} = 1$ [$\because xy+yz+zx+2xyz = \gamma$]

96. c. $D^r = \sin A + \sin B - \sin C$

$= 2\sin\frac{A+B}{2}\cos\frac{A-B}{2} - 2\sin\frac{C}{2}\cos\frac{C}{2}$

$= 2\cos\frac{C}{2}\left[\cos\left(\frac{A-B}{2}\right) - \sin\frac{C}{2}\right]$

$= 2\cos\frac{C}{2}\left[\cos\frac{A-B}{2} - \cos\frac{A+B}{2}\right]$

$= 2\cos\frac{C}{2}\left[2\sin\frac{A}{2}\sin\frac{B}{2}\right]$

$= 4\sin\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}$

Also $\sin A + \sin B + \sin C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

$\Rightarrow \quad \dfrac{\sin A + \sin B + \sin C}{\sin A + \sin B - \sin C} = \cot\frac{A}{2}\cot\frac{B}{2}$

97. a $\dfrac{\sin 2A + \sin 2B + \sin 2C}{\sin A + \sin B + \sin C} = \dfrac{4\sin A\sin B\sin C}{4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}} = 8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$ $\left[\because \sin A = 2\sin\frac{A}{2}\cos\frac{A}{2}\right]$

98. c. We have $\cos^2 A + \cos^2 B - (1 - \cos^2 C) = 0$

$\Rightarrow \quad \cos^2 A + \cos^2 B - \sin^2 C = 0$

$\Rightarrow \quad \cos^2 A + \cos(B + C)\cos(B - C) = 0$

$\Rightarrow \quad 2\cos A \cos B \cos C = 0$

Hence, either A or B or C is 90°.

99. a. In a triangle, $\tan A + \tan B + \tan C = \tan A \tan B \tan C$ (i)

$\Rightarrow \quad 6 = 2\tan C$

$\Rightarrow \quad \tan C = 3$

Also $\tan A + \tan B = 6 - 3 = 3$ (ii)

$\Rightarrow \quad \tan A$ and $\tan B$ are roots $x^2 - 3x + 2 = 0$ by Eqs. (i) and (ii).

$\Rightarrow \quad \tan A, \tan B = 2, 1$ or $1, 2$ and $\tan C = 3$.

100. a. We have $\tan 6° \tan 42° \tan 66° \tan 78°$

$= \tan 6° \tan(60° - 18°) \tan(60° + 6°) \tan(60° + 18°)$

$$= \frac{\tan 6° \tan(60°+6°) \tan 18° \tan(60°-18°) \tan(60°+18°)}{\tan 18°} = \frac{\tan 6° \tan(60°+6°) \tan(3\times 18°)}{\tan 18°}$$

$$= \frac{\tan 6° \tan(60°-6) \tan(60°+6)}{\tan 18°} = \frac{\tan 18°}{\tan 18°} = 1$$

101. c.

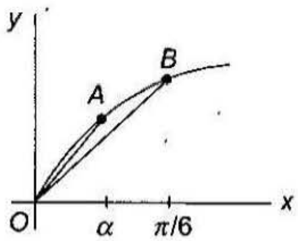


Fig. 2.38

In the graph of $y = \sin x$, let $A \equiv (\alpha, \sin\alpha)$, $B = \left(\frac{\pi}{6}, \sin\frac{\pi}{6}\right)$

Clearly, slope of $OA >$ slope of OB, so $\frac{\sin\alpha}{\alpha} > \frac{\sin \pi/6}{\pi/6} = \frac{3}{\pi} \Rightarrow \frac{\alpha}{\sin\alpha} < \frac{\pi}{3}$.

102.d. $(\alpha - \beta) = (\theta - \beta) - (\theta - \alpha)$

$\Rightarrow \cos(\alpha - \beta) = \cos(\theta - \beta)\cos(\theta - \alpha) + \sin(\theta - \beta)\sin(\theta - \alpha)$

$$= \frac{y}{b} \times \frac{x}{a} + \sqrt{1 - \frac{x^2}{a^2}}\sqrt{1 - \frac{y^2}{b^2}}$$

$$\Rightarrow \left[\frac{xy}{ab} - \cos(\alpha - \beta)\right]^2 = \left(1 - \frac{x^2}{a^2}\right)\left(1 - \frac{y^2}{b^2}\right)$$

$$\Rightarrow \frac{x^2y^2}{a^2b^2}+\cos^2(\alpha-\beta)-\frac{2xy}{ab}\cos(\alpha-\beta) = 1-\frac{y^2}{b^2}-\frac{x^2}{a^2}+\frac{x^2y^2}{a^2b^2}$$

$$\Rightarrow \frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{2xy}{ab}\cos(\alpha-\beta)=\sin^2(\alpha-\beta)$$

103. b. Let $u=\cos\theta\left\{\sin\theta+\sqrt{\sin^2\theta+\sin^2\alpha}\right\}$

$\Rightarrow \quad (u-\sin\theta\cos\theta)^2=\cos^2\theta(\sin^2\theta+\sin^2\alpha)$

$\Rightarrow \quad u^2\tan^2\theta-2u\tan\theta+u^2-\sin^2\alpha=0$

Since $\tan\theta$ is real, $4u^2-4u^2(u^2-\sin^2\alpha)\geq 0$.

$\Rightarrow \quad u^2\leq 1+\sin^2\alpha$

$\Rightarrow \quad |u|\leq\sqrt{1+\sin^2\alpha}$

104. a. $\sin\theta_1\sin\theta_2-\cos\theta_1\cos\theta_2=-1$

$\Rightarrow \quad \cos(\theta_1+\theta_2)=1$

$\Rightarrow \quad \theta_1+\theta_2=2n\pi,\ n\in I$

$\Rightarrow \quad \frac{\theta_1}{2}+\frac{\theta_2}{2}=n\pi$

Thus, $\tan\frac{\theta_1}{2}\cot\frac{\theta_2}{2}=\tan\frac{\theta_1}{2}\cot\left(n\pi-\frac{\theta_1}{2}\right)=-\tan\frac{\theta_1}{2}\cot\frac{\theta_1}{2}=-1$

105. a. $\tan\frac{\pi}{3}+2\tan\frac{2\pi}{3}+4\tan\frac{4\pi}{3}+8\tan\frac{8\pi}{3}$

$=\tan\frac{\pi}{3}+2\tan\left(\pi-\frac{\pi}{3}\right)+4\tan\left(\pi+\frac{\pi}{3}\right)+8\tan\left(3\pi-\frac{\pi}{3}\right)$

$=\tan\frac{\pi}{3}-2\tan\frac{\pi}{3}+4\tan\frac{\pi}{3}-8\tan\frac{\pi}{3}=-5\tan\frac{\pi}{3}=-5\sqrt{3}$

106. c. Since $\tan 3\theta=\frac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}$

Putting $\theta=\frac{\pi}{9}$, we get $\tan\frac{\pi}{3}=\frac{3\tan\frac{\pi}{9}-\tan^3\frac{\pi}{9}}{1-3\tan^2\frac{\pi}{9}}$

$\Rightarrow \quad 3\left(1-3\tan^2\frac{\pi}{9}\right)^2=\left(3\tan\frac{\pi}{9}-\tan^3\frac{\pi}{9}\right)^2$

$\Rightarrow \quad \tan^6\frac{\pi}{9}-33\tan^4\frac{\pi}{9}+27\tan^2\frac{\pi}{9}=3$

107. c. $\cos x+\cos y-\cos(x+y)=\frac{3}{2}$

$\Rightarrow \quad 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)-2\cos^2\left(\frac{x+y}{2}\right)+1=\frac{3}{2}$

$$\Rightarrow \quad 2\cos^2\left(\frac{x+y}{2}\right)-2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)+\frac{1}{2}=0$$

Now $\cos\left(\frac{x+y}{2}\right)$ is always real, then discriminant ≥ 0.

$$\Rightarrow \quad 4\cos^2\left(\frac{x-y}{2}\right)-4\geq 0$$

$$\Rightarrow \quad \cos^2\left(\frac{x-y}{2}\right)\geq 1$$

$$\Rightarrow \quad \cos^2\left(\frac{x-y}{2}\right)=1$$

$$\Rightarrow \quad \frac{x-y}{2}=0 \Rightarrow x=y$$

108. a. $a\sin x+b\cos(x+\theta)+b\cos(x-\theta)=d$

$$\Rightarrow \quad a\sin x+2b\cos x\cos\theta=d$$

$$\Rightarrow \quad |d|\leq\sqrt{a^2+4b^2\cos^2\theta}$$

$$\Rightarrow \quad \frac{d^2-a^2}{4b^2}\leq\cos^2\theta$$

$$\Rightarrow \quad |\cos\theta|\geq\frac{\sqrt{d^2-a^2}}{2|b|}$$

109. d. $\frac{\sin x}{\sin y}=\frac{1}{2},\ \frac{\cos x}{\cos y}=\frac{3}{2}\Rightarrow\frac{\tan x}{\tan y}=\frac{1}{3}\Rightarrow\tan(x+y)=\frac{\tan x+\tan y}{1-\tan x\tan y}=\frac{4\tan x}{1-3\tan^2 x}$

Also $\sin y=2\sin x$, $\cos y=\frac{2}{3}\cos x$

$$\Rightarrow \quad \sin^2 y+\cos^2 y=4\sin^2 x+\frac{4\cos^2 x}{9}=1$$

$$\Rightarrow \quad 36\tan^2 x+4=9\sec^2 x=9(1+\tan^2 x)$$

$$\Rightarrow \quad 27\tan^2 x=5$$

$$\Rightarrow \quad \tan x=\frac{\sqrt{5}}{3\sqrt{3}}$$

$$\Rightarrow \quad \tan(x+y)=\frac{\frac{4\sqrt{5}}{3\sqrt{3}}}{1-\frac{15}{27}}=\frac{4\sqrt{5}\times 27}{12\times 3\sqrt{3}}=\sqrt{15}$$

110. a. $\sqrt{1+\cos x}=\sqrt{2\cos^2\frac{x}{2}}=\sqrt{2}\left|\cos\frac{x}{2}\right|$ and $\sqrt{1-\cos x}=\sqrt{2\sin^2\frac{x}{2}}=\sqrt{2}\left|\sin\frac{x}{2}\right|$

$$\Rightarrow \quad \frac{\sqrt{1+\cos x}+\sqrt{1-\cos x}}{\sqrt{1+\cos x}-\sqrt{1-\cos x}}=\frac{\left|\cos\frac{x}{2}\right|+\left|\sin\frac{x}{2}\right|}{\left|\cos\frac{x}{2}\right|-\left|\sin\frac{x}{2}\right|}$$

$$= \frac{-\cos\frac{x}{2} + \sin\frac{x}{2}}{-\cos\frac{x}{2} - \sin\frac{x}{2}} \qquad \left(\because \quad \frac{\pi}{2} < \frac{x}{2} < \pi\right)$$

$$= \frac{\cos\frac{x}{2} - \sin\frac{x}{2}}{\cos\frac{x}{2} + \sin\frac{x}{2}}$$

$$= \frac{1 - \tan\frac{x}{2}}{1 + \tan\frac{x}{2}}$$

$$= \tan\left(\frac{\pi}{4} - \frac{x}{2}\right) = \cot\left(\frac{\pi}{2} - \left(\frac{\pi}{4} - \frac{x}{2}\right)\right) = \cot\left(\frac{\pi}{4} + \frac{x}{2}\right)$$

111. d. $\tan x = n \tan y$, $\cos(x-y) = \cos x \cos y + \sin x \sin y$

$\Rightarrow \quad \cos(x-y) = \cos x \cos y\,(1 + \tan x \,.\, \tan y) = \cos x \cos y\,(1 + n \tan^2 y)$

$$\Rightarrow \quad \sec^2(x-y) = \frac{\sec^2 x \sec^2 y}{(1 + n\tan^2 y)^2}$$

$$= \frac{(1 + \tan^2 x)(1 + \tan^2 y)}{(1 + n\tan^2 y)^2}$$

$$= \frac{(1 + n^2\tan^2 y)(1 + \tan^2 y)}{(1 + n\tan^2 y)^2}$$

$$= 1 + \frac{(n-1)^2 \tan^2 y}{(1 + n\tan^2 y)^2}$$

Now, $\left(\frac{1 + n\tan^2 y}{2}\right)^2 \geq n\tan^2 y \qquad (\because \text{A.M.} \geq \text{G.M.})$

$$\Rightarrow \quad \frac{\tan^2 y}{(1 + n\tan^2 y)^2} \leq \frac{1}{4n}$$

$$\Rightarrow \quad \sec^2(x-y) \leq 1 + \frac{(n-1)^2}{4n} = \frac{(n+1)^2}{4n}$$

112. b. $\cot^2 x = \cot(x-y)\cot(x-z)$

$$\Rightarrow \quad \cot^2 x = \left(\frac{\cot x \cot y + 1}{\cot y - \cot x}\right)\left(\frac{\cot x \cot z + 1}{\cot z - \cot x}\right)$$

$\Rightarrow \quad \cot^2 x \cot y \cot z - \cot^3 x \cot y - \cot^3 x \cot z + \cot^4 x$

$= \cot^2 x \cot y \cot z + \cot x \cot y + \cot x \cot z + 1$

$\Rightarrow \quad \cot^3 x(\cot y + \cot z) + \cot x(\cot y + \cot z) + 1 - \cot^4 x = 0$

$\Rightarrow \quad \cot x(\cot y + \cot z)(1 + \cot^2 x) + (1 - \cot^2 x)(1 + \cot^2 x) = 0$

$\Rightarrow \quad [\cot x(\cot y + \cot z) + (1 - \cot^2 x)] = 0$

$$\Rightarrow \quad \frac{\cot^2 x - 1}{2\cot x} = \frac{1}{2}(\cot y + \cot z) = \cot 2x$$

113. a. $A+B+C=\pi \quad \Rightarrow \quad \tan A+\tan B+\tan C=\tan A\tan B\tan C$

Now, A.M. ≥ G.M.

$$\Rightarrow \quad \frac{\tan A+\tan B+\tan C}{3} \geq (\tan A\tan B\tan C)^{1/3}$$

$$\Rightarrow \quad \frac{\tan A+\tan B+\tan C}{3} \geq (\tan A\tan B\tan C)^{1/3}$$

$$\Rightarrow \quad (\tan A\tan B\tan C)^{2/3} \geq 3$$

$$\Rightarrow \quad \left(\frac{1}{K}\right)^{2/3} \geq 3$$

$$\Rightarrow \quad K \leq \frac{1}{3\sqrt{3}}$$

114. d. $u^2=(a^2\cos^2\theta+b^2\sin^2\theta)+(a^2\sin^2\theta+b^2\sin^2\theta)$

$$=a^2+b^2+2\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}\,\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}$$

$$=a^2+b^2+2\sqrt{\sin^2\theta\cos^2\theta\left(a^4+b^4\right)+a^2b^2\left(\sin^4\theta+\cos^4\theta\right)}$$

$$=a^2+b^2+2\sqrt{a^2b^2\left(1-2\sin^2\theta\cos^2\theta\right)+\left(a^4+b^4\right)\sin^2\theta\cos^2\theta}$$

$$=(a^2+b^2)+2\sqrt{a^2b^2+\left(a^2-b^2\right)^2\sin^2\theta\cos^2\theta}$$

$$=(a^2+b^2)+2\sqrt{a^2b^2+\frac{\left(a^2-b^2\right)^2}{4}\sin^2 2\theta}$$

Max. $u^2=(a^2+b^2)=2\sqrt{a^2b^2+\dfrac{\left(a^2-b^2\right)^2}{4}}$

Min. $u^2=(a^2+b^2)+2ab$

Therefore, the difference $=2\sqrt{a^2b^2+\dfrac{\left(a^2-b^2\right)^2}{4}}-2ab=\sqrt{4a^2b^2+a^4+b^4-2a^2b^2}-2ab$

$$=\sqrt{\left(a^2+b^2\right)^2}-2ab=a^2+b^2-2ab=(a-b)^2$$

115. c. $(\sin x+\cos x)^2+k\sin x\cos x=1$

$$\Rightarrow 1+\sin 2x+\frac{k}{2}\sin 2x=1$$

$$\Rightarrow \sin 2x\left[1+\frac{k}{2}\right]=0$$

For this to be an identity, $1+\dfrac{k}{2}=0 \Rightarrow k=-2$

116. b. $k\cos^2 x - k\cos x + 1 \ge 0\ \forall x \in (-\infty, \infty)$

$\Rightarrow k(\cos^2 x - \cos x) + 1 \ge 0$

Now $\cos^2 x - \cos x = \left(\cos x - \dfrac{1}{2}\right)^2 - \dfrac{1}{4}$

$\Rightarrow -\dfrac{1}{4} \le \cos^2 x - \cos x \le 2$

We have $2k + 1 \ge 0$ and $-\dfrac{k}{4} + 1 \ge 0$

Hence, $-\dfrac{1}{2} \le k \le 4$.

117. d. $\min(2 + \sin x - \cos x) = \min\left[2 + \sqrt{2}\sin\left(x - \dfrac{\pi}{4}\right)\right] = 2 - \sqrt{2}$

118. b. $a\operatorname{cosec}\alpha - b\sec\alpha = \dfrac{a}{\sin\alpha} + \dfrac{b}{\cos\alpha}$

$$= \frac{\sqrt{a^2+b^2}}{\sin\alpha\cos\alpha}\left[\frac{a}{\sqrt{a^2+b^2}}\cos\alpha\ \frac{b}{\sqrt{a^2+b^2}}\sin\alpha\right]$$

Now $\sin 3\alpha = \dfrac{a}{\sqrt{a^2+b^2}}$ gives $\sqrt{a^2+b^2}\left[\dfrac{\sin 3\alpha\cos\alpha - \cos 3\alpha\sin\alpha}{\sin\alpha\cos\alpha}\right] = 2\sqrt{a^2+b^2}$

119. d. $\cot^4 x - 2(1 + \cot^2 x) + a^2 = 0$

$\Rightarrow \cot^4 x - 2\cot^2 x + a^2 - 2 = 0$

$\Rightarrow (\cot^2 x - 1)^2 = 3 - a^2$

To have at least one solution, $3 - a^2 \ge 0$

$\Rightarrow a^2 - 3 \le 0$

$a \in [-\sqrt{3}, \sqrt{3}]$

Integral values are $-1, 0, 1$; therefore, the sum is 0.

120. b. $\sin^2 x + a\cos x + a^2 > 1 + \cos x$

Putting $x = 0$, we have

$a + a^2 > 2$

$\Rightarrow a^2 + a - 2 > 0$

$\Rightarrow (a + 2)(a - 1) > 0$

$\Rightarrow a < -2$ or $a > 1$

Therefore, the largest negative integral value of 'a' $= -3$.

121. d. $\Delta = \frac{1}{2}ab \Rightarrow ab = 60$

$b = c\cos\theta$; $a = c\sin\theta$

$\Rightarrow \Delta = \frac{1}{2}c^2\sin\theta\cos\theta = \frac{c^2\sin 2\theta}{4} = 30$

$\Rightarrow c^2 = 120\operatorname{cosec} 2\theta$

$\Rightarrow c^2_{\min} = 120$

$\Rightarrow c_{\min} = 2\sqrt{30}$

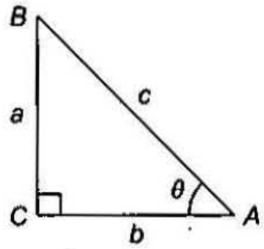


Fig. 2.39

122. b. From the figure,

$x\cos(\theta + 30°) = \theta$ (i)

and $x\sin\theta = 1 - d$ (ii)

Dividing $\sqrt{3}\cos\theta = \frac{1+d}{1-d}$, squaring Eq. (ii) and

putting the value of $\cot\theta$, we get $x^2 = \frac{1}{3}(4d^2 - 4d + 4)$

$\Rightarrow x = 2\frac{1}{2}$

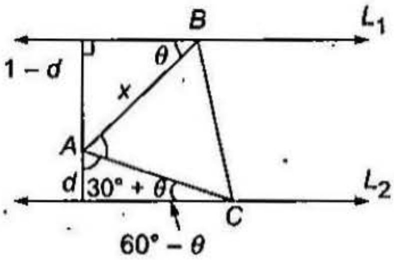


Fig. 2.40

123. d. $a = c\sin\theta$, $b = c\cos\theta$

$\Rightarrow \left(\frac{c}{a} + \frac{c}{b}\right)^2 = \left(\frac{1}{\sin\theta} + \frac{1}{\cos\theta}\right)^2 = \frac{4(1+\sin 2\theta)}{\sin^2 2\theta}$

$= 4\left(\frac{1}{\sin^2 2\theta} + \frac{1}{\sin 2\theta}\right)$ where $0 < \theta < \frac{\pi}{2}$

$\Rightarrow \left(\frac{c}{a} + \frac{c}{b}\right)^2_{\min} = 8$ when $2\theta = 90°$.

$\Rightarrow \theta = 45°$

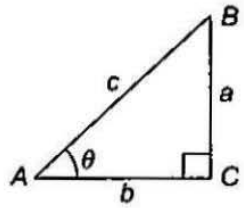


Fig. 2.41

124. c. $y = (\sin^2 x + \cos^2 x) + 2(\sin x\operatorname{cosec} x + \cos x\sec x) + \sec^2 x + \operatorname{cosec}^2 x$

$= 5 + 2 + \tan^2 x + \cot^2 x$

$= 7 + (\tan x - \cot x)^2 + 2$

$\therefore y_{\min} = 9$

Multiple Correct Answers Type

1. a, b. $2\sin\alpha\cos\alpha = 2\cos^2\beta$

$\sin 2\alpha = 1 + \cos 2\beta$

$\therefore \cos 2\beta = -(1 - \sin 2\alpha) = -\left(1 - \cos\left(\frac{\pi}{2} - 2\alpha\right)\right) = -2\sin^2\left(\frac{\pi}{4} - \alpha\right) = -2\cos^2\left(\frac{\pi}{4} + \alpha\right)$

2. **a, b, c, d.** $\cos^2\theta = 1-\sin^2\theta$. Let $81^{\sin^2\theta} = z$, we get $z+\dfrac{81}{z}=30$ or $z^2-30z+81=0$

$\Rightarrow\ (z-27)(z-3)=0$, i.e., $81^{\sin^2\theta} = 3^{4\sin^2\theta} = 3^3$ or 3^1

$\therefore\ \sin^2\theta = \dfrac{3}{4}, \dfrac{1}{4}$ or $\sin\theta = \pm\dfrac{\sqrt{3}}{2}, \pm\dfrac{1}{2}$

$\therefore\ \theta = 30°, 60°, 120°, 150°$

3. **b, c, d.** Opposite angles of a cyclic quadrilateral are supplementary.

4. **a, b, c.**

a. $\tan\theta = \dfrac{1-\cos 2\theta}{\sin 2\theta} \Rightarrow$ (a) is correct.

b. $\sin 2\theta = \dfrac{2\tan\theta}{1+\tan^2\theta}$

$\cos 2\theta = \dfrac{1-\tan^2\theta}{1+\tan^2\theta}$

$\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$

$\Rightarrow$ (b) is correct.

c. $\tan 3\theta = \dfrac{\sin 3\theta}{\cos 3\theta} \Rightarrow$ (c) is correct.

d. $\sin\theta = 1/3$ which is rational but $\cos 3\theta = \cos\theta(4\cos^2\theta - 3)$ which is irrational.
$\Rightarrow$ (d) is incorrect.

5. **a, b, c, d.**

a. $\sin\left(\dfrac{11\pi}{12}\right)\sin\left(\dfrac{5\pi}{12}\right) = \sin\left(\dfrac{\pi}{12}\right)\cos\left(\dfrac{\pi}{12}\right)$

$$= \frac{1}{2}\sin\left(\frac{\pi}{6}\right) = \frac{1}{4} \in Q$$

b. $\operatorname{cosec}\left(\dfrac{9\pi}{10}\right)\sec\left(\dfrac{4\pi}{5}\right) = -\operatorname{cosec}\left(\dfrac{\pi}{10}\right)\sec\left(\dfrac{\pi}{5}\right)$

$$= \frac{1}{\sin 18° \cos 36°}$$

$$= -\frac{16}{(\sqrt{5}-1)\ (\sqrt{5}+1)} = -4 \in Q$$

c. $\sin^4\left(\dfrac{\pi}{8}\right) + \cos^4\left(\dfrac{\pi}{8}\right) = 1 - 2\sin^2\left(\dfrac{\pi}{8}\right)\cos^2\left(\dfrac{\pi}{8}\right)$

$$= 1 - \frac{1}{2}\sin^2\left(\frac{\pi}{4}\right) = 1 - \frac{1}{4} = \frac{3}{4} \in Q$$

d. $2\cos^2\frac{\pi}{9}\ 2\cos^2\frac{2\pi}{9}\ 2\cos^2\frac{4\pi}{9} = 8\,(\cos 20°\cos 40°\cos 80°) = \frac{1}{8} \in Q$

6. a, b, d. $(\sin^2 x + \cos^2 x)^3 - 3\sin^2 x\cos^2 x(\sin^2 x + \cos^2 x) > \frac{5}{8}$

$\Rightarrow\quad 1 - 3\sin^2 x\cos^2 x > \frac{5}{8}$

$\Rightarrow\quad \frac{3}{8} > 3\sin^2 x\cos^2 x$

$\Rightarrow\quad 1 - 2\sin^2 2x > 0$

$\Rightarrow\quad \cos 4x > 0$

$\Rightarrow\quad 4x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\Rightarrow\quad 4x \in \left(2n\pi - \frac{\pi}{2}, 2n\pi + \frac{\pi}{2}\right)$, $n \in Z$, generalizing now verify.

7. a, b, d.

c. $\sum\sin^2\frac{A}{2} = \frac{1}{2}\,[3 - (\cos A + \cos B + \cos C)]$

$$= \frac{3}{2} - \frac{1}{2}\,(\cos A + \cos B + \cos C)$$

but $[\cos A + \cos B + \cos C]_{\max} = \frac{3}{2}$

$\therefore\quad \left.\sum\sin^2\frac{A}{2}\right]_{\min} = \frac{3}{2} - \frac{3}{4} = \frac{3}{4}$

$\therefore\quad \sum\sin^2\frac{A}{2} \ge \frac{3}{4}$

$\Rightarrow$ (c) is incorrect

a, b, d are correct and hold good in an equilateral triangle as the maximum values.

8. a, b, c.

a. $\tan\alpha\tan 2\alpha\tan 3\alpha = \tan 3\alpha - \tan 2\alpha - \tan\alpha$ always holds good.

b. R.H.S. $= \frac{\sin 4\alpha + \sin 2\alpha}{\sin 2\alpha\sin 4\alpha} = \frac{2\sin 3\alpha\cos\alpha}{\sin 2\alpha\sin 4\alpha} = \frac{1}{\sin\alpha} = \operatorname{cosec}\alpha$ (using $\pi/7 = \alpha$)

$\Rightarrow$ (b) is correct

c. $\cos\alpha + \cos 3\alpha + \cos 5\alpha = \dfrac{\sin 3\alpha}{\sin\alpha}\cos 3\alpha = \dfrac{\sin 6\alpha}{2\sin\alpha} = \dfrac{\sin\frac{6\pi}{7}}{2\sin\frac{\pi}{7}} = \dfrac{\sin\left(\pi+\frac{\pi}{7}\right)}{2\sin\frac{\pi}{7}} = \dfrac{1}{2}$

Also $\cos 2\alpha = \cos\dfrac{2\pi}{7} = -\cos\left(\pi - \dfrac{5\pi}{7}\right) = -\cos\left(\dfrac{5\pi}{7}\right) = -\cos 5\alpha$

d. $8\cos\alpha\cos 2\alpha\cos 4\alpha = \dfrac{\sin 8\alpha}{\sin\alpha} = \dfrac{\sin\frac{8\pi}{7}}{\sin\frac{\pi}{7}} = -\dfrac{\sin\frac{\pi}{7}}{\sin\frac{\pi}{7}} = -1$

9. a, b, c, d.

a. For $x \in \left(0, \dfrac{\pi}{4}\right)$, $\tan x < \cot x$

Also $\ln(\sin x) < 0$

$\Rightarrow (\tan x)^{\ln(\sin x)} > (\cot x)^{\ln(\sin x)}$

b. For $x \in \left(0, \dfrac{\pi}{2}\right)$, $\operatorname{cosec} x \geq 1$

$\Rightarrow \ln(\operatorname{cosec} x) \geq 0$

$\Rightarrow 4^{\ln \operatorname{cosec} x} < 5^{\ln(\operatorname{cosec} x)}$

c. $x \in \left(0, \dfrac{\pi}{2}\right) \Rightarrow \cos x \in (0, 1)$

$\Rightarrow \ln(\cos x) < 0$

Also $\dfrac{1}{2} > \dfrac{1}{3}$

$\Rightarrow \left(\dfrac{1}{2}\right)^{\ln(\cos x)} < \left(\dfrac{1}{3}\right)^{\ln(\cos x)}$

d. For $x \in \left(0, \dfrac{\pi}{2}\right)$

Since $\sin x < \tan x$, we get $\ln(\sin x) < \ln(\tan x)$

$\Rightarrow 2^{\ln(\sin x)} < 2^{\ln(\tan x)}$

10. a, c, d.

a. 1

b. 3

c. $\dfrac{\sin 24^\circ \cos 6^\circ - \cos 24^\circ \sin 6^\circ}{\sin 21^\circ \cos 39^\circ - \sin 39^\circ \cos 21^\circ} = \dfrac{\sin(18^\circ)}{\sin(-18^\circ)} = -1$

d. -1

11. a, c.

a. $\dfrac{\cos^2\alpha - \sin^2\alpha}{\sin\alpha\cos\alpha} = 2\cot 2\alpha$

b. $\frac{1+t}{1-t}-\frac{1-t}{1+t}$ where $t=\tan\alpha$

$$=\frac{(1+t)^2-(1-t)^2}{1-t^2}=\frac{4t}{1-t^2}=\frac{2\times 2\tan\alpha}{1-\tan^2\alpha}=2\tan 2\alpha$$

$\Rightarrow$ (b) is incorrect.

c. $\frac{1+t}{1-t}+\frac{1-t}{1+t}=\frac{(1+t)^2+(1-t)^2}{1-t^2}=\frac{2(1+t^2)}{1-t^2}$ (where $t=\tan\alpha$)

$$=\frac{2}{\cos 2\alpha}=2\sec 2\alpha$$

$\Rightarrow$ (c) is correct.

d. $\tan\alpha+\cot\alpha=\frac{1}{\cos\alpha\sin\alpha}=\frac{2}{\sin 2\alpha}=2\operatorname{cosec} 2\alpha$

$\Rightarrow$ (d) is incorrect.

12. a, c, d.

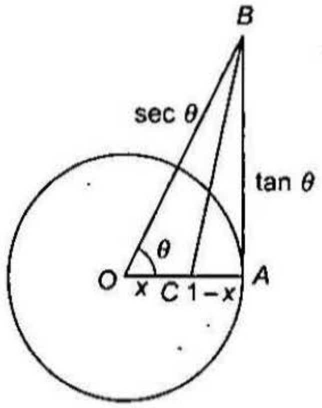


Fig. 2.42

Using property of angle bisector, we get

$$\frac{\sec\theta}{\tan\theta}=\frac{x}{1-x}$$

$$\Rightarrow\quad x=\frac{\sec\theta}{\sec\theta+\tan\theta}=\sec\theta(\sec\theta-\tan\theta)=\frac{1}{1+\sin\theta}$$

13. a, d. $y=\frac{(1+\tan^2 x)^2}{1+\tan^2 x}=1+\tan^2 x$

$$=1+\left(2-\sqrt{3}\right)^2$$

$$=8-4\sqrt{3}=4(2-\sqrt{3})$$

$$=4\left[\left(\sqrt{\frac{3}{2}}-\frac{1}{\sqrt{2}}\right)^2\right]$$

$$= 4\left[4\left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right)^2\right]$$

$$= 16\sin^2\frac{\pi}{12}$$

14. **b, d.** $\tan(\alpha+\beta) = \frac{15}{8}$ and $\operatorname{cosec}\gamma = \frac{17}{8} \Rightarrow \tan\gamma = \frac{8}{15}$

$\therefore \quad \alpha+\beta+\gamma = \frac{\pi}{2} \Rightarrow$ (b) is true.

Also $\tan(\alpha+\beta) = \dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$

$\Rightarrow \quad \cot\gamma = \dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$

$\Rightarrow \quad \tan\alpha\tan\beta+\tan\beta\tan\gamma+\tan\gamma\tan\alpha = 1$

15. **b, d.** Divide by $\cos\alpha$ and square both sides and let $\tan\alpha = t$ so that $\sec^2\alpha = 1+t^2$

$\Rightarrow \quad [(a+2)t+(2a-1)]^2 = [(2a+1)^2(1+t^2)]$

$\Rightarrow \quad t^2[(a+2)^2-(2a+1)^2]+2(a+2)(2a-1)t+[(2a-1)^2-(2a+1)^2] = 0$

$\Rightarrow \quad 3(1-a^2)t^2+2(2a^2+3a-2)t-4\times 2a = 0$

$\Rightarrow \quad 3(1-a^2)t^2-4(1-a^2)t+6at-8a = 0$

$\Rightarrow \quad t(1-a^2)(3t-4)+2a(3t-4) = 0$

$\Rightarrow \quad (3t-4)[(1-a^2)t+2a] = 0$

$\Rightarrow \quad t = \tan\alpha = \frac{4}{3}$ or $\frac{2a}{a^2-1}$

16. **a, b, c.** $\log_{1/3}\log_7(\sin x+a) > 0$

$\Rightarrow 0 < \log_7(\sin x+a) < 1$

$\Rightarrow 1 < (\sin x+a) < 7, \forall x\in R$

It is found that 'a' should be less than the minimum value of $7-\sin x$ and 'a' must be greater than the maximum value of $1-\sin x$

$\Rightarrow 1-\sin x < a < 7-\sin x \ \forall x\in R$

$\Rightarrow 2 < a < 6$

17. **a, c.** $\log_b \sin t = x \Rightarrow \sin t = b^x$

Let $\log_b(\cos t) = y$, then $b^y = \cos t$

$\Rightarrow b^{2y} = \cos^2 t = 1-\sin^2 t = 1-b^{2x}$

$\Rightarrow 2y = \log_b(1-b^{2x})$

$\Rightarrow y = \frac{1}{2}\log_b(1-b^{2x}) = \log_b\sqrt{1-b^{2x}}$

10. a. Statement 2 is true as it is one of the standard results of multiple angles.
Putting $A = \pi/18$ in the formula $\sin 3A = 3\sin A - 4\sin^3 A$, we get $8x^3 - 6x + 1 = 0$, where $x = \sin \pi/18$. Hence, statement 1 is also true because of statement 2.

11. a. Statement 2 is true, because each trigonometric function has a principle period of π or 2π and hence 2π is one of the periods of every trigonometric function.
Thus $f(2A) = f(2B)$
$\Rightarrow$ $2A = 2n\pi + 2B$, for some $n \in Z$
$\Rightarrow$ $A = n\pi + B$
$\Rightarrow$ Statement 1 is true because of statement 2.

12. b. From Fig. 2.43, $\sin 3 < \sin 1 < \sin 2$
But statement 2 is not sufficient to ensure this.
Hence, answer is (b).

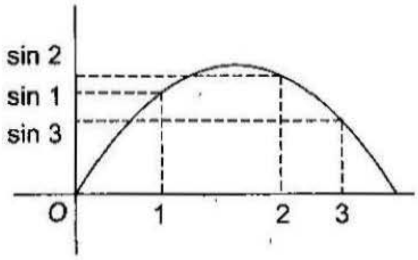


Fig. 2.43

13. a. Let $g(x) = 3\sin x + 4\cos x - 2$

Maximum value of $g(x) = \sqrt{3^2 + 4^2} - 2 = 3$

Minimum value of $g(x) = -\sqrt{3^2 + 4^2} - 2 = -7$

Therefore, the range of $f(x) = \dfrac{1}{g(x)}$ is $R - \left(-\dfrac{1}{7}, \dfrac{1}{3}\right)$

Hence, it is an unbounded function, and $f(x)$ has no maximum and no minimum values.

14. b. Statement 2 is true as it is one of the identities in triangle.
R.H.S. in statement 2 is always positive as $\alpha, \beta, \gamma \in (0, \pi/2)$
Statement 1 is true as select $\alpha = 2\pi$, $\beta = -\pi/2$, $\gamma = -\pi/2$
Then $\sin\alpha + \sin\beta + \sin\gamma = 0 - 1 - 1 = -2$, which shows that minimum value will be negative.
But statement 2 is not the correct explanation of statement 1, as $\alpha + \beta + \gamma = \pi$ does not follow that α, β, γ are angles of a triangle.

15. a. $\sin^2 A + \sin^2 B + \sin^2 C = \dfrac{1-\cos 2A}{2} + \dfrac{1-\cos 2B}{2} + \dfrac{1-\cos 2C}{2}$

$$= \frac{3 - (\cos 2A + \cos 2B + \cos 2C)}{2}$$

$$= \frac{3 - (-1 - 4\cos A\cos B\cos C)}{2}$$

$$= 2 + 2\cos A\cos B\cos C \qquad \text{(i)}$$

Hence, statement 2 is true.

From statement 1 using Eq. (i), we get $\cos A \cos B \cos C = 0$, then either A, B or C is 90°.

Both statement 1 and statement 2 are true and statement 2 is the correct explanation of statement 1.

16. d. Statement 2 is true as it is one of the conditional identities in the triangle. Since R.H.S. > 1 in statement 2, statement 1 is false.

17. d. Given $\cos x \sum \cos\alpha - \sin x \sum \sin\alpha = 0 \; \forall \; x \in R$

Hence, $\cos\alpha + \cos\beta + \cos\gamma = 0$

and $\sin\alpha + \sin\beta + \sin\gamma = 0$

Hence, statement 2 is true.

Now $(\cos\alpha + \cos\beta)^2 = (-\cos\gamma)^2$

and $(\sin\alpha + \sin\beta)^2 = (-\sin\gamma)^2$

Adding, we get

$2 + 2\cos(\alpha - \beta) = 1$

$\Rightarrow \cos(\alpha - \beta) = -1/2$

Similarly, $\cos(\beta - \gamma) = -1/2$ and $\cos(\gamma - \alpha) = -1/2$

Now $0 < \alpha < \beta < \gamma < 2\pi$

$\Rightarrow \beta - \alpha < \gamma - \alpha$

Hence, $\beta - \alpha = \dfrac{2\pi}{3}$ and $\gamma - \alpha = \dfrac{4\pi}{3}$

Statement 1 is false.

18. a. $\cos^2 A + \cos^2 B + \cos^2 C = 1 - 2\cos A \cos B \cos C$

$$\therefore \sum \cos^2 A\big|_{min} = 1 - 2 \times \frac{1}{8} = 1 - \frac{1}{4} = \frac{3}{4}$$

19. d. Let $x = \cot A$; $y = \cot B$ and $z = \cot C$

$\Rightarrow \sum \cot A \cot B = 1$

$\Rightarrow A + B + C = n\pi$

In statement 1

$$\therefore \text{L.H.S.} = \frac{1}{2}[\sin 2A + \sin 2B + \sin 2C]$$
$$= 2\sin A \sin B \sin C$$

$$\text{R.H.S.} = \frac{2}{|\operatorname{cosec} A \operatorname{cosec} B \operatorname{cosec} C|}$$
$$= \pm 2\sin A \sin B \sin C = \text{L.H.S.}$$

Statement 2 is true as it is one of the conditional identities in the triangle.

20. b. In triangle ABC, $A + B + C = \pi$

and $\cot\dfrac{A}{2} + \cot\dfrac{B}{2} + \cot\dfrac{C}{2} = \cot\dfrac{A}{2}\cot\dfrac{B}{2}\cot\dfrac{C}{2}$

Therefore, $\ln\left(\cot\dfrac{A}{2} + \cot\dfrac{B}{2} + \cot\dfrac{C}{2}\right) = \ln\cot\dfrac{A}{2} + \ln\cot\dfrac{B}{2} + \ln\cot\dfrac{C}{2}$

Hence, statement 1 is true.

In statement 2, R.H.S. $= \ln 1 + \ln\sqrt{3} + \ln(2 + \sqrt{3}) = \ln(1\sqrt{3}(2 + \sqrt{3})) = \ln(2\sqrt{3} + 3) =$ R.H.S.

But statement 2 does not explain statement 1.

18. **a, b, d.** Let $x = \cos\theta$, then $4\cos^3\theta - 3\cos\theta = -\frac{\sqrt{3}}{2}$

$$\Rightarrow \cos 3\theta = \cos\frac{5\pi}{6}$$

$$\Rightarrow 3\theta = 2n\pi \pm \frac{5\pi}{6}$$

$$\Rightarrow \theta = \frac{2n\pi}{3} \pm \frac{5\pi}{18}$$

Putting $n = 0$, we get $\theta = 5\pi/18$

$$n = 1 \Rightarrow \theta = \frac{2\pi}{3} \pm \frac{5\pi}{18} = \frac{17\pi}{18}$$

and $\theta = \frac{2\pi}{3} - \frac{5\pi}{18} = \frac{17\pi}{18}$

19. **a, c, d.** $\sin x\cos 20° + \cos x\sin 20° = 2\sin x\cos 40°$

$\Rightarrow \sin 20°\cos x = \sin x(2\cos 40° - \cos 20°)$

$$\Rightarrow \tan x = \frac{\sin 20°}{2\cos 40° - \cos 20°} = \frac{\sin 20°}{\cos 40° + \cos 40° - \cos 20°} = \frac{\sin 20°}{\cos 40° + 2\sin 30°\sin(-10°)}$$

$$= \frac{\sin 20°}{\sin 50° - \sin 10°} = \frac{\sin 20°}{2\cos 30°\sin 20°}$$

$$\Rightarrow \tan x = \frac{1}{\sqrt{3}} \Rightarrow x = 30°$$

Reasoning Type

1. d. Statement 1 is wrong as z can be written as $\frac{-(x+y)}{1-xy}$.

It implies that for any values of xy ($xy \neq 1$), we get a value of z and statement 2 is correct.

2. b. $\cos 7 = \cos(2\pi + 7 - 2\pi) = \cos(7 - 2\pi) = \cos(0.72)$

Now 1 rad and 0.72 rad angles are for first quadrant where $\cos x$ is decreasing, hence $\cos 1 < \cos 0.72$ or $\cos 1 < \cos 7$.

But statement 2 is not the correct explanation for $\cos 1 < \cos 7$.

Note that $\cos 0.5 > \cos 7$.

3. b. $\tan 4 = \tan(\pi + (4 - \pi)) = \tan(4 - \pi) = \tan 0.86$

$\tan 7.5 = \tan(2\pi + (7.5 - 2\pi)) = \tan(7.5 - 2\pi) = \tan 1.22$

Now both angles, i.e., 0.86 rad and 1.22 rad are for first quadrant, hence $\tan 0.86 < \tan 1.22$ as $\tan x$ is an increasing function. But this is not always true, as $3 > 1$ but $\tan 3 < \tan 1$.

Hence, both statements are true but statement 2 is not the correct explanation of statement 1.

4. b. In first quadrant, $\cos\theta > \sin\theta$ for $\theta \in (\pi/4, \pi/2)$.
Hence, $\cos 1 < \sin 1$.
Also in first quadrant, cosine is decreasing and sine is increasing, but this is not the correct reason for which $\cos 1 < \sin 1$. Thus, the correct answer is (b).

5. a. $f(\theta) = (\sin\theta + \operatorname{cosec}\theta)^2 + (\cos\theta + \sec\theta)^2$

$$= \sin^2\theta + \cos^2\theta + \operatorname{cosec}^2\theta + \sec^2\theta + 4$$

$$= 5 + \frac{1}{\sin^2\theta} + \frac{1}{\cos^2\theta} = 5 + \frac{4}{\sin^2 2\theta} \geq 9$$

Hence, the correct answer is (a).

6. d. $\sin^2\theta_1 + \sin^2\theta_2 + \cdots + \sin^2\theta_n = 0$

$\Rightarrow \sin\theta_1 = \sin\theta_2 = \cdots = \sin\theta_n = 0$

$\Rightarrow \cos^2\theta_1, \cos^2\theta_2, \ldots, \cos^2\theta_n = 1$

$\Rightarrow \cos\theta_1, \cos\theta_2, \ldots, \cos\theta_n = \pm 1$

Now $\cos\theta_1 + \cos\theta_2 + \cdots + \cos\theta_n = n - 4$ means two of $\cos\theta_1, \cos\theta_2, \ldots, \cos\theta_n$ must be -1 and the others are 1. Now two values from $\cos\theta_1, \cos\theta_2, \ldots, \cos\theta_n$ can be selected in ${}^nC_2$ ways. Hence, the number of solutions is ${}^nC_2 = \frac{n(n-1)}{2}$.

Hence, statement 1 is false, but statement 2 is correct.

7. a. Let $y = 27^{\cos 2x} \times 81^{\sin 2x} = 3^{3\cos 2x + 4\sin 2x}$

Now $-\sqrt{3^2+4^2} \leq 3\cos 2x + 4\sin 2x \leq \sqrt{3^2+4^2}$

or $-5 \leq 3\cos 2x + 4\sin 2x \leq 5$

$\Rightarrow 3^{-5} \leq 3^{3\cos 2x + 4\sin 2x} \leq 3^5$

Hence, the correct answer is (a).

8. d. Obviously in triangle ABC,

$$\tan A = \tan(\pi - (B+C))$$
$$= -\tan(B+C)$$
$$= \frac{\tan B + \tan C}{\tan B \tan C - 1}$$

If angle A is obtuse, then $\tan A < 0$

$\Rightarrow \frac{\tan B + \tan C}{\tan B \tan C - 1} < 0$

$\Rightarrow \tan B \tan C < 1$ (as B and C will be acute)

Thus statement 1 is false and statement 2 is true.

9. a. We know that $\tan 15° = 2 - \sqrt{3}$ which is an irrational number. Hence, statement 2 is true.

Statement 1 is also true as if $\tan 5°$ is a rational number, then $\tan 15° = \frac{3\tan 5° - \tan^3 5°}{1 - 3\tan^2 5°}$ should be a rational number, which is not true.
Hence, $\tan 5°$ is an irrational number.
Obviously, statement 2 is the correct reasoning for statement 1.

Linked Comprehension Type

For Problems 1 – 3

1. a., 2. b., 3. c.

Sol 1. a. $\sin\alpha = A\sin(\alpha+\beta) = A(\sin\alpha\cos\beta + \sin\beta\cos\alpha)$

$$\Rightarrow \quad \sin\alpha(1 - A\cos\beta) = A\sin\beta\cos\alpha \qquad \text{(i)}$$

$$\Rightarrow \quad \tan\alpha = \frac{A\sin\beta}{(1-A\cos\beta)} \qquad \text{(ii)}$$

2. b. $\tan\beta = \dfrac{\sin\beta}{\cos\beta} = \dfrac{(1-A\cos\beta)\tan\alpha}{A\cos\beta} = \dfrac{(1-A\cos\beta)\sin\alpha}{A\cos\alpha\cos\beta}$ [from Eqs. (i) and (ii)]

3. c. $\tan(\alpha+\beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$

$$= \frac{\dfrac{A\sin\beta}{1-A\cos\beta} + \dfrac{\sin\beta}{\cos\beta}}{1 - \dfrac{A\sin\beta\sin\beta}{(1-A\cos\beta)\cos\beta}}$$

$$= \frac{A\sin\beta\cos\beta + \sin\beta - A\sin\beta\cos\beta}{\cos\beta - A\cos^2\beta - A\sin^2\beta} = \frac{\sin\beta}{\cos\beta - A}$$

Also $\tan(\alpha+\beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$

$$= \frac{\dfrac{\sin\alpha}{\cos\alpha} + \dfrac{\sin\alpha(1-A\cos\beta)}{A\cos\alpha\cos\beta}}{1 - \dfrac{\sin^2\alpha(1-A\cos\beta)}{A\cos^2\alpha\cos\beta}} \qquad \text{[from Eq. (ii)]}$$

$$= \frac{[A\sin\alpha\cos\beta + \sin\alpha - A\sin\alpha\cos\beta]\cos\alpha}{A\cos^2\alpha\cos\beta - \sin^2\alpha + A\sin^2\alpha\cos\beta}$$

$$= \frac{\sin\alpha\cos\alpha}{A\cos\beta - \sin^2\alpha}$$

For Problems 4 – 6

4. d, 5. a, 6. b.

Sol. We have $\tan\left(\theta + \dfrac{\pi}{4}\right) = 3\tan 3\theta$

$$\Rightarrow \quad \frac{1+\tan\theta}{1-\tan\theta} = 3 \times \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$$

$$\Rightarrow \quad \frac{1+t}{1-t} = 3\left(\frac{3t-t^3}{1-3t^2}\right) \text{ (putting } t = \tan\theta)$$

$$\Rightarrow \quad 3t^4 - 6t^2 + 8t - 1 = 0$$

Hence,

S_1 = sum of roots = $t_1 + t_2 + t_3 + t_4 = 0$

S_2 = sum of product of roots taken two at a time = -2

S_3 = sum of product of roots taken three at time = $-8/3$

S_4 = product of all roots = $-1/3$

$$\frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3} + \frac{1}{t_4} = \frac{\sum t_1t_2t_3}{t_1t_2t_3t_4} = \frac{-8}{3}$$

For Problems 7 – 9

7. a, 8. b, 9. d.

Sol. $\sin\alpha + \sin\beta = 3$ (i)

$$\Rightarrow \quad 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) = 3 \quad \text{(ii)}$$

$\cos\alpha + \cos\beta = 4$ (iii)

$$\Rightarrow \quad 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) = 4 \quad \text{(iv)}$$

Dividing Eq. (ii) by Eq. (iv), we have

$$\tan\left(\frac{\alpha+\beta}{2}\right) = \frac{3}{4}$$

$$\Rightarrow \quad \sin(\alpha+\beta) = \frac{2\tan\left(\frac{\alpha+\beta}{2}\right)}{1+\tan^2\left(\frac{\alpha+\beta}{2}\right)} = \frac{2\times\frac{3}{4}}{1+\left(\frac{3}{4}\right)^2} = \frac{24}{25}$$

$$\text{and} \quad \cos(\alpha+\beta) = \frac{1-\tan^2\left(\frac{\alpha+\beta}{2}\right)}{1+\tan^2\left(\frac{\alpha+\beta}{2}\right)} = \frac{1-\left(\frac{3}{4}\right)^2}{1+\left(\frac{3}{4}\right)^2} = \frac{7}{25}$$

For Problems 10 – 12

10. a, 11. c, 12. b.

Sol. Let $\theta = \frac{n\pi}{7}$ (so that $7\theta = n\pi$)

$$\Rightarrow \quad 4\theta + 3\theta = n\pi$$

$$\Rightarrow \quad \tan 4\theta = \tan(n\pi - 3\theta) = -\tan 3\theta$$

$$\Rightarrow \quad \frac{4\tan\theta - 4\tan^3\theta}{1-6\tan^2\theta + \tan^4\theta} = -\frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}$$

$$\Rightarrow \quad \frac{4z-4z^3}{1-6z^2+z^4} = -\frac{3z-z^3}{1-3z^2} \qquad \text{[where } \tan\theta = z \text{ (say)]}$$

$$\Rightarrow \quad (4-4z^2)(1-3z^2) = -(3-z^2)(1-6z^2+z^4)$$

$$\Rightarrow \quad z^6-21z^4+35z^2-7=0 \qquad \text{(i)}$$

This is a cubic equation in z^2, i.e., in $\tan^2\theta$.

The roots of this equation are therefore $\tan^2 \pi/7$, $\tan^2 2\pi/7$ and $\tan^2 3\pi/7$ from Eq. (i), sum of the

roots $= \dfrac{-(-21)}{1} = 21$

$$\Rightarrow \quad \tan^2\frac{\pi}{7}+\tan^2\frac{2\pi}{7}+\tan^2\frac{3\pi}{7}=21 \qquad \text{(ii)}$$

Putting $1/y$ in place of z in Eq. (i), we get $-7y^6+35y^4-21y^2+1=0$

or $\quad 7y^6-35y^4=21y^2-1=0$ (iii)

This is a cubic equation in y^2, i.e., in $\cot^2\theta$.

The roots of this Eq. are therefore $\cot^2 \pi/7$, $\cot^2 2\pi/7$ and $\cot^2 3\pi/7$.

Sum of the roots of Eq. (iii) = 35/7 = 5

$$\Rightarrow \quad \cot^2\frac{\pi}{7}+\cot^2\frac{2\pi}{7}+\cot^2\frac{3\pi}{7}=5 \qquad \text{(iv)}$$

By multiplying Eqs. (ii) and (iv), we get

$$\left(\tan^2\frac{\pi}{7}+\tan^2\frac{2\pi}{7}+\tan^2\frac{3\pi}{7}\right)\left(\cot^2\frac{\pi}{7}+\cot^2\frac{2\pi}{7}+\cot^2\frac{3\pi}{7}\right)=21\times5=105$$

For Problems 13 – 15

13. b, 14. b, 15. d.

Sol. Angles BEC, ABD, ABE and BAC are in A.P.

Let $\angle BEC = \alpha-3\beta$, $\angle ABD = \alpha-\beta$, $\angle ABE = \alpha+\beta$ and $\angle BAC = \alpha+3\beta$

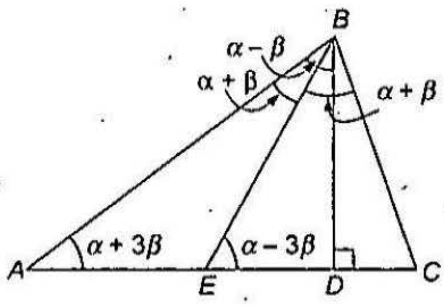


Fig. 2.44

From ΔABD, $\alpha-\beta+\alpha+3\beta = \dfrac{\pi}{2}$

$\Rightarrow \quad 2\beta + 2\beta = \frac{\pi}{2} \Rightarrow \alpha + \beta = \frac{\pi}{4}$

Now, $\alpha - 3\beta = (\alpha + 3\beta) + (\alpha + \beta)$ [using exterior angle theorem]

$\Rightarrow \quad \alpha = -7\beta$

$\therefore \quad \beta = -\frac{\pi}{24}, \ \alpha = \frac{7\pi}{24}$

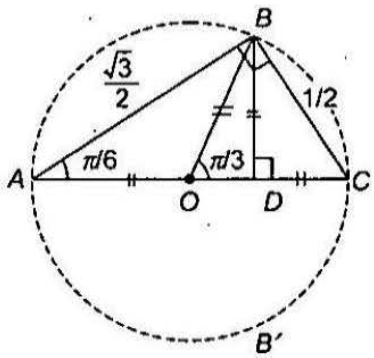


Fig. 2.45

$\therefore \quad \angle B = 2(\alpha + \beta) = \frac{\pi}{2}, \ \angle A = \frac{\pi}{6}, \angle C = \frac{\pi}{3}$

$\Rightarrow \quad ABC$ is a 30°–90°–60° triangle.

13. Area of the circle circumscribing $\Delta ABC = \pi\left(\frac{1}{2}\right)^2 = \frac{\pi}{4}$.

14. ΔBOC is equilateral $\Rightarrow r = \frac{\Delta}{s} = \dfrac{\frac{\sqrt{3}}{2}\left(\frac{1}{2}\right)^2}{\frac{1}{2}\left(\frac{3}{2}\right)} = \frac{1}{4\sqrt{3}}$

15. $BD = OB \sin\frac{\pi}{3} = \frac{1}{2}\sin\frac{\pi}{3} = \frac{\sqrt{3}}{4}$

$\therefore \quad BB' = 2BD = \frac{\sqrt{3}}{2}$

Matrix-Match Type

1. **a → q; b → r; c → s; d → p**

$\cos\theta - \sin\theta = \frac{1}{5}$, where $0 < \theta < \frac{\pi}{2}$. (i)

Squaring both sides of Eq. (i), we get

$1 - \sin 2\theta = \frac{1}{25}$

$\Rightarrow \quad \sin 2\theta = \frac{24}{25} \qquad \Rightarrow \quad \cos 2\theta = \frac{7}{25}$

Also, $(\cos\theta + \sin\theta)^2 = (\cos\theta - \sin\theta)^2 + 4\cos\theta\sin\theta = \frac{1}{25} + 2\sin 2\theta = \frac{1}{25} + \frac{48}{25} = \frac{49}{25}$

$\Rightarrow \quad \cos\theta + \sin\theta = \dfrac{7}{5}$ (ii)

$\Rightarrow \quad (\cos\theta + \sin\theta)/2 = \dfrac{7}{10}$

Also solving Eqs. (i) and (ii), we get $\cos\theta = 4/5$.

2. **a $\rightarrow$ q; b $\rightarrow$ s; c $\rightarrow$ p; d $\rightarrow$ r**

a. $A = \sin^2\theta + \cos^4\theta$

$$= \frac{1-\cos 2\theta}{2} + \left(\frac{1+\cos 2\theta}{2}\right)^2$$

$$= \frac{1}{2} - \frac{1}{2}\cos 2\theta + \frac{1}{4} + \frac{1}{2}\cos 2\theta + \frac{1}{4}\cos^2 2\theta$$

$$= \frac{3}{4} + \frac{1}{4}\left(\frac{\cos 4\theta + 1}{2}\right) = \frac{3}{4} + \frac{1}{8} + \frac{1}{8}\cos 4\theta$$

Now, $-1 \le \cos 4\theta \le 1$

$$\Rightarrow \quad \frac{-1}{8} \le \frac{\cos 4\theta}{8} \le \frac{1}{8}$$

$$\Rightarrow \quad \frac{3}{4} + \frac{1}{8} - \frac{1}{8} \le \frac{3}{4} + \frac{1}{4}\left(\frac{1+\cos 4\theta}{2}\right) \le \frac{3}{4} + \frac{1}{8} + \frac{1}{8}$$

$$\Rightarrow \quad \frac{3}{4} \le A \le 1$$

b. $A = 3\cos^2\theta + \sin^4\theta = 3\dfrac{1+\cos 2\theta}{2} + \left(\dfrac{1-\cos 2\theta}{2}\right)^2$

$$= \frac{3+3\cos 2\theta}{2} + \frac{1-2\cos 2\theta + \cos^2 2\theta}{4}$$

$$= \frac{7+4\cos 2\theta + \cos^2 2\theta}{4} = \frac{(\cos 2\theta + 2)^2 + 3}{4}$$

Now, $1 \le \cos 2\theta + 2 \le 3$

$$\Rightarrow \quad 1 \le \frac{(\cos 2\theta + 2)^2 + 3}{4} \le 3$$

c. $A = \sin^2\theta - \cos^4\theta$

$$= \frac{1-\cos 2\theta}{2} - \left(\frac{1+\cos 2\theta}{2}\right)^2$$

$$= \frac{1}{2} - \frac{1}{2}\cos 2\theta - \frac{1}{4} - \frac{1}{2}\cos 2\theta - \frac{1}{4}\cos^2 2\theta$$

$$= \frac{1}{4} - \cos 2\theta - \frac{1}{4}\cos^2 2\theta$$

$$= -\left(\frac{1}{4}\cos^2 2\theta + \cos 2\theta - \frac{1}{4}\right)$$

$$= \frac{5}{4} - \left(\frac{1}{2}\cos 2\theta + 1\right)^2$$

Now, $-\frac{1}{2} \le \frac{1}{2}\cos 2\theta \le \frac{1}{2}$

$$\Rightarrow \quad \frac{1}{2} \le \frac{1}{2}\cos 2\theta + 1 \le \frac{3}{2}$$

$$\Rightarrow \quad \frac{1}{4} \le \left(\frac{1}{2}\cos 2\theta + 1\right)^2 \le \frac{9}{4}$$

$$\Rightarrow \quad -\frac{9}{4} \le -\left(\frac{1}{2}\cos 2\theta + 1\right)^2 \le -\frac{1}{4}$$

$$\Rightarrow \quad -1 \le \frac{5}{4} - \left(\frac{1}{2}\cos 2\theta + 1\right)^2 \le 1$$

d. $A = \tan^2\theta + 2\cot^2\theta = (\tan\theta - \sqrt{2}\cot\theta)^2 + 2\sqrt{2} \ge 2\sqrt{2}$

3. **a → r; b → p; c → q; d → s**

$\cos\alpha + \cos\beta = 1/2$

$$\Rightarrow \quad 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) = \frac{1}{2} \qquad \text{(i)}$$

$\sin\alpha + \sin\beta = 1/3$

$$\Rightarrow \quad 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) = \frac{1}{3} \qquad \text{(ii)}$$

Dividing Eq. (ii) by Eq. (i), we get

$$\tan\left(\frac{\alpha+\beta}{2}\right) = \frac{2}{3}$$

$$\Rightarrow \quad \cos\left(\frac{\alpha+\beta}{2}\right) = \pm\frac{3}{\sqrt{13}}$$

Squaring and adding the given results, we have

$$2 + 2\cos(\alpha-\beta) = \frac{13}{36}$$

$$\Rightarrow \quad \cos(\alpha-\beta) = -\frac{59}{72}$$

Now, $2\cos^2\left(\frac{\alpha-\beta}{2}\right) - 1 = \cos(\alpha-\beta)$

$$\Rightarrow \quad 2\cos^2\left(\frac{\alpha-\beta}{2}\right) = 1 - \frac{59}{72} = \frac{13}{72}$$

$$\Rightarrow \cos\left(\frac{\alpha-\beta}{2}\right) = \pm\frac{\sqrt{13}}{12}$$

$$\Rightarrow \tan\left(\frac{\alpha-\beta}{2}\right) = \pm\sqrt{\frac{131}{13}}$$

4. a → s; b → r; c → p; d → q

a. $\sin(410° + 400°) = \sin 810° = \sin(720° + 90°) = \sin 90° = 1$

b. $\dfrac{\sin^2 2° - \sin^2 1°}{2\sin 3° \sin 1°} = \dfrac{\sin 3° \sin 1°}{2\sin 3° \sin 1°} = \dfrac{1}{2}$

c. $\sin(-870°) + \operatorname{cosec}(-660°) + \tan(-855°) + 2\cot(840°) + \cos(480°) + \sec(900°)$

$= -\sin(810° + 60°) - \operatorname{cosec}(720° - 60°) - \tan(810° + 45°) + 2\cot 120° + \cos 120° + \sec 180°$

$= -\dfrac{1}{2} + \dfrac{2}{\sqrt{3}} + 1 - \dfrac{2}{\sqrt{3}} - \dfrac{1}{2} - 1 = -1$

d. $\cos(\theta - \phi) = \cos\theta\cos\phi + \sin\theta\sin\phi = \dfrac{4}{5}\times\dfrac{3}{5} - \dfrac{3}{5}\times\dfrac{4}{5} = 0$

5. a → s; b → r; c → q; d → p

a. $\{\cos(2A+\theta) + \cos(2B+\theta)\} = 2\cos(A-B)\cos(A+B+\theta)$

Maximum value is $2\cos(A-B)$ when $\cos(A+B+\theta) = 1$

b. $\{\cos 2A + \cos 2B\}$

$2\cos(A+B)\cos(A-B)$

Maximum value is $2\cos(A-B)$ when $\cos(A+B) = 1$

c. For $y = \sec x$, $x \in (0, \pi/2)$, tangent drawn to it at any point lies completely below the graph of $y = \sec x$, thus $\dfrac{\sec 2A + \sec 2B}{2} \geq \sec(A+B)$

$\Rightarrow \sec 2A + \sec 2B \geq \sec(A+B)$

Hence, the minimum value is $2\sec(A+B)$.

d. $\sqrt{\{\tan\theta + \cot\theta - 2\cos 2(A+B)\}} = \sqrt{\left(\sqrt{\tan\theta} - \sqrt{\cot\theta}\right)^2 + 2 - 2\cos 2(A+B)}$

$$= \sqrt{\left(\sqrt{\tan\theta} - \sqrt{\cot\theta}\right)^2 + 4\sin^2(A+B)}$$

Minimum value occurs when $\sqrt{\tan\theta} = \sqrt{\cot\theta}$ and

minimum value is $\sqrt{4\sin^2(A+B)} = 2\sin(A+B)$

6. a → s; b → r; c → p; d → q

a. $\cos 20° + \cos 80° - \sqrt{3}\cos 50° = 2\cos 30°\cos 50° - \sqrt{3}\cos 50°$

$$= \sqrt{3}\cos 50° - \sqrt{3}\cos 50° = 0$$

b. $\cos 0° + \cos\dfrac{\pi}{7} + \cos\dfrac{2\pi}{7} + \cos\dfrac{3\pi}{7} + \cos\dfrac{4\pi}{7} + \cos\dfrac{5\pi}{7} + \cos\dfrac{6\pi}{7}$

$= 1 + \left(\cos\dfrac{\pi}{7} + \cos\dfrac{6\pi}{7}\right) + \left(\cos\dfrac{2\pi}{7} + \cos\dfrac{5\pi}{7}\right) + \left(\cos\dfrac{3\pi}{7} + \cos\dfrac{4\pi}{7}\right)$

$$= 1 + \left(\cos\frac{\pi}{7} + \cos\left(\pi - \frac{\pi}{7}\right)\right) + \left(\cos\frac{2\pi}{7} + \cos\left(\pi - \frac{2\pi}{7}\right)\right) + \left(\cos\frac{3\pi}{7} + \cos\left(\pi - \frac{3\pi}{7}\right)\right)$$

$= 1 + 0 + 0 + 0 = 1$

c. $\cos 20° + \cos 40° + \cos 60° - 4\cos 10° \cos 20° \cos 30°$

$= \cos 20° + \cos 40° + \cos 60° - 4\cos 10° \cos 20° \cos 30°$

$= 2\cos 30° \cos 10° + 2\cos^2 30° - 1 - 4\cos 10° \cos 20° \cos 30°$

$= 2\cos 30° (\cos 10° + \cos 30°) - 1 - 4\cos 10° \cos 20° \cos 30°$

$= 2\cos 30° (2\cos 10° \cos 20°) - 1 - 4\cos 10° \cos 20° \cos 30° = -1$

d. $\cos 20° \cos 100° + \cos 100° \cos 140° - \cos 140° \cos 200°$

$$= \frac{1}{2}(\cos 120° + \cos 80° + \cos 240° + \cos 40° - \cos 340° - \cos 60°)$$

$$= \frac{1}{2}\left(-\frac{1}{2} + \cos 80° - \frac{1}{2} + \cos 40° - \cos 340° - \frac{1}{2}\right) = \frac{1}{2}\left(-\frac{3}{2} + \cos 80° + \cos 40° - \cos 20°\right)$$

$$= \frac{1}{2}\left(-\frac{3}{2} + 2\cos 60° \cos 20° - \cos 20°\right) = \frac{1}{2}\left(-\frac{3}{2}\right) = -\frac{3}{4}$$

7. $a \to q$; $b \to q$; $c \to p, r$; $d \to p, s$

a. Since angles, A, B and C are acute angles

$\therefore A + B > \pi/2$

$A > \frac{\pi}{2} - B$

$\sin A - \cos B > 0$

$\Rightarrow \cos B - \sin A < 0$ (i)

Again, $B > \frac{\pi}{2} - A$

$\sin B > \cos A$

$\sin B - \cos A > 0$ (ii)

From Eq. (i) and (ii), we get that x-coordinates is –ve and y-coordinate is +ve.

Therefore, line is in 2nd quadrant only

b. $2^{\sin\theta} > 1 \Rightarrow \sin\theta > 0 \Rightarrow \theta \in$ 1st or 2nd quadrant

$3^{\cos\theta} < 1 \Rightarrow \cos\theta < 0 \Rightarrow \theta \in$ 2nd or 3rd quadrant

Hence, $\theta \in$ 2nd quadrant

c. $|\cos x + \sin x| = |\sin x| + |\cos x|$

$\Rightarrow \cos x$ and $\sin x$ must have same sign or at least one is zero.

$\Rightarrow x \in$ 2nd or 4th quadrant

d. L.H.S $= \frac{1 - \sin A}{|\cos A|} + \frac{\sin A}{\cos A} = \frac{1}{\cos A}$ which is true only if $|\cos A| = \cos A$

8. $a \to p$; $b \to p$; $c \to q$; $d \to s$

a. $x = \sin\theta$, $y = \cos\theta$

$P = (3\sin\theta - 4\sin^3\theta)^2 + (3\cos\theta - 4\cos^3\theta)^2 = \sin^2 3\theta + \cos^2 3\theta = 1$

b. On adding, we get $a = \frac{3 - \cos 4\theta + 4\sin 2\theta}{2} = (1 + \sin 2\theta)^2$

On subtracting, we get $b = (1 - \sin 2\theta)^2 \Rightarrow ab = \cos^4 2\theta \le 1$

c. $3\cos\theta = x^2 - 8x + 19$

$\Rightarrow 3\cos\theta = (x - 4)^2 + 3$

Now L.H.S. = $3\cos\theta \le 3$ or L.H.S. has the greatest value 3.

But R.H.S. $(x-4)^2+3 \ge 3$ or R.H.S. has the least value 3.

Hence, L.H.S. = R.H.S. when $3\cos\theta = (x-4)^2+3=3$

$\Rightarrow \cos\theta = 1$ and $x-4=0$

$\Rightarrow \theta = 2n\pi$ and $x=4$, where $n \in Z$.

d. $\lambda = \tan\theta$

$x = 2\sin 2\theta$ and $y = 2\cos 2\theta$

$E = x^2 - xy + y^2 = 4 - 4\sin 2\theta \cos 2\theta = 4 - 2\sin 4\theta$

$E \in [2, 6] \Rightarrow a + b = 8$

9. **a → q; b → p; c → s; d → r**

a. $9 + 16 + 24\sin(A+B) = 37$ (on squaring and adding)

$24\sin(A+B) = 12$

$\sin(A+B) = \frac{1}{2} \Rightarrow \sin C = \frac{1}{2}$

$C = 30°$ or $150°$

$\Rightarrow C = 30°$ ($\because$ for $C = 150°$)

b. $(\sin A + \sin B)^2 - \sin^2 C = 3\sin A \sin B$

$\Rightarrow \sin^2 A - \sin^2 C + \sin^2 B = \sin A \sin B$

$\Rightarrow \sin(A+C)\sin(A-C) + \sin^2 B = \sin A \sin B$

$\Rightarrow \sin B[\sin(A-C) + \sin(A+C)] = \sin A \sin B$

$\Rightarrow 2\sin A\cos C = \sin A$ (as $\sin B \ne 0$)

$\Rightarrow \cos C = 1/2$

$\Rightarrow C = 60°$

c. $2\sin x\cos x[4\cos^4 x - 4\sin^4 x] = 1$

$\Rightarrow (\sin 2x)[2(\cos^2 x + \sin^2 x)][2\cos^2 x - 2\sin^2 x] = 1$

$\Rightarrow (\sin 2x)2 \times 2\cos 2x = 1$

$\Rightarrow 2\sin 4x = 1$

$\Rightarrow \sin 4x = \frac{1}{2} \Rightarrow 4x = 30° \Rightarrow x = 7.5°$

d.

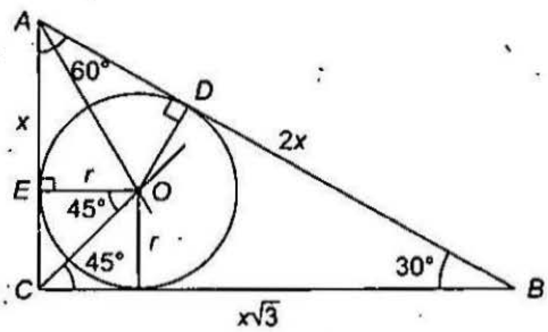


Fig. 2.46

Obviously, $AEOD$ is a cyclic quadrilateral, we have

$\angle COD = 120° + 45° = 165°$

Integer Type

1. (4) $f(\theta)=\dfrac{1-\sin 2\theta+\cos 2\theta}{2\cos 2\theta}$

$$=\frac{(\cos\theta-\sin\theta)^2+(\cos^2\theta-\sin^2\theta)}{2(\cos\theta-\sin\theta)(\cos\theta+\sin\theta)}$$

$$=\frac{\cos\theta}{\cos\theta+\sin\theta}$$

$$=\frac{1}{1+\tan\theta}$$

$$f(11^\circ)\cdot f(34^\circ)=\frac{1}{(1+\tan 11^\circ)}\times\frac{1}{(1+\tan 34^\circ)}$$

$$=\frac{1}{(1+\tan 11^\circ)}\times\frac{1}{(1+\tan(45^\circ-11^\circ))}$$

$$=\frac{1}{(1+\tan 11^\circ)}\times\frac{1}{\left(1+\dfrac{1-\tan 11^\circ}{1+\tan 11^\circ}\right)}$$

$$=\frac{1}{(1+\tan 11^\circ)}\times\frac{(1+\tan 11^\circ)}{2}=\frac{1}{2}$$

2. (5) $f(x)=2(7\cos x+24\sin x)(7\sin x-24\cos x)$

$r\cos\theta=7;\ r\sin\theta=24$

$r^2=625;\ \tan\theta=\dfrac{24}{7}$

$f(x)=2r\cos(x-\theta)\times r\sin(x-\theta)$

$\quad = r^2(\sin 2(x-\theta))$

$f(x)_{max}=25^2\Rightarrow(f(x))^{1/4}=5$

3. (1) $\tan\left(\dfrac{A}{2}\right)+\tan\left(\dfrac{B}{2}\right)=-\dfrac{b}{a};\ \tan\left(\dfrac{A}{2}\right)\times\tan\left(\dfrac{B}{2}\right)=\dfrac{c}{a}$

$A+B=90^\circ\Rightarrow\dfrac{A+B}{2}=45^\circ$

$$\Rightarrow\tan\left(\frac{A+B}{2}\right)=1=\frac{-\dfrac{b}{a}}{1-\dfrac{c}{a}}$$

$\Rightarrow 1-\dfrac{c}{a}=-\dfrac{b}{a}$

$\Rightarrow a+b=c$

$\Rightarrow\dfrac{a+b}{c}=1$

4. (5) $(1+\tan\theta)[1+\tan(45°-\theta)] = (1+\tan\theta)\left(1+\dfrac{1-\tan\theta}{1+\tan\theta}\right)$

$$= (1+\tan\theta)\left(\frac{2}{1+\tan\theta}\right) = 2$$

Hence, L.H.S. is equal to

$2(1+\tan 5°)(1+\tan 40°)(1+\tan 10°)(1+\tan 35°)(1+\tan 15°)(1+\tan 30°)(1+\tan 20°)(1+\tan 25°)$

$= 2\times 2^4 = 2^5$

5. (3) $$\sqrt{3}\left|\frac{\dfrac{-2\sin(40°)\cos(40°)}{\cos(80°)}+\dfrac{\sin(20°)}{\cos(20°)}}{\dfrac{\cot(20°)+\tan(80°)}{\cot(20°)}}\right| = \sqrt{3}\left|\frac{\tan(20°)-\tan(80°)}{1+\tan 20°\tan 80°}\right|$$

$$= \sqrt{3}\tan(60°) = 3$$

6. (1) Let $x+5 = 14\cos\theta$ and $y-12 = 14\sin\theta$

$\therefore x^2+y^2 = (14\cos\theta-5)^2+(14\sin\theta+12)^2$

$= 196+25+144+28(12\sin\theta-5\cos\theta)$

$= 365+28(12\sin\theta-5\cos\theta)$

$$\therefore \left.\sqrt{x^2+y^2}\right|_{\min} = \sqrt{365-28\times 13} = \sqrt{365-364} = 1$$

7. (5) $\cot x+\cot y = 49$

$$\Rightarrow \frac{1}{\tan x}+\frac{1}{\tan y} = 49$$

$$\Rightarrow \frac{\tan y+\tan x}{\tan x\tan y} = 49$$

$$\Rightarrow \tan x\tan y = \frac{\tan x+\tan y}{49} = \frac{42}{49} = \frac{6}{7}$$

$\Rightarrow \tan(x+y) = \dfrac{42}{1-(6/7)} = \dfrac{42}{1/7} = 294$ which is divisible by 2, 3 and 7 but not by 5.

8. (7) From the given equations, we have

$(2\cos a+9\cos d)^2 = (6\cos b+7\cos c)^2$

And $(2\sin a-9\sin d)^2 = (6\sin b-7\sin c)^2$

Adding, we have $36\cos(a+d) = 84\cos(b+c)$

$$\Rightarrow \frac{\cos(a+d)}{\cos(b+c)} = \frac{7}{3}$$

9. (8) Since $\cos A + \cos B = 0$

$\Rightarrow A + B = \pi,$

$\therefore B = \pi - A$

$\Rightarrow \sin A + \sin(\pi - A) = 1$

$\Rightarrow \sin A = \dfrac{1}{2}$

$\Rightarrow A = 30°$ and $B = 150°$ or $A = 150°$ and $B = 30°$

$\Rightarrow 12 \cos 60° + 4 \cos 300° = 8$

10.(4) $5\dfrac{2 \tan \beta}{1+\tan^2 \beta} = 3\dfrac{2 \tan \alpha}{1+\tan^2 \alpha}$

$$\Rightarrow \frac{5 \tan \beta}{1+\tan^2 \beta} = \frac{3 \tan \alpha}{1+\tan^2 \alpha} \quad \text{(i)}$$

Substituting $\tan \beta = 3 \tan \alpha$, we have

$$\frac{5 \times 3 \tan \alpha}{1+9 \tan^2 \alpha} = \frac{3 \tan \alpha}{1+\tan^2 \alpha}$$

$\Rightarrow 5 + 5 \tan^2\alpha = 1 + 9 \tan^2\alpha$

$\Rightarrow 4 \tan^2\alpha = 4$

$\Rightarrow \tan \alpha = 1$, i.e., $\tan \beta = 3$

$\therefore \tan \alpha + \tan \beta = 4$

11. (4) Let $\theta = \dfrac{\pi}{16} \Rightarrow 8\theta = \dfrac{\pi}{2}$

$y = \tan \theta + \tan 5\theta + \tan 9\theta + \tan 13\theta$

$\therefore y = (\tan \theta - \cot \theta) + (\tan 5\theta - \cot 5\theta)$

[as $\tan 13\theta = \tan(8\theta + 5\theta) = -\cot 5\theta$ and $\tan 9\theta = \tan(8\theta + \theta) = -\cot \theta$]

$= (\tan \theta - \cot \theta) + (\cot 3\theta - \tan 3\theta)$

$$= \frac{\sin^2 \theta - \cos^2 \theta}{\sin \theta \cos \theta} + \frac{\cos^2 3\theta - \sin^2 3\theta}{\sin 3\theta \cos 3\theta}$$

$$\Rightarrow y = 2\left[\frac{\cos 6\theta}{\sin 6\theta} - \frac{\cos 2\theta}{\sin 2\theta}\right]$$

$$= 2\left[\frac{\sin 2\theta \cos 6\theta - \cos 2\theta \sin 6\theta}{\sin 6\theta \sin 2\theta}\right]$$

$$= -2\left[\frac{\sin 4\theta}{\cos 2\theta \sin 2\theta}\right] = -4 \quad \left(\because\ 6\theta = \frac{\pi}{2} - 2\theta\right)$$

Hence, absolute value = 4.

12. (2) $\cos 290° = \sin 20°$; $\sin 250° = -\sin 70° = -\cos 20°$

$$\Rightarrow \frac{1}{\sin 20°} - \frac{1}{\sqrt{3} \cos 20°}$$

$$= \frac{\sqrt{3}\cos 20° - \sin 20°}{\sqrt{3}\sin 20° \cos 20°}$$

$$= \frac{2[\sin 60° \cos 20° - \sin 20° \cos 60°]}{\sqrt{3}\sin 20° \cos 20°}$$

$$= \frac{4\sin 40°}{\sqrt{3}\sin 40°} = \frac{4\sqrt{3}}{3}$$

Hence, the greatest integer less than or equal to is 2

13. (4) $\sin^6 x + \cos^6 x = (\sin^2 x + \cos^2 x)(\sin^4 x + \cos^4 x - \sin^2 x \cos^2 x)$

$$= 1 - 3\sin^2 x \cos^2 x = 1 - \frac{3(\sin 2x)^2}{4}$$

$$\Rightarrow y = \frac{4}{4 - 3(\sin 2x)^2}$$

$$\Rightarrow y_{\max} = \frac{4}{4 - 3(1)} = 4$$

14. (3) $\cos^2(45° + x) + (\sin x - \cos x)^2 = \left[\frac{\cos x}{\sqrt{2}} - \frac{\sin x}{\sqrt{2}}\right]^2 + (\sin x - \cos x)^2$

$$= \frac{3}{2}(1 - \sin 2x) = \frac{3}{2}(1 - (-1))$$

Hence, the maximum value is $= \frac{3}{2}(1 - (-1)) = 3$

15. (6) Nr. $= (\sin^2 t + \cos^2 t)^2 - 2\sin^2 t \cos^2 t - 1 = -2\sin^2 t \cos^2 t$

Dr. $= (\sin^2 t + \cos^2 t)^3 - 3\sin^2 t \cos^2 t - 1 = -3\sin^2 t \cos^2 t$

16. (6) $\frac{1}{\sin 10°} + \frac{1}{\sin 50°} - \frac{1}{\sin 70°} = \frac{1}{\cos 80°} + \frac{1}{\cos 40°} - \frac{1}{\cos 20°}$

$$= \frac{\cos 40° \cos 20° + \cos 80° \cos 20° - \cos 40° \cos 80°}{\cos 20° \cos 40° \cos 80°}$$

$$= 8[\cos 20°(\cos 40° + \cos 80°) - \cos 40° \cos 80°]$$

$$= 8[2\cos 20° \cos 60° \cos 20° - \cos 40° \cos 80°]$$

$$= 4[2\cos^2 20° - 2\cos 40° \cos 80°]$$

$$= 4[1 + \cos 40° - (\cos 120° + \cos 40°)]$$

$$= 4 \times \frac{3}{2} = 6$$

17. (7) $f(x) = 9\sin^2 x - 16\cos^2 x - 10(3\sin x - 4\cos x) - 10(3\sin x + 4\cos x) + 100$

$= 25\sin^2 x - 60\sin x + 84$

$= (5\sin x - 6)^2 + 48$

The minimum value of $f(x)$ occurs when $\sin x = 1$.

Therefore, the minimum value of $\sqrt{f(x)}$ is 7.

18. (0) In ΔABC, $\tan A + \tan B + \tan C = \tan A \tan B \tan C$

$\Rightarrow x + x + 1 + 1 - x = x(1+x)(1-x)$

$\Rightarrow 2 + x = x - x^3$

$\Rightarrow x^3 = -2 \Rightarrow x = -2^{1/3}$

$\Rightarrow \tan A = x < 0 \Rightarrow$ A is obtuse

$\Rightarrow \tan B = x + 1 = 1 - 2^{1/3} < 0$

Hence, A and B are obtuse, which is not possible in a triangle.

Hence, no such triangle can exist.

19. (4) Given $\log_{10}\left(\frac{\sin 2x}{2}\right) = -1$

$$\Rightarrow \frac{\sin 2x}{2} = \frac{1}{10}$$

$$\Rightarrow \sin 2x = \frac{1}{5} \quad \text{(i)}$$

Also $\log_{10}(\sin x + \cos x) = \dfrac{\log_{10}\left(\frac{n}{10}\right)}{2}$

$$\Rightarrow \log_{10}(\sin x + \cos x)^2 = \log_{10}\left(\frac{n}{10}\right)$$

$$\Rightarrow 1 + \sin 2x = \frac{n}{10}$$

$$\Rightarrow 1 + \frac{1}{5} = \frac{n}{10}$$

$$\Rightarrow \frac{6}{5} = \frac{n}{10}$$

$$\Rightarrow \frac{n}{3} = 4$$

20. (4) $\dfrac{2\sin 4^\circ \cos 3^\circ + 2\sin 4^\circ \cos 1^\circ}{\cos 1^\circ \cos 2^\circ \sin 4^\circ} = \dfrac{2\sin 4^\circ[\cos 3^\circ + \cos 1^\circ]}{\cos 1^\circ \cos 2^\circ \sin 4^\circ}$

$$= \frac{4\cos 2^\circ \cos 1^\circ}{\cos 1^\circ \cos 2^\circ} = 4$$

21. (7) $2\sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2} = \dfrac{1}{16}$

$$\Rightarrow \left(\cos\frac{A-B}{2} - \cos\frac{A+B}{2}\right)\sin\frac{C}{2} = \frac{1}{16}$$

$$\Rightarrow \sin\frac{C}{2}\left(\frac{1}{2} - \sin\frac{C}{2}\right) = \frac{1}{16}$$

$$\Rightarrow \sin^2\frac{C}{2} - \frac{1}{2}\sin\frac{C}{2} + \frac{1}{16} = 0$$

$$\Rightarrow \left(\frac{1}{4} - \sin\frac{C}{2}\right)^2 = 0$$

$$\Rightarrow \sin\frac{C}{2} = \frac{1}{4}$$

$$\Rightarrow \cos C = 1 - 2\sin^2\frac{C}{2} = 1 - \frac{1}{8} = \frac{7}{8}$$

22. (2) In the triangle,

$\tan A + \tan B + \tan C = \tan A \tan B \tan C$

$$\Rightarrow \frac{1}{2}\frac{(2k+1)}{2}\frac{(4k+1)}{2} = \frac{3}{2} + 3k$$

$$\Rightarrow \frac{8k^2 + 6k + 1}{8} = \frac{3+6k}{2}$$

$\Rightarrow 8k^2 + 6k + 1 = 12 + 24k$

$\Rightarrow 8k^2 - 18k - 11 = 0$

$\Rightarrow 8k^2 - 22k + 4k - 11 = 0$

$\Rightarrow (2k+1)(4k-11) = 0$

$\Rightarrow k = -1/2$ or $11/4$

For $k = -1/2$, $\tan B = 0$ (not possible)

$\therefore k = 11/4$

23. (4) $4\sin^3 x \cos 3x + 4\cos^3 x \sin 3x = \dfrac{3}{2}$

$$\Rightarrow (3\sin x - \sin 3x)\cos 3x + (3\cos x + \cos 3x)\sin 3x = \frac{3}{2}$$

$$\Rightarrow 3[\sin x \cos 3x + \cos x \sin 3x] = \frac{3}{2}$$

$$\Rightarrow \sin 4x = \frac{1}{2}$$

Archives

Subjective

1. We have $\tan(\alpha+\beta)=\dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$

$$=\frac{\frac{m}{m+1}+\frac{1}{2m+1}}{1-\frac{m}{m+1}\times\frac{1}{2m+1}}$$

$$=\frac{2m^2+2m+1}{2m^2+2m+1}=1$$

$\Rightarrow\quad \alpha+\beta=n\pi+\pi/4$, where $n\in Z$.

2. a. To draw the graph of $y=\dfrac{1}{\sqrt{2}}(\sin x+\cos x)$ from $x=-\dfrac{\pi}{2}$ to $x=\dfrac{\pi}{2}$

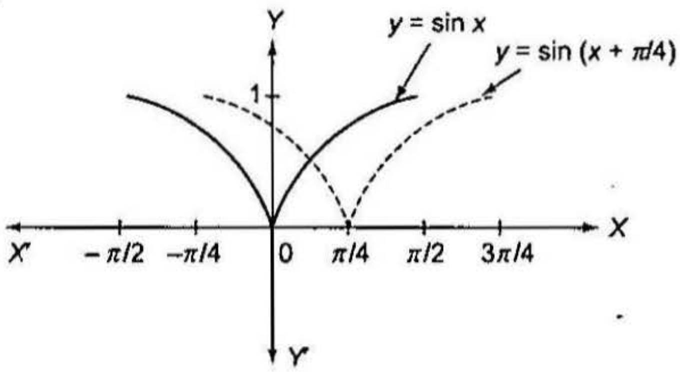


Fig. 2.47

$$y=\frac{1}{\sqrt{2}}(\sin x+\cos x)=\sin\left(x+\frac{\pi}{4}\right)$$

b. We have $\cos(\alpha+\beta)=\dfrac{4}{5}$ and $\sin(\alpha-\beta)=\dfrac{5}{13}$

$\Rightarrow\quad \tan(\alpha+\beta)=\dfrac{3}{4}$ and $\tan(\alpha-\beta)=\dfrac{5}{12}$

$$\tan 2\alpha=\tan[(\alpha+\beta)+(\alpha-\beta)]=\frac{\tan(\alpha-\beta)+\tan(\alpha+\beta)}{1-\tan(\alpha-\beta)\tan(\alpha+\beta)}=\frac{\frac{5}{12}+\frac{3}{4}}{1-\left(\frac{5}{12}\right)\left(\frac{3}{4}\right)}=\frac{56}{33}$$

3. We have,

$$5\cos\theta+3\cos\left(\theta+\frac{\pi}{3}\right)+3=5\cos\theta+3\cos\theta\cos\frac{\pi}{3}-3\sin\theta\sin\frac{\pi}{3}+3=\frac{13}{2}\cos\theta-\frac{3\sqrt{3}}{2}\sin\theta+3$$

Now, $-\sqrt{\left(\frac{13}{2}\right)^2+\left(\frac{3\sqrt{3}}{2}\right)^2} \le \frac{13}{2}\cos\theta - \frac{3\sqrt{3}}{2}\sin\theta \le \sqrt{\left(\frac{13}{2}\right)^2+\left(\frac{3\sqrt{3}}{2}\right)^2}$

$$\Rightarrow \quad -7 \le \frac{13}{2}\cos\theta - \frac{3\sqrt{3}}{2}\sin\theta \le 7$$

$$\Rightarrow \quad -4 \le \frac{13}{2}\cos\theta - \frac{3\sqrt{3}}{2}\sin\theta + 3 \le 10$$

4. Given $\alpha+\beta-\gamma=\pi$ and to prove that
$\sin^2\alpha+\sin^2\beta-\sin^2\gamma = 2\sin\alpha\sin\beta\cos\gamma$
L.H.S. $=\sin^2\alpha+\sin^2\beta-\sin^2\gamma$
$=\sin^2\alpha+\sin(\beta+\gamma)\sin(\beta-\gamma)$
$=\sin^2\alpha+\sin(\beta+\gamma)\sin(\pi-\alpha)$ $\quad (\because \alpha+\beta-\gamma=\pi)$
$=\sin^2\alpha+\sin(\beta+\gamma)\sin\alpha=\sin\alpha[\sin\alpha+\sin(\beta+\gamma)]$
$=\sin\alpha[\sin[\pi-(\beta-\gamma)]+\sin(\beta+\gamma)]$
$=\sin\alpha[\sin(\beta-\gamma)+\sin(\beta+\gamma)]=\sin\alpha[2\sin\beta\cos\gamma]$
$=2\sin\alpha\sin\beta\cos\gamma$
$=$R.H.S.

5. We have,

$$\cos\theta+\sin\theta=\sqrt{2}\left[\frac{1}{\sqrt{2}}\cos\theta+\frac{1}{\sqrt{2}}\sin\theta\right]=\sqrt{2}\sin(\pi/4+\theta)$$

$\therefore \quad \cos\theta+\sin\theta \le \sqrt{2} < \pi/2$ $\quad \left(\begin{matrix}\because \sqrt{2}=1.414\\ \pi/2=1.57\end{matrix}\right)$

$\therefore \quad \cos\theta+\sin\theta<\pi/2 \Rightarrow \quad \cos\theta<\pi/2-\sin\theta$ (i)
As $\theta\in[0,\pi/2]$ in which $\sin\theta$ increases, taking sin on both the sides of Eq. (i) we get
$\sin(\cos\theta)<\sin(\pi/2-\sin\theta)\Rightarrow\sin(\cos\theta)<\cos(\sin\theta)$
$\Rightarrow \quad \cos(\sin\theta)>\sin(\cos\theta)$ (ii)

6. L.H.S. $=\sin 12°\sin 48°\sin 54°$

$$=\frac{1}{2}[2\sin 12°\cos 42°]\sin 54°$$

$$=\frac{1}{2}\left[\sin^2 54°-\frac{1}{2}\sin 54°\right]$$

$$=\frac{1}{4}\left[2\sin^2 54°-\sin 54°\right]$$

$$=\frac{1}{4}\left[2\left(\frac{1+\sqrt{5}}{4}\right)^2-\left(\frac{1+\sqrt{5}}{4}\right)\right]$$

$$=\frac{1}{4}\left[2\left(\frac{1+5+2\sqrt{5}}{16}\right)-\left(\frac{1+\sqrt{5}}{4}\right)\right]$$

$$=\frac{1}{4}\times\frac{1}{8}[6+2\sqrt{5}-2-2\sqrt{5}]=\frac{1}{32}\times 4=\frac{1}{8}=\text{R.H.S.}$$

7. We know that

$$\cos A \cos 2A \cos 4A \cdots \cos 2^n A = \frac{1}{2^{n+1} \sin A} \sin\left(2^{n+1} A\right)$$

$$\therefore \quad 16 \cos \frac{2\pi}{15} \cos 2\left(\frac{2\pi}{15}\right) \cos 2^2\left(\frac{2\pi}{15}\right) \cos 2^3\left(\frac{2\pi}{15}\right)$$

$$= 16 \frac{\sin\left(2^4 A\right)}{2^4 \sin A} \qquad \text{(where } A = 2\pi/15\text{)}$$

$$= 16 \frac{\sin(32\pi/15)}{16 \sin 2\pi/15} = \frac{\sin(32\pi/15)}{\sin(2\pi + 2\pi/15)} = \frac{\sin(32\pi/15)}{\sin(32\pi/15)} = 1$$

8. We know that

$$\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha}$$

$$\Rightarrow \quad \frac{1-\tan^2\alpha}{\tan\alpha} = 2\cot 2\alpha \quad \Rightarrow \quad \cot\alpha - \tan\alpha = 2\cot 2\alpha \qquad \text{(i)}$$

Now we have to prove $\tan\alpha + 2\tan 2\alpha + 4\tan 4\alpha + 8\cot 8\alpha = \cot\alpha$

L.H.S.

$= \tan\alpha + 2\tan 2\alpha + 4\tan 4\alpha + 4(2\cot 8\alpha)$

$= \tan\alpha + 2\tan 2\alpha + 4\tan 4\alpha + 4(\cot 4\alpha - \tan 4\alpha)$ [using Eq. (i)]

$= \tan\alpha + 2\tan 2\alpha + 4\tan 4\alpha + 4\cot 4\alpha - 4\tan 4\alpha = \tan\alpha + 2\tan 2\alpha + 2(2\cot 4\alpha)$

$= \tan\alpha + 2\tan 2\alpha + 2(\cot 2\alpha - \tan 2\alpha)$ [using Eq. (i)]

$= \tan\alpha + 2\cot 2\alpha = \tan\alpha + (\cot\alpha - \tan\alpha)$ [using Eq. (i)]

$= \cot\alpha =$ R.H.S.

9. Given that in ΔABC, A, B and C are in A.P.

$\therefore \quad A + B = 2B$

Also $A + B + C = 180° \quad \Rightarrow \quad B + 2B = 180° \quad \Rightarrow \quad B = 60°$

Also given that $\sin(2A + B) = \sin(C - A) = -\sin(B + 2C) = \dfrac{1}{2}$

$$\Rightarrow \quad \sin(2A + 60°) = \sin(C - A) = -\sin(60 + 2C) = \frac{1}{2} \qquad \text{(i)}$$

From Eq. (i), we have

$$\sin(2A + 60°) = \frac{1}{2}$$

$\Rightarrow \quad 2A + 60° = 150°$

$\Rightarrow \quad 2A = 90°$

$\Rightarrow \quad A = 45°$

$\Rightarrow \quad C = \pi - A - B = 75°$

10. Let $y = \dfrac{\tan x}{\tan 3x}$

$$= \frac{\tan x\left(1 - 3\tan^2 x\right)}{3\tan x - \tan^3 x}$$

$$= \frac{1 - 3\tan^2 x}{3 - \tan^2 x}$$

$\Rightarrow \quad 3y-(\tan^2 x)y = 1-3\tan^2 x$

$\Rightarrow \quad (y-3)\tan^2 x = 3y-1$

$$\Rightarrow \quad \tan^2 x = \frac{3y-1}{y-3}$$

$$\Rightarrow \quad \frac{3y-1}{y-3} \geq 0 \qquad \text{(L.H.S. is a prefect square)}$$

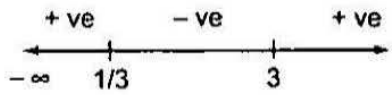


Fig. 2.48

$$\Rightarrow \quad y < \frac{1}{3} \text{ or } y \geq 3$$

Thus, y never lies between 1/3 and 3.

11. $$S = \sum_{k=1}^{n-1}(n-k)\cos\frac{2k\pi}{n}$$

$$= (n-1)\cos\frac{2\pi}{n} + (n-2)\cos 2\frac{2\pi}{n} + \cdots + 1\cos(n-1)\frac{2\pi}{n} \qquad \text{(i)}$$

We know that $\cos\theta = \cos(2\pi-\theta)$. Replacing each angle θ by $2\pi-\theta$ in Eq. (i), we get

$$S = (n-1)\cos(n-1)\frac{2\pi}{n} + (n-2)\cos(n-2)\frac{2\pi}{n} + \cdots + 1\cos\frac{2\pi}{n} \quad \text{[using Eq. (i)]} \qquad \text{(ii)}$$

Adding terms having the same angle and taking n common, we have

$$2S = n\left[\cos\frac{2\pi}{n} + \cos\frac{4\pi}{n} + \cos\frac{6\pi}{n} + \cdots + \cos(n-1)\frac{2\pi}{n}\right]$$

$$= \frac{2\pi}{n} = n\left[\frac{\sin(n-1)\frac{\pi}{n}}{\sin\frac{\pi}{n}}\cos\frac{\frac{2\pi}{n}+(n-1)\frac{2\pi}{n}}{2}\right]$$

$= n\,1\cos\pi = -n \quad (\because \sin(\pi-\theta) = \sin\theta)$

$\therefore \quad S = -n/2$

12. Given that

$$2\sin t = \frac{1-2x+5x^2}{3x^2-2x-1},\ t \in [-\pi/2, \pi/2]$$

This can be written as

$(6\sin t - 5)x^2 + 2(1-2\sin t)x - (1+2\sin t) = 0$

For the given equation to hold, x should be a real number, therefore the above equation should have real roots, i.e., $D \geq 0$

$\Rightarrow \quad 4(1-2\sin t)^2 + 4(6\sin t - 5)(1+2\sin t) \geq 0$

$\Rightarrow \quad 16\sin^2 t - 8\sin t - 4 \geq 0$

$\Rightarrow \quad (4\sin^2 t - 2\sin t - 1) \geq 0$

$$\Rightarrow \quad 4\left(\sin t-\frac{\sqrt{5}+1}{4}\right)\left(\sin t+\frac{\sqrt{5}-1}{4}\right)\geq 0$$

$$\Rightarrow \quad \sin t \leq -\left(\frac{\sqrt{5}-1}{4}\right) \quad \Rightarrow \quad \sin t \geq \frac{\sqrt{5}+1}{4}$$

$\Rightarrow \quad \sin t \leq \sin(-\pi/10)$ or $\sin t \geq \sin(3\pi/10) \quad \Rightarrow \quad t \leq -\pi/10$ or $t \geq 3\pi/10$

(Note that $\sin x$ is an increasing function from $-\pi/2$ to $\pi/2$.

Therefore, the range of t is $[-\pi/2, -\pi/10] \cup [3\pi/10, \pi/2]$.

13. $$\frac{1}{4\cos^2\theta+1+\frac{3}{2}\sin 2\theta}=\frac{1}{2[1+\cos 2\theta]+1+\frac{3}{2}\sin 2\theta}$$

$$=\frac{1}{2\cos 2\theta+\frac{3}{2}\sin 2\theta+3}$$

Now $-\sqrt{2^2+\left(\frac{3}{2}\right)^2} \leq 2\cos 2\theta+\frac{3}{2}\sin 2\theta \leq \sqrt{2^2+\left(\frac{3}{2}\right)^2}$

or $-\frac{5}{2} \leq 2\cos 2\theta+\frac{3}{2}\sin 2\theta \leq \frac{5}{2}$

$\Rightarrow \frac{1}{2} \leq 2\cos 2\theta+\frac{3}{2}\sin 2\theta+3 \leq \frac{11}{2}$

$\Rightarrow \frac{2}{11} \leq \frac{1}{2\cos 2\theta+\frac{3}{2}\sin 2\theta+3} \leq 2$

Hence, the maximum value is 2.

Objective

Fill in the blanks

1. According to the given question, we have expressed L.H.S. in the form $C_0+C_1\cos x+C_2\cos 2x+\cdots+C_n\cos nx$.

   Now,

   $$\sin^3 x\sin 3x=\frac{3\sin x-\sin 3x}{4}\sin 3x=\frac{3\sin x\sin 3x-\sin^2 3x}{4}=\frac{3(\cos 2x-\cos 4x)-(1-\cos 6x)}{8}$$

   Hence, $n=6$.

2. We know that A.M. $\geq$ G.M.

   It implies that the minimum value of A.M. is obtained when A.M. = G.M.

   Therefore, the quantities whose A.M. is being taken are equal.

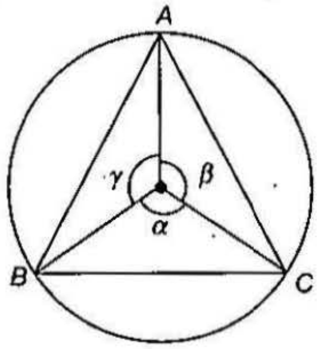


Fig. 2.49

That is, $\cos\left(\alpha+\frac{\pi}{2}\right)=\cos\left(\beta+\frac{\pi}{2}\right)=\cos\left(\gamma+\frac{\pi}{2}\right)$

$\Rightarrow \quad \sin\alpha=\sin\beta=\sin\gamma$

Also, $\alpha+\beta+\gamma=360°$

$\Rightarrow \quad \alpha=\beta=\gamma=120°=2\pi/3$

Therefore, the minimum value of A.M. $=\dfrac{\cos\left(\frac{2\pi}{3}+\frac{\pi}{2}\right)+\cos\left(\frac{2\pi}{3}+\frac{\pi}{2}\right)+\cos\left(\frac{2\pi}{3}+\frac{\pi}{2}\right)}{3}$

$$=\frac{-3\sin\frac{2\pi}{3}}{3}=-\frac{\sqrt{3}}{2}$$

3. $\sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14}\sin\frac{7\pi}{14}\sin\frac{9\pi}{14}\sin\frac{11\pi}{14}\sin\frac{13\pi}{14}$

$$=\sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14}\sin\frac{\pi}{2}\sin\left(\pi-\frac{5\pi}{14}\right)\sin\left(\pi-\frac{3\pi}{14}\right)\sin\left(\pi-\frac{\pi}{14}\right)=\sin^2\frac{\pi}{14}\sin^2\frac{3\pi}{14}\sin^2\frac{5\pi}{14}$$

$$=\left(\sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14}\right)^2$$

$$=\left[\cos\left(\frac{\pi}{2}-\frac{\pi}{14}\right)\cos\left(\frac{\pi}{2}-\frac{3\pi}{14}\right)\cos\left(\frac{\pi}{2}-\frac{5\pi}{14}\right)\right]^2$$

$$=\left[\cos\frac{3\pi}{7}\cos\frac{2\pi}{7}\cos\frac{\pi}{7}\right]^2$$

$$=\left[\frac{1}{2\sin\pi/7}\left\{2\cos\frac{\pi}{7}\sin\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{3\pi}{7}\right\}\right]^2$$

$$=\left[\frac{1}{2^2\sin\pi/7}\left\{2\sin\frac{2\pi}{7}\cos\frac{2\pi}{7}\cos\frac{3\pi}{7}\right\}\right]^2$$

$$=\left[\frac{1}{2^3\sin\pi/7}\left(2\sin\frac{4\pi}{7}\cos\left(\frac{\pi-3\pi}{7}\right)\right)\right]^2$$

$$=\left(\frac{1}{8\sin\pi/7}\sin\frac{8\pi}{7}\right)^2$$

$$=\left(\frac{\sin(\pi+\pi/7)}{8\sin\pi/7}\right)^2=\left(\frac{-\sin\pi/7}{8\sin\pi/7}\right)^2=\left(\frac{1}{8}\right)^2=\frac{1}{64}$$

4. $k=\sin\frac{\pi}{14}\sin\frac{5\pi}{18}\sin\frac{7\pi}{18}$

$$=\cos\left(\frac{\pi}{2}-\frac{\pi}{18}\right)\cos\left(\frac{\pi}{2}-\frac{5\pi}{18}\right)\cos\left(\frac{\pi}{2}-\frac{7\pi}{18}\right)$$

$$=\cos\frac{\pi}{9}\cos\frac{2\pi}{9}\cos\frac{4\pi}{9}$$

$$=\frac{1}{2^3\sin\frac{\pi}{9}}\sin\frac{8\pi}{9}=\frac{1}{8\sin\pi/9}\sin\pi/9=\frac{1}{8} \qquad \left[\because \sin\frac{8\pi}{9}=\sin(\pi-\pi/9)=\sin\pi/9\right]$$

5. $A+B=\pi/3 \quad \Rightarrow \quad \tan(A+B)=\sqrt{3}$

$$\Rightarrow \quad \frac{\tan A+\tan B}{1-\tan A\tan B}=\sqrt{3}$$

$$\Rightarrow \quad \frac{\tan A+\frac{y}{\tan A}}{1-y}=\sqrt{3} \quad [\text{where } y=\tan A\tan B]$$

$$\Rightarrow \quad \tan^2 A+\sqrt{3}(y-1)\tan A+y=0$$

For real value of $\tan A$,

$$3(y-1)^2-4y\ge 0$$

$$\Rightarrow \quad 3y^2-10y+3\ge 0$$

$$\Rightarrow \quad (y-3)\left(y-\frac{1}{3}\right)\ge 0$$

$$\Rightarrow \quad y\le\frac{1}{3} \text{ or } y\ge 3$$

But $A, B>0$ and $A+B=\pi/3 \Rightarrow A, B<\pi/3$

$$\Rightarrow \quad \tan A\tan B<3$$

$\therefore \quad y\le\frac{1}{3}$, i.e., the maximum value of y is 1/3.

6. We have $\frac{2}{\cos x}=\frac{1}{\cos(x-y)}+\frac{1}{\cos(x+y)}=\frac{2\cos x\cos y}{\cos^2 x-\sin^2 y}$

$$\Rightarrow \quad \cos^2 x-2\sin^2 y=\cos^2 x\cos y$$

$\Rightarrow \quad \cos^2 x(1-\cos y) = \sin^2 y$

$\Rightarrow \quad \cos^2 x \, 2\sin^2 \dfrac{y}{2} = 4\sin^2 \dfrac{y}{2} \cos^2 \dfrac{y}{2}$

$\Rightarrow \quad \cos^2 x = 2\cos^2 \dfrac{y}{2}$

$\Rightarrow \quad \cos^2 x \sec^2 \dfrac{y}{2} = 2$

$\Rightarrow \quad \cos x \sec \dfrac{y}{2} = \pm\sqrt{2}$

True or false

1. $\tan A = \dfrac{1-\cos B}{\sin B} = \dfrac{2\sin^2 \frac{B}{2}}{2\sin\frac{B}{2}\cos\frac{B}{2}} = \tan\dfrac{B}{2}$

Hence, $\tan 2A = \dfrac{2\tan A}{1-\tan^2 A} = \dfrac{2\tan\frac{B}{2}}{1-\tan^2\frac{B}{2}} = \tan B$

Therefore, the statement is true.

Multiple choice questions with one correct answer

1. d. From the given relations, $m+n = 2\tan\theta$, $m-n = 2\sin\theta$

$\Rightarrow \quad m^2 - n^2 = 4\tan\theta\sin\theta$ (i)

Also $4\sqrt{mn} = 4\sqrt{\tan^2\theta - \sin^2\theta} = 4\sin\theta\tan\theta$ (ii)

From Eqs. (i) and (ii), we get $m^2 - n^2 = 4\sqrt{mn}$.

2. b. $\tan\theta = \dfrac{-4}{3} \quad \Rightarrow \quad \theta \in$ II quadrant or IV quadrant $\quad \Rightarrow \quad \sin\theta = \pm 4/5$

If $\theta \in$ II quadrant, $\sin\theta = 4/5$

If $\theta \in$ IV quadrant, $\sin\theta = -4/5$

3. a. $\alpha+\beta+\gamma = 2\pi \Rightarrow \dfrac{\alpha}{2}+\dfrac{\beta}{2}+\dfrac{\gamma}{2} = \pi$

$\therefore \quad \tan\left(\dfrac{\alpha}{2}+\dfrac{\beta}{2}\right) = \tan\left(\pi - \dfrac{\gamma}{2}\right) = -\tan\dfrac{\gamma}{2}$

$\Rightarrow \quad \dfrac{\tan\alpha/2 + \tan\beta/2}{1-\tan\alpha/2\tan\beta/2} = \tan\gamma/2$

$\Rightarrow \quad \tan\alpha/2 + \tan\beta/2 + \tan\gamma/2 = \tan\alpha/2\tan\beta/2\tan\gamma/2$

4. b. We have $\sin^2\theta + \cos^4\theta = \sin^2\theta + \cos^2\cos^2\theta \le \sin^2\theta + \cos^2\theta$ $[\because \cos^2\theta \le 1]$

Thus, $A = \sin^2\theta + \cos^4\theta \le 1$

Again, $A = \sin^2\theta + \cos^4\theta = 1 - \cos^2\theta + \cos^4\theta = 1 + (\cos^4\theta - \cos^2\theta)$

$$= 1 + \left(\cos^2\theta - \frac{1}{2}\right)^2 - \frac{1}{4} = \frac{3}{4} + \left(\cos^2\theta - \frac{1}{2}\right)^2 \geq \frac{3}{4}$$

Hence, $3/4 \leq A \leq 1$.

5. c. We have $\cos\frac{7\pi}{8} = \cos\left(\pi - \frac{\pi}{8}\right) = -\cos\frac{\pi}{8}$

and $\cos\frac{5\pi}{8} = \cos\left(\pi - \frac{3\pi}{8}\right) = -\cos\frac{3\pi}{8}$

$$\therefore \quad \text{L.H.S.} = \left(1 + \cos\frac{\pi}{8}\right)\left(1 + \cos\frac{3\pi}{8}\right)\left(1 - \cos\frac{3\pi}{8}\right)\left(1 - \cos\frac{\pi}{8}\right)$$

$$= \left(1 - \cos^2\frac{\pi}{8}\right)\left(1 - \cos^2\frac{3\pi}{8}\right)$$

$$= \sin^2\frac{\pi}{8}\sin^2\frac{3\pi}{8}$$

$$= \frac{1}{4}\left(2\sin^2\frac{\pi}{8}\right)\left(2\sin^2\frac{3\pi}{8}\right)$$

$$= \frac{1}{4}\left[\left(1 - \cos\frac{\pi}{4}\right)\left(1 - \cos\frac{3\pi}{4}\right)\right] \qquad \left[\because 1 - \cos\theta = 2\sin^2\frac{\theta}{2}\right]$$

$$= \frac{1}{4}\left[\left(1 - \frac{1}{\sqrt{2}}\right)\left(1 + \frac{1}{\sqrt{2}}\right)\right] = \frac{1}{4}\left(1 - \frac{1}{2}\right) = \frac{1}{8} = \frac{1}{4}\left(1 - \frac{1}{2}\right) = \frac{1}{8} = \text{R.H.S.}$$

6. c. The given expression is $\sqrt{3}\operatorname{cosec}20° - \sec 20° = \frac{\sqrt{3}}{\sin 20°} - \frac{1}{\cos 20°}$

$$= \frac{\sqrt{3}\cos 20° - \sin 20°}{\sin 20°\cos 20°}$$

$$= 4\left[\frac{\frac{\sqrt{3}}{2}\cos 20° - \frac{1}{2}\sin 20°}{2\sin 20°\cos 20°}\right]$$

$$= 4\left[\frac{\sin 60°\cos 20° - \cos 60°\sin 20°}{\sin 2\times 20°}\right]$$

$$= \frac{4\sin(60° - 20°)}{\sin 40°} = \frac{4\sin 40°}{\sin 40°} = 4$$

7. c. $3(\sin x - \cos x)^4 + 6(\sin x + \cos x)^2 + 4(\sin^6 x + \cos^6 x)$

$= 3(1 - \sin 2x)^2 + 6(1 + \sin 2x) + 4[(\sin^2 x + \cos^2 x)^3 - 3\sin^2 x\cos^2 x(\sin^2 x + \cos^2 x)]$

$$= 3(1 - 2\sin 2x + \sin^2 2x) + (6 + 6\sin 2x) + 4\left[1 - \frac{3}{4}\sin^2 2x\right]$$

$$= 13 + 3\sin^2 2x - 3\sin^2 2x = 13$$

8. b. Given, $\sec^2\theta = \dfrac{4xy}{(x+y)^2}$

Now, $\sec^2\theta \geq 1 \Rightarrow \dfrac{4xy}{(x+y)^2} \geq 1$

$\Rightarrow \quad (x+y)^2 \leq 4xy$

$\Rightarrow \quad (x+y)^2 - 4xy \leq 0$

$\Rightarrow \quad (x-y)^2 \leq 0$

But for real values of x and y, $(x-y)^2 \geq 0 \quad \Rightarrow \quad (x-y)^2 = 0$

$\therefore \quad x = y$

Also $\quad x+y \neq 0 \Rightarrow x \neq 0, y \neq 0$

9. c. $f(\theta) = \sin\theta(\sin\theta + \sin 3\theta) = (\sin\theta + 3\sin\theta - 4\sin^3\theta)\sin\theta = (4\sin\theta - 4\sin^3\theta)\sin\theta$

$= 4\sin^2\theta(1-\sin^2\theta) = 4\sin^2\theta\cos^2\theta = (2\sin\theta\cos\theta)^2 = (\sin 2\theta)^2 \geq 0$

which is true for all θ.

10. a. We are given that

$(\cot\alpha_1)(\cot\alpha_2)\cdots(\cot\alpha_n) = 1$

$\Rightarrow \quad (\cos\alpha_1)(\cos\alpha_2)\cdots(\cos\alpha_n) = (\sin\alpha_1)(\sin\alpha_2)\cdots(\sin\alpha_n) \quad$ (i)

Let $y = (\cos\alpha_1)(\cos\alpha_2)\cdots(\cos\alpha_n)$ (to be maximum)

Squaring both sides, we get

$y^2 = (\cos^2\alpha_1)(\cos^2\alpha_2)\cdots(\cos^2_n)$

$= \cos\alpha_1\sin\alpha_1\cos\alpha_2\sin\alpha_2\cdots\cos\alpha_n\sin\alpha_n \quad$ [using Eq. (i)]

$= \dfrac{1}{2^n}[\sin 2\alpha_1 \sin 2\alpha_2 \cdots \sin 2\alpha_n]$

As $0 \leq \alpha_1, \alpha_2, \cdots, \alpha_n \leq \pi/2$

$\therefore \quad 0 \leq 2\alpha_1, 2\alpha_2, \cdots, 2\alpha_2 \leq \pi$

$\Rightarrow \quad 0 \leq \sin 2\alpha_1, \sin 2\alpha_2, \cdots, \sin 2\alpha_n \leq 1$

$\therefore \quad y^2 \leq \dfrac{1}{2^n}1 \Rightarrow y \leq \dfrac{1}{2^{n/2}}$

Therefore, the maximum value of y is $1/2^{n/2}$.

11. c. $\alpha + \beta = \dfrac{\pi}{2} \quad \Rightarrow \quad \alpha = \dfrac{\pi}{2} - \beta \quad \Rightarrow \quad \tan\alpha = \cot\beta \quad \Rightarrow \quad \tan\alpha\tan\beta = 1 \quad$ (i)

Again, $\beta + \gamma = \alpha \Rightarrow \gamma = \alpha - \beta$

$\Rightarrow \quad \tan\gamma = \dfrac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta} = \dfrac{\tan\alpha - \tan\beta}{2} \quad$ [using Eq. (i)]

$\Rightarrow \quad \tan\alpha = \tan\beta + 2\tan\gamma$

12. b. Given that $\sin\theta = 1/2$ and $\cos\phi = 1/3$, and θ and ϕ are acute angles.

$\therefore \quad \theta = \pi/6$ and $0 < \dfrac{1}{3} < \dfrac{1}{2}$

or $\cos\pi/2 < \cos\phi < \cos\pi/3$ or $\pi/3 < \phi < \pi/2$

$\therefore \quad \dfrac{\pi}{3} + \dfrac{\pi}{6} < \theta + \phi < \dfrac{\pi}{2} + \dfrac{\pi}{6}$ or $\dfrac{\pi}{2} < \theta + \phi < \dfrac{2\pi}{3} \quad \Rightarrow \quad \theta + \phi \in \left(\dfrac{\pi}{2}, \dfrac{2\pi}{3}\right)$

13. b. $\sec 2x - \tan 2x = \dfrac{1-\sin 2x}{\cos 2x} = \dfrac{1-\cos 2\left(\frac{\pi}{4}-x\right)}{\sin 2\left(\frac{\pi}{4}-x\right)} = \dfrac{2\sin^2\left(\frac{\pi}{4}-x\right)}{2\sin\left(\frac{\pi}{4}-x\right)\cos\left(\frac{\pi}{4}-x\right)} = \tan\left(\frac{\pi}{4}-x\right)$

14. d. $\sin\frac{\pi}{2n}+\cos\frac{\pi}{2n}=\frac{\sqrt{n}}{2} \Rightarrow \sin^2\frac{\pi}{2n}+\cos^2\frac{\pi}{2n}+2\sin\frac{\pi}{2n}\cos\frac{\pi}{2n}=\frac{n}{4}$

$\Rightarrow 1+\sin\frac{\pi}{n}=\frac{n}{4} \Rightarrow \sin\frac{\pi}{n}=\frac{n-4}{4}$

For $n=2$, the given equation is not satisfied.

Considering that $n>1$ and $n\neq 2$, $0<\sin\frac{\pi}{n}<1 \Rightarrow 0<\frac{n-4}{4}<1 \Rightarrow 4<n<8$

15. b. $\theta\in\left(0,\frac{\pi}{4}\right) \Rightarrow \tan\theta<1$ and $\cot\theta>1$.

Let $\tan\theta=1-x$ and $\cot\theta=1+y$, where $x, y>0$ and are very small, then

$t_1=(1-x)^{1-x}, t_2=(1-x)^{1+y}, t_3=(1+y)^{1-x}, t_4=(1+y)^{1+y}$

Clearly, $t_4>t_3$ and $t_1>t_2$. Also $t_3>t_1$.

Thus, $t_4>t_3>t_1>t_2$.

Multiple choice questions with one or more than one correct answers

1. b. $3\left[\sin^4\left(\frac{3}{2}\pi-\alpha\right)+\sin^4(3\pi+\alpha)\right]-2\left[\sin^6\left(\frac{1}{2}\pi+\alpha\right)+\sin^6(5\pi-\alpha)\right]$

$=3(\cos^4\alpha+\sin^4\alpha)-2(\cos^6\alpha+\sin^6\alpha)$

$=3(1-2\sin^2\alpha\cos^2\alpha)-2[(\sin^2\alpha\cos^2\alpha)^3-3\sin^2\alpha\cos^2\alpha(\sin^2\alpha+\cos^2\alpha)]$

$=3(1-2\sin^2\alpha\cos^2\alpha)-2[1-3\sin^2\alpha\cos^2\alpha]=1$

2. b, c. All are infinite G.P.'s with common ratio < 1

$x=\dfrac{1}{1-\cos^2\phi}=\dfrac{1}{\sin^2\phi}, y=\dfrac{1}{1-\sin^2\phi}=\dfrac{1}{\cos^2\phi}, z=\dfrac{1}{1-\cos^2\phi\sin^2\phi}$

Now, $xy+z=\dfrac{1}{\sin^2\phi\cos^2\phi}+\dfrac{1}{1-\sin^2\phi\cos^2\phi}=\dfrac{1}{\sin^2\phi\cos^2\phi(1-\sin^2\phi\cos^2\phi)}$

or $xy+z=xyz$ (i)

Clearly, $x+y=\dfrac{\sin^2\phi+\cos^2\phi}{\sin^2\phi\cos\phi}=xy$

$\therefore\ x+y+z=xyz$ [using Eq. (i)]

3. c. We know that $\sin 15°=\dfrac{\sqrt{3}-1}{2\sqrt{2}}$ (irrational)

$\cos 15°\ \dfrac{\sqrt{3}+1}{2\sqrt{2}}$ (irrational)

$$\sin 15° \cos 15° = \frac{1}{2}(2\sin 15° \cos 15°)$$

$$= \frac{1}{2}\sin 30° = \frac{1}{4} \text{ (rational)}$$

$$\sin 15° \cos 75° = \sin 15° \cos (90 - 15°)$$

$$= \sin 15° \sin 15° = \sin^2 15° = \frac{1}{2}(1 + \cos 30°) = \frac{1}{2}\left(1 - \frac{\sqrt{3}}{2}\right) \text{ (irrational)}$$

4. a, b, c, d.

$$f_n(\theta) = \frac{\sin(\theta/2)}{\cos(\theta/2)}\left[\frac{2\cos^2(\theta/2)}{\cos\theta}\frac{2\cos^2\theta}{\cos 2\theta}\frac{2\cos^2 2\theta}{\cos 4\theta}\cdots\right]$$

$$= \frac{\sin\theta}{\cos\theta}\left[\frac{2\cos^2\theta}{\cos 2\theta}\frac{2\cos^2 2\theta}{\cos 4\theta}\cdots\right]$$

$$= \frac{\sin 2\theta}{\cos 2\theta}\left[\frac{2\cos^2 2\theta}{\cos 4\theta}\cdots\right] = \tan 2^n\theta$$

$$f_2\left(\frac{\pi}{16}\right) = \tan 4\frac{\pi}{16} = \tan\frac{\pi}{4} = 1$$

Similarly, $f_3\left(\frac{\pi}{32}\right)$; $f_4\left(\frac{\pi}{64}\right)$ and $f_5\left(\frac{\pi}{128}\right)$ are found to be $\tan\frac{\pi}{4} = 1$

5. c. For $\theta = -\pi/2$, $\beta = -\pi/2$ and $\gamma = 2\pi$

$\sin\alpha + \sin\beta + \sin\gamma = -2$

Hence, the minimum value of the expression is negative.

CHAPTER 3

Trigonometric Equations

TRIGONOMETRIC EQUATIONS

An equation involving one or more trigonometrical ratios of unknown angle is called a trigonometric equation, e.g., $\cos^2 x - 4 \sin x = 1$. It is to be noted that a trigonometrical identity is satisfied for every value of the unknown angle, whereas a trigonometric equation is satisfied only for some values (finite or infinite in number) of unknown angle, e.g., $\sin^2 x + \cos^2 x = 1$ is a trigonometrical identity as it is satisfied for every value of $x \in R$.

Solution or Root of a Trigonometric Equation

The value of an unknown angle which satisfies the given trigonometric equation is called a solution or root of the equation. For example, $2 \sin\theta = 1$, clearly $\theta = 30°$ and $\theta = 150°$ satisfies the equation; therefore, 30° and 150° are solutions of the equation $2 \sin\theta = 1$ between 0° and 360°.

Principal Solution of a Trigonometric Equation

The solutions of a trigonometric equation lie in the interval $[0, 2\pi)$. For example, $\sin\theta = 1/2$, then the two values of θ between 0 and 2π are $\pi/6$ and $5\pi/6$. Thus, $\pi/6$ and $5\pi/6$ are the principal solutions of equation $\sin \theta = 1/2$.

General Solution of a Trigonometric Equation

It is known that trigonometric ratios are periodic functions. In fact, $\sin x$, $\cos x$, $\sec x$ and $\operatorname{cosec} x$ are periodic functions with a period 2π, and $\tan x$ and $\cot x$ are periodic functions with a period π. Therefore, solutions of trigonometric equations can be generalized with the help of period of trigonometric functions. The solution consisting of all possible solutions of a trigonometric equation is called its general solution.

Clearly, general solution of a trigonometric equation will involve integral $n \in Z$. General solution of a trigonometric equation is also called a 'solution'.

Here set of all integers is denoted by Z. $n \in Z$ means $n = 0, \pm 1, \pm 2, \ldots$. For example, general solution of the equation $\cos\theta = 1$ is $\theta = 2n\pi$.

Some Important General Solutions of Equations

Equation	Solution
$\sin\theta = 0$	$\theta = n\pi, n \in Z$
$\cos\theta = 0$	$\theta = (2n+1)\dfrac{\pi}{2}, n \in Z$
$\tan\theta = 0$	$\theta = n\pi, n \in Z$
$\sin\theta = 1$	$\theta = (4n+1)\dfrac{\pi}{2}, n \in Z$
$\sin\theta = -1$	$\theta = (4n-1)\dfrac{\pi}{2}, n \in Z$

Equation	Solution
$\cos\theta = 1$	$\theta = 2n\pi, n \in Z$
$\cos\theta = -1$	$\theta = (2n+1)\pi, n \in Z$
$\cot\theta = 0$	$\theta = (2n+1)\dfrac{\pi}{2}, n \in Z$

Points to Remember

1. While solving a trigonometric equation, squaring the equation at any step should be avoided as far as possible. If squaring is necessary, check the solution for extraneous values. Also see Example 3.1 for explanation.
2. Never cancel terms containing unknown terms on the two sides which are in product. It may cause the loss of a genuine solution.
3. The answer should not contain such values of angles which make any of the terms undefined or infinite. Also see Example 3.2 for explanation.
4. Domain should not change while simplifying the equation. If it changes, necessary corrections must be made.
5. Check that denominator is not zero at any stage while solving the equations.

Example 3.1 **Solve the equation $\sin x + \cos x = 1$.**

Sol. If we square we have $(\sin x + \cos x)^2 = 1$

$\Rightarrow \quad 1 + \sin 2x = 1$

$\Rightarrow \quad \sin 2x = 0$

$\Rightarrow \quad 2x = n\pi, n \in Z$

$\Rightarrow \quad x = n\pi/2, n \in Z$

But for $n = 2, 6, 10, \ldots$

$\sin x + \cos + = -1$ which contradicts the given equation.

Also for $x = 3, 7, 11, \ldots$

$\sin x + \cos x = -1$

Hence, the solution is $x = 2n\pi$ or $x = (4n+1)\dfrac{\pi}{2}$.

Example 3.2 **Solve $\dfrac{\tan 3x - \tan 2x}{1 + \tan 3x \tan 2x} = 1$.**

Sol. $\tan(3x - 2x) = \tan x = 1$

Therefore, $x = n\pi + (\pi/4)$ but this values does not define $\tan 2x$. Hence, there is no solution.

Example 3.3 **Find the values of θ which satisfy $r\sin\theta = 3$ and $r = 4(1 + \sin\theta)$, $0 \le \theta \le 2\pi$.**

Sol. $0 \le \theta \le 2\pi$

Eliminating r, we have $4\sin^2\theta + 4\sin\theta - 3 = 0$

$\Rightarrow \quad \sin\theta = \dfrac{1}{2}, -\dfrac{3}{2}$ (not possible) $\qquad \Rightarrow \quad \theta = \dfrac{\pi}{6}, \pi - \dfrac{\pi}{6} = \dfrac{5\pi}{6}$

Example 3.4 **Solve $16^{\sin^2 x} + 16^{\cos^2 x} = 10, 0 \le x < 2\pi$.**

Sol. $16^{\sin^2 x} + 16^{1-\sin^2 x} = 10$ (i)

If $16^{\sin^2 x} = t$, then $t + \dfrac{16}{t} = 10$

Then Eq. (i) becomes

$\Rightarrow \quad t^2 - 10t + 16 = 0$

$\Rightarrow \quad t = 2, 8$

$\Rightarrow \quad 16^{\sin^2 x} = 16^{1/4}$ or $16^{3/4}$

$\Rightarrow \quad \sin x = \pm\dfrac{1}{2}, \pm\dfrac{\sqrt{3}}{2}$

Now $\sin x = \dfrac{1}{2}$, then $x = \dfrac{\pi}{6}, \dfrac{5\pi}{6}$

$\sin x = -\dfrac{1}{2}$, then $x = \dfrac{7\pi}{6}$ or $\dfrac{11\pi}{6}$

$\sin x = \dfrac{\sqrt{3}}{2} \quad \Rightarrow \quad x = \dfrac{\pi}{3}, \dfrac{2\pi}{3}$

$\sin x = -\dfrac{\sqrt{3}}{2} \quad \Rightarrow \quad x = \dfrac{4\pi}{3}, \dfrac{5\pi}{3}$

Hence, there will be eight solutions in all.

Example 3.5 **Find general value of θ which satisfies both $\sin\theta = -1/2$ and $\tan\theta = 1/\sqrt{3}$, simultaneously.**

Sol. Here $\sin\theta < 0$ and $\tan\theta > 0$, then θ lies in the third quadrant.

Now $\sin\theta = -\dfrac{1}{2} \quad \Rightarrow \quad \theta = \pi + \dfrac{\pi}{6} = \dfrac{7\pi}{6}$

Generalizing, we have $\theta = 2n\pi + \dfrac{7\pi}{6}, n \in Z$.

Example 3.6 **If $\sin A = \sin B$ and $\cos A = \cos B$, then find the value of A in terms of B.**

Sol. $\sin A - \sin B = 0$ and $\cos A - \cos B = 0$

$\Rightarrow \quad 2\sin\dfrac{A-B}{2}\cos\dfrac{A+B}{2} = 0$ and $2\sin\dfrac{A+B}{2}\sin\dfrac{B-A}{2} = 0$

We observe that the common factor gives $\sin\dfrac{A-B}{2} = 0$

$\Rightarrow \quad \dfrac{A-B}{2} = n\pi, n \in Z$

$\Rightarrow \quad A - B = 2n\pi, n \in Z$

$\Rightarrow \quad A = 2n\pi + B, n \in Z$

Example 3.7 **Find the number of solutions of $\sin^2 x - \sin x - 1 = 0$ in $[-2\pi, 2\pi]$.**

Sol. $\sin^2 x - \sin x - 1 = 0$

$$\Rightarrow \sin x = \frac{1 \pm \sqrt{5}}{2}$$

$$= \frac{1-\sqrt{5}}{2} \left[\sin x = \frac{1+\sqrt{5}}{2} > 1 \text{ not possible}\right]$$

$\Rightarrow$ x can attain two values in $[0, 2\pi]$ and two more values in $[-2\pi, 0)$. Thus, there are four solutions.

Example 3.8 **Find the number of solutions of the equation $e^{\sin x} - e^{-\sin x} - 4 = 0$.**

Sol. Put $e^{\sin x} = t \Rightarrow t^2 - 4t - 1 = 0$

$\Rightarrow t = e^{\sin x} = 2 \pm \sqrt{5}$

Now $\sin x \in [-1, 1]$

$\Rightarrow e^{\sin x} \in [e^{-1}, e^{1}]$ and $2 \pm \sqrt{5} \notin [e^{-1}, e^{1}]$

Hence, there does not exist any solution.

Example 3.9 **If the equation $a \sin x + \cos 2x = 2a - 7$ possesses a solution, then find the values of a.**

Sol. The given equation can be written as $a\sin x + (1 - 2\sin^2 x) = 2a - 7$

$\Rightarrow 2\sin^2 x - a\sin x + 2a - 8 = 0$

$$\Rightarrow \sin x = \frac{a \pm \sqrt{a^2 - 8(2a-8)}}{4} = \frac{a \pm (a-8)}{4}$$

$= (a-4)/2$ (∵ $\sin x = 2$ is not possible)

Equation has solution if $-1 \le (a-4)/2 \le 1$

$\Rightarrow -2 \le (a-4) \le 2$

$\Rightarrow 2 \le a \le 6$

Concept Application Exercise 3.1

1. Solve $\sin^2\theta - \cos\theta = \frac{1}{4}, 0 \le \theta \le 2\pi$.
2. Solve $\cos^7 x + \sin^4 x = 1$ in the interval $(-\pi, \pi)$.
3. Find the general solution of $(1 - 2\cos\theta)^2 + (\tan\theta + \sqrt{3})^2 = 0$.
4. Solve $\sin 3\theta - \sin\theta = 4\cos^2\theta - 2$.

GENERAL SOLUTION OF SOME STANDARD EQUATIONS

General Solution of the Equation $\sin\theta = \sin\alpha$

Given, $\sin\theta = \sin\alpha \Rightarrow \sin\theta - \sin\alpha = 0$

$$\Rightarrow 2\cos\frac{\theta+\alpha}{2}\sin\frac{\theta-\alpha}{2} = 0$$

$$\Rightarrow \cos\frac{\theta+\alpha}{2} = 0 \text{ or } \sin\frac{\theta-\alpha}{2} = 0$$

$\Rightarrow \quad \frac{\theta+\alpha}{2}=(2m+1)\frac{\pi}{2}, \frac{\theta-\alpha}{2}=m\pi, m \in Z$

$\Rightarrow \quad \theta=(2m+1)\pi-\alpha$ or $\theta=2m\pi+\alpha, m \in Z$

$\Rightarrow \quad \theta=(2m+1)\pi+(-1)^{2m+1}\alpha, m \in Z$ (i)

or $\quad \theta=2m\pi+(-1)^{2m}\alpha, m \in Z$ (ii)

Combining Eqs. (i) and (ii), we have

$\theta=n\pi+(-1)^n\alpha, n \in Z$

Note:

- *For general solution of the equation* $\sin\theta = k$, *where* $-1 \le k \le 1$. *We have* $\sin\theta = \sin(\sin^{-1} k)$ $\Rightarrow \theta = n\pi + (-1)^n (\sin^{-1} k), n \in Z.$

Example 3.10 **Solve $2\cos^2\theta+3\sin\theta=0$.**

Sol. We have $2\cos^2\theta+3\sin\theta=0$

$\Rightarrow \quad 2(1-\sin^2\theta)+3\sin\theta=0$

$\Rightarrow \quad 2\sin^2\theta-3\sin\theta-2=0$

$\Rightarrow \quad (\sin\theta-2)(2\sin\theta+1)=0$

$\Rightarrow \quad 2\sin\theta+1=0$ $[\because \sin\theta\neq 2]$

$\Rightarrow \quad \sin\theta=-\frac{1}{2}=\sin\left(\frac{-\pi}{6}\right)$

$\Rightarrow \quad \theta=n\pi+(-1)^n\left(\frac{-\pi}{6}\right), n\in Z$

$\quad =n\pi+(-1)^{n+1}\frac{\pi}{6}, n\in Z$

Example 3.11 **Solve $4\cos\theta-3\sec\theta=\tan\theta$.**

Sol. We have $4\cos\theta-3\sec\theta=\tan\theta$

$\Rightarrow \quad 4\cos\theta-\frac{3}{\cos\theta}=\frac{\sin\theta}{\cos\theta}$

$\Rightarrow \quad 4\cos^2\theta-3=\sin\theta$

$\Rightarrow \quad 4(1-\sin^2\theta)-3=\sin\theta$

$\Rightarrow \quad 4\sin^2\theta+\sin\theta-1=0$

$\Rightarrow \quad \sin\theta=\frac{-1\pm\sqrt{1+16}}{8}$

$\quad =\frac{-1\pm\sqrt{17}}{8}$

$\quad =\frac{-1+\sqrt{17}}{8}$ or $=\frac{-1-\sqrt{17}}{8}$

Now, $\sin\theta = \dfrac{-1+\sqrt{17}}{8}$

$\Rightarrow \quad \sin\theta = \sin\alpha$, where $\sin\alpha = \dfrac{-1+\sqrt{17}}{8}$

$\Rightarrow \quad \theta = n\pi + (-1)^n \alpha$, where $\sin\alpha = \dfrac{-1+\sqrt{17}}{8}$ and $n \in Z$

and $\sin\theta = \dfrac{-1-\sqrt{17}}{8}$

$\Rightarrow \quad \sin\theta = \sin\beta$, where $\sin\beta = \dfrac{-1-\sqrt{17}}{8}$

$\Rightarrow \quad \theta = n\pi + (-1)^n \beta$, where $\sin\beta = \dfrac{-1-\sqrt{17}}{8}$

Example 3.12 **Solve $\sin^3\theta\cos\theta - \cos^3\theta\sin\theta = 1/4$.**

Sol. $\sin^3\theta\cos\theta - \cos^3\theta\sin\theta = 1/4$

$\Rightarrow \quad 4\sin\theta\cos\theta(\sin^2\theta - \cos^2\theta) = 1$

$\Rightarrow \quad 2\sin 2\theta(-\cos 2\theta) = 1$

$\Rightarrow \quad -\sin 4\theta = 1$

$\Rightarrow \quad \sin 4\theta = -1$

$\Rightarrow \quad 4\theta = 2n\pi - \dfrac{\pi}{2}$

$\Rightarrow \quad \theta = (\pi/2) + (-\pi/8), n \in Z$

Concept Application Exercise 3.2

1. Solve $2\sin\theta + 1 = 0$.
2. Solve $\sin^2 n\theta - \sin^2(n-1)\theta = \sin^2\theta$.

General Solution of Equation $\cos\theta = \cos\alpha$

Given, $\cos\theta = \cos\alpha$

$\Rightarrow \quad \cos\alpha - \cos\theta = 0$

$\Rightarrow \quad 2\sin\dfrac{\alpha+\theta}{2}\sin\dfrac{\theta-\alpha}{2} = 0$

$\Rightarrow \quad \sin\dfrac{\alpha+\theta}{2} = 0$ or $\sin\dfrac{\theta-\alpha}{2} = 0$

$\Rightarrow \quad \dfrac{\alpha+\theta}{2} = n\pi$ or $\dfrac{\theta-\alpha}{2} = n\pi, n \in Z$

$\Rightarrow \quad \theta = 2n\pi - \alpha$ or $\theta = 2n\pi + \alpha, n \in Z$

$= 2n\pi \pm \alpha, n \in Z$

Note:

- *For general solution of the equation* $\sin\theta = k$, *where* $-1 \le k \le 1$. *We have* $\cos\theta = \cos(\cos^{-1}k)$ $\Rightarrow \theta = 2n\pi \pm (\cos^{-1}k), n \in Z$.

Example 3.13 **Solve $\sqrt{3}\sec 2\theta = 2$.**

Sol. We have $\sqrt{3}\sec 2\theta = 2$

$$\Rightarrow \cos 2\theta = \frac{\sqrt{3}}{2}$$

$$= \cos\frac{\pi}{6}$$

$$\Rightarrow 2\theta = 2n\pi \pm \frac{\pi}{6}, n \in Z$$

$$\Rightarrow \theta = n\pi \pm \frac{\pi}{12}, n \in Z$$

Example 3.14 **Solve $\sin 2\theta + \cos\theta = 0$.**

Sol. We have $\sin 2\theta + \cos\theta = 0$

$$\Rightarrow \cos\theta = -\sin 2\theta$$

$$= \cos\left(\frac{\pi}{2} + 2\theta\right)$$

$$\Rightarrow \theta = 2n\pi \pm \left(\frac{\pi}{2} + 2\theta\right) n \in Z$$

Taking positive sign, we have

$$\theta = 2n\pi + \frac{\pi}{2} + 2\theta$$

$$= 2n\pi - \frac{\pi}{2}, n \in Z.$$

Taking negative sign, we have

$$\theta = 2n\pi - \left(\frac{\pi}{2} + 2\theta\right) \Rightarrow \theta = \frac{2n\pi}{3} - \frac{\pi}{6}, n \in Z$$

Example 3.15 **Solve $\cos\theta + \cos 3\theta - 2\cos 2\theta = 0$.**

Sol. We have $\cos\theta + \cos 3\theta - 2\cos 2\theta = 0$

$$\Rightarrow 2\cos 2\theta\cos\theta - 2\cos 2\theta = 0$$

$$\Rightarrow 2\cos 2\theta(\cos\theta - 1) = 0$$

$$\Rightarrow \cos 2\theta = 0 \text{ or, } \cos\theta - 1 = 0$$

$$\Rightarrow 2\theta = (2n+1)\frac{\pi}{2}, n \in Z \text{ or } \theta = 2m\pi, m \in Z$$

$$\Rightarrow \theta = (2n+1)\frac{\pi}{4}, n \in Z \text{ or } \theta = 2m\pi, m \in Z$$

Example 3.16 **Solve $\sec 4\theta - \sec 2\theta = 2$.**

Sol. $\dfrac{1}{\cos 4\theta} - \dfrac{1}{\cos 2\theta} = 2$

$\Rightarrow$ $\cos 2\theta - \cos 4\theta = 2\cos 2\theta \cos 4\theta = \cos 2\theta + \cos 6\theta$

$\Rightarrow$ $\cos 6\theta + \cos 4\theta = 0$

$\Rightarrow$ $2\cos 5\theta \cos \theta = 0$

$\Rightarrow$ $\cos 5\theta = 0$ or $\cos\theta = 0$

$\Rightarrow$ $5\theta = (2n+1)\dfrac{\pi}{2}$ or $\theta = (2n+1)\dfrac{\pi}{2}, n \in Z$

$\Rightarrow$ $\theta = \left(n + \dfrac{1}{2}\right)\dfrac{\pi}{5}$ or $\theta = \left(n + \dfrac{1}{2}\right)\pi, n \in Z$

Example 3.17 **Solve $5\cos 2\theta + 2\cos^2 \dfrac{\theta}{2} + 1 = 0, -\pi < \theta < \pi$.**

Sol. Changing all the values in terms of $\cos\theta$, we get

$5(2\cos^2\theta - 1) + (1 + \cos\theta) + 1 = 0 \Rightarrow 10\cos^2\theta + \cos\theta - 3 = 0$

$\Rightarrow$ $(5\cos\theta + 3)(2\cos\theta - 1) = 0$

$\Rightarrow$ $\theta = \dfrac{\pi}{3}, -\dfrac{\pi}{3}, \cos^{-1}\left(-\dfrac{3}{5}\right) = \pi - \cos^{-1}\dfrac{3}{5}$ and $-\pi + \cos^{-1}\dfrac{3}{5}$ $\quad [\because -\pi < \theta < \pi]$

Example 3.18 **Solve $\cos x \cos 2x \cos 3x = 1/4$.**

Sol. $\cos x \cos 2x \cos 3x = 1/4$

$\Rightarrow$ $2(2\cos x \cos 3x)\cos 2x = 1$

$\Rightarrow$ $2(\cos 4x + \cos 2x)\cos 2x = 1$

$\Rightarrow$ $2(2\cos^2 2x - 1 + \cos 2x)\cos 2x = 1$

$\Rightarrow$ $4\cos^3 2x + 2\cos^2 2x - 2\cos 2x - 1 = 0$

$\Rightarrow$ $(2\cos^2 2x - 1)(2\cos 2x + 1) = 0$

$\Rightarrow$ $\cos 4x(2\cos 2x + 1) = 0$

$\Rightarrow$ $\cos 4x = 0$ or $\cos 2x = -1/2$

$\Rightarrow$ $4x = (2n+1)\dfrac{\pi}{2}$ or $2x = 2m\pi \pm \dfrac{2\pi}{3}$, $m, n \in Z$

$\Rightarrow$ $x = (2n+1)\dfrac{\pi}{8}$ or $x = m\pi \pm \dfrac{\pi}{3}$

Concept Application Exercise 3.3

1. Solve $\cos\theta = 1/3$.
2. Solve $\tan\theta \tan 4\theta = 1$ for $0 < \theta < \pi$.
3. Solve $\cot(x/2) - \text{cosec}(x/2) = \cot x$.
4. Solve $\cot\theta + \tan\theta = 2\,\text{cosec}\,\theta$.
5. Solve $\sin 6\theta = \sin 4\theta - \sin 2\theta$.
6. Solve $\cos\theta + \cos 2\theta + \cos 3\theta = 0$.
7. Solve $\cos 6\theta + \cos 4\theta + \cos 2\theta + 1 = 0, 0 \le \theta \le \pi$.

General Solutions of the Equation tan θ = tan α

Given, $\tan\theta = \tan\alpha$

$\Rightarrow \quad \dfrac{\sin\theta}{\cos\theta} = \dfrac{\sin\alpha}{\cos\alpha}$

$\Rightarrow \quad \sin\theta\cos\alpha - \cos\theta\sin\alpha = 0$

$\Rightarrow \quad \sin(\theta - \alpha) = 0$

$\Rightarrow \quad \theta - \alpha = n\pi$

$\Rightarrow \quad \theta = n\pi + \alpha$, where $n \in Z$

Note:

- *For general solution of the equation* $\tan\theta = k$, *where* $k \in R$. *We have* $\tan\theta = \tan(\tan^{-1}k)$
  $\Rightarrow \quad \theta = n\pi + (\tan^{-1}k), n \in Z$

Example 3.19 **Solve $\tan 3\theta = -1$.**

Sol. $\tan 3\theta = -1$

$$= \tan\left(\frac{-\pi}{4}\right)$$

$$\Rightarrow \quad 3\theta = n\pi + \left(\frac{-\pi}{4}\right), n \in Z$$

$$\Rightarrow \quad \theta = \frac{n\pi}{3} - \frac{\pi}{12}, n \in Z$$

Example 3.20 **Solve $2\tan\theta - \cot\theta = -1$.**

Sol. $2\tan\theta - \cot\theta = -1$

$$\Rightarrow \quad 2\tan\theta - \frac{1}{\tan\theta} = -1$$

$\Rightarrow \quad 2\tan^2\theta + \tan\theta - 1 = 0$

$\Rightarrow \quad (\tan\theta + 1)(2\tan\theta - 1) = 0$

$$\Rightarrow \quad \tan\theta = -1 \text{ or } \tan\theta = \frac{1}{2}$$

$$\Rightarrow \quad \tan\theta = \tan\left(\frac{-\pi}{4}\right) \text{ or } \tan\theta = \tan\left(\tan^{-1}\frac{1}{2}\right)$$

$$\Rightarrow \quad \theta = n\pi + \left(\frac{-\pi}{4}\right) \text{ or } \theta = m\pi + \alpha, \text{ where } m, n \in Z \text{ and } \tan\alpha = \frac{1}{2}$$

Example 3.21 **Solve $\tan 5\theta = \cot 2\theta$.**

Sol. $\tan 5\theta = \cot 2\theta = \tan\left(\dfrac{\pi}{2} - 2\theta\right)$

$$\Rightarrow \quad 5\theta = n\pi + \frac{\pi}{2} - 2\theta$$

$\Rightarrow \quad 7\theta = n\pi + \dfrac{\pi}{2}$

$\Rightarrow \quad \theta = \dfrac{n\pi}{7} + \dfrac{\pi}{14}$, where $n \in Z$, but $n \neq 3, 10, 17, \ldots$ where $\tan 5\theta$ is not defined

Example 3.22 **Solve $\tan\theta + \tan 2\theta + \sqrt{3}\tan\theta\tan 2\theta = \sqrt{3}$.**

Sol. $\tan\theta + \tan 2\theta + \sqrt{3}\tan\theta\tan 2\theta = \sqrt{3}$

$\Rightarrow \quad \tan\theta + \tan 2\theta = \sqrt{3}\,(1 - \tan\theta\tan 2\theta)$

$\Rightarrow \quad \dfrac{\tan\theta + \tan 2\theta}{1 - \tan\theta\tan 2\theta} = \sqrt{3}$

$\Rightarrow \quad \tan 3\theta = \sqrt{3}$

$= \tan\dfrac{\pi}{3}$

$\Rightarrow \quad 3\theta = n\pi + \dfrac{\pi}{3}, n \in Z$

$\Rightarrow \theta = \dfrac{n\pi}{3} + \dfrac{\pi}{9}, n \in Z$

Concept Application Exercise 3.4

1. If $\tan a\theta - \tan b\theta = 0$, then prove that the values of θ forms an A.P.
2. Solve $\tan^2\theta + 2\sqrt{3}\tan\theta = 1$.
3. Solve $\tan^2 x + (1 - \sqrt{3})\tan x - \sqrt{3} = 0$.
4. Solve $3\cos^2\theta - 2\sqrt{3}\sin\theta\cos\theta - 3\sin^2\theta = 0$.
5. Solve $\tan\theta + \tan(\theta + \pi/3) + \tan(\theta + 2\pi/3) = 3$.
6. Solve $2\sin^3 x = \cos x$.

General Solutions of the Equation $\sin^2\theta = \sin^2\alpha$ or $\cos^2\theta = \cos^2\alpha$

Here the both given the equations are same as $\cos^2\theta = \cos^2\alpha$

$\Rightarrow \quad (1 - \sin^2\theta) - (1 - \sin^2\alpha) = 0$

$\Rightarrow \quad \sin^2\theta = \sin^2\alpha$

$\Rightarrow \quad \sin(\theta + \alpha)\sin(\theta - \alpha) = 0$

$\Rightarrow \quad \sin(\theta + \alpha) = 0$ or $\sin(\theta - \alpha) = 0$

$\Rightarrow \quad \theta + \alpha = n\pi$ or $\theta - \alpha = n\pi, n \in Z$

$\Rightarrow \quad \theta = n\pi \pm \alpha, n \in Z$

General Solutions of the Equation $\tan^2\theta = \tan^2\alpha$

$\tan^2\theta = \tan^2\alpha \quad \Rightarrow \quad \tan\theta = \pm\tan\alpha \quad \Rightarrow \quad \tan\theta = \tan(\pm\alpha) \quad \Rightarrow \quad \theta = n\pi \pm \alpha, n \in Z$

Example 3.23 **Solve $7\cos^2\theta + 3\sin^2\theta = 4$.**

Sol. We have $7\cos^2\theta + 3\sin^2\theta = 4$

$\Rightarrow \quad 7(1 - \sin^2\theta) + 3\sin^2\theta = 4$

$\Rightarrow \quad 4\sin^2\theta = 3$

$\Rightarrow \quad \sin^2\theta = \frac{3}{4} = \left(\frac{\sqrt{3}}{2}\right)^2$

$\Rightarrow \quad \sin^2\theta = \sin^2\frac{\pi}{3}$

$\Rightarrow \quad \theta = n\pi \pm \frac{\pi}{3}, n \in Z$

Example 3.24 **Solve $2\sin^2 x + \sin^2 2x = 2$.**

Sol. We have $2\sin^2 x + \sin^2 2x = 2$

$\Rightarrow \quad 2\sin^2 x + (2\sin x\cos x)^2 = 2$

$\Rightarrow \quad 2\sin^2 x\cos^2 x + \sin^2 x = 1$

$\Rightarrow \quad 2\sin^2 x\cos^2 x - (1 - \sin^2 x) = 0$

$\Rightarrow \quad 2\sin^2 x\cos^2 x - \cos^2 x = 0$

$\Rightarrow \quad \cos^2 x(2\sin^2 x - 1) = 0$

$\Rightarrow \quad \cos^2 x = 0 \text{ or } \sin^2 x = \frac{1}{2}$

$\Rightarrow \quad x = 2n\pi + \frac{\pi}{2} \text{ or } \sin^2 x = \sin^2\frac{\pi}{4}$

$= 2n\pi + \frac{\pi}{2} \text{ or } x = m\pi \pm \frac{\pi}{4}, m \in Z$, where $m, n \in Z$

Example 3.25 **Solve $4\cot 2\theta = \cot^2\theta - \tan^2\theta$.**

Sol. $\frac{4}{\tan 2\theta} = \frac{1}{\tan^2\theta} - \tan^2\theta$

$\Rightarrow \quad \frac{4(1-\tan^2\theta)}{2\tan\theta} = \frac{1-\tan^4\theta}{\tan^2\theta}$ $\left[\text{put } \tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}\right]$

$\Rightarrow \quad (1-\tan^2\theta)[2\tan\theta - (1+\tan^2\theta)] = 0$

$\Rightarrow \quad (1-\tan^2\theta)(\tan^2\theta - 2\tan\theta + 1) = 0$

$\Rightarrow \quad (1-\tan^2\theta)(\tan\theta - 1)^2 = 0$

$\Rightarrow \quad \tan\theta = \pm 1$

$\Rightarrow \quad \theta = n\pi \pm \frac{\pi}{4}, n \in Z$

Example 3.26 **Find the most general solution of $2^{1+|\cos x|+\cos^2 x+|\cos x|^3+\cdots\infty} = 4$.**

Sol. We have $2^{1+|\cos x|+\cos^2 x+|\cos x|^3+\cdots\infty} = 4$

$\Rightarrow \quad 2^{1+|\cos x|+|\cos x|^2+|\cos x|^3+\cdots\infty} = 4$

$\Rightarrow \quad 2^{\frac{1}{1-|\cos x|}} = 2^2$

$$\Rightarrow \quad \frac{1}{1-|\cos x|}=2$$

$$\Rightarrow \quad |\cos x|=\frac{1}{2} \text{ or } \cos x=\pm\frac{1}{2}$$

$$\Rightarrow \quad x=n\pi\pm\frac{\pi}{3}, n\in Z$$

Concept Application Exercise 3.5

1. Solve $\tan^2\theta+\cot^2\theta=2$.
2. Solve $3(\sec^2\theta+\tan^2\theta)=5$.
3. Solve $4\cos^2x+6\sin^2x=5$.

Solutions of Equations of the Form $a\cos\theta + b\sin\theta = c$

To solve equation, let us convert the equation to the form $\cos\theta=\cos\alpha$ or $\sin\theta=\sin\alpha$, etc.

For this let us suppose that $\left.\begin{matrix} a=r\cos\phi \\ b=r\sin\phi \end{matrix}\right\} \Rightarrow \begin{cases} r=\sqrt{a^2+b^2} \\ \tan\phi=\dfrac{b}{a} \end{cases}$ and

Substituting these values in the equation $a\cos\theta+b\sin\theta=c$, we have
$r\cos\phi\cos\theta+r\sin\phi\sin\theta=c$

$$\Rightarrow \quad r\cos(\theta-\phi)=c$$

$$\Rightarrow \quad \cos(\theta-\phi)=\frac{c}{r}=\frac{c}{\sqrt{a^2+b^2}}=\cos\beta \text{ (suppose)}$$

$$\Rightarrow \quad \theta-\phi=2n\pi\pm\beta$$

$$\Rightarrow \quad \theta=2n\pi+\phi\pm\beta, n\in Z$$

Here ϕ and β are known as a, b and c are given.
Hence, we can solve the equation of this type by putting

$a=r\cos\phi$ and $b=r\sin\phi$, provided $\left|\dfrac{c}{\sqrt{a^2+b^2}}\right|\le 1$ [$\because$ $\cos\beta$ lies between -1 and 1]

or $\dfrac{|c|}{\sqrt{a^2+b^2}}\le 1$ or $|c|\le\sqrt{a^2+b^2}$

WORKING RULES for solving such equations

1. First of all check whether $|c|\le\sqrt{a^2+b^2}$ or not.
2. If $|c|>\sqrt{a^2+b^2}$, then the given equation has no real solution.
3. If $|c|\le\sqrt{a^2+b^2}$, then divide both sides of the equation by $\sqrt{a^2+b^2}$.

4. Take $\cos\alpha = \frac{a}{\sqrt{a^2+b^2}}$ and $\sin\alpha = \frac{b}{\sqrt{a^2+b^2}}$, then the given equation will become $\cos(\theta-\alpha)=\cos\beta$, where $\tan\alpha=\frac{b}{a}$ and $\cos\beta=\frac{c}{\sqrt{a^2+b^2}}$.

We can also take $\sin\alpha = \frac{a}{\sqrt{a^2+b^2}}$, $\cos\alpha = \frac{b}{\sqrt{a^2+b^2}}$ and $\sin\beta = \frac{c}{\sqrt{a^2+b^2}}$, then the given equation will reduce to the form $\sin(\theta+\alpha)=\sin\beta$.

Example 3.27 Solve $\sqrt{3}\cos\theta+\sin\theta=\sqrt{2}$.

Sol. We have $\sqrt{3}\cos\theta+\sin\theta=\sqrt{2}$ (i)

This is of the form $a\cos\theta+b\sin\theta=c$, where $a=\sqrt{3}$, $b=1$ and $c=\sqrt{2}$

Let $\sqrt{3}=r\cos\alpha$ and $1=r\sin\alpha$

$$\Rightarrow \quad r=\sqrt{a^2+b^2}=\sqrt{\left(\sqrt{3}\right)^2+1^2}=2 \text{ and } \tan\alpha=\frac{1}{\sqrt{3}} \quad \Rightarrow \quad \alpha=\frac{\pi}{6}$$

Substituting $\sqrt{3}=r\cos\alpha$ and $1=r\sin\alpha$ in Eq. (i), it reduces to $r\cos\alpha\cos\theta+r\sin\theta=\sqrt{2}$

$$\Rightarrow \quad r\cos(\theta-\alpha)=\sqrt{2}$$

$$\Rightarrow \quad 2\cos\left(\theta-\frac{\pi}{6}\right)=\sqrt{2}$$

$$\Rightarrow \quad \cos\left(\theta-\frac{\pi}{6}\right)=\frac{1}{\sqrt{2}}=\cos\frac{\pi}{4}$$

$$\Rightarrow \quad \theta-\frac{\pi}{6}=2n\pi\pm\frac{\pi}{4}, n\in Z$$

$$\Rightarrow \quad \theta=2n\pi\pm\frac{\pi}{4}+\frac{\pi}{6}, n\in Z$$

$$=2n\pi+\frac{\pi}{4}+\frac{\pi}{6} \text{ or, } \theta=2n\pi-\frac{\pi}{4}+\frac{\pi}{6}$$

$$=2n\pi+\frac{5\pi}{12} \text{ or, } \theta=2n\pi-\frac{\pi}{12}, \text{ where } n\in Z$$

Example 3.28 Solve $\sqrt{3}\cos\theta-3\sin\theta=4\sin 2\theta\cos 3\theta$.

Sol. We have $\sqrt{3}\cos\theta-3\sin\theta=2(\sin 5\theta-\sin\theta)$

$$\Rightarrow \quad \left(\sqrt{3}/2\right)\cos\theta-(1/2)\sin\theta=\sin 5\theta$$

$\Rightarrow \cos(\theta + \pi/6) = \sin 5\theta = \cos(\pi/2 - 5\theta)$

$\Rightarrow \theta + \pi/6 = 2n\pi \pm (\pi/2 - 5\theta)$

$\Rightarrow \theta = n\pi/3 + \pi/18$ or $\theta = -n\pi/2 + \pi/6,\ \forall\ n \in Z$

Example 3.29 **Find the total number of integral values of n so that $\sin x(\sin x + \cos x) = n$ has at least one solution.**

Sol. $\sin x(\sin x + \cos x) = n$

$\Rightarrow \sin^2 x + \sin x \cos x = n$

$$\Rightarrow \frac{1-\cos 2x}{2} + \frac{\sin 2x}{2} = n$$

$\Rightarrow \sin 2x - \cos 2x = 2n - 1$

$\Rightarrow -\sqrt{2} \le 2n - 1 \le \sqrt{2}$

$$\Rightarrow \frac{1-\sqrt{2}}{2} \le n \le \frac{1+\sqrt{2}}{2}$$

$\Rightarrow n = 0, 1$

Concept Application Exercise 3.6

1. Solve $\cot\theta + \operatorname{cosec}\theta = \sqrt{3}$.
2. Solve $\sin\theta + \cos\theta = \sqrt{2}\cos A$.
3. Solve $\sqrt{2}\sec\theta + \tan\theta = 1$.
4. Find the number of integral values of k for which the equation $7\cos x + 5\sin x = 2k + 1$ has at least one solution.

PROBLEMS BASED ON EXTREME VALUES OF FUNCTIONS

Example 3.30 **If $x, y \in [0, 2\pi]$, then find the total number of ordered pairs (x, y) satisfying the equation $\sin x \cos y = 1$.**

Sol. $\sin x \cos y = 1$

$\Rightarrow \sin x = 1, \cos y = 1$ or $\sin x = -1, \cos y = -1$

If $\sin x = 1, \cos y = 1 \quad \Rightarrow \quad x = \pi/2, y = 0, 2\pi$

If $\sin x = -1, \cos y = -1 \quad \Rightarrow \quad x = 3\pi/2, y = \pi$

Thus, the possible ordered pairs are $\left(\frac{\pi}{2}, 0\right), \left(\frac{\pi}{2}, 2\pi\right)$ and $\left(\frac{3\pi}{2}, \pi\right)$.

Example 3.31 **If $3\sin x + 4\cos ax = 7$ has at least one solution, then find the possible values of a.**

Sol. We have $3\sin x + 4\cos ax = 7$ which is possible only when $\sin x = 1$ and $\cos ax = 1$

$$\Rightarrow x = (4n+1)\frac{\pi}{2} \text{ and } ax = 2m\pi;\ m, n \in Z$$

$$\Rightarrow (4n+1)\frac{\pi}{2} = \frac{2m\pi}{a}$$

$$\Rightarrow a = \frac{4m}{4n+1}$$

Example 3.32 **Solve $\cos^{50} x - \sin^{50} x = 1$.**

Sol. $\cos^{50} x - \sin^{50} x = 1 \quad \Rightarrow \quad \cos^{50} x = 1 + \sin^{50} x$

L.H.S. ≤ 1 and R.H.S. ≥ 1

Hence, we must have $\cos^{50} x = 1 + \sin^{50} x = 1 \quad \Rightarrow \quad \sin x = 0 \quad \Rightarrow \quad x = n\pi$

Example 3.33 **Solve $\sin^2 x + \cos^2 y = 2\sec^2 z$ for x, y and z.**

Sol. L.H.S. $= \sin^2 x + \cos^2 y \leq 2$ $\quad [\because \sin^2 x \leq$ and $\cos^2 y \leq 1]$

R.H.S. $= 2\sec^2 z \geq 2$

Hence, L.H.S. = R.H.S. only when $\sin^2 x = 1$, $\cos^2 y = 1$ and $2\sec^2 z = 2$

$\Rightarrow \quad \cos^2 x = 0, \sin^2 y = 0, \cos^2 z = 1$

$\Rightarrow \quad \cos x = 0, \sin y = 0, \sin z = 0$

$x = (2m+1)\dfrac{\pi}{2}, y = n\pi$ and $z = t\pi$, where m, n and t are integers.

Example 3.34 **Solve $1 + \sin x \sin^2 \dfrac{x}{2} = 0$.**

Sol. $1 + \sin x \sin^2 \dfrac{x}{2} = 0$

$\Rightarrow \quad 2 + 2\sin x \sin^2 \dfrac{x}{2} = 0$

$\Rightarrow \quad 2 + \sin x(1 - \cos x) = 0$

$\Rightarrow \quad 4 + 2\sin x - \sin 2x = 0$

$\Rightarrow \quad \sin 2x = 2\sin x + 4$

Above is not possible for any value of x as L.H.S. has maximum value 1 and R.H.S. has minimum value 2.

Hence, there is no solution.

Example 3.35 **Solve $\cos 4\theta + \sin 5\theta = 2$.**

Sol. The equation $\cos 4\theta + \sin 5\theta = 2$ is valid only when $\cos 4\theta = 1$ and $\sin 5\theta = 1$.

$\Rightarrow \quad 4\theta = 2n\pi$ and $5\theta = 2m\pi + \pi/2, n, m \in Z$

$\Rightarrow \quad \theta = \dfrac{2n\pi}{4}$ and $\theta = \dfrac{2m\pi}{5} + \dfrac{\pi}{10}, n, m \in Z$

Putting $n; m = 0, \pm 1, \pm 2, \ldots$, the common value in $[0, 2\pi]$ is $\theta = \pi/2$.

Therefore, the solution is $\theta = 2k\pi + \pi/2, k \in Z$.

Concept Application Exercise 3.7

1. Show that $x = 0$ is the only solution satisfying the equation $1 + \sin^2 ax = \cos x$, where a is irrational.
2. Solve $\sin^4 x = 1 + \tan^8 x$.
3. Solve $\sin x\left(\cos\dfrac{x}{4} - 2\sin x\right) + \left(1 + \sin\dfrac{x}{4} - 2\cos x\right)\cos x = 0$.
4. Solve $12\sin x + 5\cos x = 2y^2 - 8y + 21$, to get the values of n and y.

INEQUALITIES

Trigonometric Inequations

To solve the trigonometric inequation of the type $f(x) \le a$, or $f(x) \ge a$, where $f(x)$ is some trigonometric ratio, the following steps should be taken:

1. Draw the graph of $f(x)$ in an interval length equal to the fundamental period of $f(x)$.
2. Draw the line $y = a$.
3. Take the portion of the graph for which the inequation is satisfied.
4. To generalize, add nT $(n \in I)$ and take union over the set of integers, where T is fundamental period of $f(x)$.

Example 3.36 **Solve $\sin x > -\dfrac{1}{2}$.**

Sol. As the function $\sin x$ has least positive period 2π; therefore, it is sufficient to solve the inequality of the form $\sin x > a$, $\sin x \ge a$, $\sin x < a$, and $\sin x \le a$ first on the interval of length 2π. Then get the solution set by adding numbers of the form $2\pi n$, $n \in z$, to each of the solutions obtained on that interval.

Thus, let us solve this inequality on the interval $\left[-\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right]$.

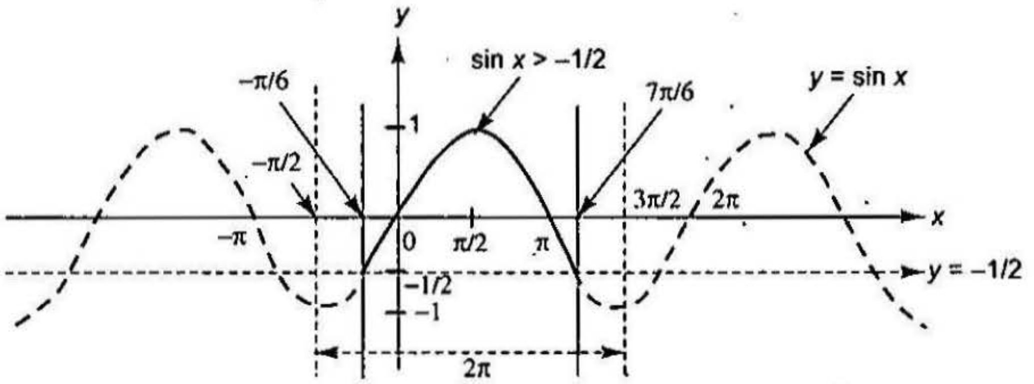


Fig. 3.1

From Fig. 3.1, $\sin x > -\dfrac{1}{2}$, when $-\dfrac{\pi}{6} < x < \dfrac{7\pi}{6}$

Thus, on generalizing, the above solution is $2n\pi - \dfrac{\pi}{6} < x < 2n\pi + \dfrac{7\pi}{6}$; $n \in Z$.

Example 3.37 **Solve $2\cos^2\theta + \sin\theta \le 2$, where $\pi/2 \le \theta \le 3\pi/2$.**

Sol. $2\cos^2\theta + \sin\theta \le 2$

$\Rightarrow\ 2(1 - \sin^2\theta) + \sin\theta \le 2$

$\Rightarrow\ -2\sin^2\theta + \sin\theta \le 0$

$\Rightarrow\ 2\sin^2\theta - \sin\theta \ge 0$

$\Rightarrow\ \sin\theta(2\sin\theta - 1) \ge 0$

$\Rightarrow\ \sin\theta(\sin\theta - 1/2) \ge 0$,

which is possible if $\sin\theta \le 0$ or $\sin\theta \ge 1/2$.

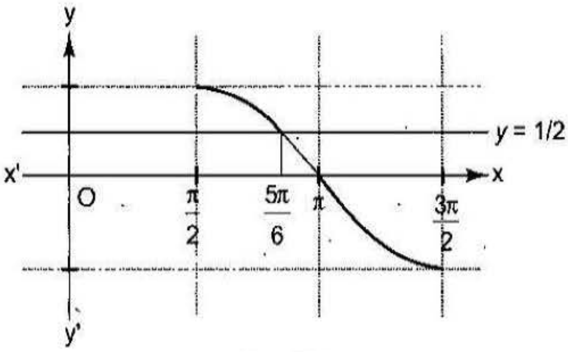


Fig. 3.2

From the graph,
Now $\sin\theta \geq 1/2 \quad \Rightarrow \quad \pi/2 \leq \theta \leq 5\pi/6$
and $\sin\theta \leq 0 \quad \Rightarrow \quad \pi \leq \theta \leq 3\pi/2$
Hence, the required values of θ are given by
$\theta \in [\pi/2, 5\pi/6] \cup [\pi, 3\pi/2]$

Example 3.38 **Solve $\sin\theta + \sqrt{3}\cos\theta \geq 1, -\pi < \theta \leq \pi$.**

Sol. The given inequation is

$$\sin\theta + \sqrt{3}\cos\theta \geq 1, -\pi < \theta \leq \pi$$

$$\Rightarrow \quad \frac{1}{2}\sin\theta + \frac{\sqrt{3}}{2}\cos\theta \geq \frac{1}{2}$$

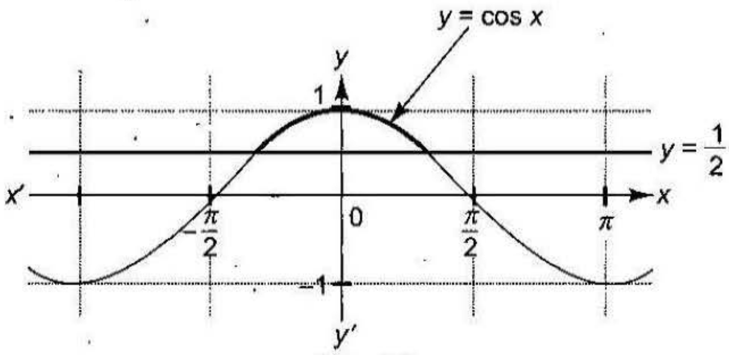


Fig. 3.3

$$\Rightarrow \quad \cos\left(\theta - \frac{\pi}{6}\right) \geq \frac{1}{2} = \cos\frac{\pi}{3} \quad \Rightarrow \quad -\frac{\pi}{3} \leq \theta - \frac{\pi}{6} \leq \frac{\pi}{3} \quad \Rightarrow \quad -\frac{\pi}{6} \leq \theta \leq \frac{\pi}{2}$$

Example 3.39 **Solve $\cos 2x > |\sin x|, x \in \left(-\frac{\pi}{2}, \pi\right)$.**

Sol. Draw the graph of $y = \cos 2x$ and $y = |\sin x|$

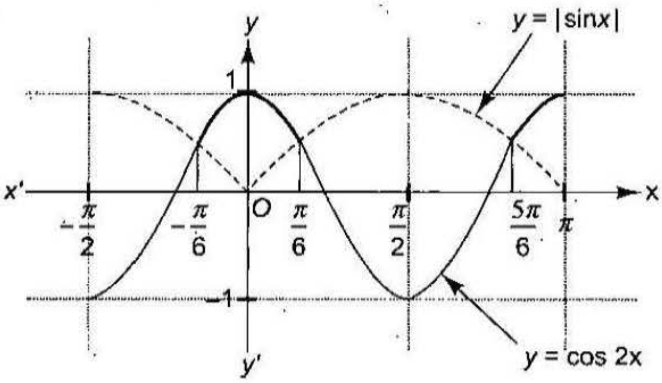


Fig. 3.4

Let $\cos 2x = \sin x$

$\Rightarrow 2\sin^2 x + \sin x - 1 = 0 \Rightarrow \sin x = -1, \frac{1}{2}$

But $\sin x \neq -1 \Rightarrow \sin x = \frac{1}{2}$

Clearly from the graph, graphs of $y = |\sin x|$ and $y = \cos 2x$ intersect at $x = \pm\frac{\pi}{6}, \frac{5\pi}{6}$.

Thus, the solution set is $x \in \left(-\frac{\pi}{6}, \frac{\pi}{6}\right) \cup \left(\frac{5\pi}{6}, \pi\right)$.

Example 3.40 **Find the number of solutions of $\sin x = \frac{x}{10}$.**

Sol. Here, let $f(x) = \sin x$ and $g(x) = \frac{x}{10}$. Also, we know that $-1 \le \sin x \le 1$.

$\Rightarrow \quad -1 \le \frac{x}{10} \quad \Rightarrow \; -10 \le x \le 10$

Thus, sketch both curves when $x \in [-10, 10]$.

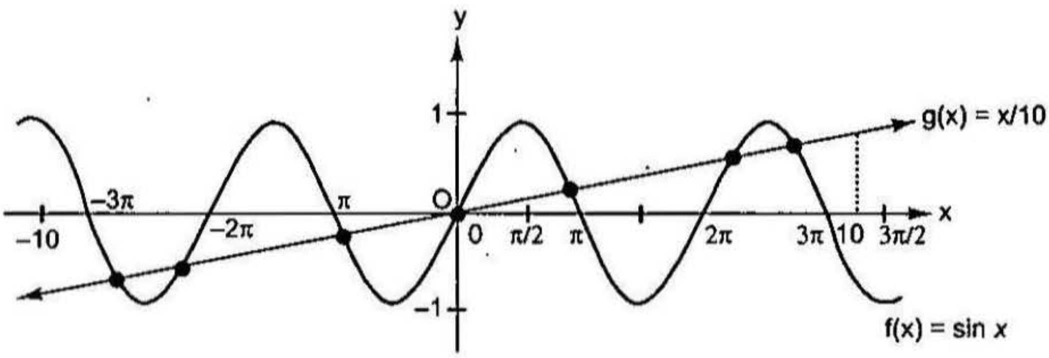


Fig. 3.5

From Fig. 3.5, $f(x) = \sin x$ and $g(x) = x/10$ intersect at seven points. So, the number of solutions is 7.

Concept Application Exercise 3.8

1. Solve $\sin^2\theta > \cos^2\theta$.
2. Find the number of solutions of the equation $\sin x = x^2 + x + 1$.
3. Solve $\tan x < 2$.
4. Prove that the least positive value of x, satisfying $\tan x = x + 1$, lies in the interval $(\pi/4, \pi/2)$.

EXERCISES

Subjective Type

Solutions on page 3.34

1. Solve $3\tan 2x - 4\tan 3x = \tan^2 3x \tan 2x$.
2. For which values of a, does the equation $4\sin(x + \pi/3)\cos(x - \pi/6) = a^2 + \sqrt{3}\sin 2x - \cos 2x$ have solutions? Find the solutions for $a = 0$, if exist.

3. Solve $\sin^4(x/3)+\cos^4(x/3)>1/2$.
4. Solve $\sin x+\sin y=\sin(x+y)$ and $|x|+|y|=1$.
5. Solve the equation $\tan^4 x+\tan^4 y+2\cot^2 x\cot^2 y=3+\sin^2(x+y)$ for the values of x and y.
6. Find the smallest positive root of the equation $\sqrt{\sin(1-x)}=\sqrt{\cos x}$.
7. Solve the equation $2\sin x+\cos y=2$ for the values of x and y.
8. Prove that the equation $2\sin x=|x|+a$ has no solution for $a\in\left(\frac{3\sqrt{3}-\pi}{3},\infty\right)$.
9. Solve $\tan\left(\frac{\pi}{2}\cos\theta\right)=\cot\left(\frac{\pi}{2}\sin\theta\right)$.
10. Solve $\sin x+\sin\left(\frac{\pi}{8}\sqrt{(1-\cos 2x)^2+\sin^2 2x}\right)=0,\ x\in\left[\frac{5\pi}{2},\frac{7\pi}{2}\right]$.
11. Solve $\sin^2 x+\frac{1}{4}\sin^2 3x=\sin x\sin^2 3x$.

Objective Type

Solutions on page 3.38

Each question has four choices a, b, c, and d, out of which *only one* answer is correct.

1. If $\sin\theta=\frac{1}{2}$ and $\cos\theta=-\frac{\sqrt{3}}{2}$, then the general value of θ is $(n\in Z)$

   a. $2n\pi+\frac{5\pi}{6}$ b. $2n\pi+\frac{\pi}{6}$ c. $2n\pi+\frac{7\pi}{6}$ d. $2n\pi+\frac{\pi}{4}$

2. The most general value for which $\tan\theta=-1$, $\cos\theta=\frac{1}{\sqrt{2}}$ is $(n\in Z)$

   a. $n\pi+\frac{7\pi}{4}$ b. $n\pi+(-1)^n\frac{7\pi}{4}$ c. $2n\pi+\frac{7\pi}{4}$ d. none of these

3. If $\cos p\theta+\cos q\theta=0$, then the different values of θ are in A.P. where the common difference is

   a. $\frac{\pi}{p+q}$ b. $\frac{\pi}{p-q}$ c. $\frac{2\pi}{p+q}$ d. $\frac{3\pi}{p\pm q}$

4. If $\cos\theta+\cos 7\theta+\cos 3\theta+\cos 5\theta=0$, then θ is equal to $(n\in Z)$

   a. $n\pi$ b. $n\pi/2$ c. $n\pi/4$ d. $n\pi/8$

5. If $3\tan^2\theta-2\sin\theta=0$, then θ is equal to $(n\in Z)$

   a. $2n\pi\pm\frac{\pi}{4}$ b. $n\pi+(-1)^n\frac{\pi}{6}$ c. $n\pi-(-1)^n\frac{\pi}{6}$ d. $n\pi+\frac{\pi}{3}$

6. If $\sin\theta$, 1, $\cos 2\theta$ are in G.P., then θ is equal to $(n\in Z)$

   a. $n\pi+(-1)^n\frac{\pi}{2}$ b. $n\pi+(-1)^{n-1}\frac{\pi}{2}$ c. $2n\pi$ d. none of these

7. The sum of all the solutions of the equation $\cos\theta\cos\left(\frac{\pi}{3}+\theta\right)\cos\left(\frac{\pi}{3}-\theta\right)=\frac{1}{4}$, $\theta\in[0,6\pi]$

   a. 15π b. 30π c. $\frac{100\pi}{3}$ d. none of these

8. If $\sec\theta - 1 = (\sqrt{2} - 1)\tan\theta$, then θ is equal to ($n \in Z$)
   a. $(2n-1)\pi$ b. $2n\pi + \frac{\pi}{4}$ c. $2n\pi - \frac{\pi}{4}$ d. $2n\pi + \frac{\pi}{3}$
9. The total number of solution of $\sin^4 x + \cos^4 x = \sin x \cos x$ in $[0, 2\pi]$ is equal to
   a. 2 b. 4 c. 6 d. none of these
10. Number of solutions of $\sin 5x + \sin 3x + \sin x = 0$ for $0 \le x \le \pi$ is
   a. 1 b. 2 c. 3 d. none of these
11. The sum of all the solution of $\cot\theta = \sin 2\theta$, ($\theta \neq n\pi$, n integer), $0 \le \theta \le \pi$ is
   a. $3\pi/2$ b. π c. $3\pi/4$ d. 2π
12. The number of solutions of $12\cos^3 x - 7\cos^2 x + 4\cos x = 9$ is
   a. 0 b. 2 c. infinite d. none of these
13. Which of the following is not the general solution of $2^{\cos 2x} + 1 = 3.2^{-\sin^2 x}$?
   a. $n\pi, n \in Z$ b. $\left(n + \frac{1}{2}\right)\pi, n \in Z$ c. $\left(n - \frac{1}{2}\right)\pi, n \in Z$ d. none of these
14. The general solution of $\cos x \cos 6x = -1$ is
   a. $x = (2n+1)\pi, n \in Z$ b. $x = 2n\pi, n \in Z$
   c. $x = n\pi, n \in Z$ d. none of these
15. The equation $\cos x + \sin x = 2$ has
   a. only one solution b. two solutions
   c. no solution d. infinite number of solutions
16. If $0 \le x \le 2\pi$, then the number of solutions of $3(\sin x + \cos x) - 2(\sin^3 x + \cos^3 x) = 8$ is
   a. 0 b. 1 c. 2 d. 4
17. If $\frac{1}{6}\sin\theta, \cos\theta, \tan\theta$ are in G.P., then θ is equal to ($n \in Z$)
   a. $2n\pi \pm \frac{\pi}{3}$ b. $2n\pi \pm \frac{\pi}{6}$ c. $n\pi + (-1)^n \frac{\pi}{3}$ d. $n\pi + \frac{\pi}{3}$
18. The number of solutions of $2\sin^2 x + \sin^2 2x = 2, x \in [0, 2\pi]$ is
   a. 4 b. 5 c. 7 d. 6
19. General solution of $\sin^2 x - 5\sin x \cos x - 6\cos^2 x = 0$ is
   a. $x = n\pi - \pi/4, n \in Z$ only b. $n\pi + \tan^{-1} 6, n \in Z$ only
   c. both (a) and (b) d. none of these
20. General solution of $\tan\theta + \tan 4\theta + \tan 7\theta = \tan\theta \tan 4\theta \tan 7\theta$ is
   a. $\theta = n\pi/12$, where $n \in Z$
   b. $\theta = n\pi/9$, where $n \in Z$
   c. $\theta = n\pi + \pi/12$, where $n \in Z$
   d. none of these
21. The number of solutions of $\sec^2\theta + \text{cosec}^2\theta + 2\,\text{cosec}^2\theta = 8$, $0 \le \theta \le \pi/2$ is
   a. 4 b. 3 c. 0 d. 2
22. The total number of solutions of $\tan x + \cot x = 2\,\text{cosec}\, x$ in $[-2\pi, 2\pi]$ is
   a. 2 b. 4 c. 6 d. 8
23. Which of the following is true for $z = (3 + 2i\sin\theta)/(1 - 2i\sin\theta)$, where $i = \sqrt{-1}$
   a. z is purely real for $\theta = n\pi \pm \pi/3, n \in Z$
   b. z is purely imaginary for $\theta = n\pi \pm \pi/2, n \in Z$
   c. z is purely real for $\theta = n\pi, n \in Z$
   d. none of these

24. Number of roots of $\cos^2 x + \frac{\sqrt{3}+1}{2}\sin x - \frac{\sqrt{3}}{4} - 1 = 0$ which lie in the interval $[-\pi, \pi]$ is

a. 2 b. 4 c. 6 d. 8

25. The complete solution of $7\cos^2 x + \sin x \cos x - 3 = 0$ is given by

a. $n\pi + \frac{\pi}{2}$ $(n \in Z)$

b. $n\pi - \frac{\pi}{2}$ $(n \in Z)$

c. $n\pi + \tan^{-1}\left(\frac{3}{4}\right)$ $(n \in Z)$

d. $n\pi + \frac{3\pi}{4}, k\pi + \tan^{-1}\left(\frac{4}{3}\right)$ $(k, n \in Z)$

26. Let $\theta \in [0, 4\pi]$ satisfy the equation $(\sin\theta + 2)(\sin\theta + 3)(\sin\theta + 4) = 6$. If the sum of all the values of θ is of the form $k\pi$, then the value of k is

a. 6 b. 5 c. 4 d. 2

27. If the inequality $\sin^2 x + a\cos x + a^2 > 1 + \cos x$ holds for any $x \in R$, then the largest negative integral value of a is

a. -4 b. -3 c. -2 d. -1

28. The number of solution of $\sin^4 x - \cos^2 x \sin x + 2\sin^2 x + \sin x = 0$ in $0 \le x \le 3\pi$ is

a. 3 b. 4 c. 5 d. 6

29. If $x, y \in [0, 2\pi]$ and $\sin x + \sin y = 2$, then the value of $x + y$ is

a. π b. $\pi/2$ c. 3π d. none of these

30. For $n \in Z$, the general solution of $(\sqrt{3} - 1)\sin\theta + (\sqrt{3} + 1)\cos\theta = 2$ is $(n \in Z)$

a. $\theta = 2n\pi \pm \frac{\pi}{4} + \frac{\pi}{12}$

b. $\theta = n\pi + (-1)^n \frac{\pi}{4} + \frac{\pi}{12}$

c. $\theta = 2n\pi \pm \frac{\pi}{4}$

d. $\theta = n\pi + (-1)^n \frac{\pi}{4} - \frac{\pi}{12}$

31. If $\sin 6\theta + \sin 4\theta + \sin 2\theta = 0$, then θ is equal to $(n \in Z)$

a. $\frac{n\pi}{4}$ or $n\pi \pm \frac{\pi}{3}$ b. $\frac{n\pi}{4}$ or $n\pi \pm \frac{\pi}{6}$ c. $\frac{n\pi}{4}$ or $2n\pi \pm \frac{\pi}{6}$ d. none of these

32. The value of $\cos y \cos\left(\frac{\pi}{2} - x\right) - \cos\left(\frac{\pi}{2} - y\right)\cos x + \sin y \cos\left(\frac{\pi}{2} - x\right) + \cos x \sin\left(\frac{\pi}{2} - y\right)$ is zero if

a. $x = 0$ b. $y = 0$ c. $x = y$ d. $n\pi + y - \frac{\pi}{4}$ $(n \in Z)$

33. The number of solution of the equation $\tan x \tan 4x = 1$ for $0 < x < \pi$ is

a. 1 b. 2 c. 5 d. 8

34. One root of the equation $\cos x - x + \frac{1}{2} = 0$ lies in the interval

a. $\left(0, \frac{\pi}{2}\right)$ b. $\left(-\frac{\pi}{2}, 0\right)$ c. $\left(\frac{\pi}{2}, \pi\right)$ d. $\left(\pi, \frac{3\pi}{2}\right)$

35. $\tan\left(\frac{p\pi}{4}\right) = \cot\left(\frac{q\pi}{4}\right)$ if $(n \in Z)$

a. $p + q = 0$ b. $p + q = 2n + 1$ c. $p + q = 2n$ d. $p + q = 2(2n + 1)$

36. The range of y such that the equation in x, $y + \cos x = \sin x$ has a real solution is

a. $[-2, 2]$ b. $[-\sqrt{2}, \sqrt{2}]$ c. $[-1, 1]$ d. $[-1/2, 1/2]$

37. One of the general solutions of $4\sin^4 x + \cos^4 x = 1$ is

a. $n\pi \pm \alpha/2$, $\alpha = \cos^{-1}(1/5)$, $\forall\ n \in Z$ b. $n\pi \pm \alpha/2$, $\alpha = \cos^{-1}(3/5)$, $\forall\ n \in Z$
c. $2n\pi \pm \alpha/2$, $\alpha = \cos^{-1}(1/3)$, $\forall\ n \in Z$ d. none of these

38. Number of roots of $(1 - \tan\theta)(1 + \sin 2\theta) = 1 + \tan\theta$ for $\theta \in [0, 2\pi]$ is

a. 3 b. 4 c. 5 d. none of these

39. The number of solutions of $\sin x + \sin 2x + \sin 3x = \cos x + \cos 2x + \cos 3x$, $0 \le x \le 2\pi$, is

a. 7 b. 5 c. 4 d. 6

40. The number of values of θ which satisfy the equation $\sin 3\theta - \sin\theta = 4\cos^2\theta - 2$, $\forall\ \theta \in [0, 2\pi]$, is

a. 4 b. 5 c. 7 d. 0

41. One of the general solutions of $4\sin\theta \sin 2\theta \sin 4\theta = \sin 3\theta$ is

a. $(3n \pm 1)\pi/12$, $\forall\ n \in Z$ b. $(4n \pm 1)\pi/9$, $\forall\ n \in Z$
c. $(3n \pm 1)\pi/9$, $\forall\ n \in Z$ d. $(3n \pm 1)\pi/3$, $\forall\ n \in Z$

42. The general solution of $\tan\theta + \tan 2\theta + \tan 3\theta = 0$ is

a. $\theta = n\pi/6$, $n \in Z$ b. $\theta = n\pi \pm \alpha$, $n \in Z$, where $\tan\alpha = 1/\sqrt{2}$
c. Both a and b d. none of these

43. The general solution of $\sin 3\alpha = 4\sin\alpha \sin(x + \alpha)\sin(x - \alpha)$ is

a. $n\pi \pm \pi/4$, $\forall\ n \in Z$ b. $n\pi \pm \pi/3$, $\forall\ n \in Z$
c. $n\pi \pm \pi/9$, $\forall\ n \in Z$ d. $n\pi \pm \pi/12$, $\forall\ n \in Z$

44. One of the general solutions of $\sqrt{3}\cos\theta - 3\sin\theta = 4\sin 2\theta \cos 3\theta$ is

a. $m\pi + \pi/18$, $m \in Z$ b. $m\pi/2 + \pi/6$, $\forall\ m \in Z$
c. $m\pi/3 + \pi/18$, $m \in Z$ d. none of these

45. The equation $\sin^4 x + \cos^4 x + \sin 2x + \alpha = 0$ is solvable for

a. $-5/2 \le \alpha \le 1/2$ b. $-3 \le \alpha \le 1$ c. $-3/2 \le \alpha \le 1/2$ d. $-1 \le \alpha \le 1$

46. Consider the system of linear equations in x, y and z:

$$(\sin 3\theta)x - y + z = 0$$
$$(\cos 2\theta)x + 4y + 3z = 0$$
$$2x + 7y + 7z = 0$$

then which of the following can be the values of θ for which the system has a non-trivial solution

a. $n\pi + (-1)^n \pi/6$, $\forall\ n \in Z$ b. $n\pi + (-1)^n \pi/3$, $\forall\ n \in Z$
c. $n\pi + (-1)^n \pi/9$, $\forall\ n \in Z$ d. none of these

47. The smallest +ve x satisfying the equation $\log_{\cos x} \sin x + \log_{\sin x} \cos x = 2$ is

a. $\pi/2$ b. $\pi/3$ c. $\pi/4$ d. $\pi/6$

48. Number of ordered pairs which satisfy the equation $x^2 + 2x\sin(xy) + 1 = 0$ are (where $y \in [0, 2\pi]$)

a. 1 b. 2 c. 3 d. 0

49. The general solution of the equation $8\cos x \cos 2x \cos 4x = \sin 6x/\sin x$ is

a. $x = (n\pi/7) + (\pi/21)$, $\forall\ n \in Z$ b. $x = (2\pi/7) + (\pi/14)$, $\forall\ n \in Z$
c. $x = (n\pi/7) + (\pi/14)$, $\forall\ n \in Z$ d. $x = (n\pi) + (\pi/14)$, $\forall\ n \in Z$

50. If $\cos 3x + \sin\left(2x - \frac{7\pi}{6}\right) = -2$, then x is equal to ($k \in Z$)

a. $\frac{\pi}{3}(6k + 1)$ b. $\frac{\pi}{3}(6k - 1)$ c. $\frac{\pi}{3}(2k + 1)$ d. none of these

51. If $(1-\tan\theta)(1+\tan\theta)\sec^2\theta + 2^{\tan^2\theta} = 0$, then the number of values of θ in the interval $(-\pi/2, \pi/2)$ are

a. 1 **b.** 2 **c.** 3 **d.** 4

52. Number of solutions of $\tan\left(\frac{\pi}{2}\sin\theta\right) = \cot\left(\frac{\pi}{2}\cos\theta\right)$, $\theta \in [0, 6\pi]$, is

a. 5 **b.** 7 **c.** 4 **d.** 5

53. The total number of solutions of $\cos x = \sqrt{1-\sin 2x}$ in $[0, 2\pi]$ is equal to

a. 2 **b.** 3 **c.** 5 **d.** none of these

54. The number of solutions of $\sum_{r=1}^{5} \cos rx = 5$ in the interval $[0, 2\pi]$ is

a. 0 **b.** 2 **c.** 5 **d.** 10

55. The number of values of x for which $\sin 2x + \cos 4x = 2$ is

a. 0 **b.** 1 **c.** 2 **d.** infinite

56. Let α and β be any two positive values of x for which $2\cos x$, $|\cos x|$ and $1 - 3\cos^2 x$ are in GP. The minimum value of $|\alpha - \beta|$ is

a. $\frac{\pi}{3}$ **b.** $\frac{\pi}{4}$ **c.** $\frac{\pi}{2}$ **d.** none of these

57. The general solution of the equation $\sin^{100} x - \cos^{100} x = 1$ is

a. $2n\pi + \frac{\pi}{3}, n \in I$ **b.** $n\pi + \frac{\pi}{2}, n \in I$ **c.** $n\pi + \frac{\pi}{4}, n \in I$ **d.** $2n\pi - \frac{\pi}{3}, n \in I$

58. The total number of solutions of $|\cot x| = \cot x + \frac{1}{\sin x}$, $x \in [0, 3\pi]$ is equal to

a. 1 **b.** 2 **c.** 3 **d.** 0

59. If $\tan(A-B) = 1$ and $\sec(A+B) = 2/\sqrt{3}$, then the smallest positive values of A and B, respectively, are

a. $\frac{25\pi}{24}, \frac{19\pi}{24}$ **b.** $\frac{19\pi}{24}, \frac{25\pi}{24}$ **c.** $\frac{31\pi}{24}, \frac{13\pi}{24}$ **d.** $\frac{13\pi}{24}, \frac{31\pi}{24}$

60. If $3\tan(\theta - 15°) = \tan(\theta + 15°)$, then θ is equal to $(n \in Z)$

a. $n\pi + \frac{\pi}{4}$ **b.** $n\pi + \frac{\pi}{8}$ **c.** $n\pi + \frac{\pi}{3}$ **d.** none of these

61. If $\tan 3\theta + \tan\theta = 2\tan 2\theta$, then θ is equal to $(n \in Z)$

a. $n\pi$ **b.** $\frac{n\pi}{4}$ **c.** $2n\pi$ **d.** none of these

62. The set of all x in $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ satisfying $|4\sin x - 1| < \sqrt{5}$ is given by

a. $\left(-\frac{\pi}{10}, \frac{3\pi}{10}\right)$ **b.** $\left(\frac{\pi}{10}, \frac{3\pi}{10}\right)$ **c.** $\left(\frac{\pi}{10}, \frac{3\pi}{10}\right)$ **d.** none of these

63. $\sin x + \cos x = y^2 - y + a$ has no value of x for any value of y if a belongs to

a. $(0, \sqrt{3})$ **b.** $(-\sqrt{3}, 0)$ **c.** $(-\infty, -\sqrt{3})$ **d.** $(\sqrt{3}, \infty)$

64. The solution of $4\sin^2 x + \tan^2 x + \operatorname{cosec}^2 x + \cot^2 x - 6 = 0$ is

a. $n\pi \pm \frac{\pi}{4}$ **b.** $2n\pi \pm \frac{\pi}{4}$ **c.** $n\pi + \frac{\pi}{3}$ **d.** $n\pi - \frac{\pi}{6}$

65. The number of solutions of $[\sin x + \cos x] = 3 + [-\sin x] + [-\cos x]$ (where [.] denotes the greatest integer function), $x \in [0, 2\pi]$, is

a. 0 b. 4 c. infinite d. 1

66. The equation $\cos^8 x + b\cos^4 x + 1 = 0$ will have a solution if b belongs to

a. $(-\infty, 2]$ b. $[2, \infty)$ c. $(-\infty, -2]$ d. none of these

67. The number of values of y in $[-2\pi, 2\pi]$ satisfying the equation $|\sin 2x| + |\cos 2x| = |\sin y|$ is

a. 3 b. 4 c. 5 d. 6

68. If both the distinct roots of the equation $|\sin x|^2 + |\sin x| + b = 0$ in $[0, \pi]$ are real, then the values of b are

a. $[-2, 0]$ b. $(-2, 0)$ c. $[-2, 0)$ d. none of these

69. If $|2\sin\theta - \operatorname{cosec}\theta| \geq 1$ and $\theta \neq \frac{n\pi}{2}$, $n \in I$, then

a. $\cos 2\theta \geq 1/2$ b. $\cos 2\theta \geq 1/4$ c. $\cos 2\theta \leq 1/2$ d. $\cos 2\theta \leq 1/4$

70. The number of solutions of the equation $\sin^3 x \cos x + \sin^2 x \cos^2 x + \sin x \cos^3 x = 1$, in the interval $[0, 2\pi]$, is

a. 4 b. 2 c. 1 d. 0

71. $e^{|\sin x|} + e^{-|\sin x|} + 4a = 0$ will have exactly four different solutions in $[0, 2\pi]$ if

a. $a \in R$ b. $a \in \left[-\frac{e}{4}, -\frac{1}{4}\right]$ c. $a \in \left[\frac{-1-e^2}{4e}, \infty\right]$ d. none of these

72. The total number of solutions of $\ln|\sin x| = -x^2 + 2x$ in $\left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]$ is equal to

a. 1 b. 2 c. 4 d. none of these

73. The total number of ordered pairs (x, y) satisfying $|x| + |y| = 4$, $\sin\left(\frac{\pi x^2}{3}\right) = 1$ is equal to

a. 2 b. 3 c. 4 d. 6

74. The total number of solutions of $\sin\{x\} = \cos\{x\}$ (where {.} denotes the fractional part) in $[0, 2\pi]$ is equal to

a. 5 b. 6 c. 8 d. none of these

75. If $a, b \in [0, 2\pi]$ and the equation $x^2 + 4 + 3\sin(ax + b) - 2x = 0$ has at least one solution, then the value of $(a + b)$ can be

a. $\frac{7\pi}{2}$ b. $\frac{5\pi}{2}$ c. $\frac{9\pi}{2}$ d. none of these

76. The equation $\tan^4 x - 2\sec^2 x + a = 0$ will have at least one solution if

a. $1 < a \leq 4$ b. $a \geq 2$ c. $a \leq 3$ d. none of these

77. Complete the set of values of x in $(0, \pi)$ satisfying the equation $1 + \log_2 \sin x + \log_2 \sin 3x \geq 0$ is

a. $\left(\frac{2\pi}{3}, \frac{3\pi}{4}\right]$ b. $\left(\frac{\pi}{3}, \frac{2\pi}{3}\right)$ c. $\left(0, \frac{\pi}{2}\right) \cup \left(\frac{2\pi}{3}, \pi\right)$ d. $\left(\frac{\pi}{2}, \frac{2\pi}{3}\right)$

78. The equation $\sin^2\theta - \frac{4}{\sin^3\theta - 1} = 1 - \frac{4}{\sin^3\theta - 1}$ has

a. no root b. one root c. two roots c. infinite roots

79. The sum of all roots of $\sin\left(\pi \log_3\left(\frac{1}{x}\right)\right) = 0$ in $(0, 2\pi)$ is

a. 3/2 b. 4 c. 9/2 d. 13/3

80. The number of pairs of integer (x, y) that satisfy the following two equations

$\begin{cases}\cos(xy) = x \\ \tan(xy) = y\end{cases}$ is

a. 1 b. 2 c. 4 d. 6

81. Sum of all the solutions in $[0, 4\pi]$ of the equation $\tan x + \cot x + 1 = \cos\left(x + \frac{\pi}{4}\right)$ is

a. 3π b. $\pi/2$ c. $7\pi/2$ d. 4π

82. Number of solutions the equation $\cos(\theta)\cdot\cos(\pi\theta) = 1$ has

a. 0 b. 2 c. 1 d. infinite

83. The general value of x satisfying the equation $2\cot^2 x + 2\sqrt{3}\cot x + 4\operatorname{cosec} x + 8 = 0$ is

a. $n\pi - \frac{\pi}{6}, n \in Z$ b. $n\pi + \frac{\pi}{6}, n \in Z$ c. $2n\pi - \frac{\pi}{6}, n \in Z$ d. $2n\pi + \frac{\pi}{6}, n \in Z$

84. Assume that θ is a rational multiple of π such that $\cos\theta$ is a distinct rational. Number of values of $\cos\theta$ is

a. 3 b. 4 c. 5 d. 6

85. Number of ordered pair(s) (a, b) for each of which the equality $a(\cos x - 1) + b^2 = \cos(ax + b^2) - 1$ holds true for all $x \in R$ are

a. 1 b. 2 c. 3 d. 4

Multiple Correct Answers Type

Solutions on page 3.54

Each question has four choices a, b, c, and d, out of which *one or more* answers are correct.

1. If $4\sin^4 x + \cos^4 x = 1$, then x is equal to $(n \in Z)$

a. $n\pi$ b. $n\pi \pm \sin^{-1}\sqrt{\frac{2}{5}}$ c. $\frac{2n\pi}{3}$ d. $2n\pi \pm \frac{\pi}{4}$

2. If $\sin^3\theta + \sin\theta\cos\theta + \cos^3\theta = 1$, then θ is equal to $(n \in Z)$

a. $2n\pi$ b. $2n\pi + \frac{\pi}{2}$ c. $2n\pi - \frac{\pi}{2}$ d. $n\pi$

3. A general solution of $\tan^2\theta + \cos 2\theta = 1$ is $(n \in Z)$

a. $n\pi - \frac{\pi}{4}$ b. $2n\pi + \frac{\pi}{4}$ c. $n\pi + \frac{\pi}{4}$ d. $n\pi$

4. If $\sin x + \cos x = \sqrt{y + \frac{1}{y}}$ for $x \in [0, \pi]$, then

a. $x = \pi/4$ b. $y = 0$ c. $y = 1$ d. $x = 3\pi/4$

5. $\sin\theta + \sqrt{3}\cos\theta = 6x - x^2 - 11, 0 \le \theta \le 4\pi, x \in R$, holds for

a. no values of x and θ b. one value of x and two values of θ
c. two values of x and two values of θ d. two point of values of (x, θ)

6. If $\sin^2 x - 2\sin x - 1 = 0$ has exactly four different solutions in $x \in [0, n\pi]$, then value/values of n is/are $(n \in N)$

a. 5 b. 3 c. 4 d. 6

7. For the smallest positive values of x and y, the equation $2(\sin x + \sin y) - 2\cos(x - y) = 3$ has a solution, then which of the following is/are true?

a. $\sin\frac{x+y}{2}=1$

b. $\cos\left(\frac{x-y}{2}\right)=\frac{1}{2}$

c. number of ordered pairs (x, y) is 2

d. number of ordered pairs (x, y) is 3

8. For the equation $1-2x-x^2=\tan^2(x+y)+\cot^2(x+y)$

a. exactly one value of x exists

b. exactly two values of x exists

c. $y=-1+n\pi+\pi/4, n\in Z$

d. $y=1+n\pi+\pi/4, n\in Z$

9. If $x+y=\pi/4$ and $\tan x+\tan y=1$, then $(n\in Z)$

a. $\sin x=0$ always

b. when $x=n\pi+\pi/4$ then $y=-n\pi$

c. when $x=n\pi$ then $y=n\pi+(\pi/4)$

d. when $x=n\pi+\pi/4$ then $y=n\pi-(\pi/4)$

10. If $x+y=2\pi/3$ and $\sin x/\sin y=2$, then

a. the number of values of $x\in[0, 4\pi]$ are 4

b. number of values of $x\in[0, 4\pi]$ are 2

c. number of values of $y\in[0, 4\pi]$ are 4

d. number of values of $y\in[0, 4\pi]$ are 8

11. Let $\tan x-\tan^2 x>0$ and $|2\sin x|<1$. Then the intersection of which of the following two sets satisfies both the inequalities?

a. $x>n\pi, n\in Z$

b. $x>n\pi-\pi/6, n\in Z$

c. $x<n\pi-\pi/4, n\in Z$

d. $x<n\pi+\pi/6, n\in Z$

12. If $\cos(x+\pi/3)+\cos x=a$ has real solutions, then

a. number of integral values of a are 3

b. sum of number of integral values of a is 0

c. when $a=1$, number of solutions for $x\in[0, 2\pi]$ are 3

d. when $a=1$, number of solutions for $x\in[0, 2\pi]$ are 2

13. For $0\le x\le 2\pi$, then $2^{\operatorname{cosec}^2 x}\sqrt{\frac{1}{2}y^2-y+1}\le\sqrt{2}$

a. is satisfied by exactly one value of y

b. is satisfied by exactly two value of x

c. is satisfied by x for which $\cos x=0$

d. is satisfied by x for which $\sin x=0$

14. If $\sin^2 x-a\sin x+b=0$ has only one solution in $(0, \pi)$, then which of the following statements are correct?

a. $a\in(-\infty, 1]\cup[2, \infty)$

b. $b\in(-\infty, 0]\cup[1, \infty)$

c. $a=1+b$

d. none of these

15. If $(\operatorname{cosec}^2\theta-4)x^2+(\cot\theta+\sqrt{3})x+\cos^2\frac{3\pi}{2}=0$ holds true for all real x, then the most general values of θ can be given by $(n\in Z)$

a. $2n\pi+\frac{11\pi}{6}$

b. $2n\pi+\frac{5\pi}{6}$

c. $2n\pi\pm\frac{7\pi}{6}$

d. $n\pi\pm\frac{11\pi}{6}$

16. If $(\sin\alpha)x^2-2x+b\ge 2$ for all the real values of $x\le 1$ and $\alpha\in(0, \pi/2)\cup(\pi/2, \pi)$, then the possible real values of b is/are

a. 2

b. 3

c. 4

d. 5

17. The value of x in $(0, \pi/2)$ satisfying $\frac{\sqrt{3}-1}{\sin x}+\frac{\sqrt{3}+1}{\cos x}=4\sqrt{2}$ is

a. $\frac{\pi}{12}$

b. $\frac{5\pi}{12}$

c. $\frac{7\pi}{24}$

d. $\frac{11\pi}{36}$

18. If $\cos 3\theta=\cos 3\alpha$, then the value of $\sin\theta$ can be given by

a. $\pm\sin\alpha$

b. $\sin\left(\frac{\pi}{3}\pm\alpha\right)$

c. $\sin\left(\frac{2\pi}{3}+\alpha\right)$

d. $\sin\left(\frac{2\pi}{3}-\alpha\right)$

19. Which of the following sets can be the subset of the general solution of $1 + \cos 3x = 2\cos 2x$ $(n \in Z)$?

a. $n\pi + \frac{\pi}{3}$ b. $n\pi + \frac{\pi}{6}$ c. $n\pi - \frac{\pi}{6}$ d. $2n\pi$

20. In a right-angled triangle, the hypotenuse is $2\sqrt{2}$ times the perpendicular drawn from the opposite vertex. Then the other acute angles of the triangle are

a. $\frac{\pi}{3}$ b. $\frac{\pi}{8}$ c. $\frac{3\pi}{8}$ d. $\frac{\pi}{6}$

Reasoning Type

Solutions on page 3.59

Each question has four choices a, b, c, and d, out of which *only one* is correct. Each question contains STATEMENT 1 and STATEMENT 2.

a. if both the statements are TRUE and STATEMENT 2 is the correct explanation of STATEMENT 1
b. if both the statements are TRUE but STATEMENT 2 is NOT the correct explanation of STATEMENT 1
c. if STATEMENT 1 is TRUE and STATEMENT 2 is FALSE
d. if STATEMENT 1 is FALSE and STATEMENT 2 is TRUE

1. **Statement 1:** The value of x for which $(\sin x + \cos x)^{1+\sin 2x} = 2$, when $0 \le x \le \pi$ is $\pi/4$ only.
Statement 2: The maximum value of $\sin x + \cos x$ occurs when $x = \pi/4$.
2. **Statement 1:** The equation $\sin^2 x + \cos^2 y = 2\sec^2 z$ is solvable when only $\sin x = 1$; $\cos y = 1$ and $\sec z = 1$, where $x, y, z \in R$.
Statement 2: The maximum value of $\sin x$ and $\cos y$ is 1 and minimum value of $\sec z$ is 1.
3. **Statement 1:** Equation $x \sin x = 1$ has four roots for $x \in (-\pi, \pi)$.
Statement 2: The graph of $y = \sin x$ and $y = 1/x$ cuts exactly two times for $x \in (0, \pi)$.
4. **Statement 1:** $\sin x = a$, where $-1 < a < 0$, then for $x \in [0, n\pi]$ has $2(n-1)$ solutions $\forall\, n \in N$.
Statement 2: $\sin x$ takes value a exactly two times when we take one complete rotation covering all the quadrant starting from $x = 0$.
5. **Statement 1:** Equation $\sqrt{1-\sin 2x} = \sin x$ has 1 solution for $x \in [0, \pi/4]$.
Statement 2: $\cos x > \sin x$ when $x \in [0, \pi/4]$.
6. **Statement 1:** The number of solution of the equation $|\sin x| = |x|$ is only one.
Statement 2: $|\sin x| \ge 0\ \forall\, x \in R$.
7. **Statement 1:** General solution of $\frac{\tan 4x - \tan 2x}{1 + \tan 4x \tan 2x} = 1$ is $x = \frac{n\pi}{2} + \frac{\pi}{8}, n \in I$.
Statement 2: General solution of $\tan \alpha = 1$ is $\alpha = n\pi + \frac{\pi}{4}, n \in I$.
8. **Statement 1:** The equation $\sin(\cos x) = \cos(\sin x)$ has no real solution.
Statement 2: $\sin x \pm \cos x \in \left[-\sqrt{2}, \sqrt{2}\right]$.
9. **Statement 1:** Equation $\sin x = e^x$ has infinite solutions.
Statement 2: $y = e^x$ is an unbounded function.
10. **Statement 1:** Number of solution of $n|\sin x| = m|\cos x|$ (where $m, n \in Z$) in $[0, 2\pi]$ is independent of m and n.
Statement 2: Multiplying trigonometric functions by constant changes only range of the function but period remains same.

Linked Comprehension Type

Solutions on page 3.61

Based upon each paragraph, three multiple choice questions have to be answered. Each question has four choices a, b, c, and d, out of which *only one* is correct.

For Problems 1 – 3

Consider the cubic equation

$x^3 - (1 + \cos\theta + \sin\theta)x^2 + (\cos\theta\sin\theta + \cos\theta + \sin\theta)x - \sin\theta\cos\theta = 0$

whose roots are x_1, x_2 and x_3.

1. The value of $x_1^2 + x_2^2 + x_3^2$ equals
   a. 1 b. 2 c. $2\cos\theta$ d. $\sin\theta(\sin\theta + \cos\theta)$
2. Number of values of θ in $[0, 2\pi]$ for which at least two roots are equal
   a. 3 b. 4 c. 5 d. 6
3. Greatest possible difference between two of the roots if $\theta \in [0, 2\pi]$ is
   a. 2 b. 1 c. $\sqrt{2}$ d. $2\sqrt{2}$

For Problems 4 – 6

Consider the equation $\sec\theta + \operatorname{cosec}\theta = a$, $\theta \in (0, 2\pi) - \{\pi/2, \pi, 3\pi/2\}$

4. If the equation has four real roots, then
   a. $|a| \ge 2\sqrt{2}$ b. $|a| < 2\sqrt{2}$ c. $a \ge -2\sqrt{2}$ d. none of these
5. If the equation has two real roots, then
   a. $|a| \ge 2\sqrt{2}$ b. $a < 2\sqrt{2}$ c. $|a| < 2\sqrt{2}$ d. none of these
6. If the equation has no real roots, then
   a. $|a| \ge 2\sqrt{2}$ b. $a < 2\sqrt{2}$ c. $|a| < 2\sqrt{2}$ d. none of these

For Problems 7 – 9

Consider the system of equations

$\sin x \cos 2y = (a^2 - 1)^2 + 1$,

$\cos x \sin 2y = a + 1$

7. Number of values of a for which the system has a solution is
   a. 1 b. 2 c. 3 d. infinite
8. Number of values of $x \in [0, 2\pi]$, when the system has solution for permissible values of a, is/are
   a. 1 b. 2 c. 3 d. 4
9. Number of values of $y \in [0, 2\pi]$, when the system has solution for permissible values of a, are
   a. 2 b. 3 c. 4 d. 5

For Problems 10 – 12

Consider the equation $\int_0^x (t^2 - 8t + 13)dt = x\sin(a/x)$

10. The number of real values of x for which the equation has solution is
    a. 1 b. 2 c. 3 d. infinite
11. If x takes the values for which the equation has a solution, then the number of values of $a \in [0, 100]$ is
    a. 2 b. 1 c. 5 d. 3
12. One of the solutions of $|y - \cos a| < x$, where x and a are values that satisfy the given equation, is
    a. $y \in [-5, 7]$ b. $y \in [-7, 5]$ c. $y \in [5, 7]$ d. none of these

For Problems 13 – 15

Consider the system of equations

$x\cos^3 y + 3x\cos y\sin^2 y = 14$

$x\sin^3 y + 3x\cos^2 y\sin y = 13$

13. The value/values of x is/are

a. $\pm 5\sqrt{5}$ b. $\pm\sqrt{5}$ c. $\pm 1/\sqrt{5}$ d. none of these

14. The number of values of $y \in [0, 6\pi]$ is

a. 5 b. 3 c. 4 d. 6

15. The value of $\sin^2 y + 2\cos^2 y$ is

a. 4/5 b. 9/5 c. 2 d. none of these

Matrix-Match Type

Solutions on page 3.64

Each question contains statements given in two columns which have to be matched. Statements (a, b, c, d) in column I have to be matched with statements (p, q, r, s) in column II. If the correct match are a → p, a → s, b → q, c → p, c → q and d → s, then the correctly bubbled 4 × 4 matrix should be as follows:

	p	q	r	s
a	p	q	r	s
b	p	q	r	s
c	p	q	r	s
d	p	q	r	s

1.

Column I (Equation)	Column II (Solution)
a. $\cos^2 2x + \cos^2 x = 1$	p. $x = \left\{n\pi + \frac{\pi}{4}\right\} \cup \left\{n\pi + \frac{\pi}{6}\right\} n \in Z$
b. $\cos x + \sqrt{3}\sin x = \sqrt{3}$	q. $x = \frac{n\pi}{3}, n \in Z$
c. $1 + \sqrt{3}\tan^2 x = (1 + \sqrt{3})\tan x$	r. $x = (2n - 1)\frac{\pi}{6}, n \in Z$
d. $\tan 3x - \tan 2x - \tan x = 0$	s. $x = \left\{2n\pi + \frac{\pi}{2}\right\} \cup \left\{2n\pi + \frac{\pi}{6}\right\}, n \in Z$

2.

Column I (Equation)	Column II (Number of solutions)
a. $x^3 + x^2 + 4x + 2\sin x = 0$ in $0 \le x \le 2\pi$	p. 4
b. $\sin e^x \cos e^x = 2^{x-2} + 2^{-x-2}$	q. 1
c. $\sin 2x + \cos 4x = 2$	r. 2
d. $30\lvert\sin x\rvert = x$ in $0 \le x \le 2\pi$	s. 0

3.

Column I (Equation)	Column II (Solution)
a. $\max_{\theta \in R} \{5 \sin \theta + 3 \sin (\theta - \alpha)\} = 7$ then the set of possible values of α is	p. $2n\pi + 3\pi/4, n \in Z$
b. $x \neq \frac{n\pi}{2}$ and $(\cos x)^{\sin^2 x - 3\sin x + 2} = 1$	q. $2n\pi \pm \frac{\pi}{3}; n \in Z$
c. $\sqrt{(\sin x)} + 2^{1/4} \cos x = 0$	r. $2n\pi + \cos^{-1}(1/3), n \in Z$
d. $\log_5 \tan x = (\log_5 4)(\log_4 (3 \sin x))$	s. no solution

Integer Type

Solutions on page 3.66

1. Number of values of p for which equation $\sin^3 x + 1 + p^3 - 3p \sin x = 0$ $(p > 0)$ has a root is ________.
2. Number of roots of the equation $|\sin x \cos x| + \sqrt{2 + \tan^2 x + \cot^2 x} = \sqrt{3}$, $x \in [0, 4\pi]$, are ________.
3. Number of roots of the equation $(3 + \cos x)^2 = 4 - 2 \sin^8 x$, $x \in [0, 5\pi]$ are ________.
4. Number of solution(s) of the equation $\frac{\sin x}{\cos 3x} + \frac{\sin 3x}{\cos 9x} + \frac{\sin 9x}{\cos 27x} = 0$ in the interval $\left(0, \frac{\pi}{4}\right)$ is ________.
5. Number of solutions of the equation $\left(\sqrt{3}+1\right)^{2x} + \left(\sqrt{3}-1\right)^{2x} = 2^{3x}$ is ________.
6. Number of integral value(s) of m for the equation $\sin x - \sqrt{3} \cos x = \frac{4m-6}{4-m}$ has solutions $x \in [0, 2\pi]$ is ________.
7. The value of a for which system of equations $\sin^2 x + \cos^2 y = \frac{3a}{2}$ and $\cos^2 x + \sin^2 y = \frac{a^2}{2}$ has a solution is ________.
8. If $\cos 4x = a_0 + a_1 \cos^2 x + a_2 \cos^4 x$ is true for all values of $x \in R$, then the value of $5a_0 + a_1 + a_2$ is ________.
9. Number of integral values of a for which the equation $\cos^2 x - \sin x + a = 0$ has roots when $x \in (0, \pi/2)$ is ________.
10. The maximum integral value of a for which the equation $a \sin x + \cos 2x = 2a - 7$ has a solution is ________.
11. Number of roots the equation $2^{\tan\left(x - \frac{\pi}{4}\right)} - 2(0.25)^{\frac{\sin^2\left(x - \frac{\pi}{4}\right)}{\cos 2x}} + 1 = 0$ is ________.
12. Number of solution of the equation $\sin^4 x - \cos^2 x \sin x + 2 \sin^2 x + \sin x = 0$ in $0 \le x \le 3\pi$ is ________.

Archives

Solutions on page 3.69

Subjective

1. Find the coordinates of the points of intersection of the curves $y = \cos x$, $y = \sin 3x$ if $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$. **(IIT-JEE, 1982)**
2. Find all the solution of $4\cos^2 x \sin x - 2\sin^2 x = 3 \sin x$. **(IIT-JEE, 1983)**
3. Find the values of $x \in (-\pi, \pi)$ which satisfy the equation $8^{(1+|\cos x|+|\cos^2 x|+|\cos^3 x|+\ldots)} = 4^3$. **(IIT-JEE, 1984)**
4. Find all values of θ in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ satisfying the equation
   $(1 - \tan\theta)(1 + \tan\theta)\sec^2\theta + 2^{\tan^2\theta} = 0$. **(IIT-JEE, 1996)**
5. Find the number of all possible value of θ, where $0 < \theta < \pi$, for which the system of equations
$$(y+z)\cos 3\theta = (xyz)\sin 3\theta$$
$$x\sin 3\theta = \frac{2\cos 3\theta}{y} + \frac{2\sin 3\theta}{z}$$
$$(xyz)\sin 3\theta = (y+2z)\cos 3\theta + y\sin 3\theta$$
   have a solution (x_0, y_0, z_0) with $y_0 z_0 \ne 0$. **(IIT-JEE, 2010)**
6. Find the number of values of θ in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ such that $\theta \ne \frac{n\pi}{5}$ for $n = 0, \pm 1, \pm 2$ and $\tan\theta = \cot 5\theta$ as well as $\sin 2\theta = \cos 4\theta$. **(IIT-JEE, 2010)**

Objective

Fill in the blanks

1. The solution set of the system of equations $x + y = \frac{2\pi}{3}$, $\cos x + \cos y = \frac{3}{2}$, where x and y are real, is ______. **(IIT-JEE, 1986)**
2. The set of all x in the interval $[0, \pi]$ for which $2\sin^2 x - 3\sin x + 1 \ge 0$ is ______. **(IIT-JEE, 1987)**
3. General value of θ satisfying the equation $\tan^2\theta + \sec 2\theta = 1$ is ______. **(IIT-JEE, 1996)**
4. The real roots of the equation $\cos^7 x + \sin^4 x = 1$ in the interval $(-\pi, \pi)$ are ______, ______ and ______. **(IIT-JEE, 1997)**

True or false

1. There exists a value of θ between 0 and 2π that satisfies the equation $\sin^4\theta - 2\sin^2\theta - 1 = 0$. **(IIT-JEE, 1984)**

Multiple choice questions with one correct answer

1. The equation $2\cos^2\frac{x}{2}\sin^2 x = x^2 + x^{-2}$; $0 < x \le \frac{\pi}{2}$ has **(IIT-JEE, 1980)**
   a. no real solution
   b. one real solution
   c. more than one solution
   d. none of these
2. The general solution of the trigonometric equation $\sin x + \cos x = 1$ is given by **(IIT-JEE, 1981)**
   a. $x = 2n\pi$; $n = 0, \pm 1, \pm 2, \ldots$
   b. $x = 2n\pi + \pi/2$; $n = 0, \pm 1, \pm 2, \ldots$
   c. $x = n\pi + (-1)^n \frac{\pi}{4} - \frac{\pi}{4}$ $n = 0, \pm 1, \pm 2, \ldots$
   d. none of these
3. The general solution of the equation $\sin x - 3\sin 2x + \sin 3x = \cos x - 3\cos 2x + \cos 3x$ is $(n \in Z)$ **(IIT-JEE, 1989)**

a. $n\pi+\frac{\pi}{8}$ b. $\frac{n\pi}{2}+\frac{\pi}{8}$ c. $(-1)^n\frac{n\pi}{2}+\frac{\pi}{8}$ d. $2n\pi+\cos^{-1}\frac{2}{3}$

4. The equation $(\cos p-1)x^2+(\cos p)x+\sin p=0$ in the variable x has real roots. Then p can take any value in the interval (IIT-JEE, 1990)
   a. $(0,2\pi)$ b. $(-\pi,0)$ c. $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ d. $(0,\pi)$
5. The number of solutions of the equation $\tan x+\sec x=2\cos x$ lying in the interval $[0, 2\pi]$ is (IIT-JEE, 1993)
   a. 0 b. 1 c. 2 d. 3
6. The general values of θ satisfying the equation $2\sin^2\theta-3\sin\theta-2=0$ is ($n\in Z$) (IIT-JEE, 1995)
   a. $n\pi+(-1)^n\pi/6$ b. $n\pi+(-1)^n\pi/2$ c. $n\pi+(-1)^n5\pi/6$ d. $n\pi+(-1)^n7\pi/6$
7. The number of distinct real roots of $\begin{vmatrix}\sin x & \cos x & \cos x\\ \cos x & \sin x & \cos x\\ \cos x & \cos x & \sin x\end{vmatrix}=0$ in the interval $-\frac{\pi}{4}\le x\le\frac{\pi}{4}$ is (IIT-JEE, 2001)
   a. 0 b. 2 c. 1 d. 3
8. The number of integral values of k for which the equation $7\cos x+5\sin x=2k+1$ has a solution is
   a. 4 b. 8 c. 10 d. 12 (IIT-JEE, 2002)
9. $\cos(\alpha-\beta)=1$ and $\cos(\alpha+\beta)=1/e$ where $\alpha,\beta\in[-\pi,\pi]$. Pairs of α,β which satisfy both the equations is/are (IIT-JEE, 2005)
   a. 0 b. 1 c. 2 d. 4
10. The value of $\theta\in(0,2\pi)$ for which the equation is $2\sin^2\theta-5\sin\theta+2>0$ is (IIT-JEE, 2006)
   a. $\left(0,\frac{\pi}{6}\right)\cup\left(\frac{5\pi}{6},2\pi\right)$ b. $\left(\frac{\pi}{8},\frac{5\pi}{6}\right)$ c. $\left(0,\frac{\pi}{8}\right)\cup\left(\frac{\pi}{6},\frac{5\pi}{6}\right)$ d. $\left(\frac{41\pi}{48},\pi\right)$
11. The number of solutions of the pair of equations (IIT-JEE, 2007)
   $2\sin^2\theta-\cos2\theta=0$
   $2\cos^2\theta-3\sin\theta=0$
   in the interval $[0, 2\pi]$ is
   a. 0 b. 1 c. 2 d. 4

Multiple choice questions with one or more than one correct answers

1. The number of all the possible triplets (a_1, a_2, a_3) such that $a_1+a_2\cos(2x)+a_3\sin^2(x)=0$ for all x is (IIT-JEE, 1987)
   a. 0 b. 1 c. 3 d. infinite
2. The values of θ lying between $\theta=0$ and $\theta=\theta/2$ and satisfying the equation
   $\begin{vmatrix}1+\sin^2\theta & \cos^2\theta & 4\sin4\theta\\ \sin^2\theta & 1+\cos^2\theta & 4\sin4\theta\\ \sin^2\theta & \cos^2\theta & 1+4\sin4\theta\end{vmatrix}=0$ are (IIT-JEE, 1988)
   a. $7\pi/24$ b. $5\pi/24$ c. $11\pi/24$ d. $\pi/24$
3. The number of values of x in the interval $[0, 5\pi]$ satisfying the equation $3\sin^2x-7\sin x+2=0$ is
   a. 0 b. 5 c. 6 d. 10 (IIT-JEE, 1998)
4. Let $2\sin^2x+3\sin x-2>0$ and $x^2-x-2<0$ (x is measured in radians). Then x lies in the interval
   a. $\left(\frac{\pi}{6},\frac{5\pi}{6}\right)$ b. $\left(-1,\frac{5\pi}{6}\right)$ c. $(-1,2)$ d. $\left(\frac{\pi}{6},2\right)$
   (IIT-JEE, 1994)

5. If $\dfrac{\sin^4 x}{2}+\dfrac{\cos^4 x}{3}=\dfrac{1}{5}$, then

a. $\tan^2 x=\dfrac{2}{3}$

b. $\dfrac{\sin^8 x}{8}+\dfrac{\cos^8 x}{27}=\dfrac{1}{125}$

c. $\tan^2 x=\dfrac{1}{3}$

d. $\dfrac{\sin^8 x}{8}+\dfrac{\cos^8 x}{27}=\dfrac{2}{125}$

(IIT-JEE, 2009)

ANSWERS AND SOLUTIONS

Subjective Type

1. We have 3 $(\tan 2x-\tan 3x)=\tan 3x\,(1+\tan 3x\tan 2x)$

$\Rightarrow\ 3(\tan 2x-\tan 3x)/(1+\tan 3x\tan 2x)=\tan 3x$

$\Rightarrow\ 3\tan(2x-3x)=\tan 3x$

$\Rightarrow\ 3\tan x+(3\tan x-\tan^3 x)/(1-3\tan^2 x)=0$

$\Rightarrow\ \tan x\,[3(1-3\tan^2 x)+3-\tan^2 x]=0$

$\Rightarrow\ \tan x\,(6-10\tan^2 x)=0$

$\Rightarrow\ \tan x=0$ or $\tan^2 x=3/5$

If $\tan x=0 \quad\Rightarrow\quad x=n\pi, n\in Z$

and if $\tan^2 x=3/5 \quad\Rightarrow\quad x=m\pi\pm\alpha=m\pi\pm\tan^{-1}\sqrt{3/5}$, $m\in Z$

Hence, $x=n\pi,\ m\pi\pm\tan^{-1}\sqrt{3/5}$, $\forall\ m, n\in Z$.

2. The given equation can be rewritten as $2[\sin(2x+\pi/6)+\sin\pi/2]=a^2+\sqrt{3}\sin 2x-\cos 2x$

$\Rightarrow\ \cos 2x=(a^2-2)/2$

$\Rightarrow\ 2\cos^2 x=a^2/2$ or $\cos^2 x=(a/2)^2$ (i)

$\Rightarrow\ a^2\le 4$ or $-2\le a\le 2$ (ii)

For $a=0$, the given equation is reduced to

$\cos x=0$, i.e., $x=n\pi+(\pi/2), n\in Z$ (iii)

3. $\sin^4(x/3)+\cos^4(x/3)>1/2\ (n\in Z)$

$\Rightarrow\ 1-2\sin^2(x/3)\cos^2(x/3)>1/2$

$\Rightarrow\ 1-\dfrac{1}{2}\sin^2(2x/3)>\dfrac{1}{2}$

$\Rightarrow\ \sin^2(2x/3)<1$

which is always true except when $\sin^2(2x/3)=1$

This means $2x/3=n\pi\pm(\pi/2)$ or $x=(3n\pi/2)\pm(3\pi/4)$, $n\in Z$

Hence, solution set of the inequality is $R-\{x: x=(3n\pi/2)\pm(3\pi/4), n\in Z\}$.

4. $\sin x+\sin y=\sin(x+y)$

$\Rightarrow\ 2\sin\dfrac{x+y}{2}\left[\cos\dfrac{x-y}{2}-\cos\dfrac{x+y}{2}\right]=0$

$\Rightarrow\ 4\sin\dfrac{x+y}{2}\sin\dfrac{x}{2}\sin\dfrac{y}{2}=0$

a. $\sin\dfrac{x+y}{2}=0 \quad\Rightarrow\quad x+y=2n\pi, n\in Z \quad\Rightarrow\quad x+y=0\ (\because |x|+|y|=1\Rightarrow -1\le x, y\le 1)$

b. $\sin\frac{x}{2}=0 \quad\Rightarrow\quad x=2m\pi, m\in Z \quad\Rightarrow\quad x=0$

c. $\sin\frac{y}{2}=0 \quad\Rightarrow\quad y=2p\pi, p\in Z \quad\Rightarrow\quad y=0$

From $|x|+|y|=1$

If $x=0$, then $|y|=1 \quad\Rightarrow\quad y=\pm 1$

If $y=0$, then $|x|=1 \quad\Rightarrow\quad x=\pm 1$

If $y=-x$, then $|x|+|-x|=2 \quad\Rightarrow\quad x=\pm\frac{1}{2}$ and $y=\mp\frac{1}{2}$

Hence, solutions are $(0, 1), (0, -1), (1, 0), (-1, 0), \left(\frac{1}{2}, -\frac{1}{2}\right)$ and $\left(-\frac{1}{2}, \frac{1}{2}\right)$.

5. $\tan^4 x+\tan^4 y+2\cot^2 x\cot^2 y=3+\sin^2(x+y)$

$\Rightarrow\quad \tan^4 x+\tan^4 y+2\cot^2 x\cot^2 y-2=1+\sin^2(x+y)$

$\Rightarrow\quad (\tan^2 x-\tan^2 y)^2+2(\tan x\tan y-\cot x\cot y)^2=-1+\sin^2(x+y)$

Now L.H.S. ≥ 0 and R.H.S. ≤ 0

$\Rightarrow\quad$ L.H.S. = R.H.S. = 0

$\Rightarrow\quad \tan^2 x=\tan^2 y$ and $\tan^2 x\tan^2 y=1$ and $\sin^2(x+y)=0$

$\qquad = \tan^2 y=1$ and $x+y=n\pi, n\in Z$

$\Rightarrow\quad x=n\pi\pm\frac{\pi}{4}, n\in Z$ and $y=n\pi\mp\frac{\pi}{4}, n\in Z$.

6. The given equation is possible if $\sin(1-x)\geq 0$ and $\cos x\geq 0$.

On squaring, we get $\sin(1-x)=\cos x$

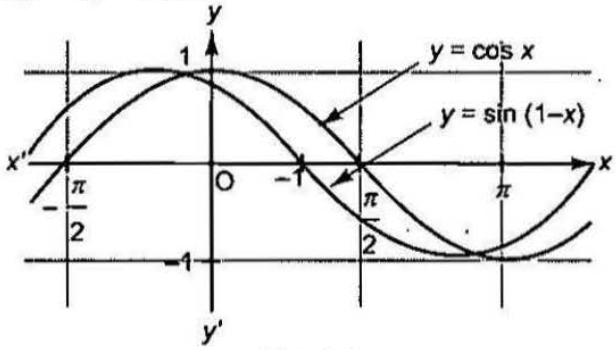


Fig. 3.6

$\Rightarrow\quad \cos\left(\frac{\pi}{2}-(1-x)\right)=\cos x$

$\Rightarrow\quad \frac{\pi}{2}-1+x=2n\pi\pm x, n\in Z$

$\Rightarrow\quad x=\dfrac{2n\pi-\frac{\pi}{2}+1}{2}, n\in Z$

For $n=2$, $x=\frac{7\pi}{4}+\frac{1}{2}$ which is the smallest positive root of the given equation.

7. $2\sin x+\cos y=2$

$\Rightarrow\quad \cos y=2(1-\sin x)$, we have $\cos y\in[-1, 1]$

$\Rightarrow\quad -\frac{1}{2}\leq 1-\sin x\leq\frac{1}{2} \quad\Rightarrow\quad \frac{1}{2}\leq\sin x\leq\frac{3}{2} \quad\Rightarrow\quad \frac{1}{2}\leq\sin x\leq 1$

Let $t = \sin x \quad \Rightarrow \quad x = \sin^{-1}(t), t \in [1/2, 1]$

$$\Rightarrow \quad \cos y = 2(1-t) \quad \Rightarrow \quad \left.\begin{array}{l} y = 2n_1\pi \pm \cos^{-1} 2(1-t) \\ \text{and } x = n_2\pi + (-1)^{n_2} \sin^{-1}(t) \end{array}\right\} t \in [1/2, 1]$$

8. $\sin x = \dfrac{1}{2}|x| + \dfrac{a}{2}$ or $2\sin x = |x| + a$. Consider graphs of $y = 2\sin x$ and $y = |x|$.

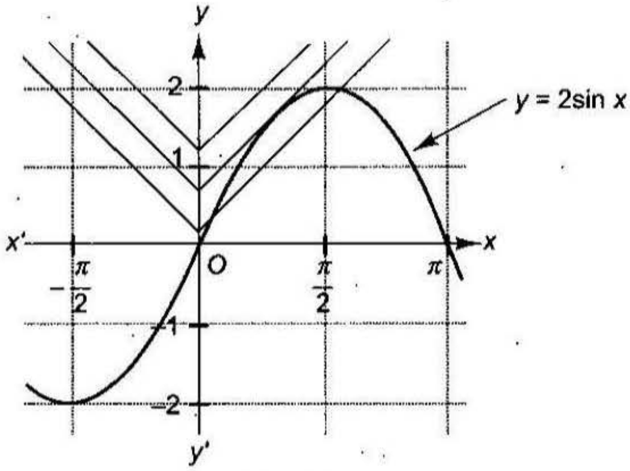


Fig. 3.7

Equation $2\sin x = |x| + a$ will have a solution so long as the line $y = |x| + a$ intersects or at least touches the curve, $y = 2\sin x$. In this case, we must have $dy/dx = 2\cos x = 1$ = the slope of the line $\Rightarrow \quad x = \pi/3$.

Hence, the solution exists if $\dfrac{\pi}{3} + a > 2\sin\dfrac{\pi}{3} \Rightarrow a > \dfrac{3\sqrt{3} - \pi}{3}$

9. $\tan\left(\dfrac{\pi}{2}\cos\theta\right) = \cot\left(\dfrac{\pi}{2}\sin\theta\right) = \tan\left(\dfrac{\pi}{2} - \dfrac{\pi}{2}\sin\theta\right)$

$\Rightarrow \quad \dfrac{\pi}{2}\cos\theta = n\pi + \dfrac{\pi}{2} - \dfrac{\pi}{2}\sin\theta,\ n \in Z$

$\Rightarrow \quad \dfrac{\pi}{2}(\sin\theta + \cos\theta) = n\pi + \dfrac{\pi}{2} = \left(n + \dfrac{1}{2}\right)\pi$

$\Rightarrow \quad \sin\theta + \cos\theta = (2n+1)$

$\Rightarrow \quad \sqrt{2}\sin\left(\theta + \dfrac{\pi}{4}\right) = (2n+1)$

$\Rightarrow \quad n = 0, -1$ are the only possibilities

So, $\sin\left(\theta + \dfrac{\pi}{4}\right) = \pm\dfrac{1}{\sqrt{2}} = \sin\left(\pm\dfrac{\pi}{4}\right)$

$\Rightarrow \quad \theta + \dfrac{\pi}{4} = m\dfrac{\pi}{2} + \dfrac{\pi}{4}, m \in Z$

$\Rightarrow \quad \theta = m\dfrac{\pi}{2},\ m \in Z$

However, for the values of $m = 2k$, $k \in Z$, the equation is not defined.

Hence, $\theta = (2k+1)\dfrac{\pi}{2}$, where $k \in Z$.

10. $\sin x+\sin\left(\frac{\pi}{8}\sqrt{(1-\cos 2x)^2+\sin^2 2x}\right)=0$

$$\sin\left(\frac{\pi}{8}\sqrt{(1-\cos 2x)^2+\sin^2 2x}\right)=\sin\frac{\pi}{8}\sqrt{2-2\cos 2x}=\sin\left(\frac{\pi}{4}|\sin x|\right)$$

Now $\sin x+\sin\left(\frac{\pi}{4}|\sin x|\right)=0$ (i)

The equation has a solution only when $\sin x\le 0$.
The graph of $f(x)=\sin x\le 0$ is shown in Fig. 3.8.

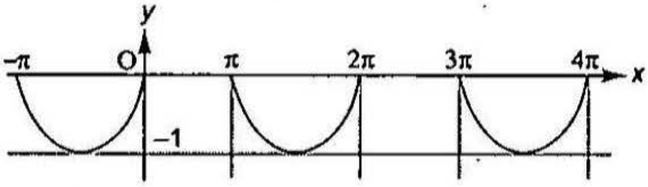


Fig. 3.8

The graph $y=\sin[\pi/4\ |\sin x|]$ is as shown in Fig. 3.9.

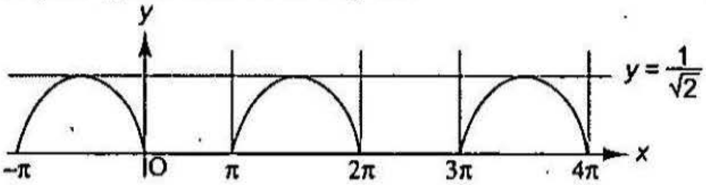


Fig. 3.9

Hence, Eq. (i) has general solution $x=n\pi,\ n\in Z$.

11. $\sin^2 x+\frac{1}{4}\sin^2 3x=\sin x\sin^2 3x$

$$\Rightarrow\quad \sin^2 x-\sin x\sin^2 3x+\frac{1}{4}\sin^2 3x=0$$

$$\Rightarrow\quad \left(\sin x-\frac{1}{2}\sin^2 3x\right)^2+\frac{1}{4}\sin^2 3x\left(1-\sin^2 3x\right)=0$$

$$\Rightarrow\quad \left(\sin x-\frac{1}{2}\sin^2 3x\right)^2+\frac{1}{4}\sin^2 3x\cos^2 3x=0$$

$$\Rightarrow\quad \left(\sin x-\frac{1}{2}\sin^2 3x\right)^2+\frac{1}{16}\sin^2 6x=0$$

$$\Rightarrow\quad \sin x-\frac{1}{2}\sin^2 3x=0 \text{ and } \sin 6x=0$$

$\Rightarrow$ $2\sin x=\sin^2 3x$ and $\sin 6x=0$ $\Rightarrow$ From $\sin 6x=0,\ x=k\pi/6,\ k\in Z$

From here, we choose those values which satisfy the equation, $2\sin x=\sin^2 3x$

Now $\sin^2 3\left(\frac{k\pi}{6}\right)=+\sin^2\frac{k\pi}{2}=\begin{cases}1, & \text{if } k \text{ is odd}\\ 0, & \text{if } k \text{ is even}\end{cases}$

$\Rightarrow$ $\sin x=0$ or $1/2$

$$\Rightarrow\quad x=n\pi \text{ or } x=n\pi+\frac{\pi}{6}(-1)^n,\ n\in Z$$

Objective Type

1. a. $\sin\theta = 1/2$ and $\cos\theta = -\sqrt{3}/2$

$\Rightarrow$ θ lies in the second quadrant.

$\Rightarrow$ $\sin\theta = \sin 5\pi/6$; $\cos\theta = \cos 5\pi/6$;

$\therefore\ \theta = 2n\pi + (5\pi/6)$

2. c. Since $\tan\theta < 0$ and $\cos\theta > 0$, θ lies in the fourth quadrant. Then $\theta = 7\pi/4$. Hence, the general value of θ is $2n\pi + 7\pi/4$, $n \in Z$.

3. c. $\cos p\theta = -\cos q\theta = \cos(\pi - q\theta)$

$\Rightarrow$ $p\theta = 2n\pi \pm (\pi - q\theta)$

$\Rightarrow$ $(p \mp q)\,\theta = (2n \pm 1)\,\pi$

$$\Rightarrow \theta = \frac{(2n \pm 1)\pi}{(p \mp q)},\ n \in Z$$

$$\Rightarrow \theta = \frac{r\pi}{p \pm q},\ \text{where } r = -3, -1, 1, 3, \ldots$$

$$\Rightarrow \theta = \cdots, \frac{-3\pi}{p \pm q}, \frac{-\pi}{p \pm q}, \frac{\pi}{p \pm q}, \frac{3\pi}{p \pm q}, \cdots$$

Shown above is an A.P. of common difference $\dfrac{2\pi}{p \pm q}$.

4. d. $(\cos\theta + \cos 7\theta) + (\cos 3\theta + \cos 5\theta) = 0$

$\Rightarrow$ $2\cos 4\theta(\cos 3\theta + \cos\theta) = 0$

$\Rightarrow$ $4\cos 4\theta \cos 2\theta \cos\theta = 0$

$$\Rightarrow 4 \times \frac{1}{2^3 \sin\theta}\left(\sin 2^3\,\theta\right) = 0$$

$\Rightarrow$ $\sin 8\theta = 0$ or $\theta = n\pi/8$, $n \in Z$

5. b. $3\dfrac{\sin^2\theta}{\cos^2\theta} - 2\sin\theta = 0$, $\cos\theta \neq 0$

$\Rightarrow$ $3\sin^2\theta - 2\sin\theta(1 - \sin^2\theta) = 0$

$\Rightarrow$ $\sin\theta(2\sin^2\theta + 3\sin\theta - 2) = 0$

$\Rightarrow$ $\sin\theta(2\sin\theta - 1)(\sin\theta + 2) = 0$

$\Rightarrow$ $\sin\theta = 0, 1$

$\Rightarrow$ $\theta = n\pi,\ n\pi + (-1)^n(\pi/6)$, $n \in Z$

6. b. We have $1^2 = \sin\theta\cos 2\theta$

$\Rightarrow$ $1 - \sin\theta(1 - 2\sin^2\theta) = 0$

$\Rightarrow$ $2\sin^3\theta - \sin\theta + 1 = 0$

$\Rightarrow$ $(\sin\theta + 1)(2\sin^2\theta - 2\sin\theta + 1) = 0$

$\Rightarrow$ $\sin\theta = -1$

The other factor gives imaginary roots.

$\Rightarrow \quad \theta = n\pi + (-1)^n\left(-\dfrac{\pi}{2}\right) = n\pi - (-1)^n\dfrac{\pi}{2} = n\pi + (-1)^{n-1}\dfrac{\pi}{2}, n \in Z.$

7. b. $2\cos\theta[\cos 120° + \cos 2\theta] = 1$

$\Rightarrow \quad 2\cos\theta\left(-\dfrac{1}{2} + 2\cos^2\theta - 1\right) = 1$

$\Rightarrow \quad 4\cos^3\theta - 3\cos\theta - 1 = 0$

$\Rightarrow \quad \cos 3\theta = 1 = \cos 0$

$\Rightarrow \quad 3\theta = 2n\pi$ or $\theta = \dfrac{2n\pi}{3}, n \in Z$

Given the values so that $2n$ does not exceed 18.

$\therefore \quad n = 0, 1, 2, 3, ..., 9$

Hence, the sum $= \dfrac{2\pi}{3}\sum_{1}^{9} n = \dfrac{2\pi}{3} \times \dfrac{9(9+1)}{2} = 30\pi.$

8. b. $\sec\theta - 1 = (\sqrt{2} - 1)\tan\theta \Rightarrow \dfrac{1-\cos\theta}{\cos\theta} = \dfrac{(\sqrt{2}-1)\sin\theta}{\cos\theta}$

$\Rightarrow \quad 2\sin^2(\theta/2) = (\sqrt{2} - 1)\, 2\sin(\theta/2)\cos(\theta/2)$

$\Rightarrow \quad \sin(\theta/2) = 0$ or $\tan(\theta/2) = (\sqrt{2} - 1) = \tan(\pi/8)$

$\Rightarrow \quad \theta/2 = n\pi$ or $\theta/2 = n\pi + (\pi/8), n \in Z$

$\Rightarrow \quad \theta = 2n\pi$ or $\theta = 2n\pi + (\pi/4), n \in Z$

9. a. $\sin^4 x + \cos^4 x = \sin x\cos x$

$\Rightarrow \quad (\sin^2 x + \cos^2 x)^2 - 2\sin^2 x\cos^2 x = \sin x\cos x$

$\Rightarrow \quad 1 - \dfrac{\sin^2 2x}{2} = \dfrac{\sin 2x}{2}$

$\Rightarrow \quad \sin^2 2x + \sin 2x - 2 = 0$

$\Rightarrow \quad (\sin 2x + 2)(\sin 2x - 1) = 0$

$\Rightarrow \quad \sin 2x = 1$

$\Rightarrow \quad 2x = (4n+1)\dfrac{\pi}{2}, n \in Z$

$\Rightarrow \quad x = (4n+1)\dfrac{\pi}{4}, n \in Z$

$\quad = \dfrac{\pi}{4}, \dfrac{5\pi}{4} \quad (\because x \in [0, 2\pi])$

Thus, there are two solutions.

10. c. $\sin 3x + (\sin 5x + \sin x) = 0$

$\Rightarrow \quad \sin 3x + (2\sin 3x\cos 2x) = 0$

$\Rightarrow \quad \sin 3x = 0$ or $\cos 2x = -\dfrac{1}{2} = \cos\dfrac{2\pi}{3}$

$\Rightarrow \quad x = n\pi/3$ or $x = n\pi \pm \pi/3, n \in Z$

Then $x = 0, \pi/3$, and $2\pi/3$, Hence, there are three solutions.

11. a. From the given relation

$\cos\theta = (2\sin\theta\cos\theta)\sin\theta,\ \sin\theta \neq 0$

$\Rightarrow \quad \sin\theta = \pm\dfrac{1}{\sqrt{2}}$ or $\cos\theta = 0$

$\Rightarrow \quad \theta = \dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{\pi}{2}\ (\because \theta \in [0, \pi])$

Then the sum of roots is $\dfrac{3\pi}{2}$.

12. c. The given equation is $(\cos x - 1)(12\cos^2 x + 5\cos x + 9) = 0$

$\Rightarrow \quad \cos x = 1$ only as the other factor gives imaginary roots

$= 1 \Rightarrow x = 2n\pi, n \in Z$

Hence, it has infinite solutions as $n \in Z$.

13. d. $\cos 2x = 1 - 2\sin^2 x$ and put $2^{-\sin^2 x} = t$

$\Rightarrow \quad 2^{\cos 2x} = 2^{1-2\sin^2 x} = 2\left(2^{-\sin^2 x}\right)^2 = 2t^2$

$\Rightarrow \quad 2t^2 - 3t + 1 = 0$

$\Rightarrow \quad t = 1, 1/2$

$\Rightarrow \quad 2^{-\sin^2 x} = 1 = 2^0$

$\Rightarrow \quad \sin^2 x = 0$ or $x = n\pi, n \in Z$

From $2^{-\sin^2 x} = \dfrac{1}{2} = 2^{-1}$, we get

$\sin^2 x = 1$ or $x = n\pi \pm \dfrac{\pi}{2}, n \in Z$

14. a. $\cos x \cos 6x = -1$

$\Rightarrow \quad 2\cos x\cos 6x = -2 \quad \Rightarrow \quad \cos 7x + \cos 5x = -2 \quad \Rightarrow \quad \cos 7x = -1$ and $\cos 5x = -1$

The value of x satisfying these two equations simultaneously and lying between 0 and 2π is π.

Therefore, the general solution is $x = 2n\pi + \pi, n \in Z$.

$\Rightarrow \quad x = (2n+1)\pi, n \in Z$

15. c. This is possible only when $\sin x = \cos x = 1$, which does not hold simultaneously.

Hence, there is no solution.

16. a. The given equation is $3(\sin x + \cos x) - 2(\sin x + \cos x)(1 - \sin x\cos x) = 8$

$\Rightarrow \quad (\sin x + \cos x)[3 - 2 + 2\sin x\cos x] = 8$

$\Rightarrow \quad (\sin x + \cos x)[\sin^2 x + \cos^2 x + 2\sin x\cos x] = 8$

$\Rightarrow \quad (\sin x + \cos x)^3 = 8$

$\Rightarrow \quad \sin x + \cos x = 2$

Above solution is not possible. Hence, the given equation has no solution.

17. a. $\cos^2\theta = \dfrac{1}{6}\sin\theta\tan\theta$

$\Rightarrow \quad 6\cos^3\theta = 1 - \cos^2\theta$

$\Rightarrow \quad 6\cos^3\theta + \cos^2\theta - 1 = 0$

$\Rightarrow \quad (2\cos\theta - 1)(3\cos^2\theta + 2\cos\theta + 1) = 0$

$\Rightarrow \quad \cos\theta = \frac{1}{2} \Rightarrow \theta = 2n\pi \pm \frac{\pi}{3}, n \in Z$

The other factor gives imaginary roots.

18. d. $(1-\cos 2x) + (1-\cos^2 2x) = 2$

$\Rightarrow \quad \cos 2x(\cos 2x + 1) = 0$

$\Rightarrow \quad \cos 2x = 0$ or $\cos 2x = -1$

$\Rightarrow \quad 2x = (2n+1)\pi/2$ or $2x = (2n \pm 1)\pi, n \in Z$

$\Rightarrow \quad x = (2n+1)\pi/4$ or $x = (2n \pm 1)\pi/2, n \in Z$

Hence, the solutions are $\pi/4, 3\pi/4, 5\pi/4, 7\pi/4, \pi/2, 3\pi/2$.

19. c. Dividing the given equation by $\cos^2 x$, as $\cos x = 0$ does not satisfy the equiation, we have

$\tan^2 x - 5\tan x - 6 = 0$

$\Rightarrow \quad (\tan x + 1)(\tan x - 6) = 0$

$\Rightarrow \quad \tan x = -1$ or $\tan x = 6$

If $\tan x = -1 = \tan(-\pi/4)$, then $x = n\pi - \pi/4, \forall n \in Z$

and, if $\tan x = 6 = \tan\alpha$ (say)

$\Rightarrow \quad \alpha = \tan^{-1} 6$, then $x = n\pi + \alpha = n\pi + \tan^{-1} 6, \forall n \in Z$

Hence, $x = n\pi - (\pi/4), n\pi + \tan^{-1} 6, n \in Z$.

20. d. From the given equation, we have $\frac{\tan\theta + \tan 4\theta}{1 - \tan\theta \tan 4\theta} = -\tan 7\theta$

$\Rightarrow \quad \tan(\theta + 4\theta) = -\tan 7\theta$

$\Rightarrow \quad \tan 5\theta = \tan(-7\theta)$

$\Rightarrow \quad 5\theta = n\pi - 7\theta$

$\Rightarrow \quad \theta = n\pi/12$, where $n \in Z$, but $n \neq 6, 18, 30, \ldots$

21. d. We have $\frac{1}{\sin^2\theta\cos^2\theta} + \frac{2}{\sin^2\theta} = 8$, $\sin\theta \neq 0$, $\cos\theta \neq 0$

$\Rightarrow \quad 1 + 2\cos^2\theta = 8\sin^2\theta\cos^2\theta = 8\cos^2\theta(1-\cos^2\theta)$

$\Rightarrow \quad 8\cos^4\theta - 6\cos^2\theta + 1 = 0$

$\Rightarrow \quad (4\cos^2\theta - 1)(2\cos^2\theta - 1) = 0$

$\Rightarrow \quad \cos^2\theta = 1/4 = \cos^2(\pi/3)$ or $\cos^2\theta = 1/2 = \cos^2(\pi/4)$

$\Rightarrow \quad \theta = n\pi \pm (\pi/3)$ or $\theta = n\pi \pm (\pi/4), n \in Z$

Hence, for $0 \le \theta \le \pi/2$, $\theta = \pi/3$, $\theta = \pi/4$

22. b. $\tan x + \cot x = 2\operatorname{cosec} x$

$\Rightarrow \quad \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} = \frac{2}{\sin x}$

$\Rightarrow \quad \frac{1}{\sin x\cos x} = \frac{2}{\sin x}$

$\Rightarrow \quad \cos x = \frac{1}{2}$

$\Rightarrow \quad x = \pm\frac{\pi}{3}, \pm\frac{5\pi}{3}$

Thus, there are four solutions.

23. c. Let $Z = \frac{3+2i\sin\theta}{1-2i\sin\theta} = \frac{(3+2i\sin\theta)(1+2i\sin\theta)}{(1-2i\sin\theta)(1+2i\sin\theta)} = \frac{(3-4\sin^2\theta)+8i\sin\theta}{1+4\sin^2\theta}$

Therefore, the real part of $Z = \frac{3-4\sin^2\theta}{1+4\sin^2\theta}$ and the imaginary part of $Z = \frac{8\sin\theta}{1+4\sin^2\theta}$

Z is real, if imaginary part $= \frac{8\sin\theta}{1+4\sin^2\theta} = 0$ or $\sin\theta = 0$ or $\theta = n\pi, \forall n \in I$

Z is purely imaginary, if real part $(3-4\sin^2\theta)/(1+4\sin^2\theta) = 0$
or $\sin^2\theta = 3/4 = \sin^2(\pi/3)$ or $\theta = n\pi \pm \pi/3, \forall n \in I$

24.b. $1-\sin^2 x + \frac{\sqrt{3}+1}{2}\sin x - \frac{\sqrt{3}}{4} - 1 = 0$

$\Rightarrow \sin^2 x - \frac{\sqrt{3}+1}{2}\sin x + \frac{\sqrt{3}}{4} = 0$

$\Rightarrow 4\sin^2 x - 2\sqrt{3}\sin x - 2\sin x + \sqrt{3} = 0$

On solving, we get $\sin x = 1/2, \sqrt{3}/2$

$\Rightarrow x = \pi/6, 5\pi/6; \pi/3, 2\pi/3$

25. d. Since, $7\cos^2 x + \sin x\cos x - 3 = 0$,
Dividing the equation by $\cos^2 x$, we get
$7 + \tan x - 3\sec^2 x = 0$

$\Rightarrow \quad 7 + \tan x - 3(1+\tan^2 x) = 0$

$\Rightarrow \quad 3\tan^2 x - \tan x - 4 = 0$

$\Rightarrow \quad (\tan x + 1)(3\tan x - 4) = 0$

$\Rightarrow \quad \tan x = -1$ or $\tan x = \frac{4}{3}$

$\Rightarrow \quad x = n\pi + \frac{3\pi}{4}$ or $x = k\pi + \tan^{-1}\left(\frac{4}{3}\right)$, where $(k, n \in Z)$

26. b. $(\sin\theta+2)(\sin\theta+3)(\sin\theta+4) = 6$

L.H.S. > 6 and R.H.S. 6

Therefore, equality only holds if $\sin\theta = -1 \Rightarrow \theta = 3\pi/2, 7\pi/2$

Therefore, sum $= 5\pi \Rightarrow k = 5$

27. b. $\sin^2 x + a\cos x + a^2 > 1 + \cos x$

Putting $x = 0$, we get

$\Rightarrow \quad a + a^2 > 2$

$\Rightarrow \quad a^2 + a - 2 > 0$

$\Rightarrow \quad (a+2)(a-1) > 0$

$\Rightarrow \quad a < -2$ or $a > 1$

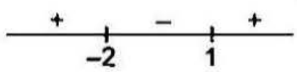


Fig. 3.10

Therefore, we have the largest negative integral value of $a = -3$.

28. b. $\sin^4 x - \cos^2 x \sin x + 2\sin^2 x + \sin x = 0$

$\Rightarrow \quad \sin x\,[\sin^3 x - \cos^2 x + 2\sin x + 1] = 0$

$\Rightarrow \quad \sin x\,[\sin^3 x - 1 + \sin^2 x + 2\sin x + 1] = 0$

$\Rightarrow \quad \sin x\,[\sin^3 x + \sin^2 x + 2\sin x] = 0$

$\Rightarrow \quad \sin^2 x = 0$ or $\sin^2 x + \sin x + 2 = 0$ (not possible for real x)

$\Rightarrow \quad \sin x = 0$

Hence, the solutions are $x = 0, \pi, 2\pi, 3\pi$.

29. a. Since, $x \in [0, 2\pi]$ and $y \in [0, 2\pi]$,
and $\sin x + \sin y = 2$
This is possible only, when $\sin x = 1$ and $\sin y = 1$
$\Rightarrow \quad x = \pi/2$ and $y = \pi/2$
Hence, $x + y = \pi$.

30. a. $(\sqrt{3} - 1)\sin\theta + (\sqrt{3} + 1)\cos\theta = 2$

$$\Rightarrow \quad \frac{(\sqrt{3}-1)}{2\sqrt{2}}\sin\theta + \left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)\cos\theta = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad \sin\frac{\pi}{12}\sin\theta + \cos\frac{\pi}{12}\cos\theta = \cos\frac{\pi}{4}$$

$$\Rightarrow \quad \cos\left(\theta - \frac{\pi}{12}\right) = \cos\frac{\pi}{4}$$

$$\Rightarrow \quad \theta - \frac{\pi}{12} = 2n\pi \pm \frac{\pi}{4}, n \in Z$$

$$= 2n\pi \pm \frac{\pi}{4} + \frac{\pi}{12}$$

31. a. $\sin 6\theta + \sin 4\theta + \sin 2\theta = 0$

$\Rightarrow \quad (\sin 6\theta + \sin 2\theta) + \sin 4\theta = 0$

$\Rightarrow \quad 2\sin 4\theta \cos 2\theta + \sin 4\theta = 0$

$\Rightarrow \quad \sin 4\theta\,(2\cos 2\theta + 1) = 0$

$$\Rightarrow \quad \sin 4\theta = 0 \text{ or } \cos 2\theta = -\frac{1}{2} = \cos\frac{2\pi}{3}$$

$$\Rightarrow \quad 4\theta = n\pi \text{ or } 2\theta = 2n\pi \pm \frac{2\pi}{3}, n \in Z$$

$$\Rightarrow \quad \theta = \frac{n\pi}{4} \text{ or } \theta = n\pi \pm \frac{\pi}{3}$$

32. d. The given expression $\cos y \sin x - \sin y \cos x + \sin y \sin x + \cos x \cos y$ is
$\sin(x-y) + \cos(x-y)$
$\therefore \sin(x-y) + \cos(x-y) = 0$

$$\Rightarrow \quad \sqrt{2}\left(\frac{1}{\sqrt{2}}\sin(x-y) + \frac{1}{\sqrt{2}}\cos(x-y)\right) = 0$$

$$\Rightarrow \quad \sin\left(x - y + \frac{\pi}{4}\right) = 0$$

$$\Rightarrow \quad \frac{\pi}{4} + x - y = n\pi, n \in Z$$

$$\Rightarrow \quad x - y = n\pi - \frac{\pi}{4}, n \in Z$$

$$\Rightarrow \quad x = n\pi - \frac{\pi}{4} + y \text{ where } n \in Z$$

33.c. $\tan x \tan 4x = 1$

$\Rightarrow \cos 4x \cos x - \sin 4x \sin x = 0$

$\Rightarrow \cos 5x = 0$

$$\Rightarrow 5x = \left(n + \frac{1}{2}\right)\pi, n \in Z$$

$$\Rightarrow x = \frac{(2n+1)}{10}\pi, 0 < x < \pi$$

$$= \frac{\pi}{10}, \frac{3\pi}{10}, \frac{5\pi}{10}, \frac{7\pi}{10}, \frac{9\pi}{10}$$

Thus, there are only five solutions.

34. a. Let $f(x) = \cos x - x + \frac{1}{2}$

$$f(0) = 1 + \frac{1}{2} > 0$$

$$f\left(\frac{\pi}{2}\right) = 0 - \frac{\pi}{2} + \frac{1}{2} = \frac{1-\pi}{2} < 0$$

Therefore, one root lies in the interval $\left(0, \frac{\pi}{2}\right)$.

35. d. $\tan\left(\frac{p\pi}{4}\right) = \cot\left(\frac{q\pi}{4}\right) = \tan\left(\frac{\pi}{2} - \frac{q\pi}{4}\right)$

$$\Rightarrow \quad \frac{p\pi}{4} = n\pi + \frac{\pi}{2} - \frac{q\pi}{4}$$

$$\Rightarrow \quad \frac{p}{4} = n + \frac{1}{2} - \frac{q}{4}$$

$$\Rightarrow \quad \frac{p+q}{4} = \frac{2n+1}{2}$$

$$\Rightarrow \quad p + q = 2(2n+1)$$

36. b. $y = \sin x - \cos x = \sqrt{2}\left[\frac{1}{\sqrt{2}} \sin - \frac{1}{\sqrt{2}} \cos x\right]$

$$= \sqrt{2} \sin\left(x - \frac{\pi}{4}\right) \quad \Rightarrow \quad -\sqrt{2} \le y \le \sqrt{2} \quad \Rightarrow \quad \text{Range of } y \text{ is } [-\sqrt{2}, \sqrt{2}]$$

37. a. $4\sin^4 x + \cos^4 x = 1$

$$\Rightarrow \quad (2\sin^2 x)^2 + \frac{1}{4}(2\cos^2 x)^2 = 1$$

$$\Rightarrow \quad (1 - \cos 2x)^2 + \frac{1}{4}(1 + \cos 2x)^2 = 1$$

$$\Rightarrow \quad 5\cos^2 2x - 6\cos 2x + 1 = 0$$

$$\Rightarrow \quad (\cos 2x - 1)(5\cos 2x - 1) = 0$$

$\Rightarrow$ $\cos 2x = 1$ or $\cos 2x = 1/5$

$\Rightarrow$ $2x = 2n\pi$ or $2x = 2n\pi \pm \alpha$, where $\alpha = \cos^{-1}(1/5)$, $\forall\, n \in Z$

38. c. $(1 - \tan\theta)[1 + 2\tan\theta/(1 + \tan^2\theta)] = 1 + \tan\theta$

$\Rightarrow$ $(1 - \tan\theta)(1 + \tan\theta)^2 = (1 + \tan\theta)(1 + \tan^2\theta)$

$\Rightarrow$ $(1 + \tan\theta)[(1 - \tan^2\theta) - (1 + \tan^2\theta)] = 0$

$\Rightarrow$ $-2\tan^2\theta = 0$, $(1 + \tan\theta) = 0$

$\Rightarrow$ $\tan\theta = 0$, or $\tan\theta = -1$

$\Rightarrow$ $\theta = n\pi$ or $n\pi - \pi/4$, $\forall\, n \in Z$, for $\theta \in [0, 2\pi]$ $\theta = 0, \pi, 2\pi, 3\pi/4, 7\pi/4$

39. d. We have $(\sin x + \sin 3x) + \sin 2x = (\cos x + \cos 3x) + \cos 2x$

$\Rightarrow$ $2\sin 2x\cos x + \sin 2x = 2\cos 2x\cos x + \cos 2x$

$\Rightarrow$ $\sin 2x(2\cos x + 1) = \cos 2x(2\cos x + 1)$

$\Rightarrow$ $(2\cos x + 1)(\sin 2x - \cos 2x) = 0$

$\Rightarrow$ $\cos x = -1/2$ or $\sin 2x - \cos 2x = 0$

$\Rightarrow$ $x = 2n\pi \pm (2\pi/3)$ or $\tan 2x = 1 = \tan(\pi/4)$

$= 2n\pi \pm (2\pi/3)$ or $x = (4n + 1)\pi/8$, $n \in Z$

But here $0 \le x \le 2\pi$

Hence, $x = \pi/8, 5\pi/8, 2\pi/3, 9\pi/8, 4\pi/3, 13\pi/8$.

40. b. $3\sin\theta - 4\sin^3\theta - \sin\theta = 2(2\cos^2\theta - 1)$

$\Rightarrow$ $2\sin\theta(1 - 2\sin^2\theta) = 2\cos 2\theta$

$\Rightarrow$ $2\cos 2\theta(\sin\theta - 1) = 0$

$\Rightarrow$ $\cos 2\theta = 0$ or $\sin\theta = 1$

$\Rightarrow$ $2\theta = (2n + 1)\pi/2$ or $\theta = 2n\pi + \pi/2$, $\forall\, n \in Z$

$\Rightarrow$ $\theta = (2n + 1)\pi/4$, or $\theta = (4n + 1)\pi/2$, $\forall\, n \in Z$

Hence, $\theta = \pi/4, 3\pi/4, 5\pi/4, 7\pi/4, \pi/2$. $(\because\ \theta \in [0, 2\pi])$

41. c. We have $4\sin\theta\sin 2\theta\sin 4\theta = 3\sin\theta - 4\sin^3\theta$

$\Rightarrow$ $\sin\theta\,[4\sin 2\theta\sin 4\theta - 3 + 4\sin^2\theta] = 0$

$\Rightarrow$ $\sin\theta\,[2(\cos 2\theta - \cos 6\theta) - 3 + 2(1 - \cos 2\theta)] = 0$

$\Rightarrow$ $\sin\theta\,(-2\cos 6\theta - 1) = 0$

$\Rightarrow$ $\sin\theta = 0$ or $\cos 6\theta = -1/2$

$\Rightarrow$ $\theta = n\pi$ or $6\theta = 2n\pi \pm 2\pi/3$, $\forall\, n \in Z$

$= n\pi$ or $\theta = (3n \pm 1)\pi/9$, $\forall\, n \in Z$

42. b. From the given equation, we have $\tan\theta + \tan 2\theta + \tan(\theta + 2\theta) = 0$

$$\Rightarrow\ (\tan\theta + \tan 2\theta) + \frac{\tan\theta + \tan 2\theta}{1 - \tan\theta\tan 2\theta} = 0$$

$$\Rightarrow\ (\tan\theta + \tan 2\theta)\left[1 + \frac{1}{1 - \tan\theta\,\tan 2\theta}\right] = 0$$

$\Rightarrow$ $(\tan\theta + \tan 2\theta)(2 - \tan\theta\tan 2\theta) = 0$

$\Rightarrow$ $\tan\theta = \tan(-2\theta)$ or $2 - \tan\theta\,[(2\tan\theta)/(1 - \tan^2\theta)] = 0$

$\Rightarrow$ $\theta = n\pi - 2\theta$ or $1 - 2\tan^2\theta = 0$

$= n\pi/3$ or $\tan^2\theta = 1/2 = \tan^2\alpha$ (say)

Therefore, $\theta = n\pi \pm \alpha$, where $\tan\alpha = 1/\sqrt{2}$.

43. b. We have $\sin 3\alpha = 4\sin\alpha(\sin^2 x - \sin^2\alpha)$

$\Rightarrow$ $3\sin\alpha - 4\sin^3\alpha = 4\sin\alpha\sin^2 x - 4\sin^3\alpha$

$\Rightarrow$ $3\sin\alpha = 4\sin\alpha\sin^2 x$ or $\sin\alpha = 0$

If $\sin\alpha \neq 0$, $\sin^2 x = 3/4 = (\sqrt{3}/2)^2 = \sin^2(\pi/3)$, therefore $x = n\pi \pm \pi/3$, $\forall\, n \in Z$
If $\sin\alpha = 0$, i.e., $\alpha = n\pi$, equation becomes an identity.

44. c. We have $\sqrt{3}\cos\theta - 3\sin\theta = 2(\sin 5\theta - \sin\theta)$

$\Rightarrow \left(\sqrt{3}/2\right)\cos\theta - (1/2)\sin\theta = \sin 5\theta$

$\Rightarrow \cos(\theta + \pi/6) = \sin 5\theta = \cos(\pi/2 - 5\theta)$

$\Rightarrow \theta + \pi/6 = 2n\pi \pm (\pi/2 - 5\theta)$

$\Rightarrow \theta = (n\pi/3) + (\pi/18)$ or $\theta = (-n\pi/2) + (\pi/6)$, $\forall\, n \in Z$

45. c. $\sin^4 x + \cos^4 x + \sin 2x + \alpha = 0$

$\Rightarrow (\sin^2 x + \cos^2 x)^2 - 2\sin^2 x\cos^2 x + \sin 2x + \alpha = 0$

$\Rightarrow \sin^2 2x - 2\sin 2x - 2 - 2\alpha = 0$

Let $\sin 2x = y$. Then the given equation becomes $y^2 - 2y - 2(1+\alpha) = 0$ where $-1 \le y \le 1$, $(\because -1 \le \sin 2x \le 1)$

For real, discriminant ≥ 0

$\Rightarrow 3 + 2\alpha \ge 0 \quad \Rightarrow \alpha \ge -\dfrac{3}{2}$

Also $-1 \le y \le 1 \quad \Rightarrow -1 \le 1 - \sqrt{3+2\alpha} \le 1$

$\Rightarrow 3 + 2\alpha \le 4 \quad \Rightarrow \alpha \le \dfrac{1}{2}$. Thus $-\dfrac{3}{2} \le \alpha \le \dfrac{1}{2}$

46. a. Since the system has a non-trivial solution, the determinant of coefficients = 0

$$\Rightarrow \begin{vmatrix} \sin 3\theta & -1 & 1 \\ \cos 2\theta & 4 & 3 \\ 2 & 7 & 7 \end{vmatrix} = 0$$

$\Rightarrow \sin 3\theta(28-21) - \cos 2\theta(-7-7) + 2(-3-4) = 0$

$\Rightarrow \sin 3\theta + 2\cos 2\theta - 2 = 0$

$\Rightarrow (3\sin\theta - 4\sin^3\theta) + 2(1 - 2\sin^2\theta) - 2 = 0$

$\Rightarrow 4\sin^3\theta + 4\sin^2\theta - 3\sin\theta = 0$

$\Rightarrow \sin\theta(2\sin\theta - 1)(2\sin\theta + 3) = 0$

$\Rightarrow \sin\theta = 0$ or $\theta = n\pi + (-1)^n \pi/6$, $\forall\, n \in Z$

47. c. Let $\log_{\cos x}\sin x = t$, then the given equation is $t + \dfrac{1}{t} = 2$

$\Rightarrow (t-1)^2 = 0 \quad \Rightarrow t = 1 \quad \Rightarrow \log_{\cos x}\sin x = 1$ or $\sin x = \cos x \quad \Rightarrow \tan x = 1$

$\Rightarrow x = \pi/4$

48. b. Given $x^2 + 2x\sin(xy) + 1 = 0$

$\Rightarrow [x + \sin(xy)]^2 + [1 - \sin^2(xy)] = 0$

$\Rightarrow x + \sin(xy) = 0$ and $\sin^2(xy) = 1$

$\sin^2(xy) = 1$ gives $\sin(xy) = 1$ or -1

If $\sin(xy) = 1 \quad \Rightarrow x = -1 \quad \Rightarrow \sin(-y) = 1 \quad \Rightarrow \sin y = -1$, then the ordered pair is $(1, 3\pi/2)$.

If $\sin(xy) = -1 \quad \Rightarrow x = 1 \quad \Rightarrow \sin y = -1 \quad \Rightarrow (-1, 3\pi/2)$

Thus, there are two ordered pairs.

49. c. The given equation is $8\sin x\cos x\cos 2x\cos 4x = \sin 6x$ $(\sin x \neq 0)$

$\Rightarrow \sin 8x = \sin 6x \quad \Rightarrow 2\cos 7x\sin x = 0$

As $\sin x \neq 0$, $\cos 7x = 0$ or $7x = n\pi + \pi/2$, $n \in Z$

i.e., $x = n\pi/7 + \pi/14$; $n \in Z$

50. d. We have $\cos 3x + \sin\left(2x - \frac{7\pi}{6}\right) = -2$

$\Rightarrow \quad 1 + \cos 3x + 1 + \sin\left(2x - \frac{7\pi}{6}\right) = 0$

$\Rightarrow \quad (1 + \cos 3x) + 1 - \cos\left(2x - \frac{2\pi}{3}\right) = 0$

$\Rightarrow \quad 2\cos^2\frac{3x}{2} + 2\sin^2\left(x - \frac{\pi}{3}\right) = 0$

$\Rightarrow \quad \cos\frac{3x}{2} = 0$ and $\sin\left(x - \frac{\pi}{3}\right) = 0$

$\Rightarrow \quad \frac{3x}{2} = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$, and $x - \frac{\pi}{3} = 0, \pi, 2\pi, \ldots \quad \Rightarrow \quad x = \frac{\pi}{3}$

Therefore, the general solution of $\cos\frac{3x}{2} = 0$ and $\sin\left(x - \frac{\pi}{3}\right) = 0$ is $x = 2k\pi + \frac{\pi}{3} = \frac{\pi}{3}(6k+1)$ where $k \in Z$.

51. b. Let $\tan^2\theta = t$

$\Rightarrow \quad 1 - t^2 + 2^t = 0$

It is clearly satisfied by $t = 3$. By inspection, we get $\tan^2\theta = 3$.

Therefore, $\theta = \pm\pi/3$ in the given interval.

52. b. $\tan\left(\frac{\pi}{2}\sin\theta\right) = \cot\left(\frac{\pi}{2}\cos\theta\right)$

$\therefore \tan\left(\frac{\pi}{2}\sin\theta\right) = \tan\left(\frac{\pi}{2} - \frac{\pi}{2}\cos\theta\right)$

$\therefore \frac{\pi}{2}\sin\theta = n\pi + \frac{\pi}{2} - \frac{\pi}{2}\cos\theta$

$\sin\theta + \cos\theta = 2n + 1$

$\Rightarrow \sin\theta + \cos\theta = 1 \pm 1$

$\Rightarrow 1 + \sin 2\theta = 1$

$\Rightarrow \sin 2\theta = 0$

$\Rightarrow \theta = n\pi$

53. a. $\cos x = \sqrt{1 - \sin 2x} = |\sin x - \cos x|$

(a) $\sin x < \cos x \quad \Rightarrow \quad x \in \left[0, \frac{\pi}{4}\right) \cup \left(\frac{5\pi}{4}, 2\pi\right]$ (i)

Then the given equation is $\cos x = \cos x - \sin x \quad \Rightarrow \quad \sin x = 0 \quad \Rightarrow \quad x = \pi, 2\pi$

$\Rightarrow \quad x = 2\pi$ [from Eq. (i)]

(b) $\sin x \geq \cos x \quad \Rightarrow \quad x \in \left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$

$\Rightarrow \quad \cos x = \sin x - \cos x$

$\Rightarrow \quad \tan x = 2$

$\Rightarrow \quad x = \tan^{-1}$ (ii)

Hence, there are two solutions.

54. b. $\sum_{r=1}^{5} \cos rx = 5$

$\Rightarrow \cos x + \cos 2x + \cos 3x + \cos 4x + \cos 5x = 5$

which is possible only, when $\cos x = \cos 2x = \cos 3x = \cos 4x = \cos 5x = 1$ and is satisfied by $x = 0$ and $x = 2\pi$.

55. a. $\sin 2x + \cos 4x = 2$

It is possible only, when $\sin 2x = 1$ and $\cos 4x = 1$

$\Rightarrow \sin 2x = 1$ and $1 - 2\sin^2 2x = 1$

$\Rightarrow \sin 2x = 1$ and $\sin 2x = 0$

Hence, there is no solution.

56. d. $\cos^2 x = 2\cos x(1 - 3\cos^2 x)$

$\Rightarrow 6\cos^3 x + \cos^2 x - 2\cos x = 0$

$\Rightarrow \cos x = 0, \frac{1}{2}, -\frac{2}{3}$

$\Rightarrow x = \frac{\pi}{2}, \frac{\pi}{3}, \cos^{-1}\left(-\frac{2}{3}\right)$ $(\because \alpha, \beta$ are $+$ve$)$

If $\alpha = \frac{\pi}{2}; \beta = \frac{\pi}{3}$, then we have $|\alpha - \beta| = \frac{\pi}{6}$.

57. b. We have $\sin^{100} x - \cos^{100} x = 1$

$\Rightarrow \sin^{100} x = 1 + \cos^{100} x$

Since the L.H.S. never exceeds 1, R.H.S. exceeds 1 unless $\cos x = 0$

Then, $x = n\pi + \frac{\pi}{2}, n \in I$

58. b. $|\cot x| = \cot x + \frac{1}{\sin x}$

If $\cot x > 0 \Rightarrow \cot x = \cot x + \frac{1}{\sin x} = 0$

$\Rightarrow \frac{1}{\sin x} = 0$, which is not possible.

If $\cot x \le 0 \quad \Rightarrow \quad -\cot x = \cot x + \frac{1}{\sin x}$

$\Rightarrow -2\cot x = \frac{1}{\sin x}$

$\Rightarrow \cos x = -\frac{1}{2}$

$\Rightarrow x = \frac{2\pi}{3}, \frac{8\pi}{3}$

59. a. $\tan(A - B) = 1$

$\Rightarrow A - B = n_1\pi + \frac{\pi}{4} = \frac{\pi}{4}, \frac{3\pi}{4}, -\frac{3\pi}{4}, \ldots$

$\sec(A + B) = \frac{2}{\sqrt{3}} \quad \Rightarrow \quad A + B = 2n_2\pi \pm \frac{\pi}{6} = \frac{\pi}{6}, \frac{11\pi}{6}, \ldots$

For the least positive values of A and B,

$$A+B=\frac{11\pi}{6}, A-B=\frac{\pi}{4} \quad \Rightarrow \quad B=\frac{19\pi}{24}, A=\frac{25\pi}{24}$$

60. a. Let $A=\theta+15°$, $B=\theta-15°$

$\Rightarrow$ $A+B=2\theta$ and $A-B=30°$

Now $\dfrac{\tan A}{\tan B}=\dfrac{3}{1}$

$\Rightarrow$ $\dfrac{\tan A+\tan B}{\tan A-\tan B}=\dfrac{3+1}{3-1}$ (applying componendo and dividendo rule)

$\Rightarrow$ $\dfrac{\sin(A+B)}{\sin(A-B)}=2$

$\Rightarrow$ $\sin 2\theta=2\sin 30°=1$

$\Rightarrow$ $2\theta=2n\pi+\dfrac{\pi}{2}$ or $\theta=n\pi+\dfrac{\pi}{4}$ $n\in Z$

61. a. $\tan 3\theta+\tan\theta=2\tan 2\theta$

$\Rightarrow$ $\tan 3\theta-\tan 2\theta=\tan 2\theta-\tan\theta$

$\Rightarrow$ $\dfrac{\sin(3\theta-2\theta)}{\cos 3\theta\cos 2\theta}=\dfrac{\sin(2\theta-\theta)}{\cos 2\theta\cos\theta}$

$\Rightarrow$ $\sin\theta(2\sin\theta\sin 2\theta)=0$

$\Rightarrow$ $\sin\theta=0$ or $\sin 2\theta=0$

$\Rightarrow$ $\theta=n\pi$ or $2\theta=n\pi, n\in Z$

But $\theta=n\pi/2$ is rejected as when n is odd, $\tan\theta$ is not defined and when n is even, i.e., $2r$, then $\theta=r\pi$.

Then $\theta=n\pi, n\in I$ is the only solution.

62. a. We have $|4\sin x-1|<\sqrt{5}$

$\Rightarrow$ $-\sqrt{5}<4\sin x-1<\sqrt{5}$

$\Rightarrow$ $-\left(\dfrac{\sqrt{5}-1}{4}\right)<\sin x<\left(\dfrac{\sqrt{5}+1}{4}\right)$

$\Rightarrow$ $-\sin\dfrac{\pi}{10}<\sin x<\cos\dfrac{\pi}{5}$

$\Rightarrow$ $\sin\left(-\dfrac{\pi}{10}\right)<\sin x<\sin\left(\dfrac{\pi}{2}-\dfrac{\pi}{5}\right)$

$\Rightarrow$ $\sin\left(-\dfrac{\pi}{10}\right)<\sin x<\sin\dfrac{3\pi}{10}$

$\Rightarrow$ $x\in\left(-\dfrac{\pi}{10},\dfrac{3\pi}{10}\right)$

63. d. $y^2-y+a=\left(y-\dfrac{1}{2}\right)^2+a-\dfrac{1}{4}$

Since $-\sqrt{2}\le\sin x+\cos x\le\sqrt{2}$, the given equation will have no real value x for any y if $a-\dfrac{1}{4}>\sqrt{2}$

i.e., $a \in \left(\sqrt{2}+\frac{1}{4}, \infty\right) \quad \Rightarrow \quad a \in (\sqrt{3}, \infty)$ (as $\sqrt{2}+\frac{1}{4} < \sqrt{3}$)

64. a. $(2\sin x - \operatorname{cosec} x)^2 + (\tan x - \cot x)^2 = 0$

$\Rightarrow \quad \sin^2 x = \frac{1}{2}$ and $\tan^2 x = 1$

$\Rightarrow \quad x = n\pi \pm \frac{\pi}{4}, n \in Z$

65. a. $[\sin x + \cos x] = 3 + [-\sin x] + [-\cos x]$

Maximum value of left-hand side is 1 and minimum of right hand side is also 1

$\Rightarrow \quad [\sin x + \cos x] = 3 + [-\sin x] + [-\cos x] = 1 \Rightarrow x \in \pi \pm \frac{\pi}{4}$

$\Rightarrow \quad [\sin x + \cos x] = 1, [-\sin x] = -1, [-\cos x] = -1$

which is not possible.

66. c. $\cos^8 x + b\cos^4 x + 1 = 0$

$\Rightarrow \quad b = -\left(\cos^4 x + \frac{1}{\cos^4 x}\right) \le -2 \ \forall x \in R$

$\Rightarrow \quad b \in (-\infty, -2]$

67. b. Here $1 \le |\sin 2x| + |\cos 2x| \le \sqrt{2}$ and $|\sin y| \le 1$

so solution is possible only when $|\sin y| = 1$

$\Rightarrow \quad \sin y = \pm 1 \quad \Rightarrow \quad y = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}$

68. b. Given that $|\sin x|^2 + |\sin x| + b = 0$

$\Rightarrow \quad |\sin x| = \frac{-1 \pm \sqrt{1-4b}}{2} \quad \Rightarrow \quad 0 \le \frac{-1 \pm \sqrt{1-4b}}{2} < 1 \quad \Rightarrow \quad -2 < b < 0$

69. a. $|2\sin\theta - \operatorname{cosec}\theta| \ge 1$

$\Rightarrow \quad |2\sin^2\theta - 1| \ge |\sin\theta|$

$\Rightarrow \quad |\cos 2\theta| \ge |\sin\theta|$

$\Rightarrow \quad 2\cos^2 2\theta \ge 1 - \cos 2\theta$

$\Rightarrow \quad 2\cos^2 2\theta + \cos 2\theta - 1 \ge 0$

$\Rightarrow \quad (2\cos 2\theta - 1)(\cos 2\theta + 1) \ge 0$

$\Rightarrow \quad \cos 2\theta \ge \frac{1}{2}$ [as $\cos\theta \ne 0$, i.e., $\cos 2\theta \ne -1$]

70. d. The given equation can be written as

$\sin x \cos x [\sin^2 x + \sin x \cos x + \cos^2 x] = 1$

$\Rightarrow \quad \sin x \cos x [1 + \sin x \cos x] = 1$

$\Rightarrow \quad \sin 2x [2 + \sin 2x] = 4$

$\Rightarrow \quad \sin 2x = \frac{-2 \pm \sqrt{4+16}}{2} = -1 \pm \sqrt{5}$

which is not possible.

71. d. $e^{|\sin x|} + e^{-|\sin x|} + 4a = 0$, let $t = e^{|\sin x|}$

$\Rightarrow \quad t \in [1, e]$

$\Rightarrow \quad t + \frac{1}{t} + 4a = 0$

$\Rightarrow \quad t^2 + 4at + 1 = 0$

This quadratic expression should have two distinct roots in $[1, e]$

$\Rightarrow \quad 16a^2 - 4 > 0, f(1) = 1 + 4a + 1 \geq 0, f(e) = e^2 + 4ae + 1 \geq 0, 1 < -2a < e$

$$\Rightarrow \quad |a| > \frac{1}{2}, a \geq -\frac{1}{2}, a \geq \frac{-1-e^2}{4e}, -\frac{e}{2} < a < -\frac{1}{2}$$

Clearly, there is no value of a satisfying the above inequalities simultaneously.

72. b.

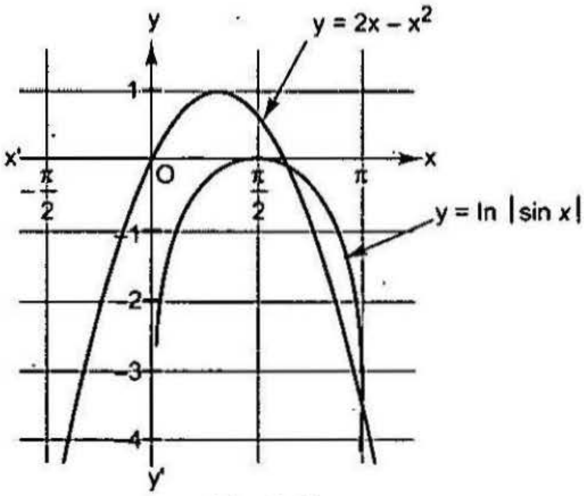


Fig. 3.11

$\ln|\sin x| = -x(x-2)$

Graphs of $y = \ln|\sin x|$ and $y = -x(x-2)$ meet exactly two times in $\left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]$.

73. c. $|x| + |y| = 4, \sin\left(\frac{\pi x^2}{3}\right) = 1$

$$\Rightarrow \quad |x|, |y| \in [0, 4], \frac{\pi x^2}{3} = (4n+1)\frac{\pi}{2}$$

$$\Rightarrow \quad x^2 = \frac{(4n+1)3}{2} = \frac{3}{2}, \text{ as } |x| \leq 4$$

$$\Rightarrow \quad |x| = \sqrt{\frac{3}{2}}, |y| = 4 - \sqrt{\frac{3}{2}}$$

Thus, there are four ordered pairs.

74. b. $\sin\{x\} = \cos\{x\}$

$\tan\{x\} = 1$

$\tan(\pi/4) = 1 < \tan 1$

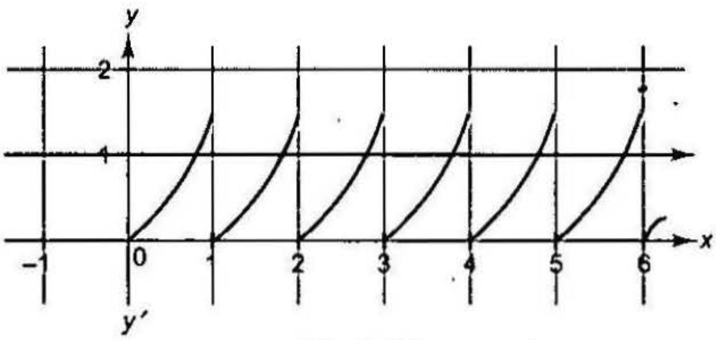


Fig. 3.12

Graphs of $y = \tan\{x\}$ and $y = 1$ meet exactly six times in $[0, 2\pi]$.

75. a. $x^2+4-2x+3\sin(ax+b)=0$

$(x-1)^2+3=-3\sin(ax+b)$

L.H.S. ≥ 3 and R.H.S. ≤ 3

$\Rightarrow$ L.H.S = R.H.S = 3

$(x-1)^2+3+3\sin(ax+b)=0$

$\Rightarrow x=1, \sin(ax+b)=-1$

$\Rightarrow \sin(a+b)=-1$

$\Rightarrow a+b=(4n-1)\dfrac{\pi}{2}, n\in I \Rightarrow a+b=\dfrac{7\pi}{2}$ (from the given options)

76. c. $\tan^4 x-2\sec^2 x+a=0$

$\Rightarrow \tan^4 x-2(1+\tan^2 x)+a=0$

$\Rightarrow \tan^4 x-2\tan^2 x+1=3-a$

$\Rightarrow (\tan^2 x-1)^2=3-a$

$\Rightarrow 3-a\geq 0 \Rightarrow a\leq 3$

77. a. $1+\log_2\sin x+\log_2\sin 3x\geq 0$

(where $\sin x, \sin 3x>0$)

$\Rightarrow \log_2(2\sin x\sin 3x)\geq 0$

$\Rightarrow 2\sin x\sin 3x\geq 1$

For $\sin x>0$

$\Rightarrow x\in(0,\pi)$ (i)

$\Rightarrow \sin 3x>0$ (ii)

$\Rightarrow 3x\in(0,\pi)\cup(2\pi,3\pi)$

$\Rightarrow x\in\left(0,\dfrac{\pi}{3}\right)\cup\left(\dfrac{2\pi}{3},\pi\right)$

For $2\sin x\sin 3x\geq 1$

$\Rightarrow 2\sin^2 x(3-4\sin^2 x)\geq 1$

$\Rightarrow 8\sin^4 x-6\sin^2+1\leq 0$

$\Rightarrow (2\sin^2 x-1)(4\sin^2 x-1)\leq 0$

$\Rightarrow \dfrac{1}{2}\leq\sin x\leq\dfrac{1}{\sqrt{2}}$

$\Rightarrow x\in\left[\dfrac{\pi}{3},\dfrac{\pi}{4}\right]\cup\left[\dfrac{2\pi}{3},\dfrac{3\pi}{4}\right]$ (iii)

Thus, $x\in\left(\dfrac{2\pi}{3},\dfrac{3\pi}{4}\right]$. [From Eqs. (i), (ii), (iii)]

78. d. $\sin^2\theta=1$ [$\sin\theta\neq 1$]

$\Rightarrow \sin\theta=-1\Rightarrow\theta=2n\pi-(\pi/2)\Rightarrow$ infinite roots

79. c. $\pi\log_3\left(\dfrac{1}{x}\right)=k\pi, k\in I$

$\log_3\left(\dfrac{1}{x}\right)=k\Rightarrow x=3^{-k}$

The possible values of k are $-1, 0, 1, 2, 3, \ldots$

$$S=3+1+\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\cdots\infty=\frac{3}{1-\frac{1}{3}}=\frac{9}{2}$$

80. a. $\dfrac{\sin(xy)}{\cos(xy)} = y$

$\Rightarrow \sin(xy) = xy$

$\Rightarrow xy = 0$

$\Rightarrow x = 0$ or $y = 0$

But $x = 0$ is not possible

$\therefore y = 0$ and $x = 1$, i.e., $(1, 0)$

81. c. $\tan x + \cot x + 1 = \cos\left(x + \dfrac{\pi}{4}\right)$

$\Rightarrow \tan x + \cot x = \cos\left(x + \dfrac{\pi}{4}\right) - 1$

Now $\tan x + \cot x \le -2$ and $\cos\left(x + \dfrac{\pi}{4}\right) - 1 \ge -2$

It implies that equality holds when both are -2.

$\Rightarrow \cos\left(x + \dfrac{\pi}{4}\right) = -1$

$\Rightarrow x + \dfrac{\pi}{4} = (2m+1)\pi,\ m \in Z$

$\Rightarrow x = \dfrac{3\pi}{4}$ or $\dfrac{11\pi}{4}$

Therefore, the sum of the solutions is $\dfrac{3\pi}{4} + \dfrac{11\pi}{4} = \dfrac{7\pi}{2}$.

82. c. $\cos(\theta)\cos(\pi\theta) = 1$

$\Rightarrow \cos(\theta) = 1$ and $\cos(\pi\theta) = 1$ (i)

or $\cos(\theta) = -1$ and $\cos(\pi\theta) = -1$ (ii)

If $\cos(\theta) = 1 \Rightarrow \theta = 2m\pi$ and $\cos(\pi\theta) = 1 \Rightarrow \theta = 2\pi$ which is possible only when $\theta = 0$.

Equation (ii) is not possible for any θ as for $\cos(\theta) = -1$, θ should be odd multiple of $\pi \Rightarrow$ irrational and for $\cos(\pi\theta) = -1 \Rightarrow \theta$ should be odd integer $\Rightarrow$ rational

Both the conditions cannot be satisfied.

Therefore, $\theta = 0$ is the only solution.

83. c. $(\cot x + \sqrt{3})^2 + \cot^2 x + 4\operatorname{cosec} x + 5 = 0$

$\Rightarrow (\cot x + \sqrt{3})^2 + \operatorname{cosec}^2 x + 4\operatorname{cosec} x + 4 = 0$

$\Rightarrow (\cot x + \sqrt{3})^2 + (\operatorname{cosec} x + 2)^2 = 0$

$\Rightarrow \cot x = -\sqrt{3}$ or $\operatorname{cosec} x = -2$

$\Rightarrow x = 2n\pi - \dfrac{\pi}{6},\ n \in Z$ $(\because x \in$ 4th quadrant$)$

84. c. $\theta = k\pi, k = \dfrac{p}{q}, p, q \in I, q \neq 0$

$\cos k\pi$ is a rational

Hence, $k = 0, 1, 1/2, 1/3, 2/3$

There are five values of $\cos\theta$ for which $\cos\theta$ is rational.

85. b. Putting $x = 0 \Rightarrow b^2 = \cos b^2 - 1 \Rightarrow \cos b^2 = 1 + b^2 \Rightarrow b = 0$

For $b = 0$, we have $a(\cos x - 1) = \cos ax - 1$

$\Rightarrow 2a\sin^2\dfrac{x}{2} = 2\sin^2\dfrac{ax}{2}$

$\Rightarrow a = 0$ or $a = 1$.

Hence, ordered pairs are $(a, b) \equiv (0, 0)$ or $(1, 0)$.

Multiple Correct Answers Type

1. a, b. We have $4\sin^4 x + \cos^4 x = 1$

$\Rightarrow \quad 4\sin^4 x = 1 - \cos^4 x = (1 - \cos^2 x)(1 + \cos^2 x) = \sin^2 x(2 - \sin^2 x)$

$\Rightarrow \quad \sin^2 x[5\sin^2 x - 2] = 0$

$\Rightarrow \quad \sin x = 0$ or $\sin x = \pm\sqrt{2/5}$

$\Rightarrow \quad x = n\pi$ or $x = n\pi \pm \sin^{-1}\sqrt{\dfrac{2}{5}}, n \in Z$

2. a, b. $(\sin^3\theta + \cos^3\theta) - (1 - \sin\theta\cos\theta) = 0$

$\Rightarrow \quad (\sin\theta + \cos\theta)(1 - \sin\theta\cos\theta) - (1 - \sin\theta\cos\theta) = 0$

$\Rightarrow \quad (1 - \sin\theta\cos\theta)(\sin\theta + \cos\theta - 1) = 0$

If $\sin\theta\cos\theta = 1 \Rightarrow 2\sin\theta\cos\theta = 2 \Rightarrow \sin 2\theta = 2$ (not possible)

$\Rightarrow \quad \sin\theta + \cos\theta = 1$

$\Rightarrow \quad \theta - \dfrac{\pi}{4} = 2n\pi \pm \dfrac{\pi}{4}, n \in Z$

$\Rightarrow \theta = 2n\pi$ or $2n\pi + \dfrac{\pi}{2}$

3. a, c, d. We have $\tan^2\theta = 1 - \cos 2\theta = 2\sin^2\theta$ or $\operatorname{cosec}^2\theta\tan^2\theta = 2$
or $(1 + \cot^2\theta)\tan^2\theta = 2$ or $\tan^2\theta + 1 = 2$

$\Rightarrow \quad \tan^2\theta = 1 \quad \Rightarrow \quad \tan\theta = \pm 1 \quad \Rightarrow \quad \theta = n\pi \pm \dfrac{\pi}{4}, n \in Z$

Moreover, $\tan^2\theta = 2\sin^2\theta \quad \Rightarrow \quad \sin^2\theta = 0 \Rightarrow \theta = n\pi$

4. a, c. $y + \dfrac{1}{y} \geq 2 \Rightarrow \sqrt{y + \dfrac{1}{y}} \geq \sqrt{2}$

But $\sin x + \cos x \leq \sqrt{2}$

$\Rightarrow \quad y + \dfrac{1}{y} = 2$ and $\sin x + \cos x = \sqrt{2}$

$\Rightarrow \quad y = 1$ and $\sin\left(x + \dfrac{\pi}{4}\right) = 1$ or $y = 1$ and $x = \dfrac{\pi}{4}$

5. b, d. $\sin\theta + \sqrt{3}\cos\theta = -2-(x^2-6x+9) = -2-(x-3)^2$

$\therefore \sin\theta + \sqrt{3}\cos\theta \ge -2$ and $-2-(x-3)^2 \le -2$

As a result, we have $\sin\theta + \sqrt{3}\cos\theta = -2$ and then $x = 3$

$\therefore x = 3$ and $\cos\left(\theta - \frac{\pi}{6}\right) = -1$, i.e., $\theta - \frac{\pi}{6} = \pi, 3\pi$

6. a, c. $\sin^2 x - 2\sin x - 1 = 0$

$\Rightarrow \quad (\sin x - 1)^2 = 2 \quad \Rightarrow \quad \sin x - 1 = \pm\sqrt{2} \quad \Rightarrow \quad \sin x = 1-\sqrt{2}$ as $\sin x \not> 1$

There are two solutions in $[0, 2\pi]$ and two more in $[2\pi, 4\pi]$.

Thus, $n = 4, 5$.

7. a, b, c. The given equation is $2(\sin x + \sin y) - 2\cos(x-y) = 3$

$$\Rightarrow \quad 2\times 2\sin\frac{x+y}{2}\cos\frac{x-y}{2} - 2\left[2\cos^2\frac{x-y}{2} - 1\right] = 3$$

$$\Rightarrow \quad 4\cos^2\left(\frac{x-y}{2}\right) - 4\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right) + 1 = 0$$

$$\Rightarrow \quad \cos\left(\frac{x-y}{2}\right) = \frac{4\sin\left(\frac{x+y}{2}\right) \pm \sqrt{16\sin^2\left(\frac{x+y}{2}\right) - 16}}{8}$$

$\therefore \sin^2\left(\frac{x+y}{2}\right) \ge 1 \quad \Rightarrow \quad \sin\frac{x+y}{2} = \pm 1$

Since x and y are smallest and positive, we have

$\sin\frac{x+y}{2} = 1$ and $\frac{x+y}{2} = \frac{\pi}{2}$

i.e., $x + y = \pi$ (i)

Also, $\cos\left(\frac{x-y}{2}\right) = \frac{1}{2}$

$\Rightarrow \quad x - y = 2\pi/3$ or $-2\pi/3$ (ii)

From Eqs. (i) and (ii), we get ($x = 5\pi/6$, $y = \pi/6$) or ($x = \pi/6$, $y = 5\pi/6$)

8. a, d. $1 - 2x - x^2 = \tan^2(x+y) + \cot^2(x+y)$

$\Rightarrow \quad -(x+1)^2 = [\tan(x+y) - \cot(x+y)]^2$

Now L.H.S. ≤ 0 and R.H.S. ≥ 0

$\Rightarrow \quad -(x+1)^2 = [\tan(x+y) - \cot(x+y)]^2 = 0$

$\Rightarrow \quad x = -1$ and $\tan(x+y) = \cot(x+y)$

$\Rightarrow \quad x = -1$ and $\tan^2(-1+y) = 1$

$\Rightarrow \quad x = -1$ and $-1 + y = n\pi \pm (\pi/4), n \in Z$

9. b, c. From $\tan x + \tan y = 1$, we have $\frac{\sin x}{\cos x} + \frac{\sin y}{\cos y} = 1$

$\Rightarrow \quad \sin x \cos y + \sin y \cos x = \cos x \cos y$

$\Rightarrow \quad 2 \sin (x + y) = 2 \cos x \cos y$

$\Rightarrow \quad 2 \sin (x + y) = \cos (x + y) + \cos (x - y)$

$\Rightarrow \quad 2 \sin (\pi/4) = \cos (\pi/4) + \cos (x - y)$

$\Rightarrow \quad \cos (x - y) = 1/\sqrt{2} = \cos (\pi/4)$

$\Rightarrow \quad x - y = 2n\pi \pm (\pi/4), \forall\, n \in Z$ (i)

Also we have $x + y = \pi/4$ (ii)

From Eqs. (i) and (ii), we have

$x = n\pi + (\pi/4)$ and $y = -n\pi, \forall\, n \in Z$

or $x = n\pi$ and $y = -n\pi + \pi/4, \forall\, n \in Z$

10. a, c. $x + y = 2\pi/3 \quad \Rightarrow \quad y = (2\pi/3) - x$

$\therefore \sin x = 2 \sin (2\pi/3 - x)$

$= 2[(\sqrt{3}/2) \cos x + (1/2) \sin x] = \sqrt{3} \cos x + \sin x$

$\Rightarrow \quad \cos x = 0 \quad \Rightarrow \quad x = n\pi + \pi/2, n \in Z \quad \Rightarrow \quad y = \dfrac{2\pi}{3} - n\pi - \dfrac{\pi}{2} = \dfrac{\pi}{6} - n\pi$

Hence, for $x \in [0, 4\pi]$, $x = \pi/2, 3\pi/2, 5\pi/2, 7\pi/2$

and for $y \in [0, 4\pi]$, $y = \pi/6, 7\pi/6, 13\pi/6, 19\pi/6$

11. a, d. $\tan x - \tan^2 x > 0$

$\Rightarrow \quad \tan x (\tan x - 1) < 0$

$\Rightarrow \quad 0 < \tan x < 1$

$\Rightarrow \quad 0 < x < \pi/4$

$\Rightarrow \quad n\pi < x < n\pi + \pi/4, n \in Z$ (generalizing)

$|\sin x| < \dfrac{1}{2} \Rightarrow \quad -\dfrac{1}{2} < \sin x < \dfrac{1}{2}$

$\Rightarrow -\pi/6 < x < \pi/6 \quad \Rightarrow -\pi/6 + n\pi < x < \pi/6 + n\pi, n \in Z$ (generalizing)

Then the common values are $n\pi < x < n\pi + \pi/6$.

12. a, b, d. $\cos (x + \pi/3) + \cos x = a$

$\Rightarrow \quad \dfrac{1}{2} \cos x - (\sqrt{3}/2) \sin x + \cos x = a$

$\Rightarrow \quad (3/2) \cos x - (\sqrt{3}/2) \sin x = a$

$\Rightarrow \quad -\sqrt{\left(\dfrac{9}{4} + \dfrac{3}{4}\right)} \le a \le \sqrt{\left(\dfrac{9}{4} + \dfrac{3}{4}\right)}$

$\Rightarrow \quad -\sqrt{3} \le a \le \sqrt{3}$ (i)

Hence, there are three integral values of $a = -1, 0, 1$ whose sum is 0.

For $a = 1$, the given equation is $(\sqrt{3}/2) \cos x - (1/2) \sin x = 1/\sqrt{3}$

$\Rightarrow \quad \cos (x + \pi/6) = 1/\sqrt{3}$

$\Rightarrow \quad x + \pi/6 = 2n\pi \pm \alpha$, where $\alpha = \cos^{-1} (1/\sqrt{3})$

$\Rightarrow \quad x = 2n\pi - \pi/6 \pm \alpha$

Hence, the solutions for $a = 1$ in $[0, 2\pi]$ are $\cos^{-1} (1/\sqrt{3}) - \pi/6$, $11/6 - \cos^{-1} (1/\sqrt{3})$.

13. a, b, c. The given inequality can be written as

$$2^{\operatorname{cosec}^2 x}\sqrt{(y-1)^2+1} \le \sqrt{2} \quad \text{(i)}$$

Since $\operatorname{cosec}^2 x \ge 1$ for all real x, we have

$$2^{\operatorname{cosec}^2 x} \ge 2 \quad \text{(ii)}$$

Also $(y-1)^2+1 \ge 1 \quad \Rightarrow \quad \sqrt{(y-1)^2+1} \ge 1$ (iii)

From Eqs. (i) and (ii), we get

$$2^{\operatorname{cosec}^2 x}\sqrt{(y+1)^2+1} \ge 2 \quad \text{(iv)}$$

Therefore, from Eqs. (i) and (iv), equality holds only when $2^{\operatorname{cosec}^2 x} = 2$ and $\sqrt{(y-1)^2+1} = 1$

$\Rightarrow \quad \operatorname{cosec}^2 x = 1$ and $(y-1)^2 + 1 = 1 \quad \Rightarrow \quad \sin x = \pm 1$ and $y = 1$

$\Rightarrow \quad x = \frac{\pi}{2}, \frac{3\pi}{2}$ and $y = 1$

Hence, the solution of the given inequality is $x = \frac{\pi}{2}, \frac{3\pi}{2}$ and $y = 1$.

14. a, b, c. $\sin^2 x - a\sin x + b = 0$ has only one solution in $(0, \pi)$.

$\Rightarrow \quad \sin x = 1$ gives one solution and $\sin x = \alpha$ gives other solution such that $\alpha > 1$ or $\alpha \le 0$

$\Rightarrow \quad (\sin x - 1)(\sin x - \alpha) = 0$ is the same equation as $\sin^2 x - a\sin x + b = 0$

$\Rightarrow \quad 1 + \alpha = a$ and $\alpha = b$

$\Rightarrow \quad 1 + b = a$ and $b > 1$ or $b \le 0$

$\Rightarrow \quad b \in (-\infty, 0] \cup [1, \infty)$ and $a \in (-\infty, 1] \cup [2, \infty)$

15. a, b. Given that the quadratic equation is an identity

$\therefore \ \operatorname{cosec}^2 \theta = 4$ and $\cot \theta = -\sqrt{3}$

$\Rightarrow \quad \operatorname{cosec} \theta = 2$ or -2 and $\tan \theta = -\frac{1}{\sqrt{3}}$

$\theta = \frac{5\pi}{6}$ or $\frac{11\pi}{6}$

16. c, d. Abscissa corresponding to the vertex is given by

$x = \frac{1}{\sin \alpha} > 1$ is the vertex

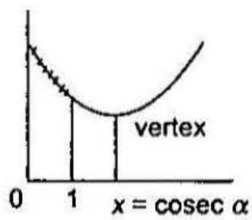


Fig. 3.13

The graph of $f(x) = (\sin \alpha)x^2 - 2x + b$ is shown as $\forall x \le 1$

Therefore, the minimum of $f(x) = (\sin \alpha)x^2 - 2x + b - 2$ must be greater than zero but minimum is at $x = 1$, i.e., $\sin \alpha - 2 + b - 2 \ge 0$

$\Rightarrow \quad b \ge 4 - \sin \alpha, \alpha \in (0, \pi) \quad \Rightarrow \quad b \ge 4$ as $\sin \alpha > 0$ in $(0, \pi)$

17.a, d. $\dfrac{\sqrt{3}-1}{2\sqrt{2}\sin x}+\dfrac{\sqrt{3}+1}{2\sqrt{2}\cos x}=2$

$$\sin\frac{\pi}{12}\cos x+\cos\frac{\pi}{12}\sin x=\sin 2x$$

$$\sin 2x=\sin\left(x+\frac{\pi}{12}\right)$$

$$\therefore 2x=x+\frac{\pi}{12} \text{ or } 2x=\pi-x-\frac{\pi}{12}$$

$$x=\frac{\pi}{12} \text{ or } 3x=\frac{11\pi}{12}$$

$$\therefore x=\frac{\pi}{12} \text{ or } \frac{11\pi}{36}$$

18. a, c, d. $\cos 3\theta=\cos 3\alpha$

Putting $n=0, 1$, we have

$3\alpha=2n\pi\pm 3\alpha$

$\therefore 3\theta=3\alpha$ or -3α or $2\pi+3\alpha$ or $2\pi-3\alpha$

$\theta=\alpha$ or $-\alpha$ or $\dfrac{2\pi}{3}+\alpha$ or $\dfrac{2\pi}{3}-\alpha \Rightarrow$ (a), (c), (d) are correct

If $n=-1$, then $3\theta=-2\pi\pm 3\alpha$

$$\Rightarrow \theta=-\frac{2\pi}{3}\pm\alpha$$

$$\Rightarrow \sin\theta=\sin\left(-\frac{2\pi}{3}\pm\alpha\right)=-\sin\left(\frac{2\pi}{3}\pm\alpha\right)=-\sin\left(\pi-\frac{\pi}{3}\pm\alpha\right)$$

$$=-\sin\left(\pi-\left(\frac{\pi}{3}\pm\alpha\right)\right)=-\sin\left(\frac{\pi}{3}\pm\alpha\right)$$

Hence, (b) is not correct.

19. b, c, d. $1+\cos 3x=2\cos 2x$

$\Rightarrow 1+4\cos^3 x-3\cos x=2(2\cos^2 x-1)$

$\Rightarrow 4\cos^3 x-4\cos^2 x-3\cos x+3=0$

Let $\cos x=y$, we have

$4y^3-4y^2-3y+3=0$

$\Rightarrow 4y^2(y-1)-3(y-1)=0$

$\Rightarrow (y-1)(4y^2-3)=0$

$\Rightarrow y=1$ or $y^2=\dfrac{3}{4}$

$\Rightarrow \cos x=1$ or $\cos^2 x=\dfrac{3}{4}$

$\Rightarrow \cos x = 1$ or $\cos^2 x = \cos^2 \pi/6$

$\Rightarrow x = 2n\pi$ or $x = n\pi \pm (\pi/6), n \in Z$

20. b, c. $p^2\sec^2\theta + p^2\text{cosec}^2\theta = (2\sqrt{2})^2\ p^2$

$$\Rightarrow \frac{1}{\sin^2\theta\cos^2\theta} = 8$$

$$\Rightarrow \sin^2 2\theta = 1/2 = \left(1/\sqrt{2}\right)^2$$

$\Rightarrow 2\theta = (n\pi) + (\pi/4), n \in Z$

$\Rightarrow \theta = (n\pi/2) + (\pi/8)$

for $n = 0$, $\theta = \pi/8$

for $n = 1$, $\theta = 3\pi/8$

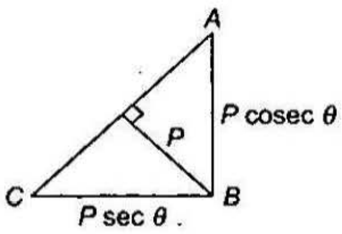


Fig. 3.14

Reasoning Type

1. a. $(\sin x + \cos x)^{1+\sin 2x} = 2 \Rightarrow (\sin x + \cos x)^{(\sin x + \cos x)^2} = 2$

Now we know that the maximum value of $\sin x + \cos x$ is $\sqrt{2}$ which occurs at $x = \pi/4$, for $0 \le x \le \pi/4$.

Also, the given equation has roots only if $\sin x + \cos x = \sqrt{2}$.

Hence, there is only one solution for $0 \le x \le \pi$.

Thus, the correct answer is (*a*).

2. d. We know that $\sin^2 x \le 1$ and $\cos^2 y \le 1$, then $\sin^2 x + \cos^2 y \le 2$

Also $\sec^2 z \ge 1$, then $2\sec^2 z \ge 2$.

Hence, the given equation is solvable only if $\sin^2 x + \cos^2 y = 2$ and $2\sec^2 z = 2$, for which $\sin^2 x$, $\cos^2 y$, $\sec^2 z = 1$.

Then $\sin x$, $\cos y$, $\sec z = \pm 1$

Hence, statement 1 is false and statement 2 is true.

3. a.

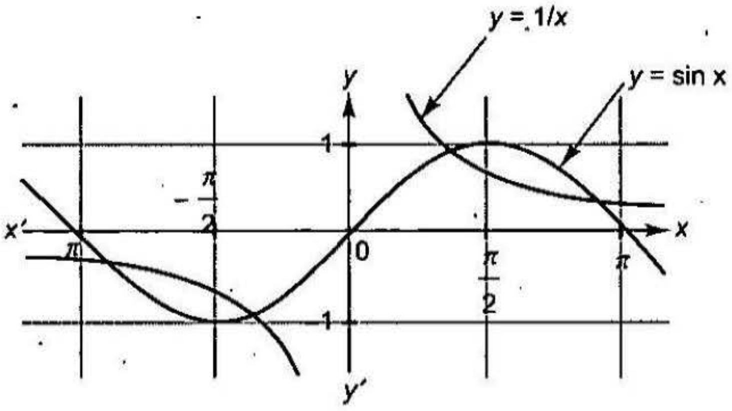


Fig. 3.15

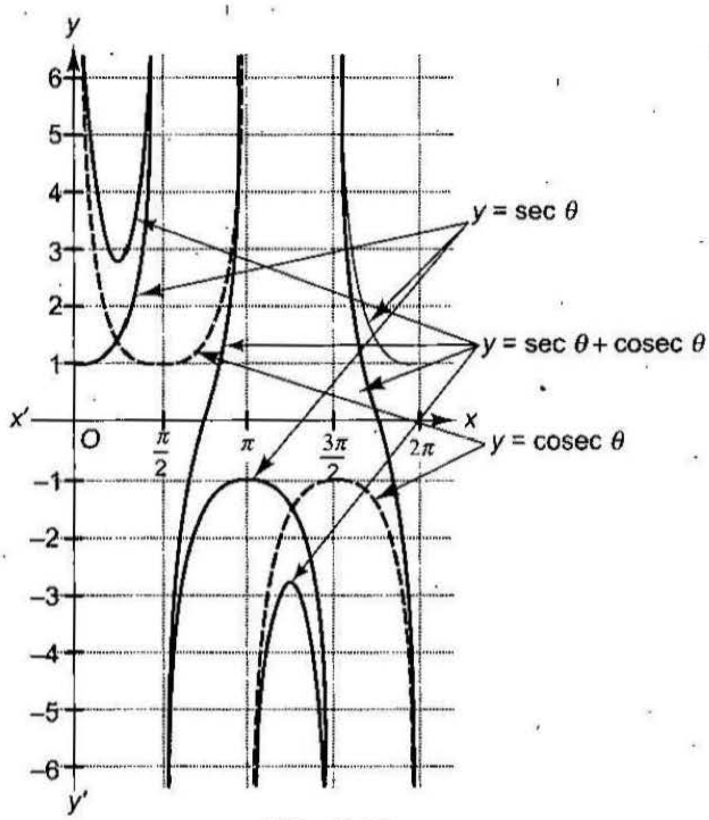


Fig. 3.17

$\sec\theta + \text{cosec}\,\theta = a$ has solution where graphs of $y = a$ and $y = \sec\theta + \text{cosec}\,\theta$ intersect. (i)

Graphs of $y = \sec\theta$, $y = \text{cosec}\,\theta$ and $y = \sec\theta + \text{cosec}\,\theta$ are as shown in Fig. 3.17.

Clearly, Eq. (i) has two solutions if $-2\sqrt{2} < y < 2\sqrt{2}$.

Equation (i) has four solutions if $y \le -2\sqrt{2}$ or $y \ge 2\sqrt{2}$.

In any case, Eq. (i) has two roots always.

For Problems 7 – 9

7. a, 8. b, 9. d.

Sol. The given system is $\sin x \cos 2y = (a^2-1)^2 + 1$, and $\cos x \sin 2y = a + 1$ (i)

Since the L.H.S. of the equations does not exceed 1, the given system may have solutions only for a's such that

$(a^2-1)^2 + 1 \le 1$ and $-1 \le a + 1 \le 1$ (ii)

$(a^2-1)^2 + 1 \le 1 \quad \Rightarrow \quad (a^2-1)^2 \le 0 \quad \Rightarrow \quad (a^2-1)^2 = 0 \quad \Rightarrow \quad a = 1$

But $a = 1$ does not satisfy $\cos x \sin 2y = a + 1$

Thus, $a = -1$ only and we get

$\sin x \cos 2y = 1$

$\cos x \sin 2y = 0$ (iii)

$\sin x \cos 2y = 1$

$\Rightarrow \sin x = 1, \cos 2y = 1$

or $\sin x = -1, \cos 2y = -1$

for which $\cos x \sin 2y = 0$

For Problems 10 – 12

10. a, 11. d, 12. b.

Sol. Given that $\int_0^x (t^2 - 8t + 13)\, dt = x \sin(a/x)$ (i)

R.H.S. shows that $x \neq 0$

Integrating L.H.S., we get

$$\left[\frac{t^3}{3} - 4t^2 + 13t\right]_0^x = x \sin(a/x)$$

or $(1/3)[x^3 - 12x^2 + 39x] = x \sin(a/x)$

or $\sin(a/x) = (1/3)[x^2 - 12x + 39]$ $\{\because x \neq 0\}$

$= (1/3)\{(x-6)^2 + 3\}$

$= (1/3)(x-6)^2 + 1$

But $\sin(a/x) \le 1$, so $\sin(a/x) = 1$, which is possible only for $x = 6$

then we have $\sin(a/6) = 1$

or $a/6 = 2n\pi + \pi/2$ or $a = 12n\pi + 3\pi, n \in Z$

Hence, $x = 6$, $a = 12n\pi + 3\pi, n \in Z$.

For $a \in [0, 100]$, there are exactly three values of $a = 3\pi$, 15π and 27π, i.e.,

$|y - \cos a| < x \Rightarrow |y + 1| < 6 \Rightarrow y \in [-7, 5]$

For Problems 13 – 15

13. a, 14. d, 15. b

Sol. The given equations are

$x\cos^3 y + 3x \cos y \sin^2 y = 14$ and (i)

$x \sin^3 y + 3x \cos^2 y \sin y = 13$ (ii)

Adding Eqs. (i) and (ii), we have

$x(\cos^3 y + 3\cos y \sin^2 y + 3\cos^2 y \sin y \sin^3 y) = 27$

$\Rightarrow x(\cos y + \sin y)^3 = 27$

$\Rightarrow x^{1/3}(\cos y + \sin y) = 3$ (iii)

Subtracting Eq. (ii) from Eq. (i), we have

$x(\cos^3 y + 3\cos y \sin^2 y - 3\cos^2 y \sin y - \sin^3 y) = 1$

$\Rightarrow x(\cos y - \sin y)^3 = 1$

$\Rightarrow x^{1/3}(\cos y - \sin y) = 1$ (iv)

Dividing Eq. (iii) by (iv), we get

$\cos y + \sin y = 3\cos y - 3\sin y$

$\Rightarrow \tan y = 1/2$

Case I:

$\sin y = 1/\sqrt{5}$ and $\cos y = 2/\sqrt{5}$

$y = 2n\pi + \alpha$, where $0 < \alpha < \pi/2$ and $\sin \alpha = 1/\sqrt{5}$

i.e., y lies in the first quadrant

From Eqs. (iii), $x^{1/3}(3/\sqrt{5}) = 3$ or $x = 5\sqrt{5}$

Case II:

$\sin y = -1/\sqrt{5}$ and $\cos y = -2/\sqrt{5}$

Draw the graphs of $y = \sin x$ and $y = 1/x$ and verify.

4. d. When $n = 1$, we have interval $[0, \pi]$, which covers only first and second quadrant, in which $\sin x = -1/2$ is not possible. Hence, the number of solutions is zero. Also from $2(n-1)$, we have zero solution when $n = 1$.
For $n = 2$, we have interval $[0, 2\pi]$ which covers all the quadrant only once. Hence, the number of solutions is two.
Also from $2(n-1)$, we have two solutions, when $n = 2$.
For $n = 3$, we have interval $[0, 3\pi]$, which covers third and fourth quadrant only once. Hence, the number of solutions is two. But from $2(n-1)$, we have four solutions which contradict.
Hence, statement 1 is false, statement 2 is true.

5. a. $\sqrt{1-\sin 2x} = \sin x$

$\Rightarrow \quad \sqrt{(\sin x - \cos x)^2} = \sin x$
$\Rightarrow \quad |\sin x - \cos x| = \sin x$
$\Rightarrow \quad \cos x - \sin x = \sin x$
$\Rightarrow \quad 2\sin x = \cos x$

$\Rightarrow \quad \tan x = \dfrac{1}{2}$ which has only one solution for $x \in [0, \pi/4]$ for these values of x.

6. b. Draw the graphs of $y = |\sin x|$ and $y = |x|$ and verify that $|\sin x| = |x|$ has only one solution $x = 0$. But statement 2 is not the correct explanation of statement 1.

7. d. Given $\tan 2x = 1$

$\therefore\ 2x = n\pi + \dfrac{\pi}{4}, n \in Z$ [note that $\tan 4x$ is not defined for these values of x]

Hence, the given equation has no solution.
Therefore, statement 1 is false and statement 2 is true.

8. a. $\cos(\sin x) = \sin(\cos x)$
$\Rightarrow \quad \cos(\sin x) = \cos[(\pi/2) - \cos x]$
$\Rightarrow \quad \sin x = 2n\pi \pm (\pi/2 - \cos x), n \in Z$ (i)
Taking + ive sign, we get $\sin x = 2n\pi + \pi/2 - \cos x$

or $(\cos x + \sin x) = \dfrac{1}{2}(4n+1)\pi$

Now L.H.S. $\in [-\sqrt{2}, \sqrt{2}]$, hence $-\sqrt{2} \le \dfrac{1}{2}(4n+1)\pi \le \sqrt{2}$.

$\Rightarrow \quad \dfrac{-2\sqrt{2}-\pi}{4\pi} \le n \le \dfrac{2\sqrt{2}-\pi}{4\pi}$, which is not satisfied by any integer n.
Similarly, taking – ve sign, we have $\sin x - \cos x = (4n-1)\pi/2$, which is also not satisfied for any integer n.
Hence, there is no solution.

9. b. Statement 1 is true.
Also statement 2 is true but does not explain statement 1.
Consider the equation $\sin x = x^3$.
Here, $y = x^3$ is an unbounded function but equation has finite number of solutions.

10. a. Let $y = n|\sin x| = m|\cos x|$

The curve $y = n|\sin x|$ and $y = m|\cos x|$ intersect at four points in $[0, 2\pi]$.

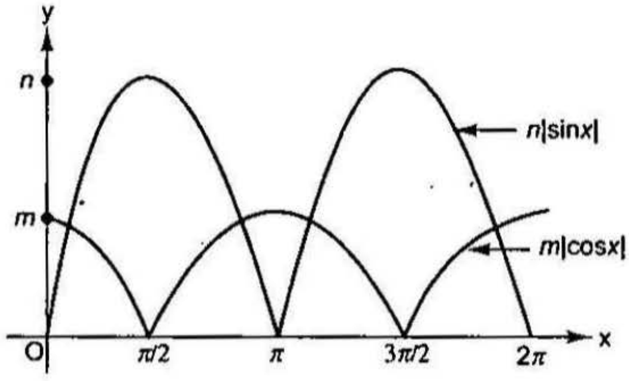


Fig. 3.16

Linked Comprehension Type

For Problems 1 – 3

1. b., 2. c., 3. a

Sol.

1. b. $x^3 - (1 + \cos\theta + \sin\theta)x^2 + (\cos\theta\sin\theta + \cos\theta + \sin\theta)x - \sin\theta\cos\theta = 0$

Given cubic function is

$f(x) = (x - 1)(x - \cos\theta)(x - \sin\theta)$

Therefore, roots are 1, $\sin\theta$ and $\cos\theta$

Hence, $x_1^2 + x_2^2 + x_3^2 = 1 + \sin^2\theta + \cos^2\theta = 2$

2. c. Now if $1 = \sin\theta$, we get $\theta = \pi/2$

If $1 = \cos\theta$, then $\theta = 0, 2\pi$

and if $\sin\theta = \cos\theta$, we get $\tan\theta = 1$

$\therefore \theta = \frac{\pi}{4}, \frac{5\pi}{4}$

Therefore, the number of values of θ in $[0, 2\pi]$ is 5.

3. a. Again the maximum possible difference between the two roots is 2.

$1 - \sin\theta = 2$ when $\theta = 3\pi/2$ or $1 - \cos\theta = 2$ when $\theta = \pi$

For Problems 4 – 6

4., a, 5. c, 6. d.

Sol. See Fig. 3.17 for the solution

$y = 2n\pi + (\pi + \alpha)$, where $0 < \alpha < \pi/2$ and $\sin\alpha = -1/\sqrt{5}$
i.e., y lies in the 3rd quadrant

Therefore, from Eq. (3), $x^{1/3}(-3/\sqrt{5}) = 3$ or $x = -5\sqrt{5}$.
Thus, $\sin^2 y + 2\cos^2 y = 1/5 + 8/5 = 9/5$.
Also there are exactly six values of $y \in [0, 6\pi]$, three in 1st quadrant and three in 3rd quadrant.

Matrix-Match Type

1. **a → r, b → s, c → p, d → q.**

a. $\cos^2 2x - \sin^2 x = 0$

$\Rightarrow \cos 3x \cos x = 0$

$\Rightarrow \cos 3x = 0$ or $\cos x = 0$

$$\Rightarrow 3x = (2n-1)\frac{\pi}{2} \text{ or } x = (2n-1)\frac{\pi}{2}, n \in Z$$

$$\Rightarrow x = (2n-1)\frac{\pi}{6} \text{ or } x = (2n-1)\frac{\pi}{2}$$

Hence, the general solution is $(2n-1)\frac{\pi}{6}$ as $(2n-1)\frac{\pi}{2}$ is contained in $(2n-1)\frac{\pi}{6}$.

b. $\cos x + \sqrt{3}\sin x = \sqrt{3}$

$$\Rightarrow \frac{\cos x}{2} + \frac{\sqrt{3}}{2}\sin x = \frac{\sqrt{3}}{2}$$

$$\Rightarrow \cos\left(x - \frac{\pi}{3}\right) = \cos\frac{\pi}{6}$$

$$\Rightarrow x - \frac{\pi}{3} = 2n\pi \pm \frac{\pi}{6}, n \in Z$$

$$\Rightarrow x - \frac{\pi}{3} = 2n\pi + \frac{\pi}{6} \text{ or } 2n\pi - \frac{\pi}{6}$$

$$\Rightarrow x = 2n\pi + \frac{\pi}{2} \text{ or } x = 2n\pi + \frac{\pi}{6}$$

c. $\sqrt{3}\tan^2 x - (\sqrt{3}+1)\tan x + 1 = 0$

$\Rightarrow \sqrt{3}\tan x(\tan x - 1) - (\tan x - 1) = 0$

$\Rightarrow (\tan x - 1)(\sqrt{3}\tan x - 1) = 0$

$\therefore \tan x = 1 \quad \Rightarrow \quad x = n\pi + \dfrac{\pi}{4}, n \in Z$

or $\tan x = \dfrac{1}{\sqrt{3}} \quad \Rightarrow \quad x = n\pi + \dfrac{\pi}{6}, n \in Z$

d. $\tan 3x - \tan 2x - \tan x = 0$

or $\tan x \tan 2x \tan 3x = 0$

$x = n\pi$ or $n\pi/2$ (rejected) or $n\pi/3$

Therefore, the general solution is $n\pi/3$ as $n\pi$ is contained in $n\pi/3$.

2. **a → q ; b → s ; c → s; d → p.**

a. Here, $x^3 + (x+2)^2 + 2\sin x = 4$.

Clearly, $x = 0$ satisfies the equation

If $0 < x \le \pi, x^3 + (x+2)^2 + 2\sin x > 4$

If $\pi < x \le 2\pi, x^3 + (x+2)^2 + 2\sin x > 27 + 25 - 2$

So, $x = 0$ is the only solution.

b. Here, $\dfrac{1}{2}\sin(2e^x) = \dfrac{1}{4}\left(2^x + 2^{-x}\right) \ge \dfrac{1}{4}2 = \dfrac{1}{2}$ $\quad$ ($\because$ A.M. $\ge$ G.M.)

$\Rightarrow \quad \sin(2e^x) \ge 1 \quad \Rightarrow \quad \sin(2e^x) = 1$

But equality can hold when $2^x = 2^{-x} = 1$, i.e., $x = 0$.

Then $\sin(2.\ e^0) = 1$, which is not true.

Hence, there is no solution.

c. $\sin 2x + \cos 4x = 2$

$\Rightarrow \quad \sin 2x = 1, \cos 4x = 1$

$\therefore 1 - 2\sin^2 2x = 1$ or $1 - 2 = 1$ (absurd)

d. The given solution is $|\sin x| = x/30$.

Therefore, the solution of this equation is the point of intersections of the curves, i.e., $y = |\sin x|$ and $y = x/30$.

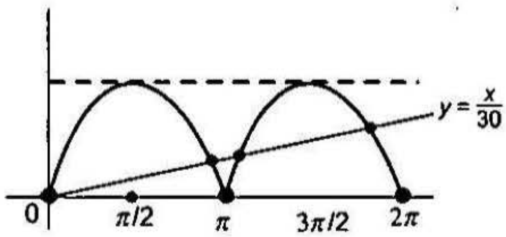


Fig. 3.18

Since there are four points of intersection in $0 \le x \le 2\pi$, the equation has four solutions.

3. **a → q; b → s; c → p; d → r.**

a. $5\sin\theta + 3(\sin\theta\cos\alpha - \cos\theta\sin\alpha)$

$= (5 + 3\cos\alpha)\sin\theta - 3\sin\alpha\cos\theta$

$$\Rightarrow \quad \max_{\theta\in R}\{5\sin\theta + 3\sin(\theta-\alpha)\} = \sqrt{(5+3\cos\alpha)^2 + 9\sin^2\alpha}$$

$$= \sqrt{34 + 30\cos\alpha}$$

Therefore, the given equation is $34 + 30\cos\alpha = 49$.

$\Rightarrow \quad \cos\alpha = 1/2 \quad \Rightarrow \quad \alpha = 2n\pi \pm \pi/3, n \in Z$

b. $(\cos x)^{\sin^2 x-3\sin x+2}=1$

$\Rightarrow \quad \sin^2 x-3\sin x+2=0 \quad \Rightarrow \quad (\sin x-2)(\sin x-1)=0 \quad \Rightarrow \quad \sin x=1$

but this does not satisfy the equation because $0^{\circ}=1$ is absurd.

c. $\sqrt{(\sin x)}+2^{1/4}\cos x=0$ (i)

$\therefore \sqrt{(\sin x)}>0$ and so $\cos x<0$,

Also $\sin x>0 \Rightarrow x$ lies in 2nd quadrant

Equation (i) can be rewritten as $2^{1/4}\cos x=-\sqrt{(\sin x)}$

On squaring, we get $\sqrt{2}\cos^2 x=\sin x$

$\Rightarrow \quad \sqrt{2}\sin^2 x+\sin x-\sqrt{2}=0 \quad \Rightarrow \quad (\sqrt{2}\sin x-1)(\sin x+\sqrt{2})=0$

$\sin x\neq-\sqrt{2}$, therefore $\sin x=1/\sqrt{2}$ (ii)

$\Rightarrow x=3\pi/4$ and so the general value of x is given by $x=2n\pi+3\pi/4,\ n\in Z$

d. $\log_5 \tan x=(\log_5 4)(\log_4(3\sin x))$

$\Rightarrow \quad \log_5 \tan x=\log_5(3\sin x)$ (i)

Since $\log x$ is real when $x>0$, we have

$\tan x>0,\ \sin x>0$

Therefoe, x lies in the first quadrant. Now Eq. (i) gives

$\tan x=3\sin x$ or $\cos x=1/3$

$\therefore x=2n\pi+\cos^{-1}(1/3),\ n\in Z$

Integer Type

1. (1) $\sin^3 x+p^3+1=3p\sin x$

$\Rightarrow (\sin x+p+1)(\sin^2 x+1+p^2-\sin x-p-p\sin x)=0$

Therefore, either $\sin x+p+1=0 \Rightarrow p=-(1+\sin x)$, or

$\sin x=1=p$

Hence, only one value of $p(p>0)$ is possible which is given by $p=1$.

2. (0) $|\sin x\cos x|+|\tan x+\cot x|=\sqrt{3}$

$$\Rightarrow |\sin x\cos x|+\frac{1}{|\sin x\cos x|}=\sqrt{3}$$

$$|\sin x\cos x|+\frac{1}{|\sin x\cos x|}\geq 2$$

Hence, there is no solution.

3. (3) $4\leq \text{L.H.S.}\leq 16$

$2\leq \text{R.H.S.}\leq 4$

Hence, equality can occur only when both sides are 4, which is possible if $x=\pi, 3\pi, 5\pi$.

That is, there are three solutions.

4. (6) $\dfrac{\sin x}{\cos 3x}+\dfrac{\sin 3x}{\cos 9x}+\dfrac{\sin 9x}{\cos 27x}=0$

$\Rightarrow \dfrac{2\sin x\cos x}{2\cos 3x\cos x}+\dfrac{2\sin 3x\cos 3x}{2\cos 9x\cos 3x}+\dfrac{2\sin 9x\cos 9x}{2\cos 27x\cos 9x}=0$

$\Rightarrow \dfrac{\sin(3x-x)}{2\cos 3x\cos x}+\dfrac{\sin(9x-3x)}{2\cos 9x\cos 3x}+\dfrac{\sin(27x-9x)}{2\cos 27x\cos 9x}=0$

$\Rightarrow (\tan 3x-\tan x)+(\tan 9x-\tan 3x)+(\tan 27x-\tan 9x)=0$

$\Rightarrow \tan 27x-\tan x=0$

$\Rightarrow \tan x=\tan 27x$

$\Rightarrow 27x=n\pi+x, n\in I$

$\Rightarrow x=\dfrac{n\pi}{26}, n\in I.$

$\Rightarrow x=\dfrac{\pi}{26},\dfrac{2\pi}{26},\dfrac{3\pi}{26},\dfrac{4\pi}{26},\dfrac{5\pi}{26},\dfrac{6\pi}{26}$

Hence, there are six solutions.

5. (1) $\left(\sqrt{3}+1\right)^{2x}+\left(\sqrt{3}-1\right)^{2x}=2^{3x}=\left(2\sqrt{2}\right)^{2x}$

$\Rightarrow \left(\dfrac{\sqrt{3}+1}{2\sqrt{2}}\right)^{2x}+\left(\dfrac{\sqrt{3}-1}{2\sqrt{2}}\right)^{2x}=1$

$\Rightarrow (\sin 75°)^{2x}+(\cos 75°)^{2x}=1$

$\Rightarrow x=1$

6. (4) Since $-2\le \sin x-\sqrt{3}\cos x\le 2$

$\Rightarrow -2\le \dfrac{4m-6}{4-m}\le 2$

or $-1\le \dfrac{2m-3}{4-m}\le 1$

if $\dfrac{2m-3}{4-m}\le 1.$

$\Rightarrow \dfrac{(2m-3)-(4-m)}{4-m}\le 0$

$\Rightarrow \dfrac{3m-7}{m-4}\ge 0$ (i)

Also, $-1\le \dfrac{2m-3}{4-m}$

$\Rightarrow \dfrac{m+1}{m-4}\le 0$ (ii)

From Eq. (i) and (ii), we get $m\in\left[-1,\dfrac{7}{3}\right]$.

Therefore, the possible integers are $-1, 0, 1, 2$.

7. (1) Adding given equations, we get

$$2 = \frac{3a}{2} + \frac{a^2}{2}$$

$$\Rightarrow a^2 + 3a - 4 = 0$$

$$\Rightarrow (a+4)(a-1) = 0$$

$\Rightarrow a = 1$ (as $a = -4$ is rejected)

8. (5) $\cos 4x = 2\cos^2 2x - 1$

$$= 2(2\cos^2 x - 1)^2 - 1$$

$$= 2(4\cos^4 x + 1 - 4\cos^2 x) - 1$$

$$= 8\cos^4 x - 8\cos^2 x + 1$$

$\therefore a_0 = 1;\ a_1 = -8,\ a_2 = 8$

$\therefore 5a_0 + a_1 + a_2 = 5$

9. (1) $1 - \sin^2 x - \sin x + a = 0$

$\Rightarrow \sin^2 x + \sin x - (a+1) = 0$ (i)

From Eq. (i), we get

$\sin^2 x + \sin x = (a+1)$

For $x \in (0, \pi/2)$, the range of $\sin^2 x + \sin x$ is $(0, 2)$.

$\Rightarrow 0 < (a+1) < 2 \Rightarrow a \in (-1, 1)$

10. (6) $a\sin x + 1 - 2\sin^2 x = 2a - 7$

$\Rightarrow 2\sin^2 x - a\sin x + (2a - 8) = 0$

$$\Rightarrow \sin x = \frac{a \pm \sqrt{a^2 - 8(2a-8)}}{4} = \frac{a \pm (a-8)}{4} = 2 \text{ or } \frac{a-4}{2}$$

For a solution $-1 \le \frac{a-4}{2} \le 1$, we have $2 \le a \le 6$.

11. (0) $$\frac{\sin^2\left(x - \frac{\pi}{4}\right)}{\cos 2x} = \frac{\frac{1}{2}(\sin x - \cos x)^2}{\cos^2 x - \sin^2 x} = \frac{-\frac{1}{2}(\sin x - \cos x)}{\cos x + \sin x} = -\frac{1}{2}\tan\left(x - \frac{\pi}{4}\right)$$

Given equation reduces to $2^{\tan\left(x - \frac{\pi}{4}\right)} - 2(0.25)^{\frac{1}{2}\tan\left(x - \frac{\pi}{4}\right)} + 1 = 0$

$$\Rightarrow 2^{\tan\left(x - \frac{\pi}{4}\right)} = 1$$

$\Rightarrow x = \pi/4$ which is not possible as $\cos 2x = 0$ for this value of x, which is not defining the original equation.

12. (4) $\sin^4 x - \cos^2 x \sin x + 2\sin^2 x + \sin x = 0$

$\Rightarrow \sin x\,[\sin^3 x - \cos^2 x + 2\sin x + 1] = 0$

$\Rightarrow \sin x\,[\sin^3 x + \sin^2 x + 2\sin x] = 0$

$\Rightarrow \sin^2 x[\sin^2 x + \sin x + 2] = 0$

$\Rightarrow \sin x = 0$, where $x = 0, \pi, 2\pi, 3\pi$

Hence, there are four solutions.

Archives

Subjective

1. At the intersection point of $y = \cos x$ and $y = \sin 3x$, we have $\cos x = \sin 3x$

$$\Rightarrow \quad \cos x = \cos\left(\frac{\pi}{2} - 3x\right) \quad \Rightarrow \quad x = 2n\pi \pm \left(\frac{\pi}{2} - 3x\right), n \in Z$$

$$\Rightarrow \quad x = \frac{\pi}{4}, \frac{\pi}{8}, -\frac{3\pi}{8} \qquad [\because -\pi/2 \le x \le \pi/2]$$

Thus, the points are $\left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right)$, $\left(\frac{\pi}{8}, \cos\frac{\pi}{8}\right)$ and $\left(-\frac{3\pi}{8}, \cos\frac{3\pi}{8}\right)$

2. The given equation is

$4\cos^2 x \sin x - 2\sin^2 x = 3\sin x$

$\Rightarrow \quad 4\cos^2 x \sin x - 2\sin^2 x - 3\sin x = 0$

$\Rightarrow \quad 4(1 - \sin^2 x)\sin x - 2\sin^2 x - 3\sin x = 0$

$\Rightarrow \quad \sin x\,[4\sin^2 x + 2\sin x - 1] = 0$

$\Rightarrow \quad$ either $\sin x = 0$ or $4\sin^2 x + 2\sin x - 1 = 0$

If $\sin x = 0 \quad \Rightarrow \quad x = n\pi, n \in Z$

If $4\sin^2 x + 2\sin x - 1 = 0$

$$\Rightarrow \quad \sin x = \frac{-1 \pm \sqrt{5}}{4}$$

$$\therefore x = m\pi + (-1)^m \sin^{-1}\left(\frac{-1 \pm \sqrt{5}}{4}\right), m \in Z$$

Thus, $x = n\pi,\ m\pi \pm (-1)^m \sin^{-1}\left(\frac{-1 \pm \sqrt{5}}{4}\right)$.

where m and n are integers.

3. The given equation is $8^{(1 + |\cos x| + |\cos^2 x| + \cos^3 x| + \cdots)} = 4^3$

$\Rightarrow \quad 2^{3(1 + |\cos x| + |\cos^2 x| + \cos^3 x| + \cdots \infty)} = 2^6$

$\Rightarrow \quad 3(1 + |\cos x| + |\cos^2 x| + \cos^3 x| + \cdots) = 6$

$\Rightarrow \quad 1 + |\cos x| + |\cos^2 x| + |\cos^3 x| + \cdots = 2$

$$\Rightarrow \quad \frac{1}{1 - |\cos x|} = 2$$

$\Rightarrow \quad |\cos x| = 1/2$

$$\Rightarrow \quad |\cos x| = \frac{1}{2}$$

$\Rightarrow \quad x = \pi/3, -\pi/3, 2\pi/3, -2\pi/3, \ldots$

The values of $x \in (-\pi, \pi)$ are $\pm\pi/3, \pm 2\pi/3$.

4. Given that $(1-\tan\theta)(1+\tan\theta)\sec^2\theta + 2^{\tan^2\theta} = 0$

$\Rightarrow \quad (1-\tan^2\theta)(1+\tan^2\theta) + 2^{\tan^2\theta} = 0$

Let us put $\tan^2\theta = t$

$\therefore (1-t)(1+t)+2^t=0 \quad \Rightarrow \quad 1-t^2+2^t=0$

It is clearly satisfied by $t = 3$

as $-8+8=0$, we get $\tan^2\theta = 3$

$\therefore\ \theta = \pm\pi/3$ in the given interval.

5. $(y+z)\cos 3\theta - (xyz)\sin 3\theta = 0$ (i)

$xyz\sin 3\theta = 2\cos(3\theta)z + (2\sin 3\theta)y$ (ii)

$\therefore (y+z)\cos 3\theta = (2\cos 3\theta)z + (2\sin 3\theta)y = (y+2z)\cos 3\theta + y\sin 3\theta$

$\therefore y(\cos 3\theta - 2\sin 3\theta) = z\cos 3\theta$ and $y(\sin 3\theta - \cos 3\theta) = 0$

$\Rightarrow \sin 3\theta - \cos 3\theta = 0$

$\therefore \sin 3\theta = \cos 3\theta$

$\therefore 3\theta = n\pi + (\pi/4), n \in Z \Rightarrow \theta = \dfrac{(4n+1)\pi}{3} \Rightarrow \theta = \dfrac{\pi}{3}$ only

6. $\tan\theta = \cot 5\theta$

$\Rightarrow \cos 6\theta = 0$

$\Rightarrow 4\cos^3 2\theta - 3\cos 2\theta = 0$

$\Rightarrow \cos 2\theta = 0$ or $\pm\dfrac{\sqrt{3}}{2}$ (i)

$\sin 2\theta = \cos 4\theta$

$\Rightarrow 2\sin^2 2\theta + \sin 2\theta - 1 = 0$

$\Rightarrow 2\sin^2 2\theta + 2\sin 2\theta - \sin 2\theta - 1 = \theta$

$\Rightarrow \sin 2\theta = -1$ or $\sin 2\theta = \dfrac{1}{2}$

$\Rightarrow \cos 2\theta = 0$ and $\sin 2\theta = -1$

$\Rightarrow 2\theta = -\dfrac{\pi}{2} \Rightarrow \theta = -\dfrac{\pi}{4}$

or $\cos 2\theta = \pm\dfrac{\sqrt{3}}{2}, \sin 2\theta = \dfrac{1}{2}$

$\Rightarrow 2\theta = \dfrac{\pi}{6}, \dfrac{5\pi}{6} \Rightarrow \theta = \dfrac{\pi}{12}, \dfrac{5\pi}{12}$

$\therefore \quad \theta = -\dfrac{\pi}{4}, \dfrac{\pi}{12}, \dfrac{5\pi}{12}$

Objective

Fill in the blanks

1. We have $\cos x + \cos y = \dfrac{3}{2}$

$\Rightarrow \quad 2\cos\left(\dfrac{x+y}{2}\right)\cos\left(\dfrac{x-y}{2}\right) = \dfrac{3}{2}$

$\Rightarrow \quad 2\cos\frac{\pi}{3}\cos\left(\frac{x-y}{2}\right)=\frac{3}{2}$ [using : $x+y=2\pi/3$]

$\Rightarrow \quad \cos\left(\frac{x-y}{2}\right)=\frac{3}{2}$, which is not possible.

Hence, the system of equations has no solution.

2. We have $2\sin^2 x-3\sin x+1\geq 0$

$\Rightarrow \quad (2\sin x-1)(\sin x-1)\geq 0$

$\Rightarrow \quad \left(\sin x-\frac{1}{2}\right)(\sin x-1)\geq 0$

$\Rightarrow \quad \sin x\leq\frac{1}{2}$ or $\sin x\geq 1$

But we know that $\sin x\leq 1$ and $\sin x\geq 0$ for $x\in[0,\pi]$.

Therefore, either $\sin x=1$ or $0\leq\sin x\leq\frac{1}{2}$ $\Rightarrow$ either $x=\pi/2$ or $x\in[0,\pi/6]\cup[5\pi/6,\pi]$

Combining, we get $x\in\left[0,\frac{\pi}{6}\right]\cup\left\{\frac{\pi}{2}\right\}\cup\left[\frac{5\pi}{6},\pi\right]$.

3. $\tan^2\theta+\sec 2\theta=1$

$t^2+\frac{1+t^2}{1-t^2}=1$, where $t=\tan\theta$

$\Rightarrow \quad t^2(t^2-3)=0 \quad \Rightarrow \quad \tan\theta=0,\pm\sqrt{3} \quad \Rightarrow \quad \theta=n\pi$ and $\theta=n\pi\pm\pi/3,\ n\in Z$.

4. $\cos^7 x=1-\sin^4 x$

$=(1-\sin^2 x)(1+\sin^2 x)$

$=\cos^2 x(1+\sin^2 x)$

$\therefore\ \cos x=0$ or $x=\pi/2,-\pi/2$, or $\cos^5 x=1+\sin^2 x$

$\cos^5 x\leq 1$ but $1+\sin^2 x\geq 1$

$\Rightarrow \quad \cos^5 x=1+\sin^2 x=1$

$\Rightarrow \quad \cos x=1$ and $\sin x=0$. [both these imply $x=0$]

Hence, $x=-\frac{\pi}{2},\frac{\pi}{2}$ and 0.

True or false

1. Given that equation is $\sin^4\theta-2\sin^2\theta-1=0$

$\therefore\ \sin^2\theta=\frac{2\pm\sqrt{8}}{2}=1\pm\sqrt{2}$

But $\sin^2\theta$ cannot be –ve

$\therefore\ \sin^2\theta=\sqrt{2}+1$

But as $-1\leq\sin\theta\leq 1$, $\sin^2\theta\neq\sqrt{2}+1$

Thus there is no value of θ which satisfies the given equation.

Therefore, statement is false.

Multiple choice questions with one correct answer

1. a. The given equation is

$$2\cos^2\left(\frac{x}{2}\right)\sin^2 x=x^2+\frac{1}{x^2}$$

Now $\cos(\alpha-\beta)=1 \Rightarrow \alpha-\beta=0 \Rightarrow \alpha=\beta$

$\therefore \cos(\alpha+\beta)=1/e \Rightarrow \cos 2\alpha=1/e$

$\because 0<1/e<1$ and $2\alpha\in[-2\pi,2\pi]$

There will be two values of 2α satisfying $\cos 2\alpha=1/e$ in $[0,2\pi]$ and two in $[-2\pi,0]$.

Therefore, there will be four values of a in $[-\pi,\pi]$ and correspondingly four values of β. Hence, there are four sets of (α,β).

10. a. $2\sin^2\theta-5\sin\theta+2>0$

$\Rightarrow (\sin\theta-2)(2\sin\theta-1)>0 \Rightarrow \sin\theta<1/2$ $\quad [\because -1\le\sin\theta\le 1]$

$\Rightarrow x\in(0,\pi/6)\cup(5\pi/6,2\pi)$

11. c. $2\sin^2\theta-\cos 2\theta=0$

$\Rightarrow 1-2\cos 2\theta=0$

$$\Rightarrow \cos 2\theta=\frac{1}{2}$$

$$\Rightarrow 2\theta=\frac{\pi}{3},\frac{5\pi}{3},\frac{7\pi}{3},\frac{11\pi}{3}$$

$$\Rightarrow \theta=\frac{\pi}{6},\frac{5\pi}{6},\frac{7\pi}{6},\frac{11\pi}{6} \quad \text{(i)}$$

where $\theta\in[0,2\pi]$.

Also $2\cos^2\theta-3\sin\theta=0$

$\Rightarrow 2\sin^2\theta+3\sin\theta-2=0$

$\Rightarrow (2\sin\theta-1)(\sin\theta+2)=0$

$\Rightarrow \sin\theta=1/2$ $\quad [\because \sin\theta\neq-2]$

$\Rightarrow \theta=\pi/6,\ 5\pi/6$ where $\theta\in[0,2\pi]$ $\quad$ (ii)

Combining Eqs. (i) and (ii), we get $\theta=\dfrac{\pi}{6},\dfrac{5\pi}{6}$

Therefore, there are two solutions.

Multiple choice questions with one or more than one correct answer

1. d. Since $a_1+a_2\cos 2x+a_3\sin^2 x=0$ for all x

Putting $x=0$ and $x=\pi/2$, we get

$a_1+a_2=0$, and $a_1-a_2+a_3=0$

$\Rightarrow a_2=-a_1$ and $a_3=-2a_1$

Therefore, the given equation becomes

$a_1-a_1\cos 2x-2a_1\sin^2 x=0,\ \forall x$

$\Rightarrow a_1(1-\cos 2x-2\sin^2 x)=0,\ \forall x \Rightarrow a_1(2\sin^2 x-2\sin^2 x)=0,\ \forall x$

The above is satisfied for all values of a_1.

Hence, the infinite number of triplets $(a_1,-a_1,-2a_1)$ is possible.

2. a, c. We have

$$\begin{vmatrix} 1+\sin^2\theta & \cos^2\theta & 4\sin 4\theta \\ \sin^2\theta & 1+\cos^2\theta & 4\sin 4\theta \\ \sin^2\theta & \cos^2\theta & 1+4\sin 4\theta \end{vmatrix} = 0$$

Operating $C_1 \to C_1 + C_2$, we get

$$\begin{vmatrix} 2 & \cos^2\theta & 4\sin 4\theta \\ 2 & 1+\cos^2\theta & 4\sin 4\theta \\ 1 & \cos^2\theta & 1+4\sin 4\theta \end{vmatrix} = 0$$

Operating $R_1 \to R_1 - R_2$; $R_2 \to R_2 - R_3$, we get

$$\begin{vmatrix} 1 & -1 & 0 \\ 1 & 1 & -1 \\ 1 & \cos^2\theta & 1+4\sin 4\theta \end{vmatrix} = 0$$

Expanding along R_1, we get $1 + 4\sin 4\theta + 1 = 0$

$\Rightarrow \; 2(1+2\sin 4\theta) = 0 \quad \Rightarrow \quad \sin 4\theta = -1/2 \quad \Rightarrow \quad 4\theta = \pi + \pi/6$ or $2\pi - \pi/6$

$\Rightarrow \; 4\theta = 7\pi/6$ or $11\pi/6 \quad \Rightarrow \quad \theta = 7\pi/24$ or $11\pi/24$. Hence, there are two correct options.

3. c. $3\sin^2 x - 7\sin x + 2 = 0$

$\Rightarrow \; (\sin x - 2)(3\sin x - 1) = 0 \quad \Rightarrow \quad \sin x = 1/3 = \sin\alpha$, say, ($\sin x = 2$, not possible)

$x = n\pi + (-1)^n \alpha$, $n = 0, 1, 2, 3, 4, 5$ in $(0, 5\pi)$

4. d. $2\sin^2 x + 3\sin x - 2 > 0$

$(2\sin x - 1)(\sin x + 2) > 0$

$\Rightarrow \; 2\sin x - 1 > 0$ $\qquad [\because -1 \le \sin x \le 1]$

$\Rightarrow \; \sin x > 1/2 \quad \Rightarrow \quad x \in (\pi/6, 5\pi/6)$ (i)

Also $x^2 - x - 2 < 0 \quad \Rightarrow \quad (x-2)(x+1) < 0 \quad \Rightarrow \quad -1 < x < 2$ (ii)

Combining Eqs. (i) and (ii), we get

$x \in (\pi/6, 2)$

5. a, b. $\dfrac{(\sin x)^4}{2} + \dfrac{(\cos x)^4}{3} = \dfrac{1}{5}$

$3 - 6\cos^2 x + 5(\cos x)^4 = \dfrac{6}{5}$. Let $\cos x = t$

$\Rightarrow \; 25t^4 - 30t^2 + 9 = 0$

$\Rightarrow \; t^2 = \dfrac{3}{5}$

$\Rightarrow \; \tan^2 x = \dfrac{2}{3}$

$\Rightarrow \; (\sin x)^8 = \left(\dfrac{2}{5}\right)^4 = \dfrac{16}{625}$

$\Rightarrow \; (\cos x)^8 = \left(\dfrac{\sqrt{3}}{\sqrt{5}}\right)^4 = \dfrac{81}{625}$

$\Rightarrow \; \dfrac{(\sin x)^8}{8} + \dfrac{(\cos x)^8}{27} = \dfrac{1}{125}$

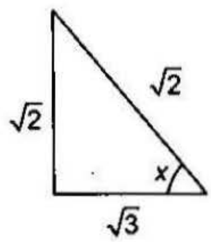


Fig. 3.19

where $0 < x \le \frac{\pi}{2}$

L.H.S. $= 2\cos^2\left(\frac{x}{2}\right)\sin^2 x = (1+\cos x)\sin^2 x$

$\because$ $1 + \cos x < 2$ and $\sin^2 x \le 1$ for $0 < x < \frac{\pi}{2}$

$\therefore$ $(1 + \cos x) \sin^2 x < 2$

and R.H.S. $= x^2 + \frac{1}{x^2} \ge 2$

$\therefore$ For $0 < x \le \frac{\pi}{2}$, given equation is not possible for any real value of x.

2. c. $\sin x + \cos x = 1$

$\Rightarrow$ $\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x = \frac{1}{\sqrt{2}}$

$\Rightarrow$ $\sin x \cos (\pi/4) + \cos x \sin (\pi/4) = \sin \pi/4$

$\Rightarrow$ $\sin (x + \pi/4) = \sin \pi/4$

$\Rightarrow$ $x + (\pi/4) = n\pi + (-1)^n \pi/4, n \in Z$

$\Rightarrow$ $x = n\pi + [(-1)^n \pi/4] - \pi/4$

where $n = 0, \pm 1, \pm 2, ...$

3. b. The given equation is

$\sin x - 3 \sin 2x + \sin 3x = \cos x - 3 \cos 2x + \cos 3x$

$\Rightarrow$ $2 \sin 2x \cos x - 3 \sin 2x = 2 \cos 2x \cos x - 3 \cos 2x$

$\Rightarrow$ $\sin 2x (2 \cos x - 3) = \cos 2x (2 \cos x - 3)$

$\Rightarrow$ $\sin 2x = \cos 2x$ (as $\cos x \ne 3/2$)

$\Rightarrow$ $\tan 2x = 1 \Rightarrow 2x = n\pi + \pi/4 \Rightarrow x = \frac{n\pi}{2} + \frac{\pi}{8}, n \in Z$

4. d. The given equation is

$(\cos p - 1) x^2 + (\cos p) x + \sin p = 0$

For this equation to have real roots $D \ge 0$

$\Rightarrow$ $\cos^2 p - 4 \sin p (\cos p - 1) \ge 0$

$\Rightarrow \cos^2 p - 4 \sin p \cos p + 4 \sin^2 p + 4 \sin p - 4 \sin^2 p \ge 0$

$\Rightarrow$ $(\cos p - 2 \sin p)^2 + 4 \sin p (1 - \sin p) \ge 0$

For every real value of p, we have

$(\cos p - 2 \sin p)^2 \ge 0$ and $\sin p(1 - \sin p) \ge 0$

$\therefore D \ge 0, \forall \sin p \in (0, \pi)$

5. c. The given equation is

$\tan x + \sec x = 2 \cos x$

$\Rightarrow$ $\frac{\sin x}{\cos x} + \frac{1}{\cos x} = 2\cos x$

$\Rightarrow$ $\sin x + 1 = 2 \cos^2 x = 2 - 2 \sin^2 x$

$\Rightarrow$ $2 \sin^2 x + \sin x - 1 = 0$

$\Rightarrow$ $(2 \sin x - 1) (\sin x + 1) = 0$

$\Rightarrow \quad \sin x = \frac{1}{2}, -1$

$\Rightarrow \quad x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2} \in [0, 2\pi]$

But for $x = 3\pi/2$, $\tan x$ and $\sec x$ are not defined.
Therefore, there are only two solutions.

6. d. The given equation is
$2\sin^2\theta - 3\sin\theta - 2 = 0$
$\Rightarrow \quad (2\sin\theta + 1)(\sin\theta - 2) = 0$

$\Rightarrow \quad \sin = -\frac{1}{2}$ [$\because \sin\theta - 2 = 0$ is not possible].

$\Rightarrow \quad \sin\theta = \sin(-\pi/6) = \sin(7\pi/6)$
$\Rightarrow \quad \theta = n\pi + (-1)^n(-\pi/6) = n\pi + [(-1)^n\, 7\pi/6]$
$\Rightarrow \quad$ Thus, $\theta = n\pi + (-1)^n\, 7\pi/6, n \in Z$

7. c. To simplify the determinant, let $\sin x = a$; $\cos x = b$. Then the equation becomes

$\begin{vmatrix} a & b & b \\ b & a & b \\ b & b & a \end{vmatrix} = 0$. Operating $C_2 \to C_2 - C_1$; $C_3 \to C_3 - C_2$, we get

$$\begin{vmatrix} a & b-a & 0 \\ b & a-b & b-a \\ b & 0 & a-b \end{vmatrix} = 0.$$

$\Rightarrow \quad a(a-b)^2 - (b-a)[b(a-b) - b(b-a)] = 0$
$\Rightarrow \quad a(a-b)^2 - 2b(b-a)(a-b) = 0$
$\Rightarrow \quad (a-b)^2(a-2b) = 0$
$\Rightarrow \quad a = b$ or $a = 2b$

$\Rightarrow \quad \frac{a}{b} = 1$ or $\frac{a}{b} = 2$

$\Rightarrow \quad \tan x = 1$ or $\tan x = 2$
But we have $-\pi/4 \le x \le \pi/4$
$\Rightarrow \quad \tan(\pi/4) \le \tan x \le \tan(\pi/4)$
$\Rightarrow \quad -1 \le \tan x \le 1$
$\therefore \tan x = 1 \quad \Rightarrow \quad x = \pi/4$
Therefore, there is only one real root.

8. b. We know that

$-\sqrt{a^2+b^2} \le a\cos\theta + b\sin\theta \le \sqrt{a^2+b^2} \quad \Rightarrow \quad -\sqrt{74} \le 2k+1 \le \sqrt{74}$
$\Rightarrow \quad -8 \le 2k+1 \le 8 \quad \Rightarrow \quad -4.5 \le k \le 3.5$
(considering only integral values) $\Rightarrow$ k can take eight integral values.

9. d. Given that $\cos(\alpha - \beta) = 1$ and $\cos(\alpha + \beta) = 1/e$
where $\alpha, \beta \in [-\pi, \pi]$

CHAPTER 4

Inverse Trigonometric Functions

INTRODUCTION

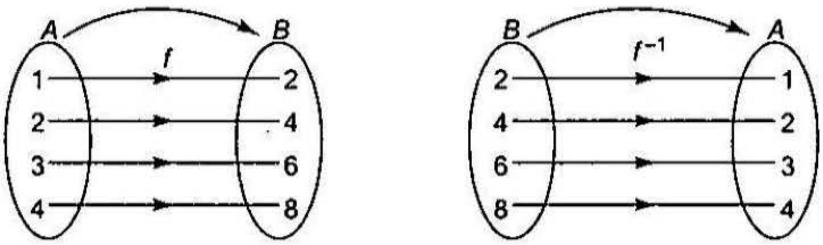


Fig. 4.1

If $f: X \to Y$ is a function defined by $y = f(x)$ such that f is both one-one and onto, then there exists a unique function $g: Y \to X$ such that for each $y \in Y$, $g(y) = x$ if and only if $y = f(x)$. The function g so defined is called the inverse of f and denoted by f^{-1}. Also if g is the inverse of f, then f is the inverse of g; and the two functions f and g are said to be inverses of each other.

The condition for existence of inverse of a function is that the function must be one-one and onto. Whenever an inverse function is defined, the range of the original function becomes the domain of the inverse function and domain of the original function becomes the range of the inverse function.

We know that trigonometric functions are many-one in their actual domain. Hence, for inverse functions to get defined, the actual domain of trigonometric functions must be restricted to make the function one-one.

Inverse Circular Functions

Since the domain of sine function is the set of all real numbers and range is $[-1, 1]$, if we restrict its domain to $[-\pi/2, \pi/2]$, then it becomes one-one and onto within the range $[-1, 1]$. Actually, sine function can be restricted to any of the intervals $[-3\pi/2, -\pi/2]$, $[-\pi/2, \pi/2]$, $[\pi/2, 3\pi/2]$, etc. It becomes one-one and its range is $[-1, 1]$. We can, therefore, define the inverse of sine function in each of these intervals. We denote the inverse of sine function by $\sin^{-1}$ (arc sine function). Thus, $\sin^{-1}$ is a function whose domain is $[-1, 1]$ and the range could be any of the intervals $[-3\pi/2, -\pi/2]$, $[-\pi/2, \pi/2]$ or $[\pi/2, 3\pi/2]$ and so on. Corresponding to each such interval, we get a branch of the function $\sin^{-1}$. The branch with range $[-\pi/2, \pi/2]$ is called the principal value branch, whereas other intervals as range give different branches of $\sin^{-1}$. When we refer to the function $\sin^{-1}$, we take it as the function whose domain is $[-1, 1]$ and range is $[-\pi/2, \pi/2]$. We write $\sin^{-1}$: $[-1, 1] \to [-\pi/2, \pi/2]$.

From the definition of the inverse functions, it follows that $\sin(\sin^{-1}x) = x$ if $-1 \le x \le 1$ and $\sin^{-1}(\sin x) = x$ if $-\pi/2 \le x \le \pi/2$.

If any point (x_1, y_1) lies on the curve $y = f(x)$, then corresponding to it (y_1, x_1) lies on $y = f^{-1}(x)$. Since points (x_1, y_1) and the (y_1, x_1) are symmetrical about the line $y = x$, the graphs of $y = f(x)$ and $y = f^{-1}(x)$ are symmetrical about the line $y = x$.

Note:

- $\sin^{-1}x$ is *entirely different from* $(\sin x)^{-1}$. *The former is the measure of an angle in radians whose* sine *is x while the latter is* $\dfrac{1}{\sin x}$.

Domain, Range and Graphs of Inverse Trigonometric Functions

With reference to the preceding discussion, the domain, range and graphs of inverse trigonometric functions can be summarized as follows.

In the following figures, dotted line graphs are of trigonometric functions and solid line graphs are of corresponding inverse trigonometric functions.

$f(x) = \sin^{-1}x$

Domain: $[-1, 1]$

Range (principal values): $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$

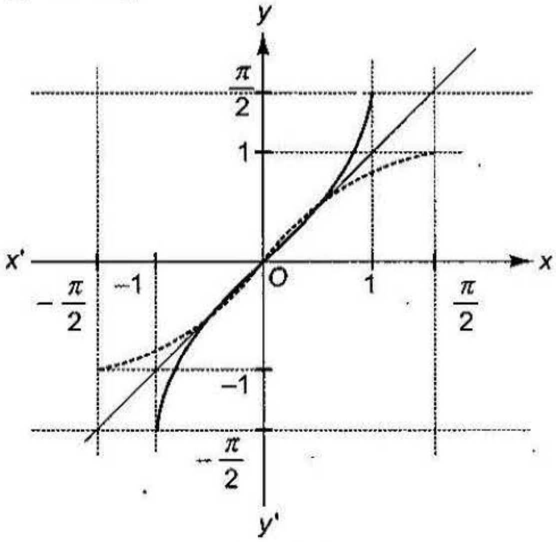


Fig. 4.2

$f(x) = \cos^{-1}x$

Domain: $[-1, 1]$

Range (principal values): $[0, \pi]$

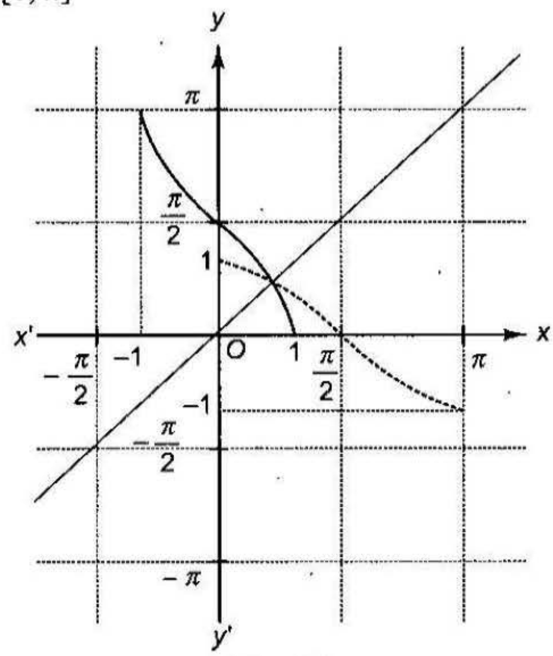


Fig. 4.3

$f(x) = \tan^{-1} x$
Domain: R

Range (principal values): $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$

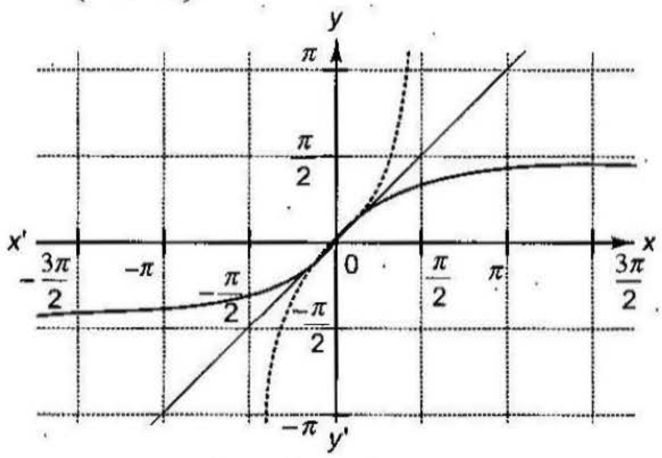


Fig. 4.4

$f(x) = \cot^{-1} x$
Domain: R
Range (principal values): $(0, \pi)$

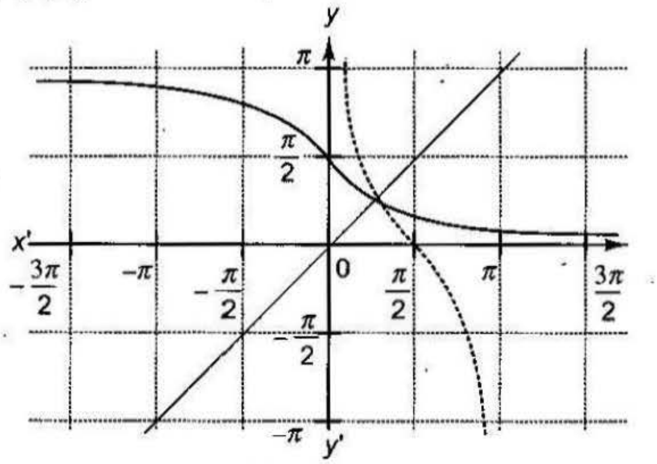


Fig. 4.5

$f(x) = \sec^{-1} x$
Domain: $(-\infty, -1] \cup [1, \infty)$
Range (principal values): $[0, \pi] - \{\pi/2\}$

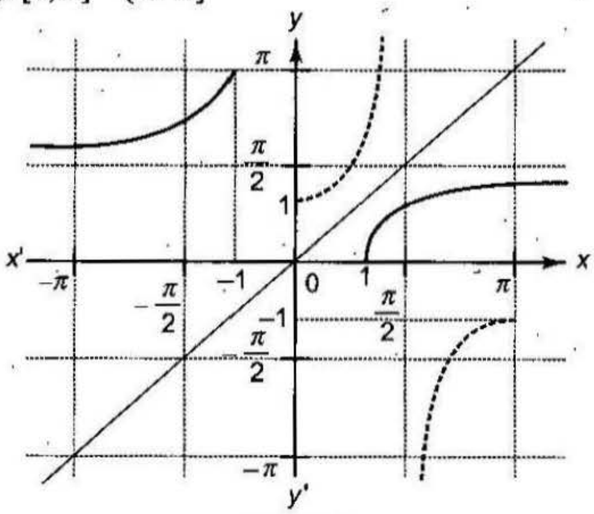


Fig. 4.6

$f(x) = \text{cosec}^{-1}x$

Domain: $(-\infty, -1] \cup [1, \infty)$

Range (principal values): $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

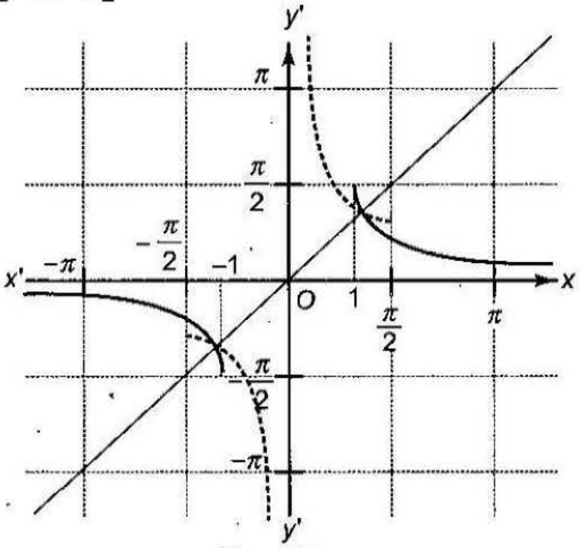
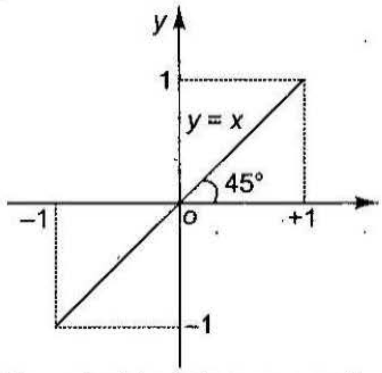


Fig. 4.7

PROPERTIES AND IMPORTANT FORMULAS OF INVERSE FUNCTIONS

Property 1

i. $\sin(\sin^{-1} x) = x, \quad \text{for all } x \in [-1, 1]$

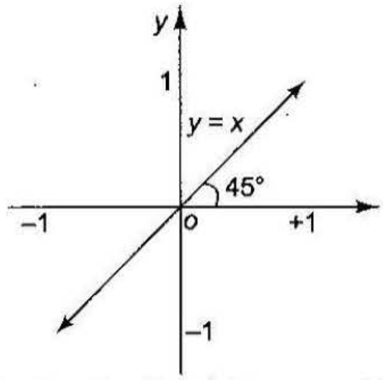


Graph of $y = \sin(\sin^{-1}x)$ or $y = \cos(\cos^{-1}x)$

Fig. 4.8

ii. $\cos(\cos^{-1} x) = x, \quad \text{for all } x \in [-1, 1]$

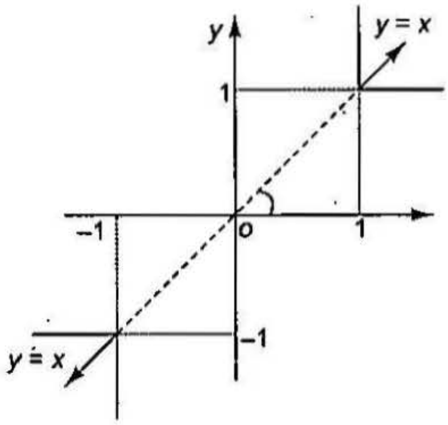
iii. $\tan(\tan^{-1} x) = x, \quad \text{for all } x \in R$



Graph of $y = \tan(\tan^{-1}x)$ or $y = \cot(\cot^{-1}x)$

Fig. 4.9

iv. $\cot(\cot^{-1} x) = x$, for all $x \in R$
v. $\operatorname{cosec}(\operatorname{cosec}^{-1} x) = x$, for all $x \in (-\infty, -1] \cup [1, \infty)$



Graph of $y = \sec(\sec^{-1}x)$
or $y = \operatorname{cosec}(\operatorname{cosec}^{-1}x)$

Fig. 4.10

vi. $\sec(\sec^{-1} x) = x$, for all $x \in (-\infty, -1] \cup [1, \infty)$

Property 2

i. $\sin^{-1}(\sin x) = x$, for all $x \in [-\pi/2, \pi/2]$
$\sin^{-1}(\sin x)$ is defined when $\sin x \in [-1, 1]$ which is true $\forall\, x \in R$.
But range of $\sin^{-1}x$ is $[-\pi/2, \pi/2]$, hence $\sin^{-1}(\sin x) = x$ is true only for $x \in [-\pi/2, \pi/2]$.
With the same reasoning, we have the following results.

ii. $\cos^{-1}(\cos x) = x$, for all $x \in [0, \pi]$
iii. $\tan^{-1}(\tan x) = x$, for all $x \in (-\pi/2, \pi/2)$
iv. $\cot^{-1}(\cot x) = x$, for all $x \in (0, \pi)$
v. $\operatorname{cosec}^{-1}(\operatorname{cosec} x) = x$, for all $x \in [-\pi/2, \pi/2] - \{0\}$
vi. $\sec^{-1}(\sec x) = x$, for all $x \in [0, \pi] - \{\pi/2\}$

Graph of $y = \sin^{-1}(\sin x)$

For x not lying in the principal domain, we have the following method to draw the graph.
Consider $y = \sin^{-1}(\sin x) \Rightarrow \sin y = \sin x \Rightarrow y = n\pi + (-1)^n x, n \in Z$
Now, keeping in mind that $y \in [-\pi/2, \pi/2]$, we have the following table:

Value of n	**Relation**	**Range of x**
...	...	...
...	...	...
$n = -2$	$y = -2\pi + x$	$x \in [3\pi/2, 5\pi/2]$
$n = -1$	$y = -\pi - x$	$x \in [-3\pi/2, -\pi/2]$
$n = 0$	$y = x$	$x \in [-\pi/2, \pi/2]$
$n = 1$	$y = \pi - x$	$x \in [\pi/2, 3\pi/2]$
$n = 2$	$y = 2\pi + x$	$x \in [-5\pi/2, -3\pi/2]$
...	...	...
...	...	...

From the preceding information, we can plot the graph of $y = \sin^{-1}(\sin x)$ as follows (Fig. 4.11):

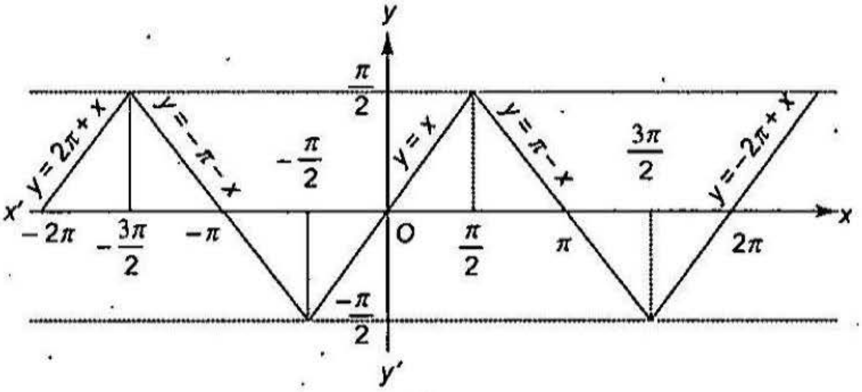


Fig. 4.11

Graph of $y = \cos^{-1}(\cos x)$

$y = \cos^{-1}(\cos x)$

$\Rightarrow \quad \cos y = \cos x \quad \Rightarrow \quad y = 2n\pi \pm x,\ n \in Z$

Now, keeping in mind that $y \in [0, \pi]$, we can plot the graph of $y = \sin^{-1}(\sin x)$ as follows (Fig. 4.12):

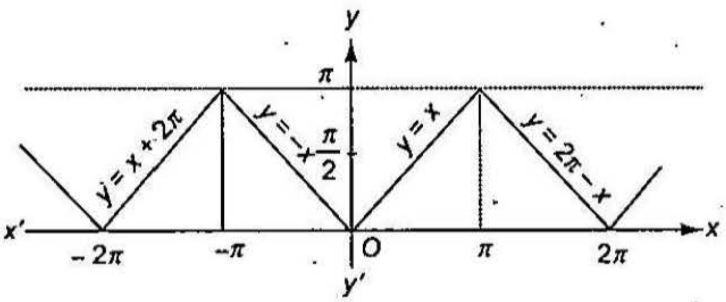


Fig. 4.12

Graph of $y = \tan^{-1}(\tan x)$

$y = \tan^{-1}(\tan x)$

$\Rightarrow \quad \tan y = \tan x \quad \Rightarrow \quad y = n\pi + x,\ n \in Z$

Now, keeping in mind that $y \in (-\pi/2, \pi/2)$, we can plot the graph of $y = \tan^{-1}(\tan x)$ as follows (Fig. 4.13):

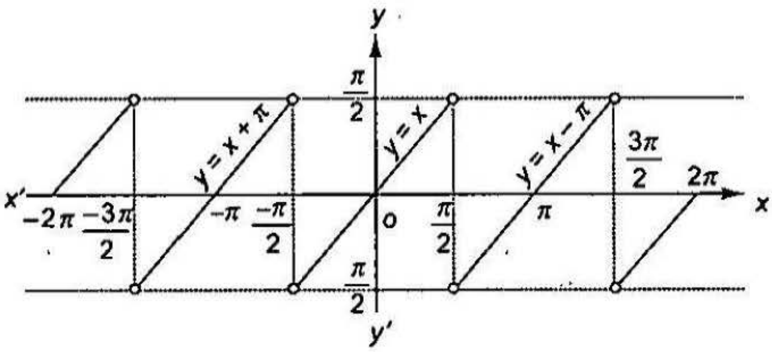


Fig. 4.13

Graph of $y = \cot^{-1}(\cot x)$

$y = \cot^{-1}(\cot x) \quad \Rightarrow \quad \cot y = \cot x$

$\Rightarrow \quad \tan y = \tan x \quad \Rightarrow \quad y = n\pi + x,\ n \in Z$

Now, keeping in mind that $y \in (0, \pi)$, we can plot the graph of $y = \cot^{-1}(\cot x)$ as follows (Fig. 4.14):

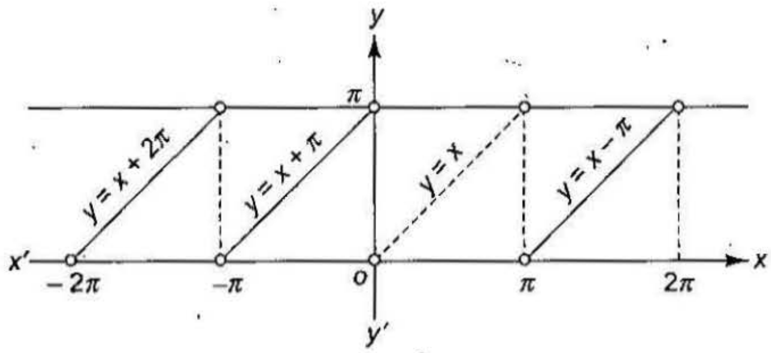


Fig. 4.14

Graph of $y = \text{cosec}^{-1}(\text{cosec}\, x)$

$y = \text{cosec}^{-1}(\text{cosec}\, x) \Rightarrow \quad \text{cosec}\, y = \text{cosec}\, x \quad \Rightarrow \quad \sin y = \sin x$

Hence, graph of $y = \text{cosec}^{-1}(\text{cosec}\, x)$ is the same as that of $y = \sin^{-1}(\sin x)$, but excluding points $x = n\pi, n \in Z$.

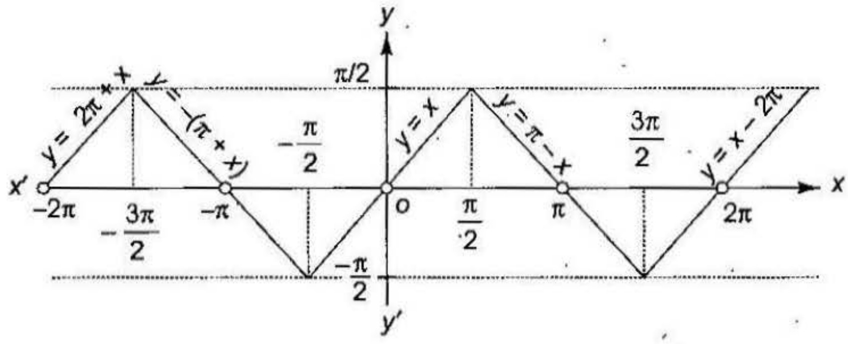


Fig. 4.15

Graph of $f(x) = \sec^{-1}(\sec x)$

$y = \sec^{-1}(\sec x) \Rightarrow \quad \sec y = \sec x \quad \Rightarrow \quad \cos y = \cos x$

Hence, graph of $y = \sec^{-1}(\sec x)$ is the same as that of $y = \cos^{-1}(\cos x)$, but excluding points $x = (2n+1)\pi/2, n \in Z$.

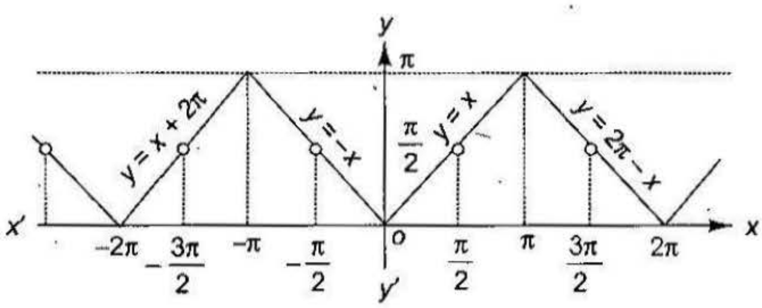


Fig. 4.16

Example 4.1 **Evaluate the following:**

i. $\sin^{-1}(\sin \pi/4)$ ii. $\cos^{-1}(2\pi/3)$ iii. $\tan^{-1}(\tan \pi/3)$

Sol. We know that

$\sin^{-1}(\sin \theta) = \theta$, if $-\pi/2 \le \theta \le \pi/2$,

$\cos^{-1}(\cos \theta) = \theta$, if $0 \le \theta \le \pi$

and $\tan^{-1}(\tan\theta)=\theta$, if $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$

Then

i. $\sin^{-1}\left(\sin\frac{\pi}{4}\right)=\frac{\pi}{4}$

ii. $\cos^{-1}\left(\cos\frac{2\pi}{3}\right)=\frac{2\pi}{3}$

iii. $\tan^{-1}\left(\tan\frac{\pi}{3}\right)=\frac{\pi}{3}$

Example 4.2 **Evaluate the following:**

i. $\sin^{-1}\left(\frac{2\pi}{3}\right)$ ii. $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)$ iii. $\tan^{-1}\left(\tan\frac{2\pi}{3}\right)$ iv. $\cos\left(\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)+\frac{\pi}{6}\right)$

Sol.

i. $\sin^{-1}\left(\sin\frac{2\pi}{3}\right)\neq\frac{2\pi}{3}$, as $\frac{2\pi}{3}$ does not lie between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$

Now, $\sin^{-1}\left(\sin\frac{2\pi}{3}\right)=\sin^{-1}\left(\sin\left(\pi-\frac{\pi}{3}\right)\right)=\sin^{-1}\left(\sin\frac{\pi}{3}\right)=\frac{\pi}{3}$

ii. $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)\neq\frac{7\pi}{6}$, as $\frac{7\pi}{6}$ does not lie between 0 and π

Now, $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)=\cos^{-1}\left(\cos\left(2\pi-\frac{5\pi}{6}\right)\right)=\cos^{-1}\left(\cos\frac{5\pi}{6}\right)=\frac{5\pi}{6}$

iii. $\tan^{-1}\left(\tan\frac{2\pi}{3}\right)\neq\frac{2\pi}{3}$, because $\frac{2\pi}{3}$ does not lie between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$

Now, $\tan^{-1}\left(\tan\frac{2\pi}{3}\right)=\tan^{-1}\left(\tan\left(\pi-\frac{\pi}{3}\right)\right)=\tan^{-1}\left(-\tan\frac{\pi}{3}\right)=\tan^{-1}\left(\tan\left(-\frac{\pi}{3}\right)\right)=-\frac{\pi}{3}$

iv. $\cos\left(\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)+\frac{\pi}{6}\right)=\cos\left(\frac{5\pi}{6}+\frac{\pi}{6}\right)=\cos(\pi)=-1$

Example 4.3 **Evaluate the following:**

i. $\sin^{-1}(\sin 10)$ ii. $\sin^{-1}(\sin 5)$ iii. $\cos^{-1}(\cos 10)$ iv. $\tan^{-1}(\tan(-6))$

Sol i. Here, $\theta=10$ rad does not lie between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.

But, $3\pi-\theta$, i.e., $3\pi-10$ lies between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.

Also, $\sin(3\pi-10)=\sin 10$

$\therefore\quad \sin^{-1}(\sin 10)=\sin^{-1}(\sin(3\pi-10))=(3\pi-10)$

ii. Here, $\theta=5$ rad. Clearly, it does not lie between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. But both $2\pi-5$ and $5-2\pi$ lie between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. Therefore,

$\sin(5-2\pi)=\sin(-(2\pi-5))=-\sin(2\pi-5)=-(-\sin 5)=\sin 5$

$\Rightarrow\quad \sin^{-1}(\sin 5)=\sin^{-1}(\sin(5-2\pi))=5-2\pi$

iii. We know that $\cos^{-1}(\cos\theta) = \theta$, if $0 \le \theta \le \pi$
Here, $\theta = 10$ rad.
Clearly, it does not lie between 0 and π.
However, $(4\pi - 10)$ lies between 0 and π such that $\cos(4\pi - 10) = \cos 10$
$\Rightarrow \quad \cos^{-1}(\cos 10) = \cos^{-1}(\cos(4\pi - 10)) = 4\pi - 10$

iv. we know that $\tan^{-1}(\tan\theta) = \theta$, if $-\pi/2 < \theta < \pi/2$.
Here, $\theta = -6$ rad does not lie between $-\pi/2$ and $\pi/2$. We find that $2\pi - 6$ lies between $-\pi/2$ and $\pi/2$ such that
$\tan(2\pi - 6) = -\tan 6 = \tan(-6)$
$\therefore \quad \tan^{-1}(\tan(-6)) = \tan^{-1}(\tan(2\pi - 6)) = 2\pi - 6$

Example 4.4 **Evaluate the following:**

i. $\sin\left(\cos^{-1}\dfrac{3}{5}\right)$ **ii.** $\cos\left(\tan^{-1}\dfrac{3}{4}\right)$ **iii.** $\sin\left(\dfrac{\pi}{2} - \sin^{-1}\left(\dfrac{1}{2}\right)\right)$

Sol.

i. Let $\cos^{-1} 3/5 = \theta$.
Then, $\cos\theta = 3/5 \quad \Rightarrow \quad \sin\theta = 4/5$
$\therefore \sin(\cos^{-1} 3/5) = \sin\theta = 4/5$

ii. Let $\tan^{-1} 3/4 = \theta$.

Then, $\tan\theta = 3/4 \quad \Rightarrow \quad \cos\theta = 4/5 \qquad \left(\because \text{as} \cos^2\theta = \dfrac{1}{1+\tan^2\theta}\right)$

$\therefore \cos(\tan^{-1}(3/4)) = \cos\theta = 4/5$

iii. $\sin\left(\dfrac{\pi}{2} - \sin^{-1}\left(-\dfrac{1}{2}\right)\right) = \sin\left(\dfrac{\pi}{2} - \left(-\dfrac{\pi}{6}\right)\right) = \sin\dfrac{2\pi}{3} = \dfrac{\sqrt{3}}{2}$

Example 4.5 **If $\cos^{-1}\lambda + \cos^{-1}\mu + \cos^{-1}\gamma = 3\pi$, then find the value of $\lambda\mu + \mu\gamma + \gamma\lambda$.**

Sol.

We know that $0 \le \cos^{-1} x \le \pi$.
Hence, from the question
$\cos^{-1}\lambda = \pi, \cos^{-1}\mu = \pi, \cos^{-1}\gamma = \pi$
[$\because \cos^{-1}\lambda + \cos^{-1}\mu + \cos^{-1}\gamma = 3\pi$ is possible only when each term attains its maximum.]
$\Rightarrow \quad \lambda = \mu = \gamma = -1 \quad \Rightarrow \quad \lambda\mu + \mu\gamma + \gamma\lambda = 3$

Example 4.6 **If $\cos(2\sin^{-1} x) = \dfrac{1}{9}$, then find the values of x.**

Sol. Let $\sin^{-1} x = \theta$

$\therefore \cos 2\theta = \dfrac{1}{9}$

$\Rightarrow \quad 1 - 2\sin^2\theta = 9 \quad \Rightarrow \quad 1 - 2x^2 = \dfrac{1}{9} \quad \Rightarrow \quad x^2 = \dfrac{4}{9} \quad \Rightarrow \quad x = \pm\dfrac{2}{3}$

Example 4.7 **Find the value of $\sin\left(\dfrac{1}{2}\cot^{-1}\left(-\dfrac{3}{4}\right)\right)$.**

Sol.

Let $\cot^{-1}(-3/4) = \theta \quad \Rightarrow \quad \cot\theta = -3/4 \quad \Rightarrow \quad \theta \in (\pi/2, \pi)$
$\Rightarrow \quad \cos\theta = -3/5\ (\theta \in (\pi/2, \pi)) \quad \Rightarrow \quad 1 - 2\sin^2(\theta/2) = -3/5$

$\therefore \quad \sin^2 \theta/2 = 4/5$ or $\sin \theta/2 = 2/\sqrt{5}$

Example 4.8 **Find the number of solutions of the equation $\cos(\cos^{-1}x) = \operatorname{cosec}(\operatorname{cosec}^{-1}x)$.**

Sol.

$\cos(\cos^{-1}x) = x$ for $x \in [-1, 1]$

$\operatorname{cosec}(\operatorname{cosec}^{-1}x) = x$ for $x \in (-\infty, -1] \cup [1, \infty)$

$\Rightarrow \cos(\cos^{-1}x) = \text{coesec}(\operatorname{cosec}^{-1}x)$ for $x = \pm 1$ only.

Hence, there are two roots only.

Example 4.9 **Find the range of $f(x) = |3\tan^{-1}x - \cos^{-1}(0)| - \cos^{-1}(-1)$.**

Sol.

$f(x) = |3\tan^{-1}x - \cos^{-1}(0)| - \cos^{-1}(-1) = |3\tan^{-1}x - (\pi/2)| - \pi$

Now $-\dfrac{\pi}{2} < \tan^{-1} x < \dfrac{\pi}{2}$

$\Rightarrow -\dfrac{3\pi}{2} < 3\tan^{-1} x < \dfrac{3\pi}{2}$

$\Rightarrow -2\pi < 3\tan^{-1} x - \dfrac{\pi}{2} < \pi$

$\Rightarrow 0 \le \left|3\tan^{-1} x - \dfrac{\pi}{2}\right| < 2\pi$

$\Rightarrow -\pi \le \left|3\tan^{-1} x - \dfrac{\pi}{2}\right| - \pi < \pi$

Concept Application Exercise 4.1

1. Find the principal of

   i. $\operatorname{cosec}^{-1}(-1)$ ii. $\cot^{-1}\left(-\dfrac{1}{\sqrt{3}}\right)$

2. Find the principal value of

   i. $\sin^{-1}(\sin 3)$ ii. $\sin^{-1}(\sin 100)$ iii. $\cos^{-1}(\cos 20)$ iv. $\cot^{-1}(\cot 4)$

3. Evaluate the following:

   i. $\sin(\cot^{-1} x)$ ii. $\sin\left(\dfrac{\pi}{2} - \sin^{-1}\left(-\dfrac{\sqrt{3}}{2}\right)\right)$

4. If $\sin^{-1} x + \sin^{-1} y + \sin^{-1} z = \dfrac{3\pi}{2}$, then find the value $x^2 + y^2 + z^2$.

5. If $(\sin^{-1} x)^2 + (\sin^{-1} y)^2 + (\sin^{-1} z)^2 = \dfrac{3\pi^2}{4}$, then find the minimum value of $x + y + z$.

6. Find the value of $\tan\left[\dfrac{1}{2}\cos^{-1}\dfrac{\sqrt{5}}{3}\right]$.

Property 3

i. $\sin^{-1}(-x) = -\sin^{-1}(x)$, for all $x \in [-1, 1]$
ii. $\cos^{-1}(-x) = \pi - \cos^{-1} x$, for all $x \in [-1, 1]$
iii. $\tan^{-1}(-x) = -\tan^{-1} x$, for all $x \in R$
iv. $\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1} x$, for all $x \in (-\infty, -1] \cup [1, \infty)$
v. $\sec^{-1}(-x) = \pi - \sec^{-1} x$, for all $x \in (-\infty, -1] \cup [1, \infty)$
vi. $\cot^{-1}(-x) = \pi - \cot^{-1} x$, for all $x \in R$

Proof:

i. Clearly, $-x \in [-1, 1]$ for all $x \in [-1, 1]$

Let $\sin^{-1}(-x) = \theta$ (i)

$\Rightarrow \quad -x = \sin\theta$

$\Rightarrow \quad x = -\sin\theta$

$\Rightarrow \quad x = \sin(-\theta)$

$\Rightarrow \quad -\theta = \sin^{-1} x$ $\quad [\because x \in [-1, 1]$ and $-\theta \in [-\pi/2, \pi/2]$ for all $\theta \in [-\pi/2, \pi/2]]$

$\Rightarrow \quad \theta = -\sin^{-1} x$ (ii)

From Eqs. (i) and (ii), we get $\sin^{-1}(-x) = -\sin^{-1} x$

Proof:

ii. Clearly, $-x \in [-1, 1]$ for all $x \in [-1, 1]$

Let $\cos^{-1}(-x) = \theta$ (i)

$\Rightarrow \quad -x = \cos\theta$

$\Rightarrow \quad x = -\cos\theta$

$\Rightarrow \quad x = \cos(\pi - \theta)$

$\Rightarrow \quad \cos^{-1} x = \pi - \theta$ $\quad [\because x \in [-1, 1]$ and $\pi - \theta \in [0, \pi]$ for all $\theta \in [0, \pi]]$

$\Rightarrow \quad \theta = \pi - \cos^{-1} x$ (ii)

From Eqs. (i) and (ii), we get

$\cos^{-1}(-x) = \pi - \cos^{-1} x$

Similarly, we can prove other results.

Property 4

i. $\sin^{-1}\left(\dfrac{1}{x}\right) = \operatorname{cosec}^{-1} x$, for all $x \in (-\infty, -1] \cup [1, \infty)$

ii. $\cos^{-1}\left(\dfrac{1}{x}\right) = \sec^{-1} x$, for all $x \in (-\infty, -1] \cup [1, \infty)$

iii. $\tan^{-1}\left(\dfrac{1}{x}\right) = \begin{cases} \cot^{-1} x, & \text{for } x > 0 \\ -\pi + \cot^{-1} x, & \text{for } x < 0 \end{cases}$

Proof:

i. Let $\operatorname{cosec}^{-1} x = \theta$ (i)

where $\theta \in [-\pi/2, \pi/2] - \{0\}$ and $x \in (-\infty, -1] \cup [1, \infty)$

$\Rightarrow \quad x = \operatorname{cosec}\theta \quad \Rightarrow \quad \dfrac{1}{x} = \sin\theta \quad \Rightarrow \quad \theta = \sin^{-1}\dfrac{1}{x}$ (ii)

From Eqs. (i) and (ii), we get $\sin^{-1}(1/x) = \operatorname{cosec}^{-1} x$.

ii. Let $\sec^{-1} x = \theta$ (i)

where $\theta \in [0, \pi] - \{\pi/2\}$ and $x \in (-\infty, -1] \cup [1, \infty)$

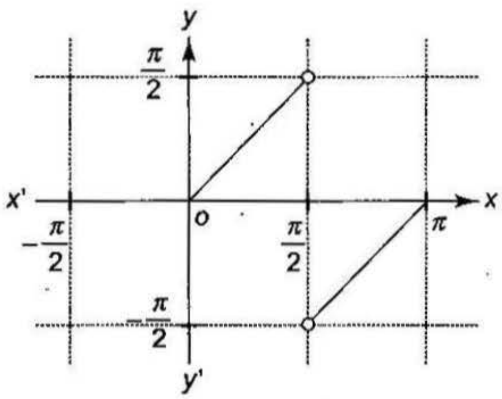
Now, $\sec^{-1} x = \theta$

$$\Rightarrow \quad x = \sec\theta \quad \Rightarrow \quad \frac{1}{x} = \cos\theta \quad \Rightarrow \quad \theta = \cos^{-1}\frac{1}{x} \tag{ii}$$

From Eqs.(i) and (ii), we get $\cos^{-1}\left(\frac{1}{x}\right) = \sec^{-1} x$.

iii. Let $\cot^{-1} x = \theta$, where $\theta \in (0, \pi)$ and $x \in R$

$$\Rightarrow \quad x = \cot\theta \quad \Rightarrow \quad \frac{1}{x} = \tan\theta \quad \Rightarrow \quad \tan^{-1}\left(\frac{1}{x}\right) = \tan^{-1}(\tan\theta)$$



Graph of $y = \tan^{-1}(\tan x)$

Fig.4.17

From the graph,

$$\tan^{-1}\left(\frac{1}{x}\right) = \begin{cases} \theta, & 0 < \theta < \pi/2 \\ -\pi + \theta, & \pi/2 < \theta < \pi \end{cases} = \begin{cases} \cot^{-1} x, & 0 < \cot^{-1} x < \pi/2 \\ -\pi + \cot^{-1} x, & \pi/2 < \cot^{-1} x < \pi \end{cases} = \begin{cases} \cot^{-1} x, & x > 0 \\ -\pi + \cot^{-1} x, & x < 0 \end{cases}$$

Example 4.10 **Prove that $\tan^{-1} x + \tan^{-1}\frac{1}{x} = \begin{cases} \pi/2, & \text{if } x > 0 \\ -\pi/2, & \text{if } x < 0 \end{cases}$**

Sol.

We know that $\tan^{-1}\left(\frac{1}{x}\right) = \begin{cases} \cot^{-1} x, & x > 0 \\ -\pi + \cot^{-1} x, & x < 0 \end{cases}$

$$\Rightarrow \quad \tan^{-1} x + \tan^{-1}\frac{1}{x} = \begin{cases} \tan^{-1} x + \cot^{-1} x, & x > 0 \\ -\pi + \cot^{-1} x + \tan^{-1} x, & x < 0 \end{cases} = \begin{cases} \frac{\pi}{2}, & x > 0 \\ -\pi + \frac{\pi}{2}, & x < 0 \end{cases} = \begin{cases} \frac{\pi}{2}, & x > 0 \\ -\frac{\pi}{2}, & x < 0 \end{cases}$$

Example 4.11 **If $x > y > z > 0$, then find the value of $\cot^{-1}\frac{xy+1}{x-y} + \cot^{-1}\frac{yz+1}{y-z} + \cot^{-1}\frac{zx+1}{z-x}$.**

Sol.

$$\cot^{-1}\frac{xy+1}{x-y} + \cot^{-1}\frac{yz+1}{y-z} + \cot^{-1}\frac{zx+1}{z-x}$$

$$= \tan^{-1}\frac{x-y}{1+xy} + \tan^{-1}\frac{y-z}{1+yz} + \pi + \tan^{-1}\frac{z-x}{1+zx} \left[\because \tan^{-1}\left(\frac{1}{x}\right) = \begin{cases} \cot^{-1}x, & \text{for } x>0 \\ -\pi+\cot^{-1}x, & \text{for } x<0 \end{cases}\right]$$

$$= \tan^{-1}x - \tan^{-1}y + \tan^{-1}y - \tan^{-1}z + \pi + \tan^{-1}z - \tan^{-1}x$$

$$= \pi$$

Example 4.12 **Find the value of x for which $\sec^{-1}x + \sin^{-1}x = \frac{\pi}{2}$.**

Sol.

We know that $\sec^{-1}x$ is defined for $x \in (-\infty, -1] \cup [1, \infty)$.
But $\sin^{-1}x$ is defined for $x \in [-1, 1]$

Hence, $\sec^{-1}x + \sin^{-1}x = \frac{\pi}{2}$ for $x = \pm 1$.

Concept Application Exercise 4.2

1. If $\tan^{-1}\left(\frac{1}{y}\right) = -\pi + \cot^{-1}y$, where $y = x^2 - 3x + 2$, then find the value of x.

2. If $\alpha \in \left(-\frac{\pi}{2}, 0\right)$, then find the value of $\tan^{-1}(\cot\alpha) - \cot^{-1}(\tan\alpha)$.

Property 5

i. $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$, for all $x \in [-1, 1]$

ii. $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$, for all $x \in R$

iii. $\sec^{-1}x + \operatorname{cosec}^{-1}x = \frac{\pi}{2}$ for all $x \in (-\infty, -1] \cup [1, \infty)$

Proof:

i. Let $\sin^{-1}x = \theta$ (i)

where $\theta \in [-\pi/2, \pi/2]$

$$\Rightarrow -\frac{\pi}{2} \le -\theta \le \frac{\pi}{2}$$

$$\Rightarrow 0 \le \frac{\pi}{2} - \theta \le \pi$$

$$\Rightarrow \frac{\pi}{2} - \theta \in [0, \pi]$$

Now, $\sin^{-1}x = \theta$

$\Rightarrow x = \sin\theta$

$$\Rightarrow x = \cos\left(\frac{\pi}{2} - \theta\right)$$

$$\Rightarrow \quad \cos^{-1}x = \frac{\pi}{2} - \theta \qquad [\because x \in [-1, 1] \text{ and } (\pi/2 - \theta) \in [0, \pi]]$$

$$\Rightarrow \quad \theta + \cos^{-1} x = \frac{\pi}{2} \qquad \text{(ii)}$$

From Eqs. (i) and (ii), we get $\sin^{-1} x + \cos^{-1} x = \pi/2$. Similarly, we get the other results.

Example 4.13 **If $\sin^{-1} x = \pi/5$, for some $x \in (-1, 1)$, then find the value of $\cos^{-1} x$.**

Sol.

$$\sin^{-1}x + \cos^{-1} x = \frac{\pi}{2} \quad \Rightarrow \quad \cos^{-1} x = \frac{\pi}{2} - \sin^{-1} x = \frac{\pi}{2} - \frac{\pi}{5} = \frac{3\pi}{10}$$

Example 4.14 **If $\sin\left(\sin^{-1}\frac{1}{5} + \cos^{-1} x\right) = 1$, then find the value of x.**

Sol.

$$\sin\left(\sin^{-1}\frac{1}{5} + \cos^{-1} x\right) = 1 = 1$$

$$\Rightarrow \quad \sin^{-1}\frac{1}{5} + \cos^{-1} x = \frac{\pi}{2}$$

$$\Rightarrow \quad \sin^{-1}\frac{1}{5} = \frac{\pi}{2} - \cos^{-1} x$$

$$\Rightarrow \quad \sin^{-1}\frac{1}{5} = \sin^{-1} x$$

$$\Rightarrow \quad x = \frac{1}{5}$$

Example 4.15 **Solve $\sin^{-1} x \le \cos^{-1} x$.**

Sol.

$$\cos^{-1} x \ge \sin^{-1} x \Rightarrow \quad \frac{\pi}{2} \ge 2\sin^{-1} x \quad \Rightarrow \quad \sin^{-1} x \le \frac{\pi}{4}$$

$$\Rightarrow \quad -1 \le x \le \sin\left(\frac{\pi}{4}\right) \quad \Rightarrow \quad x \in \left[-1, \frac{1}{\sqrt{2}}\right]$$

Example 4.16 **Find the range of $f(x) = \sin^{-1} x + \tan^{-1} x + \cos^{-1} x$.**

Sol.

Clearly, the domain of the function is $[-1, 1]$.

Also, $\tan^{-1} x \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$ for $x \in [-1, 1]$.

Now, $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$ for $x \in [-1, 1]$.

Thus, $f(x) = \tan^{-1} x + \frac{\pi}{2}$, where $x \in [-1, 1]$.

Hence, the range is $\left[-\frac{\pi}{4} + \frac{\pi}{2}, \frac{\pi}{4} + \frac{\pi}{2}\right] = \left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$.

Example 4.17 **Find the minimum value of $(\sec^{-1}x)^2+(\text{cosec}^{-1}x)^2$.**

Sol.

Let $I=(\sec^{-1}x)^2+(\text{cosec}^{-1}x)^2$

$$=(\sec^{-1}x+\text{cosec}^{-1}x)^2-2\sec^{-1}x\,\text{cosec}^{-1}x$$

$$=\frac{\pi^2}{4}-2\sec^{-1}x\left(\frac{\pi}{2}-\sec^{-1}x\right)$$

$$=\frac{\pi^2}{4}+2\left(\sec^{-1}x\right)^2-\pi\sec^{-1}x$$

$$=\frac{\pi^2}{4}+2\left[\left(\sec^{-1}x\right)^2-2\frac{\pi}{4}\sec^{-1}x+\left(\frac{\pi}{4}\right)^2\right]-\frac{\pi^2}{8}$$

$$=2\left(\sec^{-1}x-\frac{\pi}{4}\right)^2+\frac{\pi^2}{8}\Rightarrow I\geq\frac{\pi^2}{8}$$

Example 4.18 **Solve $\sin^{-1}\frac{14}{|x|}+\sin^{-1}\frac{2\sqrt{15}}{|x|}=\frac{\pi}{2}$.**

Sol.

$$\sin^{-1}\frac{14}{|x|}+\sin^{-1}\frac{2\sqrt{15}}{|x|}=\frac{\pi}{2}$$

$$\Rightarrow\quad\sin^{-1}\frac{14}{|x|}=\frac{\pi}{2}-\sin^{-1}\frac{2\sqrt{15}}{|x|}=\cos^{-1}\frac{2\sqrt{15}}{|x|}=\sin^{-1}\sqrt{1-\left(\frac{2\sqrt{15}}{|x|}\right)^2}$$

For $0\leq\frac{2\sqrt{15}}{|x|}\leq 1$ or $|x|\geq 2\sqrt{15}$, we have

$$\left(\frac{14}{|x|}\right)^2=1-\left(\frac{2\sqrt{15}}{|x|}\right)^2\quad\Rightarrow\quad|x|=16\quad\Rightarrow\quad x=\pm 16$$ which satisfy $|x|\geq 2\sqrt{15}$.

Example 4.19 **If $\alpha=\sin^{-1}(\cos(\sin^{-1}x))$ and $\beta=\cos^{-1}(\sin(\cos^{-1}x))$, then find $\tan\alpha\cdot\tan\beta$.**

Sol.

$$\beta=\cos^{-1}\left(\sin\left(\frac{\pi}{2}-\sin^{-1}x\right)\right)=\cos^{-1}[\cos(\sin^{-1}x)]$$ also $\alpha=\sin^{-1}[\cos(\sin^{-1}x)]$

$\alpha+\beta=\pi/2\Rightarrow\tan\alpha=\cot\beta\Rightarrow\tan\alpha\cdot\tan\beta=1$

Example 4.20 **Find the value of $\sin^{-1}x+\sin^{-1}\frac{1}{x}+\cos^{-1}x+\cos^{-1}\frac{1}{x}$.**

Sol.

$\sin^{-1}x+\sin^{-1}\frac{1}{x}+\cos^{-1}x+\cos^{-1}\frac{1}{x}$ is defined only when $x,\frac{1}{x}\in[-1,1]$

which is possible only when $x=\pm 1$

for which $\sin^{-1}x+\sin^{-1}\frac{1}{x}+\cos^{-1}x+\cos^{-1}\frac{1}{x}=\pi$

Concept Application Exercise 4.3

1. If $\sin^{-1} x + \sin^{-1} y = \dfrac{2\pi}{3}$, then find the value of $\cos^{-1} x + \cos^{-1} y$.

2. Solve $\cot^{-1} x + \tan^{-1} 3 = \dfrac{\pi}{2}$.

3. Solve $\sec^{-1} x > \operatorname{cosec}^{-1} x$.

4. Solve $\tan^{-1} x > \cot^{-1} x$.

5. Solve $2\cos^{-1} x + \sin^{-1} x = \dfrac{11\pi}{6}$.

6. Solve $(\tan^{-1} x)^2 + (\cot^{-1} x)^2 = \dfrac{5\pi^2}{8}$.

Property 6

i. For $x > 0$,

$$\sin^{-1} x = \cos^{-1}\sqrt{1-x^2} = \tan^{-1}\frac{x}{\sqrt{1-x^2}} = \cot^{-1}\frac{\sqrt{1-x^2}}{x} = \sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right) = \operatorname{cosec}^{-1}\left(\frac{1}{x}\right)$$

Refer the following diagram for the proof.

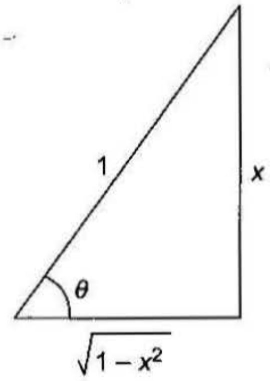


Fig. 4.18

ii. For $x > 0$,

$$\cos^{-1} x = \sin^{-1}\sqrt{1-x^2} = \tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right) = \cot^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right) = \sec^{-1}\frac{1}{x} = \operatorname{cosec}^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)$$

Refer the following diagram for the proof.

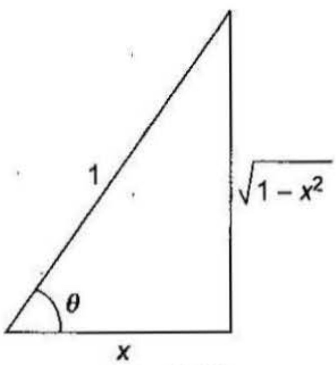


Fig. 4.19

iii. For $x > 0$,

$$\tan^{-1} x = \sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right) = \cot^{-1}\left(\frac{1}{x}\right)$$

$$= \sec^{-1}\sqrt{1+x^2} = \operatorname{cosec}^{-1}\left(\frac{\sqrt{1+x^2}}{x}\right)$$

Refer the following diagram for the proof.

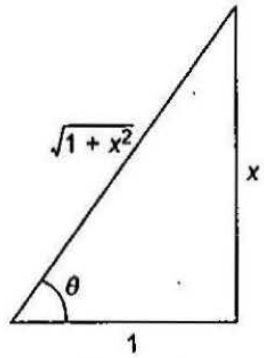


Fig. 4.20

Example 4.21 **Find $\tan^{-1}\frac{x}{\sqrt{a^2-x^2}}$, in terms of $\sin^{-1}$ where $x \in (0, a)$.**

Sol.

$$\tan^{-1}\frac{x}{\sqrt{a^2-x^2}} = \tan^{-1}\left(\frac{a\sin\theta}{a\cos\theta}\right) \quad \text{[putting } x = a\sin\theta\text{]}$$

$$= \tan^{-1}(\tan\theta) = \theta = \sin^{-1}\left(\frac{x}{a}\right)$$

Example 4.22 **Prove that $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right) = \frac{1}{2}\tan^{-1}x$.**

Sol.

$$\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right) = \tan^{-1}\left[\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right] \quad \text{[putting } x = \tan\theta\text{]}$$

$$= \tan^{-1}\left[\frac{\sec\theta-1}{\tan\theta}\right]$$

$$= \tan^{-1}\left[\frac{1-\cos\theta}{\sin\theta}\right]$$

$$= \tan^{-1}\left[\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right]$$

$$= \tan^{-1}\left(\tan\frac{\theta}{2}\right) = \frac{\theta}{2} = \frac{1}{2}\tan^{-1}x$$

Example 4.23 **Simplify $\sin \cot^{-1} \tan \cos^{-1} x$.**

Sol.

Let $\cos^{-1} x = \theta$

$\Rightarrow \quad x = \cos\theta$

$\Rightarrow \quad \sec\theta = \dfrac{1}{x}$

$\Rightarrow \quad \tan\theta = \sqrt{\sec^2\theta - 1} = \sqrt{\dfrac{1}{x^2} - 1} = \dfrac{1}{|x|}\sqrt{1 - x^2}$

Now, $\sin \cot^{-1} \tan\theta = \sin \cot^{-1}\left(\dfrac{1}{|x|}\sqrt{1 - x^2}\right)$.

Again, putting $x = \sin\theta$, we get

$\sin \cot^{-1}\left(\dfrac{1}{|x|}\sqrt{1 - x^2}\right) = \sin \cot^{-1}\left(\dfrac{\sqrt{1 - \sin^2\theta}}{|\sin\theta|}\right) = \sin \cot^{-1} |\cot\theta| = \sin\theta = x$

Example 4.24 **Prove that $\operatorname{cosec}(\tan^{-1}(\cos(\cot^{-1}(\sec(\sin^{-1} a))))) = \sqrt{3 - a^2}$, where $a \in [0, 1]$.**

Sol.

Here $x = \operatorname{cosec}(\tan^{-1}(\cos(\cot^{-1}(\sec(\sin^{-1} a)))))$

$$= \operatorname{cosec}\left(\tan^{-1}\left(\cos\left(\cot^{-1}\left(\frac{1}{\sqrt{1-a^2}}\right)\right)\right)\right)$$

$$= \operatorname{cosec}\left(\tan^{-1}\left(\frac{1}{\sqrt{2-a^2}}\right)\right)$$

$$= \sqrt{3 - a^2} \qquad \text{(i)}$$

Example 4.25 **If $x < 0$, then prove that $\cos^{-1} x = \pi - \sin^{-1}\sqrt{1 - x^2}$.**

Sol.

Let $x = \cos\theta \quad \Rightarrow \quad \cos\theta = x$

Since $x < 0$, $\theta \in [\pi/2, \pi]$

Now, $\sin^{-1}\sqrt{1 - x^2} = \sin^{-1}\sqrt{1 - \cos^2\theta}$

$= \sin^{-1}(\sin\theta) \neq \theta \qquad (\because \theta \notin [-\pi/2, \pi/2])$

$= \sin^{-1}(\sin(\pi - \theta)) = \pi - \theta$

$\Rightarrow \quad \cos^{-1} x = \pi - \sin^{-1}\sqrt{1 - x^2}$

Example 4.26 **Prove that $\cos^{-1}\left\{\sqrt{\dfrac{1+x}{2}}\right\} = \dfrac{\cos^{-1} x}{2}$, $-1 < x < 1$.**

Sol.

Let $x = \cos\theta$, where $\theta \in [0, \pi]$

$\Rightarrow \cos^{-1}\left\{\sqrt{\dfrac{1+x}{2}}\right\} = \cos^{-1}\left\{\sqrt{\dfrac{1+\cos\theta}{2}}\right\}$

$$= \cos^{-1}\left\{\sqrt{\frac{2\cos^2\frac{\theta}{2}}{2}}\right\}$$

$$= \cos^{-1}\left(\cos\frac{\theta}{2}\right)$$

$$= \frac{\theta}{2}$$

$$= \frac{\cos^{-1}x}{2}$$

Example 4.27 **Prove that** $\tan^{-1}\left\{\frac{x}{a+\sqrt{a^2-x^2}}\right\} = \frac{1}{2}\sin^{-1}\frac{x}{a}, -a < x < a.$

Sol.

Let $x = a\sin\theta$, since $-a < x < a$

$\Rightarrow -a < a\sin\theta < a \Rightarrow -1 < \sin\theta < 1 \Rightarrow \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$$\Rightarrow \tan^{-1}\left\{\frac{x}{a+\sqrt{a^2-x^2}}\right\} = \tan^{-1}\left\{\frac{a\sin\theta}{a+\sqrt{a^2-\sin^2\theta}}\right\}$$

$$= \tan^{-1}\left\{\frac{\sin\theta}{1+\cos\theta}\right\}$$

$$= \tan^{-1}\left\{\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}\right\}$$

$$= \tan^{-1}\left\{\tan\frac{\theta}{2}\right\}$$

$$= \frac{\theta}{2}$$

$$= \frac{1}{2}\sin^{-1}\frac{x}{a}$$

Example 4.28 **Prove that** $\sin^{-1}\left\{\frac{\sqrt{1+x}+\sqrt{1-x}}{2}\right\} = \frac{\pi}{4} + \frac{\sin^{-1}x}{2}, 0 < x < 1.$

Sol.

Let $x = \sin\theta$. Since $0 < x < 1 \Rightarrow 0 < \sin\theta < 1 \Rightarrow \theta \in \left(0, \frac{\pi}{2}\right)$

$$\Rightarrow \sin^{-1}\left\{\frac{\sqrt{1+x}+\sqrt{1-x}}{2}\right\} = \sin^{-1}\left\{\frac{\sqrt{1+\cos\theta}+\sqrt{1-\cos\theta}}{2}\right\}$$

$$= \sin^{-1}\left\{\frac{\sqrt{2\cos^2\frac{\theta}{2}}+\sqrt{2\sin^2\frac{\theta}{2}}}{2}\right\}$$

$$= \sin^{-1}\left\{\frac{\cos\frac{\theta}{2}+\sin\frac{\theta}{2}}{\sqrt{2}}\right\}$$

$$= \sin^{-1}\left\{\sin\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\right\}$$

$$= \frac{\pi}{4}+\frac{\theta}{2} \qquad \left[\because \theta \in \left(0, \frac{\pi}{2}\right) \Rightarrow \left(\frac{\pi}{4}+\frac{\theta}{2}\right) \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)\right]$$

$$= \frac{\pi}{4}+\frac{\sin^{-1} x}{2}$$

Example 4.29 **Prove that $\cos^{-1}\left(\frac{1-x^{2n}}{1+x^{2n}}\right) = 2\tan^{-1} x^n, 0 < x < \infty$.**

Sol.

Since $0 < x < \infty$; $0 < x^n < \infty$

Let $x^n = \tan\theta \Rightarrow \theta \in (0, \pi/2)$

$$\cos^{-1}\left(\frac{1-x^{2n}}{1+x^{2n}}\right) = \cos^{-1}\left(\frac{1-\tan^2\theta}{1-\tan^2\theta}\right)$$

$$= \cos^{-1}(\cos 2\theta)$$

$$= 2\theta$$

$$= 2\tan^{-1} x^n$$

Concept Application Exercise 4.4

1. Evaluate $\tan^{-1}\left(\frac{\sqrt{1+a^2x^2}-1}{ax}\right)$, where $x \neq 0$.

2. Express $\sin^{-1}\frac{\sqrt{x}}{\sqrt{x+a}}$ as a function of $\tan^{-1}$.

3. If $x < 0$, then prove that $\cos^{-1}x = \pi + \tan^{-1}\frac{\sqrt{1-x^2}}{x}$.

4. If $\tan(\cos^{-1}x) = \sin\left(\cot^{-1}\frac{1}{2}\right)$, then find the value of x.

5. Evaluate $\sin^{-1}\left(\dfrac{x+\sqrt{1-x^2}}{\sqrt{2}}\right)$, where $-\dfrac{1}{\sqrt{2}} < x < \dfrac{1}{\sqrt{2}}$.

6. Prove that $\sin\left[2\tan^{-1}\left\{\sqrt{\dfrac{1-x}{1+x}}\right\}\right] = \sqrt{1-x^2}$.

Property 7

i. $$\tan^{-1}x+\tan^{-1}y = \begin{cases} \tan^{-1}\left(\dfrac{x+y}{1-xy}\right), \text{if } xy<1 & \text{if } xy<1 \\ \pi+\tan^{-1}\left(\dfrac{x+y}{1-xy}\right), & \text{if } x>0,\ y>0 \text{ and } xy>1 \\ -\pi+\tan^{-1}\left(\dfrac{x+y}{1-xy}\right), & \text{if } x<0, y<0 \text{ and } xy>1 \end{cases}$$

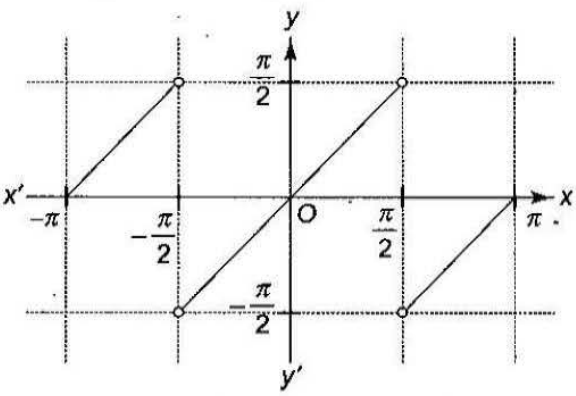
ii. $$\tan^{-1}x-\tan^{-1}y = \begin{cases} \tan^{-1}\left(\dfrac{x-y}{1+xy}\right), & \text{if } xy>-1 \\ \pi+\tan^{-1}\left(\dfrac{x-y}{1+xy}\right), & \text{if } x>0,\ y<0 \text{ and } xy<-1 \\ -\pi+\tan^{-1}\left(\dfrac{x-y}{1+xy}\right), & \text{if } x<0,\ y>0 \text{ and } xy<-1 \end{cases}$$

Proof:

i. Let $\tan^{-1}x = A$ and $\tan^{-1}y = B$, where $A, B \in (-\pi/2, \pi/2)$.

Now, $\tan(A+B) = \dfrac{\tan A+\tan B}{1-\tan A\tan B} = \dfrac{x+y}{1-xy}$

$$\Rightarrow \quad \tan^{-1}\left(\frac{x+y}{1-xy}\right) = \tan^{-1}\tan(A+B)$$
$$= \tan^{-1}\tan\alpha, \text{ where } \alpha \in (-\pi, \pi)$$



Graph of $y = \tan^{-1}(\tan x)$

Fig. 4.21

From the graph,

$$\tan^{-1}\left(\frac{x+y}{1-xy}\right)=\tan^{-1}(\tan\alpha)=\begin{cases}\alpha+\pi, & -\pi<\alpha<(-\pi/2)\\ \alpha, & (-\pi/2)\le\alpha\le(\pi/2)\\ \alpha-\pi, & (\pi/2)<\alpha<\pi\end{cases}$$

$$=\begin{cases}\tan^{-1}x+\tan^{-1}y+\pi, & -\pi<\tan^{-1}x+\tan^{-1}y<(-\pi/2)\\ \tan^{-1}x+\tan^{-1}y, & (-\pi/2)\le\tan^{-1}x+\tan^{-1}y\le(\pi/2)\\ \tan^{-1}x+\tan^{-1}y-\pi, & (\pi/2)<\tan^{-1}x+\tan^{-1}y<\pi\end{cases}$$

Case I

$$-\pi<\tan^{-1}x+\tan^{-1}y<(-\pi/2) \quad\Rightarrow\quad x<0, y<0$$

Also, $\tan^{-1}x<(-\pi/2)-\tan^{-1}y$

$$\Rightarrow\quad \tan^{-1}x<-\left((\pi/2)-\tan^{-1}(-y)\right) \quad\Rightarrow\quad x<-\tan(-1/y) \quad\Rightarrow\quad x<(1/y) \quad\Rightarrow\quad xy>1$$

Case II

$$(\pi/2)<\tan^{-1}x+\tan^{-1}y<\pi \quad\Rightarrow\quad x,y>0$$

Also, $\tan^{-1}x>(\pi/2)-\tan^{-1}y \quad\Rightarrow\quad \tan^{-1}x>\tan^{-1}(1/y) \quad\Rightarrow\quad x>(1/y) \quad\Rightarrow\quad xy>1.$

Case III

$$(-\pi/2)\le\tan^{-1}x+\tan^{-1}x\le(\pi/2) \quad\Rightarrow\quad xy<1$$

This property can be proved by replacing y by $-y$.

Example 4.30 **Find the value of $\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}$.**

Sol.

Here, $\frac{1}{2}\times\frac{1}{3}=\frac{1}{6}<1\Rightarrow\quad \tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}=\tan^{-1}\left(\frac{(1/2)+(1/3)}{1-(1/2)\times(1/3)}\right)=\tan^{-1}1=\frac{\pi}{4}$

Example 4.31 **If two angles of a triangle are $\tan^{-1}(2)$ and $\tan^{-1}(3)$, then find the third angle.**

Sol.

Given two angles are $\tan^{-1}(2)$ and $\tan^{-1}(3)$. Now $(2)(3)>1$

$\Rightarrow\quad \tan^{-1}(2)$ and $\tan^{-1}(3)=\pi+\tan^{-1}\left(\frac{2+3}{1-2\times3}\right)=\pi+\tan^{-1}(-1)=\pi-\frac{\pi}{4}=\frac{3\pi}{4}$. Hence, the third angle is $\pi-\frac{3\pi}{4}=\frac{\pi}{4}$.

Example 4.32 **Solve $\tan^{-1}\frac{x-1}{x+2}+\tan^{-1}\frac{x+1}{x+2}=\frac{\pi}{4}$.**

Sol.

$$\tan^{-1}\frac{x-1}{x+2}+\tan^{-1}\frac{x+1}{x+2}=\frac{\pi}{4}$$

$$\Rightarrow \quad \tan^{-1}\left[\frac{\frac{x-1}{x+2}+\frac{x+1}{x+2}}{1-\left(\frac{x-1}{x+2}\right)\left(\frac{x+1}{x+2}\right)}\right]=\frac{\pi}{4}$$

$$\Rightarrow \quad \left[\frac{2x(x+2)}{x^2+4+4x-x^2+1}\right]=\tan\frac{\pi}{4}$$

$$\Rightarrow \quad \frac{2x(x+2)}{4x+5}=\tan\frac{\pi}{4}=1 \quad \Rightarrow \quad 2x^2+4x=4x+5$$

$$\Rightarrow \quad x=\pm\sqrt{\frac{5}{2}}$$

But for $x=-\sqrt{\frac{5}{2}}$, L.H.S. is negative. Hence $x=\sqrt{\frac{5}{2}}$.

Example 4.33 **Find the value of $\tan^{-1}\left(\frac{1}{2}\tan 2A\right)+\tan^{-1}(\cot A)+\tan^{-1}(\cot^3 A)$, for $0<A<\frac{\pi}{4}$.**

Sol.

For $0<A<(\pi/4)$, $\cot A>1 \quad \Rightarrow \quad (\cot A)(\cot^3 A)>1$

Then $\tan^{-1}\left(\frac{1}{2}\tan 2A\right)+\tan^{-1}(\cot A)+\tan^{-1}(\cot^3 A)$

$$=\tan^{-1}\left(\frac{\tan A}{1-\tan^2 A}\right)+\pi+\tan^{-1}\left(\frac{\cot A+\cot^3 A}{1-\cot^4 A}\right)$$

$$=\tan^{-1}\left(\frac{\tan A}{1-\tan^2 A}\right)+\pi+\tan^{-1}\left(\frac{\cot A}{1-\cot^2 A}\right)$$

$$=\tan^{-1}\left(\frac{\tan A}{1-\tan^2 A}\right)+\pi+\tan^{-1}\left(\frac{\tan A}{\tan^2 A-1}\right)=\pi$$

Example 4.34 **Simplify $\tan^{-1}\left[\frac{3\sin 2\alpha}{5+3\cos 2\alpha}\right]+\tan^{-1}\left[\frac{\tan\alpha}{4}\right]$, where $-\frac{\pi}{2}<\alpha<\frac{\pi}{2}$.**

Sol.

$$\tan^{-1}\left[\frac{3\sin 2\alpha}{5+3\cos 2\alpha}\right]+\tan^{-1}\left[\frac{\tan\alpha}{4}\right]=\tan^{-1}\left(\frac{6\tan\alpha}{8+2\tan^2\alpha}\right)+\tan^{-1}\left(\frac{\tan\alpha}{4}\right)$$

$$=\tan^{-1}\left(\frac{\frac{3\tan\alpha}{4+\tan^2\alpha}+\frac{\tan\alpha}{4}}{1-\frac{3\tan^2\alpha}{16+4\tan^2\alpha}}\right) \qquad \left[\because \frac{3\tan^2\alpha}{16+4\tan^2\alpha}<1\right]$$

$$=\tan^{-1}\left(\frac{12\tan\alpha+4\tan\alpha+\tan^3\alpha}{16+\tan^2\alpha}\right)$$

$$=\tan^{-1}(\tan\alpha)=\alpha$$

Example 4.35 If $a_1, a_2, a_3, \ldots, a_n$ is an A.P. with common difference d, then prove that

$$\tan\left[\tan^{-1}\left(\frac{d}{1+a_1a_2}\right)+\tan^{-1}\left(\frac{d}{1+a_2a_3}\right)+\cdots+\tan^{-1}\left(\frac{d}{1+a_{n-1}a_n}\right)\right]=\frac{(n-1)d}{1+a_1a_n}.$$

Sol.

We have

$$\tan^{-1}\left(\frac{d}{1+a_1a_2}\right)+\tan^{-1}\left(\frac{d}{1+a_2a_3}\right)+\cdots+\tan^{-1}\left(\frac{d}{1+a_{n-1}a_n}\right)$$

$$=\tan^{-1}\left(\frac{a_2-a_1}{1+a_1a_2}\right)+\tan^{-1}\left(\frac{a_3-a_2}{1+a_2a_3}\right)+\cdots+\tan^{-1}\left(\frac{a_n-a_{n-1}}{1+a_{n-1}a_n}\right)$$

$$=(\tan^{-1}a_2-\tan^{-1}a_1)+(\tan^{-1}a_3-\tan^{-1}a_2)+\cdots+(\tan^{-1}a_n-\tan^{-1}a_{n-1})$$

$$=\tan^{-1}a_n-\tan^{-1}a_1=\tan^{-1}\left(\frac{a_n-a_1}{1+a_na_1}\right)=\tan^{-1}\left(\frac{(n-1)d}{1+a_1a_n}\right).$$

$$\Rightarrow \quad \tan\left[\tan^{-1}\left(\frac{d}{1+a_1a_2}\right)+\tan^{-1}\left(\frac{d}{1+a_2a_3}\right)+\cdots+\tan^{-1}\left(\frac{d}{1+a_{n-1}a_n}\right)\right]=\frac{(n-1)d}{1+a_1a_n}$$

Example 4.36 Solve the equation $\tan^{-1}2x+\tan^{-1}3x=\pi/4$.

Sol. $\tan^{-1}2x+\tan^{-1}3x=\dfrac{\pi}{4}$

$$\tan^{-1}\left(\frac{2x+3x}{1-6x^2}\right)=\frac{\pi}{4}$$

$$\Rightarrow \quad \frac{5x}{1-6x^2}=1$$

$$\Rightarrow \quad 6x^2+5x-1=0 \qquad \text{(i)}$$

$\Rightarrow$ $(6x-1)(x+1)=0$ $\Rightarrow$ $x=1/6$ or -1, but $x=-1$ does not satisfy Eq. (i). Hence, $x=1/6$.

Concept Application Exercise 4.5

1. Find the value of $\sin^{-1}\left(\dfrac{3}{5}\right)+\tan^{-1}\left(\dfrac{1}{7}\right)$.
2. Find the value of $\tan\left[\cos^{-1}\left(\dfrac{4}{5}\right)+\tan^{-1}\left(\dfrac{2}{3}\right)\right]$.
3. If $x>y>0$, then find the value of $\tan^{-1}\dfrac{x}{y}+\tan^{-1}\left[\dfrac{x+y}{x-y}\right]$.
4. Find the sum: $\tan^{-1}\dfrac{c_1x-y}{c_1y+x}+\tan^{-1}\dfrac{c_2-c_1}{1+c_2c_1}+\tan^{-1}\dfrac{c_3-c_2}{1+c_3c_2}+\cdots+\tan^{-1}\dfrac{1}{c_n}$.
5. If $x+y+z=xyz$, and $x, y, z>0$, then find the value of $\tan^{-1}x+\tan^{-1}y+\tan^{-1}z$.
6. Find the value of $\tan^{-1}\left(\sqrt{\dfrac{a\lambda}{bc}}\right)+\tan^{-1}\left(\sqrt{\dfrac{b\lambda}{ca}}\right)+\tan^{-1}\left(\sqrt{\dfrac{c\lambda}{ab}}\right)$, where $a, b, c\in R^+$ and $\lambda=a+b+c$.

Property 8

i. $$\sin^{-1}x+\sin^{-1}y=\begin{cases}\sin^{-1}\left(x\sqrt{1-y^2}+y\sqrt{1-x^2}\right), & x\ge 0,\ y\ge 0 \text{ and } x^2+y^2\le 1\\ \pi-\sin^{-1}\left(x\sqrt{1-y^2}+y\sqrt{1-x^2}\right), & x\ge 0,\ y\ge 0 \text{ and } x^2+y^2>1\end{cases}$$

Proof:

Let $\sin^{-1}x = A$ and $\sin^{-1}y = B$, where $x \ge 0$ and $y \ge 0$

$\Rightarrow \quad A, B \in [0, \pi/2] \quad \Rightarrow \quad A + B \in [0, \pi]$

Now, $\sin(A + B) = \sin A \cos B + \sin B \cos A = x\sqrt{1-y^2}+y\sqrt{1-x^2}$

$$\Rightarrow \quad \sin^{-1}(\sin(A+B))=\sin^{-1}\left(x\sqrt{1-y^2}+y\sqrt{1-x^2}\right)$$

$$\Rightarrow \quad \sin^{-1}\left(x\sqrt{1-y^2}+y\sqrt{1-x^2}\right)=\begin{cases}A+B, & 0\le A+B\le(\pi/2)\\ \pi-(A+B), & (\pi/2)<A+B\le\pi\end{cases} \qquad \text{(i)}$$

Now, $A + B \le (\pi/2) \Rightarrow A \le (\pi/2) - B$

$\Rightarrow \quad \sin A \le \cos B \quad \Rightarrow \quad x \le \sqrt{1-y^2} \quad \Rightarrow \quad x^2+y^2\le 1$

And $A+B>\dfrac{\pi}{2} \Rightarrow x^2+y^2>1$

Hence from Eq. (i), we get

$$\sin^{-1}\left(x\sqrt{1-y^2}+y\sqrt{1-x^2}\right)=\begin{cases}\sin^{-1}x+\sin^{-1}y, & x^2+y^2\le 1\\ \pi-(\sin^{-1}x+\sin^{-1}y), & x^2+y^2>1\end{cases}$$

$$\Rightarrow \quad \sin^{-1}x+\sin^{-1}y=\begin{cases}\sin^{-1}\left(x\sqrt{1-y^2}+y\sqrt{1-x^2}\right), & x\ge 0,\ y\ge 0 \text{ and } x^2+y^2\le 1\\ \pi-\sin^{-1}\left(x\sqrt{1-y^2}+y\sqrt{1-x^2}\right), & x\ge 0,\ y\ge 0 \text{ and } x^2+y^2>1\end{cases}$$

Note:

For $x < 0$ and $y < 0$, these identities can be used with the help of property 3, i.e., change x and y to $-x$ and $-y$ which are positive.

ii. $\cos^{-1}x+\cos^{-1}y=\cos^{-1}\left(xy-\sqrt{1-x^2}\sqrt{1-y^2}\right), x\ge 0, y\ge 0$

Proof:

Let $\cos^{-1}x = A$ and $\cos^{-1}y = B$, where $x \ge 0$ and $y \ge 0$

$\Rightarrow \quad A, B \in [0, \pi/2] \quad \Rightarrow \quad A + B \in [0, \pi]$

Now, $\cos(A + B) = \cos A \cos B - \sin B \sin A = xy-\sqrt{1-y^2}\sqrt{1-x^2}$

$$\Rightarrow \quad \cos^{-1}(\cos(A+B)) = \cos^{-1}\left(xy - \sqrt{1-x^2}\sqrt{1-y^2}\right)$$

$$\Rightarrow \quad A+B = \cos^{-1}x + \cos^{-1}y = \cos^{-1}\left(xy - \sqrt{1-x^2}\sqrt{1-y^2}\right)$$

iii. $$\cos^{-1}x - \cos^{-1}y = \begin{cases} \cos^{-1}\left(xy + \sqrt{1-x^2}\sqrt{1-y^2}\right), & x \ge 0,\ y \ge 0 \text{ and } x \le y \\ -\cos^{-1}\left(xy + \sqrt{1-x^2}\sqrt{1-y^2}\right), & x \ge 0,\ y \ge 0 \text{ and } x > y \end{cases}$$

Proof:

Let $\cos^{-1}x = A$ and $\cos^{-1}y = B$, where $x \ge 0$ and $y \ge 0$

$\Rightarrow \quad A, B \in [0, \pi/2]$

If $x \le y$, then $\cos^{-1}x \ge \cos^{-1}y$ ($\because \cos^{-1}$ is a decreasing function)

$\Rightarrow \quad A \ge B \quad \Rightarrow \quad A - B \in [0, \pi/2]$

Now, $\cos(A-B) = \cos A \cos B + \sin B \sin A = xy - \sqrt{1-y^2}\sqrt{1-x^2}$

$$\Rightarrow \quad \cos^{-1}(\cos(A-B)) = \cos^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)$$

$$\Rightarrow \quad A-B = \cos^{-1}x - \cos^{-1}y = \cos^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)$$

If $x > y$, then $\cos^{-1}x < \cos^{-1}y$

$\Rightarrow \quad A < B \quad \Rightarrow \quad A - B \in [-\pi/2, 0)$

$$\Rightarrow \quad \cos^{-1}x - \cos^{-1}y = -\cos^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)$$

Note:

For $x < 0$ and $y < 0$, these identities can be used with the help of property 3. i.e., change x and y to $-x$ and $-y$ which are positive.

Example 4.37 Find the value of $\cot^{-1}\frac{3}{4} + \sin^{-1}\frac{5}{13}$.

Sol. $\cot^{-1}\frac{3}{4} + \sin^{-1}\frac{5}{13} = \sin^{-1}\frac{4}{5} + \sin^{-1}\frac{5}{13}$

$$= \sin^{-1}\left(\frac{4}{5}\sqrt{1-\left(\frac{5}{13}\right)^2} + \frac{5}{13}\sqrt{1-\left(\frac{4}{5}\right)^2}\right) = \sin^{-1}\left(\frac{4}{5}\frac{12}{13} + \frac{5}{13}\frac{3}{5}\right) = \sin^{-1}\frac{63}{65}$$

Example 4.38 Solve $\sin^{-1}x + \sin^{-1}2x = \frac{\pi}{3}$.

Sol. $\sin^{-1}x + \sin^{-1}2x = \frac{\pi}{3}$ (i)

$$\sin^{-1}2x = \sin^{-1}\frac{\sqrt{3}}{2} - \sin^{-1}x = \sin^{-1}\left[\frac{\sqrt{3}}{2}\sqrt{1-x^2} - x\sqrt{1-\frac{3}{4}}\right]$$

$$\Rightarrow \quad 2x = \frac{\sqrt{3}}{2}\sqrt{1-x^2} - \frac{x}{2}$$

$$\Rightarrow \quad \left(\frac{5x}{2}\right)^2 = \frac{3}{4}\left(1-x^2\right) \quad \Rightarrow \quad 28x^2 = 3$$

$$\Rightarrow \quad x = \sqrt{\frac{3}{28}} = \frac{1}{2}\sqrt{\frac{3}{7}} \qquad \left(\because x = -\frac{1}{2}\sqrt{\frac{3}{7}} \text{ makes L.H.S. of Eq. (i) negative}\right)$$

Property 9

i. $$2\sin^{-1}x = \begin{cases} \sin^{-1}\left(2x\sqrt{1-x^2}\right), & -\frac{1}{\sqrt{2}} \le x \le \frac{1}{\sqrt{2}} \\ \pi - \sin^{-1}\left(2x\sqrt{1-x^2}\right), & x > \frac{1}{\sqrt{2}} \\ -\pi - \sin^{-1}\left(2x\sqrt{1-x^2}\right), & x < -\frac{1}{\sqrt{2}} \end{cases}$$

ii. $$3\sin^{-1}x = \begin{cases} \sin^{-1}\left(3x-4x^3\right), & -\frac{1}{2} \le x \le \frac{1}{2} \\ \pi - \sin^{-1}\left(3x-4x^3\right), & x > \frac{1}{2} \\ -\pi - \sin^{-1}\left(3x-4x^3\right), & x < -\frac{1}{2} \end{cases}$$

Proof:

Let $x = \sin\theta$, $\theta \in [-\pi/2, \pi/2] \quad \Rightarrow \quad \theta = \sin^{-1}x$

Now, $\sin^{-1}\left(2x\sqrt{1-x^2}\right) = \sin^{-1}(2\sin\theta\cos\theta)$

$$= \sin^{-1}(\sin 2\theta)$$

$$= \sin^{-1}(\sin\alpha), \text{ where } \alpha \in [-\pi, \pi]$$

Now, consider the graph of $y = \sin^{-1}(\sin\alpha)$, where $\alpha \in [-\pi, \pi]$

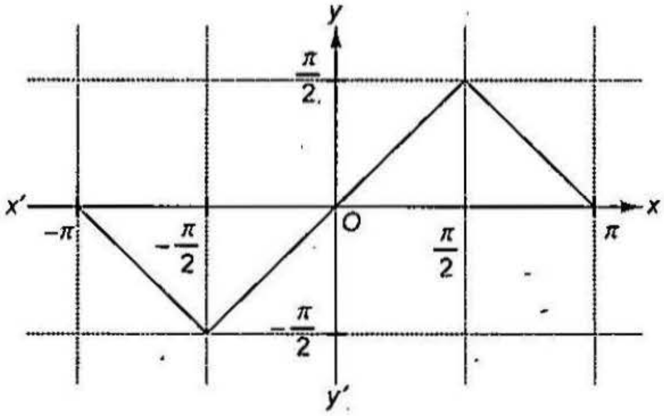


Fig. 4.22

From the graph,

$$\sin^{-1}\left(2x\sqrt{1-x^2}\right)=\sin^{-1}(\sin\alpha)$$

$$=\begin{cases}-\alpha-\pi, & -\pi<\alpha<(-\pi/2)\\ \alpha, & (-\pi/2)\le\alpha\le(\pi/2)\\ -\alpha+\pi, & (\pi/2)<\alpha<\pi\end{cases}$$

$$=\begin{cases}-2\sin^{-1}(x-\pi), & -\pi\le 2\sin^{-1}x<(-\pi/2)\\ 2\sin^{-1}x, & -(\pi/2)\le 2\sin^{-1}x\le(\pi/2)\\ -2\sin^{-1}(x+\pi), & (\pi/2)<2\sin^{-1}x<\pi\end{cases}$$

$$=\begin{cases}-2\sin^{-1}(x-\pi), & (-\pi/2)\le\sin^{-1}x<(-\pi/4)\\ 2\sin^{-1}x, & (-\pi/4)\le\sin^{-1}x\le(\pi/4)\\ -2\sin^{-1}(x+\pi), & (\pi/4)<\sin^{-1}x<(\pi/2)\end{cases}$$

$$=\begin{cases}-2\sin^{-1}x-\pi, & x<-\dfrac{1}{\sqrt{2}}\\ 2\sin^{-1}x, & -\dfrac{1}{\sqrt{2}}\le x\le\dfrac{1}{\sqrt{2}}\\ -2\sin^{-1}(x+\pi), & x>\dfrac{1}{\sqrt{2}}\end{cases}$$

$$\Rightarrow\ 2\sin^{-1}x=\begin{cases}\sin^{-1}\left(2x\sqrt{1-x^2}\right), & -\dfrac{1}{\sqrt{2}}\le x\le\dfrac{1}{\sqrt{2}}\\ \pi-\sin^{-1}\left(2x\sqrt{1-x^2}\right), & x>\dfrac{1}{\sqrt{2}}\\ -\pi-\sin^{-1}\left(2x\sqrt{1-x^2}\right), & x<-\dfrac{1}{\sqrt{2}}\end{cases}$$

Property 10

i. $$2\tan^{-1}x=\begin{cases}\tan^{-1}\left(\dfrac{2x}{1-x^2}\right), & \text{if } -1<x<1\\ \pi+\tan^{-1}\left(\dfrac{2x}{1-x^2}\right), & \text{if } x>1\\ -\pi+\tan^{-1}\left(\dfrac{2x}{1-x^2}\right), & \text{if } x<-1\end{cases}$$

ii. $$3\tan^{-1}x=\begin{cases}\tan^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right), & \text{if } -\dfrac{1}{\sqrt{3}}<x<\dfrac{1}{\sqrt{3}}\\ \pi+\tan^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right), & \text{if } x>\dfrac{1}{\sqrt{3}}\\ -\pi+\tan^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right), & \text{if } x<-\dfrac{1}{\sqrt{3}}\end{cases}$$

Proof:

i. In Property 7 (i) replace y by x.

From this information, we can also draw the graph of $y = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$ as follows (Fig. 4.23).

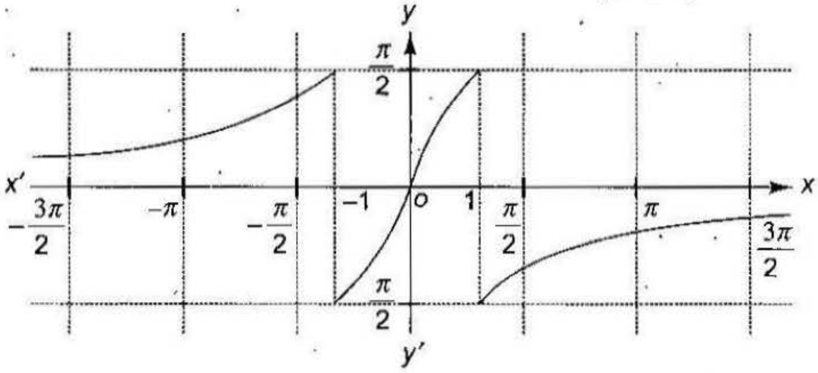


Fig. 4.23

Example 4.39 **Find the value of $4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{70} + \tan^{-1}\frac{1}{99}$.**

Sol.

$$4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{70} + \tan^{-1}\frac{1}{99} = 2\tan^{-1}\left[\frac{\frac{2}{5}}{1-\frac{1}{25}}\right] - \tan^{-1}\frac{1}{70} + \tan^{-1}\frac{1}{99}$$

$$= 2\tan^{-1}\left(\frac{5}{12}\right) + \tan^{-1}\left[\frac{\frac{1}{99}-\frac{1}{70}}{1+\frac{1}{99}\times\frac{1}{70}}\right]$$

$$= \tan^{-1}\left[\frac{\frac{5}{6}}{1-\frac{25}{144}}\right] + \tan^{-1}\left(\frac{-29}{6931}\right)$$

$$= \tan^{-1}\left(\frac{120}{119}\right) - \tan^{-1}\left(\frac{1}{239}\right)$$

$$= \tan^{-1}\left[\frac{\frac{120}{119}-\frac{1}{239}}{1+\frac{120}{119}\times\frac{1}{239}}\right] = \tan^{-1}(1) = \frac{\pi}{4}$$

Property 11

i. $$2\tan^{-1}x = \begin{cases} \sin^{-1}\left(\frac{2x}{1+x^2}\right), & \text{if } -1 \le x \le 1 \\ \pi - \sin^{-1}\left(\frac{2x}{1+x^2}\right), & \text{if } x > 1 \\ -\pi - \sin^{-1}\left(\frac{2x}{1+x^2}\right), & \text{if } x < -1 \end{cases}$$

ii. $2\tan^{-1}x = \begin{cases} \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right), & \text{if } 0 \le x < \infty \\ -\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right), & \text{if } -\infty < x \le 0 \end{cases}$

Proof:

i. Let $x = \tan\theta$, $\theta \in (-\pi/2, \pi/2)$ $\Rightarrow$ $\theta = \tan^{-1}x$

Now, $\sin^{-1}\left(\dfrac{2x}{1+x^2}\right) = \sin^{-1}\left(\dfrac{2\tan\theta}{1+\tan^2\theta}\right) = \sin^{-1}(\sin 2\theta) = \sin^{-1}(\sin\alpha)$, where $\alpha \in (-\pi, \pi)$.

Now, consider the graph of $y = \sin^{-1}(\sin\alpha)$, where $\alpha \in (-\pi, \pi)$.

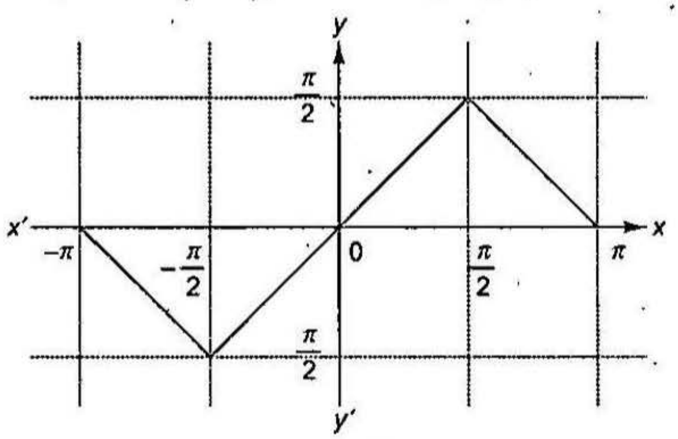


Fig. 4.24

From the graph,

$$\sin^{-1}\left(\frac{2x}{1+x^2}\right) = \sin^{-1}(\sin\alpha)$$

$$= \begin{cases} -\alpha-\pi, & -\pi < \alpha < (-\pi/2) \\ \alpha, & (-\pi/2) \le \alpha \le (\pi/2) \\ -\alpha+\pi, & (\pi/2) < \alpha < \pi \end{cases}$$

$$= \begin{cases} -2\tan^{-1}(x-\pi) & -\pi < 2\tan^{-1}x < -(\pi/2) \\ 2\tan^{-1}x, & (-\pi/2) \le 2\tan^{-1}x \le (\pi/2) \\ -2\tan^{-1}(x+\pi), & (\pi/2) < 2\tan^{-1}x < \pi \end{cases}$$

$$= \begin{cases} -2\tan^{-1}(x-\pi), & (-\pi/2) < \tan^{-1}x < (-\pi/4) \\ 2\tan^{-1}x, & (-\pi/4) \le \tan^{-1}x \le (\pi/4) \\ -2\tan^{-1}(x+\pi), & (\pi/4) < \tan^{-1}x < (\pi/2) \end{cases}$$

$$= \begin{cases} -2\tan^{-1}(x-\pi), & x < -1 \\ 2\tan^{-1}x, & -1 \le x \le 1 \\ -2\tan^{-1}(x+\pi), & x > 1 \end{cases}$$

From this information, we can also draw the graph of $y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$ as follows (Fig. 4.25).

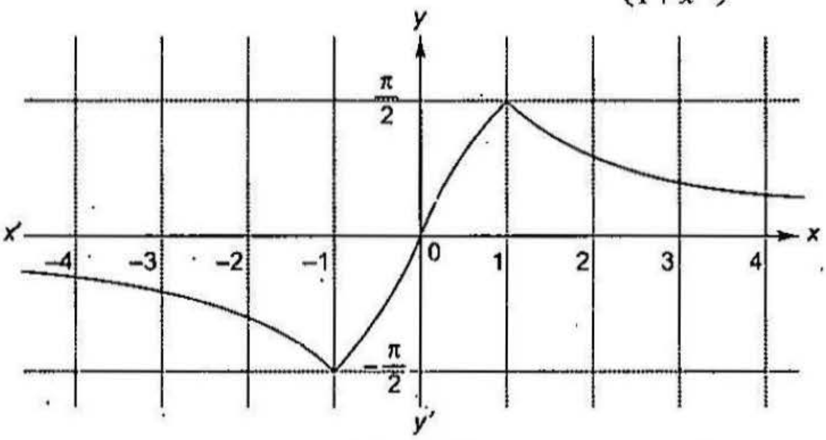


Fig. 4.25

ii. Let $x = \tan\theta$, $\theta \in (-\pi/2, \pi/2)$ $\Rightarrow$ $\theta = \tan^{-1}x$

Now, $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = \cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right) = \cos^{-1}(\cos 2\theta) = \cos^{-1}(\cos\alpha)$, where $\alpha \in (-\pi, \pi)$.

Now, consider the graph of $y = \cos^{-1}(\cos\alpha)$, where $\alpha \in (-\pi, \pi)$.

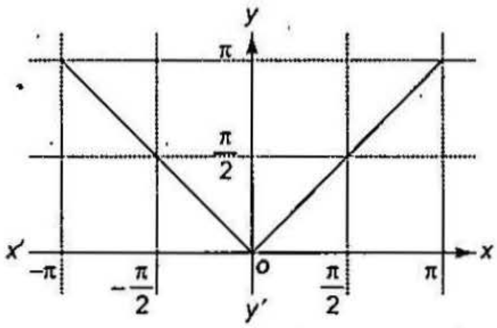


Fig. 4.26

From the graph,

$$\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = \cos^{-1}(\cos\alpha)$$

$$= \begin{cases} -\alpha, & -\pi < \alpha < 0 \\ \alpha, & 0 \le \alpha < \pi \end{cases}$$

$$= \begin{cases} -2\tan^{-1}x, & -\pi < 2\tan^{-1}x < 0 \\ 2\tan^{-1}x, & 0 \le 2\tan^{-1}x < \pi \end{cases}$$

$$= \begin{cases} -2\tan^{-1}x, & (-\pi/2) < \tan^{-1}x < 0 \\ 2\tan^{-1}x, & 0 \le \tan^{-1}x < (\pi/2) \end{cases}$$

$$= \begin{cases} -2\tan^{-1} x, & x < 0 \\ 2\tan^{-1} x, & x \geq 0 \end{cases}$$

From this information, we can also draw the graph of $y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ as follows (Fig. 4.27).

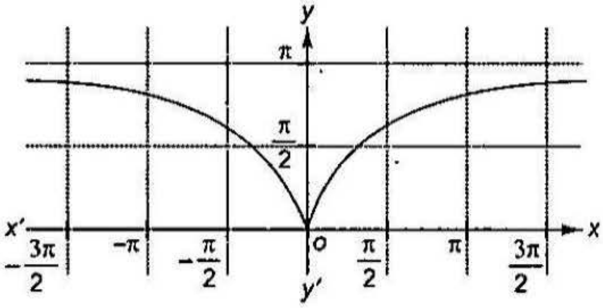


Fig. 4.27

Example 4.40 If $\sin^{-1}\frac{2x}{1+x^2} = \tan^{-1}\frac{2x}{1-x^2}$, then find the values of x.

Sol.

By referring to the graphs of $y = \sin^{-1}\frac{2x}{1+x^2}$ and $y = \sin^{-1}\frac{2x}{1+x^2}$, we get $-1 < x < 1$.

Example 4.41 If $\sin^{-1}\left(\frac{4x}{x^2+4}\right) + 2\tan^{-1}\left(-\frac{x}{2}\right)$ is independent of x, find the values of x.

Sol.

$$\sin^{-1}\left(\frac{4x}{x^2+4}\right) + 2\tan^{-1}\left(-\frac{x}{2}\right) = \sin^{-1}\left(\frac{2\times\frac{x}{2}}{\left(\frac{x}{2}\right)^2+1}\right) - 2\tan^{-1}\frac{x}{2}$$

$$= 2\tan^{-1}\frac{x}{2} - 2\tan^{-1}\frac{x}{2} = 0$$

$$\Rightarrow \left|\frac{x}{2}\right| \leq 1 \quad \Rightarrow \quad |x| \leq 2 \quad \Rightarrow \quad -2 \leq x \leq 2$$

Example 4.42 If $\cos^{-1}\frac{6x}{1+9x^2} = -\frac{\pi}{2} + 2\tan^{-1}3x$, then find the values of x.

Sol.

$$\cos^{-1}\frac{6x}{1+9x^2} = -\frac{\pi}{2} + 2\tan^{-1}3x$$

$$\Rightarrow \frac{\pi}{2} - \sin^{-1}\frac{6x}{1+9x^2} = -\frac{\pi}{2} + 2\tan^{-1}3x$$

$$\sin^{-1}\frac{6x}{1+9x^2}=\pi-2\tan^{-1}3x$$

$$\sin^{-1}\frac{2\times 3x}{1+(3x)^2}=\pi-2\tan^{-1}3x$$

It is true when $3x>1 \quad \Rightarrow \quad x>\frac{1}{3}$

i.e., $x\in\left(\frac{1}{3},\infty\right)$

Example 4.43 **If $(x-1)(x^2+1)>0$, then find the value of $\sin\left(\frac{1}{2}\tan^{-1}\frac{2x}{1-x^2}-\tan^{-1}x\right)$.**

Sol.

$(x-1)(x^2+1)>0 \quad \Rightarrow \quad x>1$

$$\therefore \quad \sin\left[\frac{1}{2}\tan^{-1}\left(\frac{2x}{1-x^2}\right)-\tan^{-1}x\right]=\sin\left[\frac{1}{2}\left(-\pi+2\tan^{-1}x\right)-\tan^{-1}x\right]=\sin\left(-\frac{\pi}{2}\right)=-1$$

Example 4.44 **Solve $\cos^{-1}\left(\frac{1}{2}x^2+\sqrt{1-x^2}\sqrt{1-\frac{x^2}{4}}\right)=\cos^{-1}\frac{x}{2}-\cos^{-1}x$.**

Sol.

$$\cos^{-1}\left(\frac{1}{2}x^2+\sqrt{1-x^2}\sqrt{1-\frac{x^2}{4}}\right)=\cos^{-1}\left(\frac{x}{2}x+\sqrt{1-x^2}\sqrt{1-\left(\frac{x}{2}\right)^2}\right)$$

For $\cos^{-1}\left(\frac{1}{2}x^2+\sqrt{1-x^2}\sqrt{1-\frac{x^2}{4}}\right)=\cos^{-1}\frac{x}{2}-\cos^{-1}x$

L.H.S. >0, hence R.H.S. $>0 \quad \Rightarrow \quad \cos^{-1}\frac{x}{2}-\cos^{-1}x>0$

Since $\cos^{-1}x$ is a decreasing function, we get

$\frac{x}{2}\le x \quad \Rightarrow \quad \frac{x}{2}\ge 0 \Rightarrow x\ge 0 \quad \Rightarrow \quad x\in[0,1]$

Example 4.45 **If $x\in\left(0,\frac{\pi}{2}\right)$, then show that**

$$\cos^{-1}\left(\frac{7}{2}(1+\cos 2x)+\sqrt{(\sin^2x-48\cos^2x)}\sin x\right)=x-\cos^{-1}(7\cos x).$$

Sol.

$$y=\cos^{-1}\left(\frac{7}{2}(1+\cos 2x)+\sqrt{(\sin^2 x-48\cos^2 x)}\sin x\right)$$

$$=\cos^{-1}\left((7\cos x)(\cos x)+\sqrt{1-49\cos^2 x}\sqrt{1-\cos^2 x}\right)$$

$$=\cos^{-1}(\cos x)-\cos^{-1}(7\cos x) \qquad [\because \cos x<7\cos x]$$

$$=x-\cos^{-1}(7\cos x)$$

Example 4.46 **Prove that** $2\cos^{-1}x=\begin{cases}2\pi-\cos^{-1}(2x^2-1), & \text{if } -1\le x<0\\ \cos^{-1}(2x^2-1), & \text{if } 0\le x\le 1\end{cases}$.

Sol.

Let $\cos^{-1}x=\theta$, where $\theta\in[0,\pi]$ $\Rightarrow$ $\cos\theta=x$

Now, $\cos^{-1}(2x^2-1)=\cos^{-1}(2\cos^2\theta-1)=\cos^{-1}(\cos 2\theta)=\cos^{-1}(\cos\alpha)$, where $\alpha\in[0,2\pi]$

Refer the graph of $y=\cos^{-1}(\cos\alpha)$, $\alpha\in[0,2\pi]$:

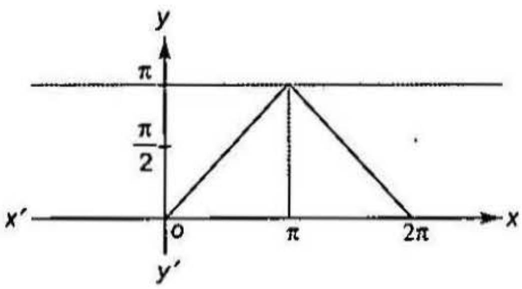


Fig. 4.28

From the graph,

$$\cos^{-1}(2x^2-1)=\begin{cases}\alpha, & \text{if } 0\le\alpha<\pi\\ 2\pi-\alpha, & \text{if } \pi\le\alpha\le 2\pi\end{cases}$$

$$=\begin{cases}2\cos^{-1}x, & \text{if } 0\le 2\cos^{-1}x\le\pi\\ 2\pi-\cos^{-1}x, & \text{if } \pi<2\cos^{-1}x\le 2\pi\end{cases}$$

$$=\begin{cases}2\cos^{-1}x, & \text{if } 0\le\cos^{-1}x\le(\pi/2)\\ 2\pi-\cos^{-1}x, & \text{if } (\pi/2)<\cos^{-1}x\le\pi\end{cases}$$

$$=\begin{cases}2\cos^{-1}x, & \text{if } 0\le x\le 1\\ 2\pi-\cos^{-1}x, & \text{if } -1\le x<0\end{cases}$$

$$\Rightarrow\ 2\cos^{-1}x=\begin{cases}2\pi-\cos^{-1}(2x^2-1), & \text{if } -1\le x<0\\ \cos^{-1}(2x^2-1), & \text{if } 0\le x\le 1\end{cases}$$

EXERCISES

Subjective Type

Solutions on page 4.54

1. Solve $2\cos^{-1}x = \sin^{-1}\left(2x\sqrt{1-x^2}\right)$.

2. Find the domain for $f(x) = \sin^{-1}\left(\dfrac{1+x^2}{2x}\right)$.

3. Find the range of $f(x) = \cot^{-1}(2x - x^2)$.

4. Find the sum $\cot^{-1}2 + \cot^{-1}8 + \cot^{-1}18 + \cdots \infty$.

5. Find the sum $\operatorname{cosec}^{-1}\sqrt{10} + \operatorname{cosec}^{-1}\sqrt{50} + \operatorname{cosec}^{-1}\sqrt{170} + \cdots + \operatorname{cosec}^{-1}\sqrt{(n^2+1)(n^2+2n+2)}$.

6. Find the number of positive integral solutions of the equation

$$\tan^{-1}x + \cos^{-1}\frac{y}{\sqrt{1-y^2}} = \sin^{-1}\frac{3}{\sqrt{10}}.$$

7. If $\tan^{-1}y = 4\tan^{-1}x \left(|x| < \tan\dfrac{\pi}{8}\right)$, find y as an algebraic function of x, and hence, prove that $\tan \pi/8$ is a root of the equation $x^4 - 6x^2 + 1 = 0$.

8. If x_1, x_2, x_3 and x_4 are the roots of the equation $x^4 - x^3\sin 2\beta + x^2\cos 2\beta - x\cos\beta - \sin\beta = 0$, prove that $\tan^{-1}x_1 + \tan^{-1}x_2 + \tan^{-1}x_3 + \tan^{-1}x_4 = n\pi + (\pi/2) - \beta$, where n is an integer.

9. Solve for real values of x: $\dfrac{(\sin^{-1}x)^3 + (\cos^{-1}x)^3}{(\tan^{-1}x + \cot^{-1}x)^3} = 7$.

10. Find the set of values of parameter a so that the equation $(\sin^{-1}x)^3 + (\cos^{-1}x)^3 = a\pi^3$ has a solution.

11. If $p > q > 0$ and $pr < -1 < qr$, then find the value of $\tan^{-1}\dfrac{p-q}{1+pq} + \tan^{-1}\dfrac{q-r}{1+qr} + \tan^{-1}\dfrac{r-p}{1+rp}$.

12. Solve the equation $\sqrt{|\sin^{-1}|\cos x|| + |\cos^{-1}|\sin x||} = \sin^{-1}|\cos x| - \cos^{-1}|\sin x|$.

13. Solve the equation $\tan^{-1}\dfrac{x+1}{x-1} + \tan^{-1}\dfrac{x-1}{x} = \tan^{-1}(-7)$.

14. Solve the equation $\sin^{-1}6x + \sin^{-1}6\sqrt{3}\,x = -\pi/2$.

15. If $0 < a_1 < a_2 < \cdots < a_n$, then prove that

$$\tan^{-1}\left(\frac{a_1x-y}{x+a_1y}\right) + \tan^{-1}\left(\frac{a_2-a_1}{1+a_2a_1}\right) + \tan^{-1}\left(\frac{a_3-a_2}{1+a_3a_2}\right) + \cdots + \tan^{-1}\left(\frac{a_n-a_{n-1}}{1+a_na_{n-1}}\right) + \tan^{-1}\left(\frac{1}{a_n}\right) = \tan^{-1}\frac{x}{y}.$$

Objective Type

Solutions on page 4.59

Each question has four choices a, b, c and d, out of which *only one* is correct.

1. The principal value of $\sin^{-1}(\sin 10)$ is
   a. 10 b. $10-3\pi$ c. $3\pi-10$ d. none of these
2. $\cos^{-1}\left(\cos\left(\frac{5\pi}{4}\right)\right)$ is given by
   a. $\frac{5\pi}{4}$ b. $\frac{3\pi}{4}$ c. $\frac{-\pi}{4}$ d. none of these.
3. The value of $\sin^{-1}(\sin 12)+\cos^{-1}(\cos 12)$ is equal to
   a. zero b. $24-2\pi$ c. $4\pi-24$ d. none of these
4. The value of the expression $\sin^{-1}\left(\sin\frac{22\pi}{7}\right)+\cos^{-1}\left(\cos\frac{5\pi}{3}\right)+\tan^{-1}\left(\tan\frac{5\pi}{7}\right)+\sin^{-1}(\cos 2)$ is
   a. $\frac{17\pi}{42}-2$ b. -2 c. $\frac{-\pi}{21}-2$ d. none of these
5. The value of $\sin^{-1}(\cos(\cos^{-1}(\cos x)+\sin^{-1}(\sin x)))$, where $x\in\left(\frac{\pi}{2},\pi\right)$, is equal to
   a. $\frac{\pi}{2}$ b. $-\pi$ c. π d. $-\frac{\pi}{2}$
6. $\cos^{-1}(\cos(2\cot^{-1}(\sqrt{2}-1)))$ is equal to
   a. $\sqrt{2}-1$ b. $\frac{\pi}{4}$ c. $\frac{3\pi}{4}$ d. none of these
7. The value of $\sin^{-1}\left(\cot\left(\sin^{-1}\sqrt{\frac{2-\sqrt{3}}{4}}+\cos^{-1}\frac{\sqrt{12}}{4}+\sec^{-1}\sqrt{2}\right)\right)$ is
   a. 0 b. $\frac{\pi}{2}$ c. $\frac{\pi}{3}$ d. none of these
8. The value of $\cos^{-1}\sqrt{\frac{2}{3}}-\cos^{-1}\frac{\sqrt{6}+1}{2\sqrt{3}}$ is equal to
   a. $\frac{\pi}{3}$ b. $\frac{\pi}{4}$ c. $\frac{\pi}{2}$ d. $\frac{\pi}{6}$
9. The value of $\cos\left(\frac{1}{2}\cos^{-1}\frac{1}{8}\right)$ is
   a. $\frac{3}{4}$ b. $-\frac{3}{4}$ c. $\frac{1}{16}$ d. $\frac{1}{4}$
10. If $\tan(x+y)=33$ and $x=\tan^{-1}3$, then y will be
   a. 0.3 b. $\tan^{-1}(1.3)$ c. $\tan^{-1}(0.3)$ d. $\tan^{-1}\left(\frac{1}{18}\right)$

11. The value of $\tan\left(\frac{1}{2}\cos^{-1}\left(\frac{\sqrt{5}}{3}\right)\right)$ is

a. $\frac{3+\sqrt{5}}{2}$ b. $3+\sqrt{5}$ c. $\frac{1}{2}(3-\sqrt{5})$ d. none of these

12. $\tan^{-1}\left[\frac{\cos x}{1+\sin x}\right]$ is equal to

a. $\frac{\pi}{4}-\frac{x}{2}$, for $x \in \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right)$ b. $\frac{\pi}{4}-\frac{x}{2}$, for $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

c. $\frac{\pi}{4}-\frac{x}{2}$, for $x \in \left(\frac{3\pi}{2}, \frac{5\pi}{2}\right)$ d. $\frac{\pi}{4}-\frac{x}{2}$, for $x \in \left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right)$

13. If $f(x) = x^{11}+x^9-x^7+x^3+1$ and $f(\sin^{-1}(\sin 8)) = \alpha$, α is a constant, then $f(\tan^{-1}(\tan 8))$ is equal to

a. α b. $\alpha-2$ c. $\alpha+2$ d. $2-\alpha$

14. The maximum value of $f(x) = \tan^{-1}\left(\frac{(\sqrt{12}-2)x^2}{x^4+2x^2+3}\right)$ is

a. 18° b. 36° c. 22.5° d. 15°

15. The value of $\sin^{-1}\left[x\sqrt{1-x}-\sqrt{x}\sqrt{1-x^2}\right]$ is equal to

a. $\sin^{-1}x+\sin^{-1}\sqrt{x}$ b. $\sin^{-1}x-\sin^{-1}\sqrt{x}$ c. $\sin^{-1}\sqrt{x}-\sin^{-1}x$ d. none of these

16. $\tan^{-1}\left(\frac{x}{y}\right)-\tan^{-1}\left(\frac{x-y}{x+y}\right)$ is

a. $\frac{\pi}{2}$ b. $\frac{\pi}{3}$ c. $\frac{\pi}{4}$ d. $\frac{\pi}{4}$ or $-\frac{3\pi}{4}$

17. If $\tan^{-1}\frac{a+x}{a}+\tan^{-1}\frac{a-x}{a}=\frac{\pi}{6}$, then $x^2=$

a. $2\sqrt{3}a$ b. $\sqrt{3}a$ c. $2\sqrt{3}a^2$ d. none of these

18. If $\cot^{-1}\frac{n}{\pi}>\frac{\pi}{6}$, $n \in N$, then the maximum value of n is

a. 6 b. 7 c. 5 d. none of these

19. $\sec^2(\tan^{-1}2)+\operatorname{cosec}^2(\cot^{-1}3)$ is equal to

a. 5 b. 13 c. 15 d. 6

20. If $A=\tan^{-1}\left(\frac{x\sqrt{3}}{2K-x}\right)$ and $B=\tan^{-1}\left(\frac{2x-K}{K\sqrt{3}}\right)$, then the value of $A-B$ is

a. 0° b. 45° c. 60° d. 30°

21. The value of $\sec\left[\tan^{-1}\frac{b+a}{b-a}-\tan^{-1}\frac{a}{b}\right]$ is

a. 2 b. $\sqrt{2}$ c. 4 d. 1

22. If $a \sin^{-1} x - b \cos^{-1} x = c$, then $a \sin^{-1} x + b \cos^{-1} x$ is equal to

a. 0 b. $\dfrac{\pi ab + c(b-a)}{a+b}$ c. $\dfrac{\pi}{2}$ d. $\dfrac{\pi ab + c(a-b)}{a+b}$

23. The number of solution of the equation $\cos^{-1}\left(\dfrac{1+x^2}{2x}\right) - \cos^{-1} x = \dfrac{\pi}{2} + \sin^{-1} x$ is given by

a. 0 b. 1 c. 2 d. 3

24. The sum of the solutions of the equation $2 \sin^{-1} \sqrt{x^2+x+1} + \cos^{-1}\sqrt{x^2+x} = \dfrac{3\pi}{2}$ is

a. 0 b. -1 c. 1 d. 2

25. The number of solutions of the equation $\tan^{-1}(1+x) + \tan^{-1}(1-x) = \dfrac{\pi}{2}$ is

a. 2 b. 3 c. 1 d. 0

26. The number of solution of the equation $\sin^{-1} x + \sin^{-1}(1-x) = \cos^{-1} x$ is

a. 1 b. 0 c. 2 d. none of these

27. If $\tan^{-1}\dfrac{1-x}{1+x} = \dfrac{1}{2}\tan^{-1} x$, then x is equal to

a. 1 b. $\sqrt{3}$ c. $\dfrac{1}{\sqrt{3}}$ d. none of these

28. For the equation $\cos^{-1} x + \cos^{-1} 2x + \pi = 0$, the number of real solution is

a. 1 b. 2 c. 0 d. ∞

29. The value of 'a', for which $ax^2 + \sin^{-1}(x^2-2x+2) + \cos^{-1}(x^2-2x+2) = 0$ has a real solution is

a. $\dfrac{\pi}{2}$ b. $-\dfrac{\pi}{2}$ c. $\dfrac{2}{\pi}$ d. $-\dfrac{2}{\pi}$

30. The number of real solutions of the equation $\tan^{-1}\sqrt{x^2-3x+2} + \cos^{-1}\sqrt{4x-x^2-3} = \pi$ is

a. one b. two c. zero d. infinite

31. If $3\tan^{-1}\left(\dfrac{1}{2+\sqrt{3}}\right) - \tan^{-1}\dfrac{1}{x} = \tan^{-1}\dfrac{1}{3}$, then x is equal to

a. 1 b. 2 c. 3 d. $\sqrt{2}$

32. If $\tan^{-1} x + 2\cot^{-1} x = \dfrac{2\pi}{3}$, then x is equal to

a. $\dfrac{\sqrt{3}-1}{\sqrt{3}+1}$ b. 3 c. $\sqrt{3}$ d. $\sqrt{2}$

33. If $3\sin^{-1}\left(\dfrac{2x}{1+x^2}\right) - 4\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right) + 2\tan^{-1}\left(\dfrac{2x}{1-x^2}\right) = \dfrac{\pi}{3}$, where $|x| < 1$, then x is equal to

a. $\dfrac{1}{\sqrt{3}}$ b. $-\dfrac{1}{\sqrt{3}}$ c. $\sqrt{3}$ d. $-\dfrac{\sqrt{3}}{4}$ e. $\dfrac{\sqrt{3}}{2}$

34. If $\sin^{-1}\left(\dfrac{5}{x}\right) + \sin^{-1}\left(\dfrac{12}{x}\right) = \dfrac{\pi}{2}$, then x is equal to

a. $\dfrac{7}{13}$ b. $\dfrac{4}{3}$ c. 13 d. $\dfrac{13}{7}$

35. If $\sin^{-1}(x-1)+\cos^{-1}(x-3)+\tan^{-1}\left(\frac{x}{2-x^2}\right)=\cos^{-1}k+\pi$, then the value of k is

a. 1 b. $-\frac{1}{\sqrt{2}}$ c. $\frac{1}{\sqrt{2}}$ d. none of these

36. If $\cot^{-1}x+\cot^{-1}y+\cot^{-1}z=\frac{\pi}{2}$, $x, y, z>0$ and $xy<1$, then $x+y+z$ is also equal to

a. $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$ b. xyz c. $xy+yz+zx$ d. none of these

37. If $\cos^{-1}x+\cos^{-1}y+\cos^{-1}z=\pi$, then

a. $x^2+y^2+z^2+xyz=0$ b. $x^2+y^2+z^2+2xyz=0$
c. $x^2+y^2+z^2+xyz=1$ d. $x^2+y^2+z^2+2xyz=1$

38. If $\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\frac{\pi}{2}$, then

a. $x+y+z-xyz=0$ b. $x+y+z+xyz=0$ c. $xy+yz+zx+1=0$ d. $xy+yz+zx-1=0$

39. If $\cos^{-1}\sqrt{p}+\cos^{-1}\sqrt{1-p}+\cos^{-1}\sqrt{1-q}=\frac{3\pi}{4}$, then the value of q is

a. 1 b. $\frac{1}{\sqrt{2}}$ c. $\frac{1}{3}$ d. $\frac{1}{2}$

40. If $\sin^{-1}a+\sin^{-1}b+\sin^{-1}c=\pi$, then the value of $a\sqrt{(1-a^2)}+b\sqrt{(1-b^2)}+c\sqrt{(1-c^2)}$ will be

a. $2abc$ b. abc c. $\frac{1}{2}abc$ d. $\frac{1}{3}abc$

41. If $\sin^{-1}x+\sin^{-1}y+\sin^{-1}z=\pi$, then $x^4+y^4+z^4+4x^2y^2z^2=K(x^2y^2+y^2z^2+z^2x^2)$, where K is equal to

a. 1 b. 2 c. 4 d. none of these

42. If $\cos^{-1}x-\cos^{-1}\frac{y}{2}=\alpha$, then $4x^2-4xy\cos\alpha+y^2$ is equal to

a. 4 b. $2\sin^2\alpha$ c. $-4\sin^2\alpha$ d. $4\sin^2\alpha$

43. The value of x which satisfies equation $2\tan^{-1}2x=\sin^{-1}\frac{4x}{1+4x^2}$ is valid in the interval

a. $\left[\frac{1}{2},\infty\right)$ b. $\left(-\infty,-\frac{1}{2}\right]$ c. $[-1,1]$ d. $\left[-\frac{1}{2},\frac{1}{2}\right]$

44. If $x\in[-1,0)$, then $\cos^{-1}(2x^2-1)-2\sin^{-1}x$ is equal to

a. $-\frac{\pi}{2}$ b. π c. $\frac{3\pi}{2}$ d. -2π

45. If $2\sin^{-1}x=\sin^{-1}(2x\sqrt{1-x^2})$, then

a. $[-1,1]$ b. $\left[-\frac{1}{\sqrt{2}},1\right]$ c. $\left[-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right]$ d. none of these

46. If $x_1=2\tan^{-1}\left(\frac{1+x}{1-x}\right)$, $x_2=\sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$, where $x\in(0,1)$, then x_1+x_2 is equal to

a. 0 b. 2π c. π d. none of these

47. The value of $\sin(2\sin^{-1}(0.8))$ is equal to

a. $\sin 1.2°$ b. $\sin 1.6°$ c. 0.48 d. 0.96

48. The value of $\tan(\sin^{-1}(\cos(\sin^{-1}x)))\tan(\cos^{-1}(\sin(\cos^{-1}x)))$, where $x \in (0, 1)$, is equal to

a. 0 b. 1 c. -1 d. none of these

49. If $\sin^{-1}\left(\frac{2a}{1+a^2}\right)+\sin^{-1}\left(\frac{2b}{1+b^2}\right)=2\tan^{-1}x$, then x is equal to $[a, b \in (0, 1)]$

a. $\frac{a-b}{1+ab}$ b. $\frac{b}{1+ab}$ c. $\frac{b}{1-ab}$ d. $\frac{a+b}{1-ab}$

50. If x takes negative permissible value, then $\sin^{-1}x$ is equal to

a. $\cos^{-1}\sqrt{1-x^2}$ b. $-\cos^{-1}\sqrt{1-x^2}$ c. $\cos^{-1}\sqrt{x^2-1}$ d. $\pi-\cos^{-1}\sqrt{1-x^2}$

51. If $x^2+y^2+z^2=r^2$, then $\tan^{-1}\left(\frac{xy}{zr}\right)+\tan^{-1}\left(\frac{yz}{xr}\right)+\tan^{-1}\left(\frac{xz}{yr}\right)$ is equal to

a. π b. $\frac{\pi}{2}$ c. 0 d. none of these

52. If $f(x)=\sin^{-1}\left(\frac{\sqrt{3}}{2}x-\frac{1}{2}\sqrt{1-x^2}\right)$, $-\frac{1}{2}\le x\le 1$, then $f(x)$ is equal to

a. $\sin^{-1}\left(\frac{1}{2}\right)-\sin^{-1}(x)$ b. $\sin^{-1}x-\frac{\pi}{6}$ c. $\sin^{-1}x+\frac{\pi}{6}$ d. none of these

53. If $x \in (0, 1)$, then the value of $\tan^{-1}\left(\frac{1-x^2}{2x}\right)+\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ is equal to

a. $-\frac{\pi}{2}$ b. zero c. $\frac{\pi}{2}$ d. π

54. The trigonometric equation $\sin^{-1}x=2\sin^{-1}a$ has a solution for

a. all real values b. $|a|<\frac{1}{2}$ c. $|a|\le\frac{1}{\sqrt{2}}$ d. $\frac{1}{2}<|a|<\frac{1}{\sqrt{2}}$

55. If $2\tan^{-1}x=\pi+\tan^{-1}\left(\frac{2x}{1-x^2}\right)$, then

a. $x>1$ b. $x<1$ c. $x>-1$ d. $-1<x<1$

56. $\sum_{r=1}^{n}\sin^{-1}\left(\frac{\sqrt{r}-\sqrt{r-1}}{\sqrt{r(r+1)}}\right)$ is equal to

a. $\tan^{-1}(\sqrt{n})-\frac{\pi}{4}$ b. $\tan^{-1}(\sqrt{n+1})-\frac{\pi}{4}$ c. $\tan^{-1}(\sqrt{n})$ d. $\tan^{-1}(\sqrt{n+1})$

57. $\sum_{r=1}^{n}\tan^{-1}\left(\frac{2^{r-1}}{1+2^{2r-1}}\right)$ is equal to

a. $\tan^{-1}(2^n)$ b. $\tan^{-1}(2^n)-\frac{\pi}{4}$ c. $\tan^{-1}(2^{n+1})$ d. $\tan^{-1}(2^{n+1})-\frac{\pi}{4}$

58. $\sum_{m=1}^{n} \tan^{-1}\left(\frac{2m}{m^4+m^2+2}\right)$ is equal to

a. $\tan^{-1}\left(\frac{n^2+n}{n^2+n+2}\right)$ b. $\tan^{-1}\left(\frac{n^2-n}{n^2-n+2}\right)$ c. $\tan^{-1}\left(\frac{n^2+n+2}{n^2+n}\right)$ d. none of these

59. The value of $\sum_{r=0}^{\infty} \tan^{-1}\left(\frac{1}{1+r+r^2}\right)$ is equal to

a. $\frac{\pi}{2}$ b. $\frac{3\pi}{4}$ c. $\frac{\pi}{4}$ d. none of these

60. If $\sin^{-1} x = \theta+\beta$ and $\sin^{-1} y = \theta-\beta$, then $1 + xy$ is equal to

a. $\sin^2\theta+\sin^2\beta$ b. $\sin^2\theta+\cos^2\beta$ c. $\cos^2\theta+\cos^2\theta$ d. $\cos^2\theta+\sin^2\beta$

61. If $u = \cot^{-1}\sqrt{\tan\alpha} - \tan^{-1}\sqrt{\tan\alpha}$, then $\tan\left(\frac{\pi}{4}-\frac{u}{2}\right)$ is equal to

a. $\sqrt{\tan\alpha}$ b. $\sqrt{\cot\alpha}$ c. $\tan\alpha$ d. $\cot\alpha$

62. of $\tan\left[\frac{\pi}{4}+\frac{1}{2}\cos^{-1}\frac{a}{b}\right] + \tan\left[\frac{\pi}{4}-\frac{1}{2}\cos^{-1}\frac{a}{b}\right]$ is equal to

a. $\frac{2a}{b}$ b. $\frac{2b}{a}$ c. $\frac{a}{b}$ d. $\frac{b}{a}$

63. The value $2\tan^{-1}\left[\sqrt{\frac{a-b}{a+b}}\tan\frac{\theta}{2}\right]$ is equal to

a. $\cos^{-1}\left(\frac{a\cos\theta+b}{a+b\cos\theta}\right)$ b. $\cos^{-1}\left(\frac{a+b\cos\theta}{a\cos\theta+b}\right)$ c. $\cos^{-1}\left(\frac{a\cos\theta}{a+b\cos\theta}\right)$ d. $\cos^{-1}\left(\frac{b\cos\theta}{a\cos\theta+b}\right)$

64. $\cot^{-1}\left[\frac{\sqrt{1-\sin x}+\sqrt{1+\sin x}}{\sqrt{1-\sin x}-\sqrt{1+\sin x}}\right]$ $\left(\text{where } x\in\left[0,\frac{\pi}{2}\right]\right)$ is equal to

a. $\pi-x$ b. $2\pi-x$ c. $\frac{x}{2}$ d. $\pi-\frac{x}{2}$

65. The value of $\tan^{-1}\left(\frac{x\cos\theta}{1-x\sin\theta}\right)-\cot^{-1}\left(\frac{\cos\theta}{x-\sin\theta}\right)$ is

a. 2θ b. θ c. $\theta/2$ d. independent of θ

66. If $x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$, then the value of $\tan^{-1}\left(\frac{\tan x}{4}\right)+\tan^{-1}\left(\frac{3\sin 2x}{5+3\cos 2x}\right)$ is

a. $x/2$ b. $2x$ c. $3x$ d. x

67. If $\cot^{-1}\left(\sqrt{\cos\alpha}\right) - \tan^{-1}\left(\sqrt{\cos\alpha}\right) = x$, then $\sin x$ is

a. $\tan^2\frac{\alpha}{2}$ b. $\cot^2\frac{\alpha}{2}$ c. $\tan a$ d. $\cot\frac{\alpha}{2}$

68. $\tan\left(\frac{\pi}{4}+\frac{1}{2}\cos^{-1}x\right)+\tan\left(\frac{\pi}{4}-\frac{1}{2}\cos^{-1}x\right), x\neq 0$, is equal to

a. x b. $2x$ c. $\frac{2}{x}$ d. none of these

69. The least and the greatest values of $(\sin^{-1}x)^3+(\cos^{-1}x)^3$ are

a. $\frac{-\pi}{2}, \frac{\pi}{2}$ b. $\frac{-\pi^3}{8}, \frac{\pi^3}{8}$ c. $\frac{\pi^3}{32}, \frac{7\pi^3}{8}$ d. none of these

70. Range of $f(x)=\sin^{-1}x+\tan^{-1}x+\sec^{-1}x$ is

a. $\left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$ b. $\left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$ c. $\left\{\frac{\pi}{4}, \frac{3\pi}{4}\right\}$ d. none of these

71. Range of $\tan^{-1}\left(\frac{2x}{1+x^2}\right)$ is

a. $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$ b. $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ c. $\left(-\frac{\pi}{2}, \frac{\pi}{4}\right]$ d. $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$

72. If $[\cot^{-1}x]+[\cos^{-1}x]=0$, where $[\cdot]$ denotes the greatest integer function, then the complete set of values of x is

a. $(\cos 1, 1]$ b. $(\cos 1, \cos 1)$ c. $(\cot 1, 1]$ d. none of these

73. $\sin^{-1}(\sin 5)>x^2-4x$ holds if

a. $x=2-\sqrt{9-2\pi}$ b. $x=2+\sqrt{9-2\pi}$

c. $x>2+\sqrt{9-2\pi}$ d. $x\in(2-\sqrt{9-2\pi}, 2+\sqrt{9-2\pi})$

74. The value of $\frac{\alpha^3}{2}\operatorname{cosec}^2\left(\frac{1}{2}\tan^{-1}\frac{\alpha}{\beta}\right)+\frac{\beta^3}{2}\sec^2\left(\frac{1}{2}\tan^{-1}\left(\frac{\beta}{\alpha}\right)\right)$ is equal to

a. $(\alpha-\beta)(\alpha^2+\beta^2)$ b. $(\alpha+\beta)(\alpha^2-\beta^2)$ c. $(\alpha+\beta)(\alpha^2+\beta^2)$ d. none of these

75. The value of $\lim_{|x|\to\infty}\cos(\tan^{-1}(\sin(\tan^{-1}x)))$ is equal to

a. -1 b. $\sqrt{2}$ c. $-\frac{1}{\sqrt{2}}$ d. $\frac{1}{\sqrt{2}}$

76. $\sin^{-1}(3x-2-x^2)+\cos^{-1}(x^2-4x+3)=\frac{\pi}{4}$ can have a solution for $x\in$

a. $[1,2]$ b. $\left(\frac{3+\sqrt{5}}{2}, 2+\sqrt{2}\right)$ c. $\left(\frac{3-\sqrt{5}}{2}, 2-\sqrt{2}\right)$ d. none of these

77. If $2^{2\pi/\sin^{-1}x}-2(a+2)2^{\pi/\sin^{-1}x}+8a<0$ for at least one real x, then

a. $\frac{1}{8}\leq a<2$ b. $a<2$ c. $a\in R-\{2\}$ d. $a\in\left[0,\frac{1}{8}\right)\cup(2,\infty)$

78. The number of integral values of k for which the equation $\sin^{-1}x+\tan^{-1}x=2k+1$ has a solution is

a. 1 b. 2 c. 3 d. 4

79. If $\tan^{-1}(\sin^2\theta - 2\sin\theta + 3) + \cot^{-1}\left(5^{\sec^2 y} + 1\right) = \frac{\pi}{2}$, then the value of $\cos^2\theta - \sin\theta$ is equal to

a. 0 b. –1 c. 1 d. none of these

80. Complete solution set of $[\cot^{-1} x] + 2[\tan^{-1} x] = 0$, where $[\cdot]$ denotes the greatest integer function, is equal to

a. $(0, \cot 1)$ b. $(0, \tan 1)$ c. $(\tan 1, \infty)$ d. $(\cot 1, \tan 1)$

81. Let $\begin{vmatrix} \tan^{-1} x & \tan^{-1} 2x & \tan^{-1} 3x \\ \tan^{-1} 3x & \tan^{-1} x & \tan^{-1} 2x \\ \tan^{-1} 2x & \tan^{-1} 3x & \tan^{-1} x \end{vmatrix} = 0$, then the number of values of x satisfying the equation is

a. 1 b. 2 c. 3 d. 4

82. Which of the following is the solution set of the equation $2\cos^{-1}x = \cot^{-1}\left(\frac{2x^2-1}{2x\sqrt{1-x^2}}\right)$?

a. $(0,1)$ b. $(-1,1)-\{0\}$ c. $(-1,0)$ d. $[-1,1]$

83. The values of x satisfying the equation $\sin(\tan^{-1} x) = \cos(\cot^{-1}(x+1))$ is

a. $\frac{1}{2}$ b. $-\frac{1}{2}$ c. $\sqrt{2}-1$ d. no finite value

84. There exists a positive real number x satisfying $\cos(\tan^{-1}x) = x$. Then the value of $\cos^{-1}\left(\frac{x^2}{2}\right)$ is

a. $\frac{\pi}{10}$ b. $\frac{\pi}{5}$ c. $\frac{2\pi}{5}$ d. $\frac{4\pi}{5}$

85. The range of values of p for which the equation $\sin\cos^{-1}(\cos(\tan^{-1}x)) = p$ has a solution is

a. $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ b. $[0,1)$ c. $\left(\frac{1}{\sqrt{2}}, 1\right)$ d. $(-1, 1)$

86. Sum of roots of the equation $\sin^{-1} x - \cos^{-1} x = \sin^{-1}(3x-2)$ is

a. 3/2 b. 1 c. 1/2 d. 2

87. The solution set of the equation $\sin^{-1}\sqrt{1-x^2} + \cos^{-1} x = \cot^{-1}\frac{\sqrt{1-x^2}}{x} - \sin^{-1} x$ is

a. $[-1, 1]-\{0\}$ b. $(0, 1] \cup \{-1\}$ c. $[-1,0) \cup \{1\}$ d. $[-1,1]$

88. The number of real solutions of the equation $\sqrt{1+\cos 2x} = \sqrt{2}\sin^{-1}(\sin x)$, $-\pi \le x \le \pi$, is

a. 0 b. 1 c. 2 d. infinite

89. The equation $3\cos^{-1} x - \pi x - \frac{\pi}{2} = 0$ has

a. one negative solution b. one positive solution
c. no solution d. more than one solution

90. If $\left|\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)\right| < \frac{\pi}{3}$, then

a. $x \in \left[-\frac{1}{3}, \frac{1}{\sqrt{3}}\right]$ b. $x \in \left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ c. $x \in \left(0, \frac{1}{\sqrt{3}}\right)$ d. none of these

91. If $\sin^{-1} x + \sin^{-1} y = \frac{\pi}{2}$, then $\frac{1+x^4+y^4}{x^2-x^2y^2+y^2}$ is equal to

a. 1 b. 2 c. $\frac{1}{2}$ d. none of these

92. The value of $\sin^{-1}(x^2-4x+6)+\cos^{-1}(x^2-4x+6)$ for all $x \in R$ is

a. $\frac{\pi}{2}$ b. π c. 0 d. none of these

93. The product of all values of x satisfying the equation

$$\sin^{-1}\cos\left(\frac{2x^2+10|x|+4}{x^2+5|x|+3}\right) = \cot\left(\cot^{-1}\left(\frac{2-18|x|}{9|x|}\right)\right)+\frac{\pi}{2} \text{ is}$$

a. 9 b. –9 c. –3 d. –1

94. The value of $2\tan^{-1}(\operatorname{cosec}\tan^{-1} x - \tan\cot^{-1} x)$ is equal to

a. $\cot^{-1} x$ b. $\cot^{-1}\frac{1}{x}$ c. $\tan^{-1} x$ d. none of these

Multiple Correct Answers Type

Solutions on page 4.80

Each question has four choices a, b, c and d, out of which *one or more* answers are correct.

1. If $\alpha, \beta\ (\alpha < \beta)$ are the roots of the equation $6x^2+11x+3=0$, then which of the following are real?

a. $\cos^{-1}\alpha$ b. $\sin^{-1}\beta$ c. $\operatorname{cosec}^{-1}\alpha$ d. Both $\cot^{-1}\alpha$ and $\cot^{-1}\beta$

2. $2\tan^{-1}(-2)$ is equal to

a. $-\cos^{-1}\left(\frac{-3}{5}\right)$ b. $-\pi+\cos^{-1}\frac{3}{5}$ c. $-\frac{\pi}{2}+\tan^{-1}\left(-\frac{3}{4}\right)$ d. $-\pi+\cot^{-1}\left(-\frac{3}{4}\right)$

3. If α, β and γ are the roots of $\tan^{-1}(x-1)+\tan^{-1}x+\tan^{-1}(x+1)=\tan^{-1}3x$, then

a. $\alpha+\beta+\gamma=0$
b. $\alpha\beta+\beta\gamma+\gamma\alpha=-1/4$
c. $\alpha\beta\gamma=1$
d. $|\alpha-\beta|_{max}=1$

4. If $f(x)=\sin^{-1}x+\sec^{-1}x$ is defined, then which of the following value/values is/are in its range?

a. $-\pi/2$ b. $\pi/2$ c. π d. $3\pi/2$

5. If $(\sin^{-1}x+\sin^{-1}w)(\sin^{-1}y+\sin^{-1}z)=\pi^2$, then $D=\begin{vmatrix} x^{N_1} & y^{N_2} \\ z^{N_3} & w^{N_4}\end{vmatrix}$ $(N_1, N_2, N_3, N_4 \in N)$

a. has a maximum value of 2
b. has a minimum value of 0
c. 16 different D are possible
d. has a minimum value of –2

6. Indicate the relation which can hold in their respective domain for infinite values of x.

a. $\tan|\tan^{-1}x|=|x|$ b. $\cot|\cot^{-1}x|=|x|$ c. $\tan^{-1}|\tan x|=|x|$ d. $\sin|\sin^{-1}x|=|x|$

7. If α is a real number for which $f(x)=\log_e\cos^{-1}x$ is defined, then a possible value of $[\alpha]$ (where $[\cdot]$ denotes the greatest integer function) is

a. 0 b. 1 c. – 1 d. –2

8. Which of the following is a rational number?

a. $\sin\left(\tan^{-1}3+\tan^{-1}\frac{1}{3}\right)$
b. $\cos\left(\frac{\pi}{2}-\sin^{-1}\frac{3}{4}\right)$
c. $\log_2\left(\sin\left(\frac{1}{4}\sin^{-1}\frac{\sqrt{63}}{8}\right)\right)$
d. $\tan\left(\frac{1}{2}\cos^{-1}\frac{\sqrt{5}}{3}\right)$

9. If $z=\sec^{-1}\left(x+\frac{1}{x}\right)+\sec^{-1}\left(y+\frac{1}{y}\right)$, where $xy<0$, then the possible values of z is (are)

a. $\frac{8\pi}{10}$
b. $\frac{7\pi}{10}$
c. $\frac{9\pi}{10}$
d. $\frac{21\pi}{20}$

10. If $f(x)=(\sin^{-1}x)^2+(\cos^{-1}x)^2$, then

a. $f(x)$ has the least value of $\frac{\pi^2}{8}$
b. $f(x)$ has the greatest value of $\frac{5\pi^2}{8}$
c. $f(x)$ has the least value of $\frac{\pi^2}{16}$
d. $f(x)$ has the greatest value of $\frac{5\pi^2}{4}$

11. If $\sin^{-1}x+\sin^{-1}y=\frac{\pi}{2}$ and $\sin 2x=\cos 2y$, then

a. $x=\frac{\pi}{8}+\sqrt{\frac{1}{2}-\frac{\pi^2}{64}}$
b. $y=\sqrt{\frac{1}{2}-\frac{\pi^2}{64}}-\frac{\pi}{12}$
c. $x=\frac{\pi}{12}+\sqrt{\frac{1}{2}-\frac{\pi^2}{64}}$
d. $y=\sqrt{\frac{1}{2}-\frac{\pi^2}{64}}-\frac{\pi}{8}$

12. If $\cot^{-1}\left(\frac{n^2-10n+21.6}{\pi}\right)>\frac{\pi}{6}$, $n\in N$, then n can be

a. 3
b. 2
c. 4
d. 8

13. If $S_n=\cot^{-1}(3)+\cot^{-1}(7)+\cot^{-1}(13)+\cot^{-1}(21)+\cdots$ n terms, then

a. $S_{10}=\tan^{-1}\frac{5}{6}$
b. $S_\infty=\frac{\pi}{4}$
c. $S_6=\sin^{-1}\frac{4}{5}$
d. $S_{20}=\cot^{-1}1.1$

14. The value of k $(k>0)$ such that the length of the longest interval in which the function $f(x)=\sin^{-1}|\sin kx|+\cos^{-1}(\cos kx)$ is constant is $\pi/4$ is/are

a. 8
b. 4
c. 12
d. 16

15. Equation $1+x^2+2x\sin(\cos^{-1}y)=0$ is satisfied by

a. exactly one value of x
b. exactly two values of x
c. exactly one value of y
d. exactly two values of y

16. To the equation $2^{2\pi/\cos^{-1}x}-\left(a+\frac{1}{2}\right)2^{\pi/\cos^{-1}x}-a^2=0$ has only one real root, then

a. $1\le a\le 3$
b. $a\ge 1$
c. $a\le -3$
d. $a\ge 3$

17. If $\sin^{-1}\left(a-\frac{a^2}{3}+\frac{a^3}{9}+\cdots\right)+\cos^{-1}(1+b+b^2+\cdots)=\frac{\pi}{2}$, then

a. $b=\frac{2a-3}{3a}$
b. $b=\frac{3a-2}{2a}$
c. $a=\frac{3}{2-3b}$
d. $a=\frac{2}{3-2b}$

18. If $2\tan^{-1}x + \sin^{-1}\dfrac{2x}{1+x^2}$ is independent of x, then

a. $x > 1$ b. $x < -1$ c. $0 < x < 1$ d. $-1 < x < 0$

19. $\cos^{-1}x + \cos^{-1}\left(\dfrac{x}{2} + \dfrac{1}{2}\sqrt{3-3x^2}\right)$ is equal to

a. $\dfrac{\pi}{3}$ for $x \in \left[\dfrac{1}{2}, 1\right]$ b. $\dfrac{\pi}{3}$ for $x \in \left[0, \dfrac{1}{2}\right]$

c. $2\cos^{-1}x - \cos^{-1}\dfrac{1}{2}$ for $x \in \left[\dfrac{1}{2}, 1\right]$ d. $2\cos^{-1}x - \cos^{-1}\dfrac{1}{2}$ for $x \in \left[0, \dfrac{1}{2}\right]$

20. Which of the following quantities is/are positive?

a. $\cos(\tan^{-1}(\tan 4))$ b. $\sin(\cot^{-1}(\cot 4))$ c. $\tan(\cos^{-1}(\cos 5))$ d. $\cot(\sin^{-1}(\sin 4))$

21. If $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = \pi$, then

a. $x^2 + y^2 + z^2 + 2xyz = 1$ b. $2(\sin^{-1}x + \sin^{-1}y + \sin^{-1}z) = \cos^{-1}x + \cos^{-1}y + \cos^{-1}z$

c. $xy + yz + zx = x + y + z - 1$ d. $\left(x + \dfrac{1}{x}\right) + \left(y + \dfrac{1}{y}\right) + \left(z + \dfrac{1}{z}\right) \ge 6$

22. Which one of the following quantities is/are positive?

a. $\cos(\tan^{-1}(\tan 4))$ b. $\sin(\cot^{-1}(\cot 4))$ c. $\tan(\cos^{-1}(\cos 5))$ d. $\cot(\sin^{-1}(\sin 4))$

23. Which of the following is/are the value of $\cos\left[\dfrac{1}{2}\cos^{-1}\left(\cos\left(-\dfrac{14\pi}{5}\right)\right)\right]$?

a. $\cos\left(-\dfrac{7\pi}{5}\right)$ b. $\sin\left(\dfrac{\pi}{10}\right)$ c. $\cos\left(\dfrac{2\pi}{5}\right)$ d. $-\cos\left(\dfrac{3\pi}{5}\right)$

Reasoning Type

Solutions on page 4.87

Each question has four choices a, b, c and d, out of which *only one* is correct. Each question contains STATEMENT 1 and STATEMENT 2.

a. Both the statements are TRUE, and STATEMENT 2 is the correct explanation of STATEMENT 1
b. Both the statements are TRUE but STATEMENT 2 is NOT the correct explanation of STATEMENT 1
c. STATEMENT 1 is TRUE and STATEMENT 2 is FALSE
d. STATEMENT 1 is FALSE and STATEMENT 2 is TRUE

1. **Statement 1:** Number of roots of the equation $\cot^{-1}x + \cos^{-1}2x + \pi = 0$ is zero.
Statement 2: Range of $\cot^{-1}x$ and $\cos^{-1}x$ is $(0, \pi)$ and $[0, \pi]$, respectively.

2. **Statement 1:** Range of $f(x) = \tan^{-1}x + \sin^{-1}x + \cos^{-1}x$ is $(0, \pi)$.
Statement 2: $f(x) = \tan^{-1}x + \sin^{-1}x + \cos^{-1}x = \dfrac{\pi}{2} + \tan^{-1}x$, for $x \in [-1, 1]$.

3. **Statement 1:** $\operatorname{cosec}^{-1}\left(\dfrac{1}{2} + \dfrac{1}{\sqrt{2}}\right) > \sec^{-1}\left(\dfrac{1}{2} + \dfrac{1}{\sqrt{2}}\right)$.
Statement 2: $\operatorname{cosec}^{-1}x < \sec^{-1}x$ if $1 \le x < \sqrt{2}$.

4. **Statement 1:** $\sin^{-1}\left(\dfrac{1}{\sqrt{e}}\right) > \tan^{-1}\left(\dfrac{1}{\sqrt{\pi}}\right)$.
Statement 2: $\sin^{-1}x > \tan^{-1}y$ for $x > y$, $\forall\ x, y \in (0, 1)$.

5. **Statement 1:** Principal value of $\cos^{-1}(\cos 30)$ is $30-9\pi$.
   Statement 2: $30-9\pi \in [0, \pi]$.

6. Let $f(x)=\sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$.

   Statement 1: $f'(2)=-\dfrac{2}{5}$

   Statement 2: $\sin^{-1}\left(\dfrac{2x}{1+x^2}\right)=\pi-2\tan^{-1}x, \forall x>1$

7. **Statement 1:** Domain of $\tan^{-1}x$ and $\cot^{-1}x$ is R.
   Statement 2: $f(x)=\tan x$ and $g(x)=\cot x$ are unbounded functions.

8. **Statement 1:** $\tan^{-1}\left(\dfrac{3}{4}\right)+\tan^{-1}\left(\dfrac{1}{7}\right)=\dfrac{\pi}{4}$

   Statement 2: For $x>0, y>0$, $\tan^{-1}\left(\dfrac{x}{y}\right)+\tan^{-1}\left(\dfrac{y-x}{y+x}\right)=\dfrac{\pi}{4}$

9. **Statement 1:** Principal value of $\sin^{-1}(\sin 3)$ can be 3 if we restrict the domain of $f(x)=\sin x$ to $[\pi/2, 3\pi/2]$.
   Statement 2: The restriction that the principal values of $\sin^{-1}(\sin x)$ is $[-\pi/2, -\pi/2]$ is a matter of convention. We could have allowed principal values $[\pi/2, 3\pi/2]$ without affecting the condition required for definition of inverse function.

Linked Comprehension Type

Solutions on page 4.88

Based upon each paragraph, three multiple choice questions have to be answered. Each question has four choices a, b, c and d, out of which only one is correct.

For Problems 1–3

For $x, y, z, t \in R$, $\sin^{-1}x+\cos^{-1}y+\sec^{-1}z \geq t^2-\sqrt{2\pi}\, t+3\pi$

1. The value of $x+y+z$ is equal to
   a. 1 b. 0 c. 2 d. -1
2. The principal value of $\cos^{-1}(\cos 5t^2)$ is
   a. $\dfrac{3\pi}{2}$ b. $\dfrac{\pi}{2}$ c. $\dfrac{\pi}{3}$ d. $\dfrac{2\pi}{3}$
3. The value of $\cos^{-1}(\min\{x, y, z\})$ is
   a. 0 b. $\dfrac{\pi}{2}$ c. π d. $\dfrac{\pi}{3}$

For Problems 4 – 6

$ax+b(\sec(\tan^{-1}x))=c$ and $ay+b(\sec(\tan^{-1}y))=c$

4. The value of xy is
   a. $\dfrac{2ab}{a^2-b^2}$ b. $\dfrac{c^2-b^2}{a^2-b^2}$ c. $\dfrac{c^2-b^2}{a^2+b^2}$ d. none of these
5. The value of $x+y$ is
   a. $\dfrac{2ac}{a^2-b^2}$ b. $\dfrac{c^2-b^2}{a^2-b^2}$ c. $\dfrac{c^2-b^2}{a^2+b^2}$ d. none of these

6. The value of $\dfrac{x+y}{1-xy}$ is

a. $\dfrac{2\,ab}{a^2-c^2}$ b. $\dfrac{2ac}{a^2-c^2}$ c. $\dfrac{c^2-b^2}{a^2+b^2}$ d. none of these

For Problems 7 – 9

Consider the system of equations $\cos^{-1}x+(\sin^{-1}y)^2=\dfrac{p\pi^2}{4}$ and $\left(\cos^{-1}x\right)\left(\sin^{-1}y\right)^2=\dfrac{\pi^4}{16}, p\in Z$.

7. The value of p for which system has a solution is

a. 1   b. 2   c. 0   d. -1

8. The value of x which satisfies the system of equations is

a. $\cos\dfrac{\pi^2}{8}$ b. $\sin\dfrac{\pi^2}{4}$ c. $\cos\dfrac{\pi^2}{2}$ d. none of these

9. Which of the following is not the value of y that satisfies the system of equations?

a. 1   b. -1 c. $\dfrac{1}{2}$ d. none of these

For Problems 10 – 12

Let $\cos^{-1}(4x^3-3x)=a+b\cos^{-1}x$.

10. If $x\in\left[-\dfrac{1}{2},-1\right)$, then the value of $a+b\pi$ is

a. 2π b. 3π c. π d. -2π

11. If $x\in\left[-\dfrac{1}{2},\dfrac{1}{2}\right]$, then the principal value of $\sin^{-1}\left(\sin\dfrac{a}{b}\right)$ is

a. $-\dfrac{\pi}{3}$ b. $\dfrac{\pi}{3}$ c. $-\dfrac{\pi}{6}$ d. $\dfrac{\pi}{6}$

12. If $x\in\left(\dfrac{1}{2},1\right]$, then the value of $\lim\limits_{y\to a} b\cos(y)$ is

a. $-1/3$ b. -3 c. $\dfrac{1}{3}$ d. 3

Matrix-Match Type

Solutions on page 4.91

Each question contains statements given in two columns which have to be matched. Statements a, b, c, d in column I have to be matched with statements p, q, r, s in column II. If the correct matches are a-p, a-s, b-q, b-r, c-p, c-q and d-s, then the correctly bubbled 4 × 4 matrix should be as follows:

	p	q	r	s
a	p	q	r	s
b	p	q	r	s
c	p	q	r	s
d	p	q	r	s

1.

Column I	Column II
a. $\cos^{-1}(4x^3-3x)=3\cos^{-1}x$, then x can take values	p. $[1/2, 1]$
b. $\sin^{-1}(3x-4x^3)=3\sin^{-1}x$, then x can take values	q. $[-1/2, 0]$
c. $\cos^{-1}(4x^3-3x)=3\sin^{-1}x$, then x can take values	r. $[0, \sqrt{3}/2]$
d. $\sin^{-1}(3x-4x^3)=3\cos^{-1}x$, then x can take values	s. $[0, 1/2]$

2.

Column I	Column II
a. $(\sin^{-1}x)^2+(\sin^{-1}y)^2=\dfrac{\pi^2}{2}$ $\Rightarrow x^3+y^3=$	p. 1
b. $(\cos^{-1}x)^2+(\cos^{-1}y)^2=2\pi^2$ $\Rightarrow x^5+y^5$	q. -2
c. $(\sin^{-1}x)^2(\cos^{-1}y)^2=\dfrac{\pi^4}{4}\Rightarrow \|x-y\|$	r. 0
d. $\|\sin^{-1}x-\sin^{-1}y\|=\pi\Rightarrow x^y$	s. 2

3.

Column I	Column II
a. $x\in[\pi, 2\pi]\Rightarrow \|\tan^{-1}(\tan x)\|$ can be	p. $\|x-2\pi\|$
b. $x\in[\pi, 2\pi]\Rightarrow \|\cot^{-1}(\cot x)\|$ can be	q. $\|x-\pi\|$
c. $x\in[-\pi, \pi]\Rightarrow \|\sin^{-1}(\sin x)\|$ can be	r. $\|x\|$
d. $x\in[-\pi, \pi]\Rightarrow \|\cos^{-1}(\cos x)\|$ can be	s. $\|x+\pi\|$

4.

Column I	Column II
a. $\sin^{-1}\dfrac{4}{5}+2\tan^{-1}\dfrac{1}{3}=$	p. $\pi/6$
b. $\sin^{-1}\dfrac{12}{13}+\cos^{-1}\dfrac{4}{5}+\tan^{-1}\dfrac{63}{16}=$	q. $\pi/2$
c. If $A=\tan^{-1}\dfrac{x\sqrt{3}}{2\lambda-x}$ and $B=\tan^{-1}\left(\dfrac{2x-\lambda}{\lambda\sqrt{3}}\right)$, then the value of $A-B$ is	r. $\pi/4$
d. $\tan^{-1}\dfrac{1}{7}+2\tan^{-1}\dfrac{1}{3}=$	s. π

5.

Column I	Column II
a. Range of $f(x) = \sin^{-1} x + \cos^{-1} x + \cot^{-1} x$ is	p. $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$
b. Range of $f(x) = \cot^{-1} x + \tan^{-1} x + \operatorname{cosec}^{-1} x$ is	q. $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$
c. Range of $f(x) = \cot^{-1} x + \tan^{-1} x + \cos^{-1} x$ is	r. $\{0, \pi\}$
d. Range of $f(x) = \sec^{-1} x + \operatorname{cosec}^{-1} x + \sin^{-1} x$ is	s. $\left[\frac{3\pi}{4}, \frac{5\pi}{4}\right]$

6.

Column I	Column II
a. $\sin^{-1} x + x > 0$, for	p. $x < 0$
b. $\cos^{-1} x - x \geq 0$, for	q. $x \in (0, 1]$
c. $\tan^{-1} x + x < 0$, for	r. $x \in [-1, 0)$
d. $\cot^{-1} x + x > 0$, for	s. $x > 0$

Integer Type

Solutions on page 4.94

1. The solution set of inequality $(\cot^{-1}x)(\tan^{-1}x) + \left(2 - \frac{\pi}{2}\right)\cot^{-1}x - 3\tan^{-1}x - 3\left(2 - \frac{\pi}{2}\right) > 0$ is (a, b), then the value of $\cot^{-1}a + \cot^{-1}b$ is ________.
2. If $x = \sin^{-1}(a^6 + 1) + \cos^{-1}(a^4 + 1) - \tan^{-1}(a^2 + 1)$, $a \in R$, then the value of $\sec^2 x$ is ________.
3. If the roots of the equation $x^3 - 10x + 11 = 0$ are u, v and w. Then the value of $3\operatorname{cosec}^2(\tan^{-1}u + \tan^{-1}v + \tan^{-1}w)$ is ________.
4. Number of values of x for which $\sin^{-1}\left(x^2 - \frac{x^4}{3} + \frac{x^6}{9} \cdots\right) + \cos^{-1}\left(x^4 - \frac{x^8}{3} + \frac{x^{12}}{9} \cdots\right) = \frac{\pi}{2}$, where $0 \leq |x| < \sqrt{3}$, is ________.
5. If the domain of the function $f(x) = \sqrt{3\cos^{-1}(4x) - \pi}$ is $[a, b]$, then the value of $4a + 64b$ is ________.
6. If $0 < \cos^{-1} x < 1$ and $1 + \sin(\cos^{-1} x) + \sin^2(\cos^{-1} x) + \sin^3(\cos^{-1} x) + \cdots \infty = 2$, then the value of $12x^2$ is ________.
7. If $\tan^{-1}\left(x + \frac{3}{x}\right) - \tan^{-1}\left(x - \frac{3}{x}\right) = \tan^{-1}\frac{6}{x}$, then the value of x^4 is ________.
8. If range of the function $f(x) = \sin^{-1}x + 2\tan^{-1}x + x^2 + 4x + 1$ is $[p, q]$, then the value of $(p + q)$ is ________.
9. If n is the number of terms of the series $\cot^{-1} 3$, $\cot^{-1} 7$, $\cot^{-1} 13$, $\cot^{-1} 21$, ..., whose sum is $\frac{1}{2}\cos^{-1}\left(\frac{24}{145}\right)$, then the value of $n - 5$ is ________.

10. If the area enclosed by the curves $f(x) = \cos^{-1}(\cos x)$ and $g(x) = \sin^{-1}(\cos x)$ in $x \in [9\pi/4, 15\pi/4]$ is $a\pi^2/b$ (where a and b are coprime), then the value of $(a - b)$ is ________.

11. Absolute value of sum of all integers in the domain of $f(x) = \cot^{-1}\sqrt{(x+3)\,x} + \cos^{-1}\sqrt{x^2+3x+1}$ is ________.

12. The least value of $(1 + \sec^{-1}x)(1 + \cos^{-1}x)$ is ________.

13. Let $\cos^{-1}(x) + \cos^{-1}(2x) + \cos^{-1}(3x)$ be π. If x satisfies the equation $ax^3 + bx^2 + cx - 1 = 0$, then the value of $(b - a - c)$ is ________.

14. Number of integral values of x satisfying the equation $\tan^{-1}(3x) + \tan^{-1}(5x) = \tan^{-1}(7x) + \tan^{-1}(2x)$ is ________.

Archives

Solutions on page 4.98

Subjective

1. Find the value of $\cos(2\cos^{-1}x + \sin^{-1}x)$ at $x = 1/5$, where $0 \le \cos^{-1}x \le \pi$ and $-\pi/2 \le \sin^{-1}x \le \pi/2$. **(IIT-JEE, 1981)**

2. Prove that $\cos\tan^{-1}\sin\cot^{-1}x = \sqrt{\dfrac{x^2+1}{x^2+2}}$. **(IIT-JEE, 2002)**

Objective

Fill in the blanks

1. Let a, b and c be positive real numbers. Let $\theta = \tan^{-1}\sqrt{\dfrac{a(a+b+c)}{bc}} + \tan^{-1}\sqrt{\dfrac{b(a+b+c)}{ca}} + \tan^{-1}\sqrt{\dfrac{c(a+b+c)}{ab}}$. Then $\tan\theta =$ ____. **(IIT-JEE, 1981)**

2. The numerical value of $\tan\left(2\tan^{-1}\left(\dfrac{1}{5}\right) - \dfrac{\pi}{4}\right)$ is equal to _____. **(IIT-JEE, 1984)**

3. The greater of the two angles $A = 2\tan^{-1}(2\sqrt{2} - 1)$ and $B = 3\sin^{-1}(1/3) + \sin^{-1}(3/5)$ is _____. **(IIT-JEE, 1989)**

Multiple choice questions with one correct answer

1. The value of $\tan\left[\cos^{-1}\left(\dfrac{4}{5}\right) + \tan^{-1}\left(\dfrac{2}{3}\right)\right]$ is **(IIT-JEE, 1983)**

   a. $\dfrac{6}{17}$ b. $\dfrac{7}{16}$ c. $\dfrac{16}{7}$ d. none of these

2. The principal value of $\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$ is **(IIT-JEE, 1986)**

   a. $-\frac{2\pi}{3}$ b. $\frac{2\pi}{3}$ c. $\frac{4\pi}{3}$ d. $\frac{5\pi}{3}$

   e. none of these

3. If we consider only the principal values of the inverse trigonometric functions, then the value of

   $\tan\left(\cos^{-1}\frac{1}{5\sqrt{2}}-\sin^{-1}\frac{4}{\sqrt{17}}\right)$ is **(IIT-JEE, 1994)**

   a. $\frac{\sqrt{29}}{3}$ b. $\frac{29}{3}$ c. $\frac{\sqrt{3}}{29}$ d. $\frac{3}{29}$ **(IIT-JEE, 1999)**

4. The number of real solutions of $\tan^{-1}\sqrt{x(x+1)}+\sin^{-1}\sqrt{x^2+x+1}=\pi/2$ is

   a. zero b. one c. two d. infinite

   (IIT-JEE, 2001)

5. If $\sin^{-1}\left(x-\frac{x^2}{2}+\frac{x^3}{4}-\cdots\right)+\cos^{-1}\left(x^2-\frac{x^4}{2}+\frac{x^6}{4}-\cdots\right)=\frac{\pi}{2}$ for $0<|x|<\sqrt{2}$, then x equals

   a. 1/2 b. 1 c. −1/2 d. −1

6. Domain of the definition of the function $f(x)=\sqrt{\sin^{-1}(2x)+\pi/6}$ is **(IIT-JEE, 2003)**

   a. [−1/4, 1/2] b. [−1/2, 1/9] c. [−1/2, 1/2] d. [−1/4, 1/4]

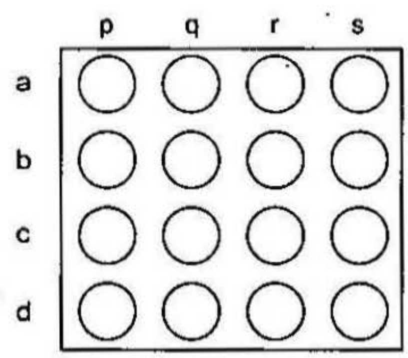
7. The value of x for which $\sin(\cot^{-1}(1+x))=\cos(\tan^{-1}x)$ is **(IIT-JEE, 2004)**

   a. 1/2 b. 1 c. 0 d. −1/2

Match the following type

1. The question contains statements given in two columns which have to be matched. Statements a, b, c, d in column I have to be matched with statements p, q, r, s in column II. The answers to these questions have to be appropriately bubbled as illustrated in the following example. If the correct matches are a-p, a-s, b-q, b-r, c-p, c-q, and d-s, then the correctly bubbled 4×4 matrix should be as follows:

   (IIT-JEE, 2007)



Let (x, y) be such that $\sin^{-1}(ax)+\cos^{-1}(y)+\cos^{-1}(bxy)=\pi/2$. Match the statements in column I with statements in column II and indicate your answer by darkening the appropriate bubbles in the 4×4 matrix given in the ORS.

Column I	Column II
a. If $a = 1$ and $b = 0$, then (x, y)	**p.** lies on the circle $x^2 + y^2 = 1$
b. If $a = 1$ and $b = 1$, then (x, y)	**q.** lies on $(x^2 - 1)(y^2 - 1) = 0$
c. If $a = 1$ and $b = 2$, then (x, y)	**r.** lies on $y = x$
d. If $a = 2$ and $b = 2$, then (x, y)	**s.** lies on $(4x^2 - 1)(y^2 - 1) = 0$

ANSWERS AND SOLUTIONS

Subjective Type

1. Let $x = \cos y$, where $0 \le y \le \pi$, $|x| \le 1$

$$2\cos^{-1} x = \sin^{-1}\left(2x\sqrt{1-x^2}\right) \quad \text{(i)}$$

$\Rightarrow$ $2\cos^{-1}(\cos y) = \sin^{-1}(2\cos y\sqrt{1-\cos^2 y})$

$= \sin^{-1}(2\cos y \sin y)$

$= \sin^{-1}(\sin 2y)$

$\Rightarrow$ $\sin^{-1}(\sin 2y) = 2y$ for $-\pi/4 \le y \le \pi/4$

and $2\cos^{-1}(\cos y) = 2y$ for $0 \le y \le \pi$

Thus, Eq. (i) holds only when

$y \in [0, \pi/4] \quad \Rightarrow \quad x \in [1/\sqrt{2}, 1]$

2. $f(x) = \sin^{-1}\left(\dfrac{1+x^2}{2x}\right)$ is defined for $-1 \le \dfrac{1+x^2}{2x} \le 1$ or $\left|\dfrac{1+x^2}{2x}\right| \le 1$

$\Rightarrow$ $|1 + x^2| \le |2x|$, for all x

$\Rightarrow$ $1 + x^2 \le |2x|$, for all x (as $1 + x^2 > 0$)

$\Rightarrow$ $x^2 - 2|x| + 1 \le 0$

$\Rightarrow$ $|x|^2 - 2|x| + 1 \le 0$ (as $x^2 = |x^2|$)

$\Rightarrow$ $(|x| - 1)^2 \le 0$

But $(|x| - 1)^2$ is always either positive or zero.

Thus, $(|x| - 1)^2 = 0$

$\Rightarrow$ $|x| = 1 \Rightarrow x = \pm 1$

Hence, domain for $f(x)$ is $\{-1, 1\}$.

3. Let $\theta = \cot^{-1}(2x - x^2)$, where $\theta \in (0, \pi)$

$\Rightarrow$ $\cot\theta = 2x - x^2$, where $\theta \in (0, \pi)$

$= 1 - (1 - 2x + x^2)$, where $\theta \in (0, \pi)$

$= 1 - (1 - x)^2$, where $\theta \in (0, \pi)$

$\Rightarrow$ $\cot\theta \le 1$, where $\theta \in (0, \pi)$ $\quad \Rightarrow \quad \dfrac{\pi}{4} \le \theta < \pi \quad \Rightarrow$ Range of $f(x)$ is $\left[\dfrac{\pi}{4}, \pi\right)$

4. Let t_n denotes the nth term of the series.

Then $t_n = \cot^{-1} 2n^2 = \tan^{-1} \dfrac{1}{2n^2} = \tan^{-1} \dfrac{(2n+1)-(2n-1)}{1+(4n^2-1)} = \tan^{-1}(2n+1) - \tan^{-1}(2n-1)$

Putting $n = 1, 2, 3, \ldots$, etc. in (1), we get

$t_1 = \tan^{-1} 3 - \tan^{-1} 1$

$t_2 = \tan^{-1} 5 - \tan^{-1} 3$

$t_3 = \tan^{-1} 7 - \tan^{-1} 5$

$\vdots$

$t_n = \tan^{-1}(2n+1) - \tan^{-1}(2n-1)$

Adding, we get

$S_n = \tan^{-1}(2n+1) - \tan^{-1} 1$

as $n \to \infty$, $\tan^{-1}(2n+1) \to \pi/2$

Hence, the required sum $= \dfrac{\pi}{4}$.

5. Let $\theta = \operatorname{cosec}^{-1} \sqrt{(n^2+1)(n^2+2n+2)}$

$\Rightarrow \quad \operatorname{cosec}^2\theta = (n^2+1)(n^2+2n+2) = (n^2+1)^2 + 2n(n^2+1) + n^2 + 1 = (n^2+n+1)^2 + 1$

$\Rightarrow \quad \cot^2\theta = (n^2+n+1)^2$

$$\Rightarrow \quad \tan\theta = \frac{1}{n^2+n+1} = \frac{(n+1)-n}{1+(n+1)n}$$

$$\Rightarrow \quad \theta = \tan^{-1}\left[\frac{(n+1)-n}{1+(n+1)n}\right] = \tan^{-1}(n+1) - \tan^{-1} n$$

Thus, sum of n terms of the given series

$= (\tan^{-1} 2 - \tan^{-1} 1) + (\tan^{-1} 3 - \tan^{-1} 2) + (\tan^{-1} 4 - \tan^{-1} 3) + \cdots + (\tan^{-1}(n+1) - \tan^{-1} n)$

$= \tan^{-1}(n+1) - \pi/4$

6. Here, $\tan^{-1} x + \cos^{-1} \dfrac{y}{\sqrt{1-y^2}} = \sin^{-1} \dfrac{3}{\sqrt{10}}$

$$\Rightarrow \quad \tan^{-1} x + \tan^{-1}\left(\frac{1}{y}\right) = \tan^{-1}(3)$$

$$\Rightarrow \quad \tan^{-1}\left(\frac{1}{y}\right) = \tan^{-1} 3 - \tan^{-1}(x)$$

$$\Rightarrow \quad \tan^{-1}\left(\frac{1}{y}\right) = \tan^{-1}\left(\frac{3-x}{1+3x}\right)$$

$$\Rightarrow \quad y = \frac{1+3x}{3-x}$$

As x, y are positive integers, $x = 1, 2$ and corresponding $y = 2, 7$.

Therefore, the solutions are $(x, y) = (1, 2)$ and $(2, 7)$, i.e., there are two solutions.

7. We have $\tan^{-1} y = 4\tan^{-1} x$

$$\Rightarrow \quad \tan^{-1} y = 2\tan^{-1} \frac{2x}{1-x^2} \text{ (as } |x| < 1)$$

$$= \tan^{-1} \frac{\dfrac{4x}{1-x^2}}{1 - \dfrac{4x^2}{\left(1-x^2\right)^2}}$$

$$= \tan^{-1} \frac{4x\left(1-x^2\right)}{x^4 - 6x^2 + 1} \quad \left(\text{as } \left|\frac{2x}{1-x^2}\right| < 1\right)$$

$$\Rightarrow \quad y = \frac{4x\left(1-x^2\right)}{x^4 - 6x^2 + 1}$$

If $x = \tan \dfrac{\pi}{8} \quad \Rightarrow \quad \tan^{-1} y = 4 \tan^{-1} x = \dfrac{\pi}{2} \quad \Rightarrow \quad y \to \infty \quad \Rightarrow \quad x^4 - 6x^2 + 1 = 0$

Hence, $\tan \dfrac{\pi}{8}$ is a root of $x^4 - 6x^2 + 1 = 0$.

8. x_1, x_2, x_3 and x_4 are the roots of the equation

$x^4 - x^3 \sin 2\beta + x^2 \cos 2\beta - x \cos \beta - \sin \beta = 0$.

$\Sigma x_1 = x_1 + x_2 + x_3 + x_4 = -\dfrac{(-\sin 2\beta)}{1} = \sin 2\beta$

$\Sigma x_1 x_2 = \cos 2\beta$

$\Sigma x_1 x_2 x_3 = \cos \beta$ and $x_1 x_2 x_3 x_4 = -\sin \beta$

Now, $\tan\left[\tan^{-1} x_1 + \tan^{-1} x_2 + \tan^{-1} x_3 + \tan^{-1} x_4\right]$

$$= \frac{\Sigma x_1 - \Sigma x_1 x_2 x_3}{1 - \Sigma x_1 x_2 + x_1 x_2 x_3 x_4} = \frac{\sin 2\beta - \cos\beta}{1 - \cos 2\beta - \sin\beta} = \frac{2\sin\beta\cos\beta - \cos\beta}{2\sin^2\beta - \sin\beta} = \frac{\cos\beta(2\sin\beta - 1)}{\sin\beta(2\sin\beta - 1)} = \cot\beta$$

or $\tan\left[\tan^{-1} x_1 + \tan^{-1} x_2 + \tan^{-1} x_3 + \tan^{-1} x_4\right] = \tan\left(\dfrac{\pi}{2} - \beta\right)$

$$\Rightarrow \quad \tan^{-1} x_1 + \tan^{-1} x_2 + \tan^{-1} x_3 + \tan^{-1} x_4 = n\pi + \frac{\pi}{2} - \beta, n \in Z$$

9. As, $\tan^{-1} x + \cot^{-1} x = \dfrac{\pi}{2} \ \forall x \in R$

So, the given equation can be written as

$$(\sin^{-1} x)^3 + (\cos^{-1} x)^3 = 7\left(\frac{\pi^3}{8}\right)$$

$$\Rightarrow \quad (\sin^{-1} x + \cos^{-1} x)^3 - 3(\sin^{-1} x)(\cos^{-1} x)(\sin^{-1} x + \cos^{-1} x) = 7\left(\frac{\pi}{2}\right)^3$$

$$\Rightarrow \quad \left(\frac{\pi}{2}\right)^3 - 3(\sin^{-1} x)(\cos^{-1} x)\left(\frac{\pi}{2}\right) = 7\left(\frac{\pi}{2}\right)^3 \quad \Rightarrow \quad (\sin^{-1} x)(\cos^{-1} x) = -\frac{\pi^2}{2}$$

Now, as the maximum value of $\cos^{-1} x$ is π ($\therefore \cos^{-1} x$ is always ≥ 0) and the minimum value of $\sin^{-1} x$ is $-\pi/2$ and this happens at the same value of x, i.e., $x = -1$.

So, the minimum value of $(\sin^{-1} x)(\cos^{-1} x) = \left(-\frac{\pi}{2}\right)(\pi) = -\frac{\pi^2}{2}$.

So, if $(\sin^{-1} x)(\cos^{-1} x) = -\frac{\pi^2}{2}$, then $x = -1$ only.

10. $(\sin^{-1}x)^3 + (\cos^{-1} x)^3 = a\pi^3$

$\Rightarrow \quad (\sin^{-1} x + \cos^{-1} x)((\sin^{-1} x + \cos^{-1}x)^2 - 3 \sin^{-1} x \cos^{-1} x) = a\pi^3$

$$\Rightarrow \quad \frac{\pi^2}{4} - 3 \sin^{-1} x \cos^{-1} x = 2a\pi^2$$

$$\Rightarrow \quad \sin^{-1} x \left(\frac{\pi}{2} - \sin^{-1} x\right) = \frac{\pi^2}{12}(1-8a)$$

$$\Rightarrow \quad (\sin^{-1} x)^2 - \frac{\pi}{2} \sin^{-1} x = -\frac{\pi^2}{12}(1-8a)$$

$$\Rightarrow \quad \left(\sin^{-1} x - \frac{\pi}{4}\right)^2 = \frac{\pi^2}{12}(8a-1) + \frac{\pi^2}{16}$$

$$= \frac{\pi^2}{48}(32a-1)$$

Now, $\sin^{-1} x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$$\Rightarrow \quad -\frac{3\pi}{4} \le \sin^{-1} x - \frac{\pi}{4} \le \frac{\pi}{4}$$

$$\Rightarrow \quad 0 \le \left(\sin^{-1} x - \frac{\pi}{4}\right)^2 \le \frac{9\pi^2}{16}$$

$$\Rightarrow \quad 0 \le \frac{\pi^2}{48}(32a-1) \le \frac{9\pi^2}{16}$$

$$\Rightarrow \quad 0 \le 32a - 1 \le 27$$

$$\Rightarrow \quad \frac{1}{32} \le a \le \frac{7}{8}$$

Thus, the required set of values of a is $\left[\frac{1}{32}, \frac{7}{8}\right]$.

11. Since $p, q > 0$, therefore $pq > 0$,

$$\tan^{-1} \frac{p-q}{1+pq} = \tan^{-1} p - \tan^{-1} q \qquad \text{(i)}$$

Since $qr > -1$,

$$\tan^{-1} \frac{q-r}{1+qr} = \tan^{-1} q - \tan^{-1} r \qquad \text{(ii)}$$

Since $pr < -1$ and $r < 0$,

$$\tan^{-1} \frac{r-p}{1+rp} = \pi + \tan^{-1} r - \tan^{-1} p \qquad \text{(iii)}$$

On adding Eqs. (i), (ii) and (iii), we get

$$\tan^{-1} \frac{p-q}{1+pq} + \tan^{-1} \frac{q-r}{1+qr} + \tan^{-1} \frac{r-p}{1+rp} = \pi$$

12. Given $\sqrt{|\sin^{-1}|\cos x|| + |\cos^{-1}|\sin x||} = \sin^{-1}(|\cos x|) - \cos^{-1}|\sin x|$

When $x \in \left[0, \frac{\pi}{2}\right]$, $0 \le \frac{\pi}{2} - x \le \frac{\pi}{2}$, we have

$$\sqrt{\sin^{-1}\sin\left(\frac{\pi}{2}-x\right)+\cos^{-1}\cos\left(\frac{\pi}{2}-x\right)} = \sin^{-1}\left(\sin\left(\frac{\pi}{2}-x\right)\right)-\cos^{-1}\left(\cos\left(\frac{\pi}{2}-x\right)\right)$$

$$\Rightarrow \quad \sqrt{2\left(\frac{\pi}{2}-x\right)} = \frac{\pi}{2}-x-\frac{\pi}{2}+x$$

$$\Rightarrow \quad x = \frac{\pi}{2}$$

When $x \in \left[-\frac{\pi}{2}, 0\right]$, $\frac{\pi}{2} \le \frac{\pi}{2} - x \le \pi$, we have

$$\sqrt{\sin^{-1}\left(\sin\left(\frac{\pi}{2}-x\right)\right)+\cos^{-1}\left(-\cos\left(\frac{\pi}{2}-x\right)\right)} = \sin^{-1}\left(\sin\left(\frac{\pi}{2}-x\right)\right)-\cos^{-1}\left(-\cos\left(\frac{\pi}{2}-x\right)\right)$$

$$\Rightarrow \quad \sqrt{\pi+2x} = 0 \qquad \Rightarrow \quad x = -\frac{\pi}{2}$$

13. Taking the tangents of both sides of the equation, we have

$$\frac{\tan\left[\tan^{-1}\frac{x+1}{x-1}\right]+\tan\left[\tan^{-1}\frac{x-1}{x}\right]}{1-\tan\left[\tan^{-1}\frac{x+1}{x-1}\right]\tan\left[\tan^{-1}\frac{x-1}{x}\right]} = \tan((\tan^{-1}(-7)) = -7$$

$$\Rightarrow \quad \frac{\frac{x+1}{x-1}+\frac{x-1}{x}}{1-\frac{x+1}{x-1}\frac{x-1}{x}} = -7$$

$$\Rightarrow \quad \frac{2x^2-x+1}{1-x} = -7$$

so that $x = 2$.

This value makes the left-hand side of the given equation positive, so there is no value of x strictly satisfying the given equation.

The value $x = 2$ is a solution of the equation $\tan^{-1}\frac{x+1}{x-1} + \tan^{-1}\frac{x-1}{x} = \pi + \tan^{-1}(-7)$.

14. Let us transfer $\sin^{-1} 6\sqrt{3}x$ to the right-hand side of the equation and calculate the sine of the both sides of the resulting equation:

$\sin(\sin^{-1} 6x) = \sin(-\sin^{-1} 6\sqrt{3}x - \pi/2)$

$\Rightarrow \quad 6x = -\sin(\sin^{-1} 6\sqrt{3}x + \sin^{-1} 1)$ [using $\sin^{-1}(-x) = -\sin^{-1}(x)$]

$\quad = -\sin(\sin^{-1}\sqrt{1-108x^2})$ (i)

Squaring both sides, we get

$36x^2 = 1 - 108x^2 \qquad \Rightarrow 144x^2 = 1$

whose roots are $x = 1/12$ and $x = -1/12$.

Let us verify:
Substituting $x = 1/12$ in the given equation, we get

$$\sin^{-1}\left(-\frac{1}{2}\right)+\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{6}-\frac{\pi}{3} = -\frac{\pi}{2}$$

Thus $x = 1/12$ is the roots of given equation.

Using Eq. (i), we get $x = \frac{-1}{12}$

L.H.S. $= \frac{1}{2}$

R.H.S. $= -\sqrt{1-108x^2} = -1/2$

Thus L.H.S. $\neq$ R.H.S. of Eq. (i)

Thus $x = -1/12$ is a root of the given equation.

15. Here,

$$\tan^{-1}\left(\frac{a_1x-y}{x+a_1y}\right) = \tan^{-1}\left(\frac{a_1-\frac{y}{x}}{1+a_1\frac{y}{x}}\right) = \tan^{-1}a_1 - \tan^{-1}\frac{y}{x}$$

$$\tan^{-1}\left(\frac{a_2-a_1}{1+a_2a_1}\right) = \tan^{-1}a_2 - \tan^{-1}a_1$$

$$\tan^{-1}\left(\frac{a_3-a_2}{1+a_3a_2}\right) = \tan^{-1}a_3 - \tan^{-1}a_2$$

$$\vdots$$

$$\tan^{-1}\left(\frac{a_n-a_{n-1}}{1+a_na_{n-1}}\right) = \tan^{-1}a_n - \tan^{-1}a_{n-1}$$

$$\tan^{-1}\left(\frac{1}{a_n}\right) = \cot^{-1}a_n$$

Adding, we get L.H.S. $= \tan^{-1}a_n + \cot^{-1}a_n - \tan^{-1}\frac{y}{x} = \frac{\pi}{2} - \tan^{-1}\frac{y}{x} =$ R.H.S.

$$= \cot^{-1}\frac{y}{x} = \tan^{-1}\frac{x}{y}$$

Objective Type

1. c. $\sin^{-1}(\sin 10) = \sin^{-1}[\sin(3\pi-10)] = 3\pi-10$

2. b. $\cos^{-1}\left(\cos\frac{5\pi}{4}\right) = \cos^{-1}\left(\cos\left(2\pi-\frac{5\pi}{4}\right)\right) = \cos^{-1}\left(\cos\frac{3\pi}{4}\right) = \frac{3\pi}{4}$

3. a. $\sin^{-1}(\sin 12)+\cos^{-1}(\cos 12) = \sin^{-1}(\sin(12-4\pi))+\cos^{-1}(\cos(4\pi-12)) = 12-4\pi+4\pi-12 = 0$

4. a. $\sin^{-1}\sin\left(\frac{22\pi}{7}\right) = \sin^{-1}\sin\left(3\pi+\frac{\pi}{7}\right) = -\frac{\pi}{7}$

$$\cos^{-1}\cos\left(\frac{5\pi}{3}\right) = \cos^{-1}\cos\left(2\pi-\frac{\pi}{3}\right) = \frac{\pi}{3}$$

$$\tan^{-1}\tan\left(\frac{5\pi}{7}\right)=\tan^{-1}\tan\left(\pi-\frac{2\pi}{7}\right)=-\frac{2\pi}{7}$$

$$\sin^{-1}\cos(2)=\frac{\pi}{2}-\cos^{-1}\cos 2=\frac{\pi}{2}-2$$

Therefore, the required value $=-\dfrac{\pi}{7}+\dfrac{\pi}{3}-\dfrac{2\pi}{7}+\dfrac{\pi}{2}-2$

$$=\frac{(-18+35)\pi}{42}-2=\frac{17\pi}{42}-2$$

5. d. $\sin^{-1}(\cos(\cos^{-1}(\cos x)+\sin^{-1}(\sin x)))=\sin^{-1}(\cos(x+\pi-x))$ [as $x\in(\pi/2,\pi)$]

$$=\sin^{-1}(\cos\pi)=\sin^{-1}(-1)=-\frac{\pi}{2}$$

6. c. $\cos^{-1}(\cos(2\cot^{-1}(\sqrt{2}-1)))=\cos^{-1}\left(\cos\left(2\left(67.5^\circ\right)\right)\right)$

$$=\cos^{-1}\left(\cos\left(135^\circ\right)\right)=135^\circ=\frac{3\pi}{4}$$

7. a. We have $\sin^{-1}\left(\cot\left(\sin^{-1}\sqrt{\dfrac{2-\sqrt{3}}{4}}+\cos^{-1}\dfrac{\sqrt{12}}{4}+\sec^{-1}\sqrt{2}\right)\right)$

$$=\sin^{-1}\left(\cot\left(\sin^{-1}\left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right)+\cos^{-1}\frac{\sqrt{3}}{2}+\cos^{-1}\frac{1}{\sqrt{2}}\right)\right)$$

$=\sin^{-1}[\cot(15^\circ+30^\circ+45^\circ)]$

$=\sin^{-1}(\cot(90^\circ))=\sin^{-1}(0)=0$

8. d. $\cos^{-1}\left(\sqrt{\dfrac{2}{3}}\right)-\cos^{-1}\left(\dfrac{\sqrt{6}+1}{2\sqrt{3}}\right)$

$$=\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)-\tan^{-1}\left(\frac{\sqrt{3}-\sqrt{2}}{1+\sqrt{6}}\right)=\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)-\left[\tan^{-1}\sqrt{3}-\tan^{-1}\sqrt{2}\right]$$

$$=\left(\tan^{-1}\frac{1}{\sqrt{2}}+\tan^{-1}\sqrt{2}\right)-\tan^{-1}\sqrt{3}$$

$$=\frac{\pi}{2}-\frac{\pi}{3}=\frac{\pi}{6}$$

9. a. Let $\cos^{-1}\left(\dfrac{1}{8}\right)=\theta$, where $0<\theta<\pi$, then $\dfrac{1}{2}\cos^{-1}\dfrac{1}{8}=\dfrac{1}{2}\theta$

$$\Rightarrow\quad\cos\left(\frac{1}{2}\cos^{-1}\frac{1}{8}\right)=\cos\frac{\theta}{2}$$

Now, $\cos^{-1}\dfrac{1}{8}=\theta \quad\Rightarrow\cos\theta=\dfrac{1}{8}\quad\Rightarrow\quad\cos^2\dfrac{\theta}{2}=\dfrac{9}{16}\qquad\Rightarrow\cos\dfrac{\theta}{2}=\dfrac{3}{4}$

10. c. $x=\tan^{-1}3 \qquad\Rightarrow\quad \tan x=3$

$\tan(x+y)=33$

$$\Rightarrow \quad \frac{\tan x + \tan y}{1 - \tan x \tan y} = 33$$

$$\Rightarrow \quad \frac{3 + \tan y}{1 - 3\tan y} = 33$$

$\Rightarrow \quad 3 + \tan y = 33 - 99 \tan y$

$\Rightarrow \quad 100 \tan y = 30$

$\Rightarrow \quad \tan y = 0.3 \quad \Rightarrow \quad y = \tan^{-1}(0.3)$

11. c. Let $\cos^{-1}\left(\frac{\sqrt{5}}{3}\right) = \alpha$. Then $\cos\alpha = \frac{\sqrt{5}}{3}$, where $0 < \alpha < \frac{\pi}{2}$

Now, $\tan\frac{\alpha}{2} = \sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}} = \sqrt{\frac{1-\sqrt{5}/3}{1+\sqrt{5}/3}} = \sqrt{\frac{3-\sqrt{5}}{3+\sqrt{5}}} = \sqrt{\frac{(3-\sqrt{5})^2}{9-5}} = \frac{1}{2}(3-\sqrt{5})$

12. a.
$$\tan^{-1}\left[\frac{\cos x}{1+\sin x}\right] = \tan^{-1}\left[\frac{\sin[(\pi/2)-x]}{1+\cos[(\pi/2)-x]}\right]$$
$$= \tan^{-1}\left[\frac{2\sin[(\pi/4)-(x/2)]\cos[(\pi/4)-(x/2)]}{2\cos^2[(\pi/4)-(x/2)]}\right]$$
$$= \tan^{-1}\tan\left(\frac{\pi}{4}-\frac{x}{2}\right) = \frac{\pi}{4}-\frac{x}{2}$$

$$\Rightarrow \quad -\frac{\pi}{2} < \frac{\pi}{4} - \frac{x}{2} < \frac{\pi}{2}$$

$$\Rightarrow \quad -\frac{3\pi}{4} < -\frac{x}{2} < \frac{\pi}{4}$$

$$\Rightarrow \quad -\frac{\pi}{4} < \frac{x}{2} < \frac{3\pi}{4}$$

$$\Rightarrow \quad -\frac{\pi}{2} < \frac{x}{2} < \frac{3\pi}{2}$$

13. d. $f(x) + f(-x) = 2$

Now $(\sin^{-1}(\sin 8)) = 3\pi - 8 = y$

and $(\tan^{-1}(\tan 8)) = (8 - 3\pi)$

Hence, $f(y) + f(-y) = 2$

Given $f(y) = \alpha$, we have $f(-y) = 2 - \alpha$.

14. d. $f(x) = \tan^{-1}\left(\frac{(\sqrt{12}-2)x^2}{x^4+2x^2+3}\right)$

$$= \tan^{-1}\left(\frac{2(\sqrt{3}-1)}{x^2+\frac{3}{x^2}+2}\right)$$

As $x^2+\frac{3}{x^2}\geq 2\sqrt{3}$ [using A.M. ≥ G.M.]

$$\Rightarrow x^2+\frac{3}{x^2}+2\geq 2+2\sqrt{3}$$

$$\therefore (f(x))_{\max}=\tan^{-1}\left(\frac{2(\sqrt{3}-1)}{2(\sqrt{3}+1)}\right)=\frac{\pi}{12}$$

15. b. Let $x=\sin\theta$ and $\sqrt{x}=\sin\phi$, where $x\in[0,1]\Rightarrow\theta,\phi\in[0,\pi/2]$

$$\Rightarrow \theta-\phi\in\left[\frac{-\pi}{2},\frac{\pi}{2}\right]$$

Now, $\sin^{-1}\left(x\sqrt{1-x}-\sqrt{x}\sqrt{1-x^2}\right)=\sin^{-1}\left(\sin\theta\sqrt{1-\sin^2\phi}-\sin\phi\sqrt{1-\sin^2\theta}\right)$

$$=\sin^{-1}(\sin\theta\cos\phi-\sin\phi\cos\theta)$$

$$=\sin^{-1}\sin(\theta-\phi)=\theta-\phi=\sin^{-1}(x)-\sin^{-1}\left(\sqrt{x}\right)$$

16. c. $\tan^{-1}\frac{x}{y}-\tan^{-1}\left(\frac{x-y}{x+y}\right)=\tan^{-1}\frac{x}{y}-\tan^{-1}\left(\frac{1-(y/x)}{1+(y/x)}\right)$

$$=\tan^{-1}\frac{x}{y}-\left(\tan^{-1}1-\tan^{-1}\frac{y}{x}\right)$$

$$=\tan^{-1}\frac{x}{y}+\tan^{-1}\frac{y}{x}-\frac{\pi}{4}$$

$$=\tan^{-1}\frac{x}{y}+\cot^{-1}\frac{x}{y}-\frac{\pi}{4}=\frac{\pi}{2}-\frac{\pi}{4}=\frac{\pi}{4}$$

17. c. Given equation is $\tan^{-1}\frac{a+x}{a}+\tan^{-1}\frac{a-x}{a}=\frac{\pi}{6}$

$$\Rightarrow \tan^{-1}\left(\frac{\frac{a+x}{a}+\frac{a-x}{a}}{1-\frac{a+x}{a}\frac{a-x}{a}}\right)=\frac{\pi}{6}$$

$$\Rightarrow \frac{2a^2}{x^2}=\tan\frac{\pi}{6}=\frac{1}{\sqrt{3}}\Rightarrow x^2=2\sqrt{3}a^2$$

18. c. $\cot^{-1}\frac{n}{\pi}>\frac{\pi}{6}$

$\Rightarrow \frac{n}{\pi}<\cot\frac{\pi}{6}$ [as $\cot^{-1}x$ is a decreasing function]

$\Rightarrow \frac{n}{\pi}<\sqrt{3} \quad\Rightarrow n<\sqrt{3}\,\pi \quad\Rightarrow n<5.46 \quad\Rightarrow$ maximum value of n is 5

19. c. Let $\tan^{-1}2=\alpha \quad\Rightarrow \tan\alpha=2$

and $\cot^{-1}3=\beta \quad\Rightarrow \cot\beta=3$

$$\sec^2(\tan^{-1}2)+\operatorname{cosec}^2(\cot^{-1}3)=\sec^2\alpha+\operatorname{cosec}^2\beta=1+\tan^2\alpha+1+\cot^2\beta$$
$$=2+(2)^2+(3)^2=15$$

20. d. $\tan(A-B)=\dfrac{\tan A-\tan B}{1+\tan A\tan B}$

$$=\frac{\dfrac{\sqrt{3}\,x}{2K-x}-\dfrac{2x-K}{\sqrt{3}\,K}}{1+\dfrac{\sqrt{3}\,x}{2K-x}\cdot\dfrac{2x-K}{\sqrt{3}\,K}}$$

$$=\frac{3Kx-(2x-K)(2K-x)}{(2K-x)\sqrt{3}\,K+\sqrt{3}x(2x-K)}$$

$$=\frac{3Kx-(4Kx-2x^2-2K^2+Kx)}{2\sqrt{3}K^2-\sqrt{3}Kx+2\sqrt{3}x^2-\sqrt{3}Kx}$$

$$=\frac{2x^2-2Kx+2K^2}{2\sqrt{3}x^2-2\sqrt{3}Kx+2\sqrt{3}K^2}=\frac{1}{\sqrt{3}}=\tan 30^\circ$$

$\therefore A-B=30^\circ$

21. b. $\tan^{-1}\dfrac{b+a}{b-a}-\tan^{-1}\dfrac{a}{b}=\tan^{-1}\dfrac{\dfrac{b+a}{b-a}-\dfrac{a}{b}}{1+\dfrac{b+a}{b-a}\dfrac{a}{b}}$

$$=\tan^{-1}\frac{b^2+ab-ab+a^2}{b^2-ab+ab+a^2}=\tan^{-1}\frac{a^2+b^2}{a^2+b^2}=\tan^{-1}1=\frac{\pi}{4}$$

Therefore, the required value $=\sec\left(\dfrac{\pi}{4}\right)=\sqrt{2}$

22. d. $a\sin^{-1}x-b\cos^{-1}x=c$

We have $b\sin^{-1}x+b\cos^{-1}x=\dfrac{b\pi}{2}$ $\quad\Rightarrow\quad(a+b)\sin^{-1}x=\dfrac{b\pi}{2}+c$

$$\Rightarrow\quad \sin^{-1}x=\frac{\dfrac{(b\pi)}{2}+c}{a+b}$$

$$\Rightarrow\quad \cos^{-1}x=\frac{\pi ab+c(a-b)}{a+b}$$

23. b. $\cos^{-1}\left(\dfrac{1+x^2}{2x}\right)=\dfrac{\pi}{2}+(\sin^{-1}x+\cos^{-1}x)$

$$\Rightarrow\quad\cos^{-1}\left(\frac{1+x^2}{2x}\right)=\pi\quad\Rightarrow\left(\frac{1+x^2}{2x}\right)=\cos\pi=-1\quad\Rightarrow x^2+1+2x=0\quad\Rightarrow x=-1$$

24. b. $0\le x^2+x+1\le 1$ and $0\le x^2+x\le 1$

$\therefore x=-1,0$

For $x=-1$

L.H.S. $=2\sin^{-1}1+\cos^{-1}0=\dfrac{3\pi}{2}$

$\therefore x = -1$ is a solution.

For $x = 0$, L.H.S. $= 2\sin^{-1} 1 + \cos^{-1} 0 = \dfrac{3\pi}{2}$

Therefore, $x = 0$ is a solution and sum of the solutions $= -1$.

25. c. $\tan^{-1}(1+x) + \tan^{-1}(1-x) = \dfrac{\pi}{2}$

$$\Rightarrow \tan^{-1}(1+x) = \frac{\pi}{2} - \tan^{-1}(1-x)$$
$$= \cot^{-1}(1-x)$$
$$= \tan^{-1}\left(\frac{1}{1-x}\right)$$

$$\Rightarrow 1+x = \frac{1}{1-x} \Rightarrow 1-x^2 = 1 \Rightarrow x = 0$$

26. c. We have $\sin^{-1} x + \sin^{-1}(1-x) = \cos^{-1} x$

$$\Rightarrow \sin(\sin^{-1} x + \sin^{-1}(1-x)) = \sin(\cos^{-1} x)$$

$$\Rightarrow x\sqrt{1-(1-x)^2} + \sqrt{1-x^2}(1-x) = \sqrt{1-x^2}$$

$$\Rightarrow x\sqrt{1-(1-x)^2} = x\sqrt{1-x^2}$$

$$\Rightarrow x = 0 \text{ or } 2x - x^2 = 1 - x^2 \quad \Rightarrow \quad x = 0 \text{ or } x = \frac{1}{2}$$

27. c. We have $\tan^{-1}\dfrac{1-x}{1+x} = \dfrac{1}{2}\tan^{-1} x$

$$\Rightarrow \tan^{-1}\left[\frac{1-\tan\theta}{1+\tan\theta}\right] = \frac{1}{2}\theta \qquad \text{(putting } x = \tan\theta\text{)}$$

$$\Rightarrow \tan^{-1}\left[\frac{\tan\frac{\pi}{4} - \tan\theta}{1 + \tan\frac{\pi}{4}\tan\theta}\right] = \frac{\theta}{2}$$

$$\Rightarrow \tan^{-1}\tan\left(\frac{\pi}{4} - \theta\right) = \frac{\theta}{2}$$

$$\Rightarrow \frac{\pi}{4} - \theta = \frac{\theta}{2}$$

$$\Rightarrow \theta = \frac{\pi}{6} = \tan^{-1} x$$

$$\Rightarrow x = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$$

28. c. We have $\cos^{-1} x + \cos^{-1}(2x) = -\pi$, which is not possible as $\cos^{-1}x$ and $\cos^{-1}2x$ never take negative values.

29. b. The given equation is $ax^2 + \sin^{-1}((x-1)^2 + 1) + \cos^{-1}((x-1)^2 + 1) = 0$.

Now, $-1 \le (x-1)^2 + 1 \le 1 \Rightarrow x = 1$

So, we have $a + \dfrac{\pi}{2} = 0 \Rightarrow a = -\dfrac{\pi}{2}$

30. c. Since $\sqrt{x^2-3x+2} \geq 0 \Rightarrow 0 \leq \tan^{-1}\sqrt{x^2-3x+2} < \frac{\pi}{2}$

and $\sqrt{4x-x^2-3} \geq 0 \Rightarrow 0 < \cos^{-1}\sqrt{4x-x^2-3} \leq \frac{\pi}{2}$

Adding, we have $0 <$ L.H.S. $< \pi$

Therefore, the given equation has no solution.

31. b. The given equation can be written as

$$3\tan^{-1}\left(2-\sqrt{3}\right) = \tan^{-1}\left(\frac{1}{x}\right) + \tan^{-1}\left(\frac{1}{3}\right)$$

$$\Rightarrow \quad 3(15°) = \tan^{-1}\frac{\frac{1}{x}+\frac{1}{3}}{1-\frac{1}{x}\,\frac{1}{3}} \quad \Rightarrow \quad 1 = \frac{3+x}{3x-1} \Rightarrow x = 2$$

32. c. $\tan^{-1}x + 2\cot^{-1}x = \frac{2\pi}{3}$

$$\Rightarrow \quad \tan^{-1}x = 2\left(\frac{\pi}{3} - \cot^{-1}x\right) = 2\left(\frac{\pi}{3} - \left(\frac{\pi}{2} - \tan^{-1}x\right)\right) = 2\left(-\frac{\pi}{6} + \tan^{-1}x\right)$$

$$\Rightarrow \quad \tan^{-1}x = \frac{\pi}{3} \quad \Rightarrow \quad x = \tan\frac{\pi}{3} = \sqrt{3}$$

33. a. Put $x = \tan\theta$

$$\therefore 3\sin^{-1}\frac{2\tan\theta}{1+\tan^2\theta} - 4\cos^{-1}\frac{1-\tan^2\theta}{1+\tan^2\theta} + 2\tan^{-1}\frac{2\tan\theta}{1-\tan^2\theta} = \frac{\pi}{3}$$

$$\Rightarrow \quad 3\sin^{-1}(\sin 2\theta) - 4\cos^{-1}(\cos 2\theta) + 2\tan^{-1}(\tan 2\theta) = \frac{\pi}{3}$$

$$\Rightarrow \quad 3(2\theta) - 4(2\theta) + 2(2\theta) = \frac{\pi}{3} \quad \Rightarrow \quad 2\theta = \frac{\pi}{3} \quad \Rightarrow \quad \theta = \frac{\pi}{6} \quad \Rightarrow \quad x = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$$

34. c. Put $\sin^{-1}\frac{5}{x} = A \Rightarrow \frac{5}{x} = \sin A$

$$\sin^{-1}\frac{12}{x} = B \Rightarrow \frac{12}{x} = \sin B \quad \Rightarrow \quad A + B = \frac{\pi}{2}$$

$$\Rightarrow \quad \sin A = \sin\left(\frac{\pi}{2} - B\right) = \cos B = \sqrt{1-\sin^2 B}$$

$$\Rightarrow \quad \frac{5}{x} = \sqrt{1 - \frac{144}{x^2}} \Rightarrow \frac{169}{x^2} = 1$$

$\Rightarrow \quad x^2 = 169 \quad \Rightarrow \quad x = 13$ [$\because x = -13$ does not satisfy the given equation]

35. c. $\sin^{-1}(x-1) \quad \Rightarrow \quad -1 \leq x-1 \leq 1 \quad \Rightarrow \quad 0 \leq x \leq 2$

$\cos^{-1}(x-3) \quad \Rightarrow \quad -1 \leq x-3 \leq 1 \quad \Rightarrow \quad 2 \leq x \leq 4$

$\therefore x = 2$

So, $\sin^{-1}(2-1) + \cos^{-1}(2-3) + \tan^{-1}\frac{2}{2-4} = \cos^{-1}k + \pi$

$$\Rightarrow \quad \sin^{-1}1 + \cos^{-1}(-1) + \tan^{-1}(-1) = \cos^{-1}k + \pi.$$

$\Rightarrow \quad \dfrac{\pi}{2}+\pi-\dfrac{\pi}{4}=\cos^{-1}k+\pi$

$\Rightarrow \quad \cos^{-1}k=\dfrac{\pi}{4} \Rightarrow k=\dfrac{1}{\sqrt{2}}$

36. b. $\cot^{-1}x+\cot^{-1}y+\cot^{-1}z=\dfrac{\pi}{2}$

$\Rightarrow \quad \dfrac{\pi}{2}-\tan^{-1}x+\dfrac{\pi}{2}-\tan^{-1}y+\dfrac{\pi}{2}-\tan^{-1}z=\dfrac{\pi}{2}$

$\Rightarrow \quad \tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\pi$

$\Rightarrow \quad \tan^{-1}x+\tan^{-1}y=\pi-\tan^{-1}z$

$\Rightarrow \quad \tan(\tan^{-1}x+\tan^{-1}y)=\tan(\pi-\tan^{-1}z)$

$\Rightarrow \quad \dfrac{x+y}{1-xy}=-z$

$\Rightarrow \quad x+y+z=xyz$

37. d. Given that $\cos^{-1}x+\cos^{-1}y+\cos^{-1}z=\pi$

$\Rightarrow \quad \cos^{-1}(x)+\cos^{-1}(y)=\pi-\cos^{-1}(z)$

$\Rightarrow \quad \cos^{-1}\left(xy-\sqrt{1-x^2}\sqrt{1-y^2}\right)=\cos^{-1}(-z)$

$\Rightarrow \quad xy-\sqrt{(1-x^2)(1-y^2)}=-z$

$\Rightarrow \quad (xy+z)=\sqrt{(1-x^2)(1-y^2)}$

Squaring both sides, we get $x^2+y^2+z^2+2xyz=1$

Trick: Put $x=y=z=\dfrac{1}{2}$ so that $\cos^{-1}\dfrac{1}{2}+\cos^{-1}\dfrac{1}{2}+\cos^{-1}\dfrac{1}{2}=\pi$.

38. d. Given that $\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\dfrac{\pi}{2}$

$\Rightarrow \quad \tan^{-1}\left[\dfrac{x+y+z-xyz}{1-xy-yz-xz}\right]=\dfrac{\pi}{2}$

Hence, $xy+yz+zx-1=0$.

39. d. Let $\alpha=\cos^{-1}\sqrt{p}$, $\beta=\cos^{-1}\sqrt{1-p}$ and $\gamma=\cos^{-1}\sqrt{1-q}$

$\Rightarrow \quad \cos\alpha=\sqrt{p}$, $\cos\beta=\sqrt{1-p}$ and $\cos\gamma=\sqrt{1-q}$

Therefore, $\sin\alpha=\sqrt{1-p}$, $\sin\beta=\sqrt{p}$ and $\sin\gamma=\sqrt{q}$

The given equation may be written as

$\alpha+\beta+\gamma=\dfrac{3\pi}{4}$

$\Rightarrow \quad \alpha+\beta=\dfrac{3\pi}{4}-\gamma$

$\Rightarrow \quad \cos(\alpha+\beta)=\cos\left(\dfrac{3\pi}{4}-\gamma\right)$

$\Rightarrow \quad \cos\alpha\cos\beta-\sin\alpha\sin\beta=\cos\left(\pi-\left(\dfrac{\pi}{4}+\gamma\right)\right)=-\cos\left(\dfrac{\pi}{4}+\gamma\right)$

$\Rightarrow \quad \sqrt{p}\sqrt{1-p}-\sqrt{1-p}\sqrt{p}=-\left(\dfrac{1}{\sqrt{2}}\sqrt{1-q}-\dfrac{1}{\sqrt{2}}\sqrt{q}\right)$

$\Rightarrow \quad 0 = \sqrt{1-q} - \sqrt{q} \Rightarrow 1 - q = q \Rightarrow q = \frac{1}{2}$

40. a. Let $\sin^{-1} a = A$, $\sin^{-1} b = B$ and $\sin^{-1} c = C$

$\Rightarrow \quad \sin A = a, \sin B = b, \sin C = c$

and $A + B + C = \pi \Rightarrow \sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$ (i)

$\Rightarrow \quad \sin A \cos A + \sin B \cos B + \sin C \cos C = 2 \sin A \sin B \sin C$

$\Rightarrow \quad \sin A \sqrt{(1-\sin^2 A)} + \sin B \sqrt{(1-\sin^2 B)} + \sin C \sqrt{1-\sin^2 C} = 2 \sin A \sin B \sin C$ (ii)

$\Rightarrow \quad a\sqrt{(1-a^2)} + b\sqrt{(1-b^2)} + c\sqrt{(1-c)^2} = 2abc$

Trick: Let $a = \frac{1}{\sqrt{2}}, b = \frac{1}{\sqrt{2}}, c = 1$.

Then $a\sqrt{1-a^2} + b\sqrt{1-b^2} + c\sqrt{1-c^2} = \frac{1}{\sqrt{2}}\sqrt{1-\frac{1}{2}} + \frac{1}{\sqrt{2}}\sqrt{1-\frac{1}{2}} + 1\sqrt{1-1} = 1.$

41. b. Since $\sin^{-1} x + \sin^{-1} y + \sin^{-1} z = \pi$

$\therefore \sin^{-1} x + \sin^{-1} y = \pi - \sin^{-1} z$

$\Rightarrow \quad \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right) = \pi - \sin^{-1}(z)$

$\Rightarrow \quad x\sqrt{1-y^2} + y\sqrt{1-x^2} = \sin(\pi - \sin^{-1}(z)) = \sin(\sin^{-1} z) = z$

$\Rightarrow \quad x^2(1-y^2) = z^2 + y^2(1-x^2) - 2zy\sqrt{1-x^2} \Rightarrow (x^2 - y^2 - z^2)^2 = 4y^2 z^2 (1-x^2)$

$\Rightarrow \quad x^4 + y^4 + z^4 - 2x^2y^2 - 2x^2z^2 + 2y^2z^2 = 4y^2z^2 - 4x^2y^2z^2$

$\Rightarrow \quad x^4 + y^4 + z^4 + 4x^2y^2z^2 = 2(x^2y^2 + y^2z^2 + z^2x^2) \quad \Rightarrow \quad K = 2$

42. d. We have $\cos^{-1} x - \cos^{-1} \frac{y}{2} = \alpha$

$\Rightarrow \quad x = \cos\left(\cos^{-1}\frac{y}{2} + \alpha\right) = \cos\left(\cos^{-1}\frac{y}{2}\right)\cos\alpha - \sin\left(\cos^{-1}\frac{y}{2}\right)\sin\alpha$

$$= \frac{y}{2}\cos\alpha - \sqrt{1-\frac{y^2}{4}}\sin\alpha$$

$\Rightarrow \quad 2x = y\cos\alpha - \sin\alpha\sqrt{4-y^2}$

$\Rightarrow \quad 2x - y\cos\alpha = -\sin\alpha\sqrt{4-y^2}$

Squaring, we get

$4x^2 + y^2\cos^2\alpha - 4xy\cos\alpha = 4\sin^2\alpha - y^2\sin^2\alpha$

$\Rightarrow \quad 4x^2 - 4xy\cos\alpha + y^2 = 4\sin^2\alpha$

43. d. $2\tan^{-1} 2x = \sin^{-1}\frac{4x}{1+4x^2}$

$\Rightarrow \quad -\frac{\pi}{2} \le 2\tan^{-1} 2x \le \frac{\pi}{2}$

$\Rightarrow \quad -\frac{\pi}{4} \le \tan^{-1} 2x \le \frac{\pi}{4}$

$\Rightarrow \quad -1 \le 2x \le 1$

$\Rightarrow \quad -\frac{1}{2} \le x \le \frac{1}{2}$

44. b. $\cos^{-1}(2x^2-1) = 2\pi - 2\cos^{-1}x$ (as $x < 0$)

$\Rightarrow \quad \cos^{-1}(2x^2-1) - 2\sin^{-1}x = 2\pi - 2\cos^{-1}x - 2\sin^{-1}x$

$= 2\pi - 2(\cos^{-1}x + \sin^{-1}x)$

$= 2\pi - 2\,\frac{\pi}{2} = \pi$

45. c. $2\sin^{-1}x = \sin^{-1}\left(2x\sqrt{1-x^2}\right)$

Range of the right-hand angle is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\Rightarrow \quad -\frac{\pi}{2} \le 2\sin^{-1}x \le \frac{\pi}{2}$

$\Rightarrow \quad \frac{-\pi}{4} \le \sin^{-1}x \le \frac{\pi}{4}$

$\Rightarrow \quad x \in \left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$

46. c. $x_1 = 2\tan^{-1}\left(\frac{1+x}{1-x}\right)$ and $x_2 = \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right) = \tan^{-1}\left(\frac{1-x^2}{2x}\right)$

Now $\frac{1+x}{1-x} > 1 \quad \Rightarrow \quad x_1 = \pi + \tan^{-1}\left(\frac{2\left(\frac{1+x}{1-x}\right)}{1-\left(\frac{1+x}{1-x}\right)^2}\right) = \pi + \tan^{-1}\left(\frac{1-x^2}{-2x}\right) = \pi - \tan^{-1}\left(\frac{1-x^2}{2x}\right)$

$\Rightarrow \quad x_1 + x_2 = \pi$

47. d. $\sin(2\sin^{-1}(0.8)) = \sin(\sin^{-1}(2\times 0.8\sqrt{1-(0.8)^2})) = \sin(\sin^{-1}0.96) = 0.96$

48. b. $\tan(\sin^{-1}(\cos(\sin^{-1}x)))\tan(\cos^{-1}(\sin(\cos^{-1}x)))$

$= \tan(\sin^{-1}(\cos(\cos^{-1}\sqrt{1-x^2})))\tan\{\cos^{-1}(\sin(\sin^{-1}\sqrt{1-x^2}))\}$

$= \tan(\sin^{-1}\sqrt{1-x^2})\tan(\cos^{-1}\sqrt{1-x^2})$

$= \tan(\cos^{-1}x)\tan(\sin^{-1}x)$

$= \tan(\cos^{-1}x)\tan(\pi/2 - \cos^{-1}x) = \tan(\cos^{-1}x)\cot(\cos^{-1}x) = 1$

49. d. Since $\sin^{-1}\left(\frac{2x}{1+x^2}\right) = 2\tan^{-1}x$ for $x \in (-1, 1)$

$\sin^{-1}\left(\frac{2a}{1+a^2}\right) + \sin^{-1}\left(\frac{2b}{1+b^2}\right) = 2\tan^{-1}x$

$\Rightarrow \quad 2\tan^{-1}a + 2\tan^{-1}b = 2\tan^{-1}x$

$\Rightarrow \quad \tan^{-1} a + \tan^{-1} b = \tan^{-1} x$

$\Rightarrow \quad \tan^{-1}\left(\frac{a+b}{1-ab}\right) = \tan^{-1} x$

$\Rightarrow \quad x = \frac{a+b}{1-ab}$

50. b. If $x < 0$, then $\sin^{-1}x < 0$ but $\cos^{-1}\sqrt{1-x^2}$ is always positive.

So, $\sin^{-1}x = -\cos^{-1}\sqrt{1-x^2}$.

51. b.

We have $\frac{xy}{zr}\frac{yz}{xr} = \frac{y^2}{r^2} = \frac{y^2}{x^2+y^2+z^2} < 1$

$$\Rightarrow \quad \tan^{-1}\left(\frac{xy}{zr}\right) + \tan^{-1}\left(\frac{yz}{xr}\right) + \tan^{-1}\left(\frac{xz}{yr}\right) = \tan^{-1}\left(\frac{\frac{xy}{zr}+\frac{yz}{xr}}{1-\frac{xy}{zr}\frac{yz}{xr}}\right) + \tan^{-1}\left(\frac{xz}{yr}\right)$$

$$= \tan^{-1}\left(\frac{\frac{y\left(x^2+z^2\right)}{xzr}}{\frac{r^2-y^2}{r^2}}\right) + \tan^{-1}\left(\frac{xz}{yr}\right)$$

$$= \tan^{-1}\left(\frac{\frac{yr\left(x^2+z^2\right)}{xz}}{\left(x^2+z^2\right)}\right) + \tan^{-1}\left(\frac{xz}{yr}\right)$$

$$= \tan^{-1}\left(\frac{yr}{xz}\right) + \tan^{-1}\left(\frac{xz}{yr}\right) = \frac{\pi}{2}$$

52. b. Let $x = \sin\theta$ where $-\frac{1}{2} \le x \le 1 \Rightarrow -\frac{\pi}{6} \le \theta \le \frac{\pi}{2}$

Then $f(x) = \sin^{-1}\left(\frac{\sqrt{3}}{2}x - \frac{1}{2}\sqrt{1-x^2}\right)$

$= \sin^{-1}\left(\frac{\sqrt{3}}{2}\sin\theta - \frac{1}{2}\cos\theta\right)$

$= \sin^{-1}\left(\sin\left(\theta - \frac{\pi}{6}\right)\right)$

$= \theta - \frac{\pi}{6} = \sin^{-1}x - \frac{\pi}{6}$ $\quad \left[\because \theta - \frac{\pi}{6} \in \left(\frac{-\pi}{3}, \frac{\pi}{3}\right)\right]$

53. c. Let $y = \tan^{-1}\left(\frac{1-x^2}{2x}\right) + \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

Put $x = \tan\theta$. As $x \in (0, 1)$, $\theta \in \left(0, \frac{\pi}{4}\right)$ and $\frac{\pi}{2} - 2\theta \in (0, \pi/2)$

$\therefore \quad y = \tan^{-1}(\cot 2\theta) + \cos^{-1}(\cos 2\theta) = \tan^{-1}\left(\tan\left(\frac{\pi}{2} - 2\theta\right)\right) + \cos^{-1}(\cos 2\theta) = \frac{\pi}{2} - 2\theta + 2\theta = \frac{\pi}{2}$

54. c. $\sin^{-1} x = 2\sin^{-1} a$

Now $-\frac{\pi}{2} \le \sin^{-1} x \le \frac{\pi}{2}$

$\Rightarrow \quad -\frac{\pi}{2} \le 2\sin^{-1} a \le \frac{\pi}{2}$

$\Rightarrow \quad -\frac{\pi}{4} \le \sin^{-1} a \le \frac{\pi}{4}$

$\Rightarrow \quad \frac{1}{\sqrt{2}} \le a \le \frac{1}{\sqrt{2}} \quad \Rightarrow \quad |a| \le \frac{1}{\sqrt{2}}$

55. a. Let $\tan^{-1} x = \theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2} \quad \Rightarrow \quad -\pi < 2\theta < \pi$

Let $\frac{\pi}{2} < 2\theta < \pi \quad \Rightarrow \quad \frac{\pi}{4} < \theta < \frac{\pi}{2} \quad \Rightarrow \quad \frac{\pi}{4} < \tan^{-1} x < \frac{\pi}{2} \quad \Rightarrow x > 1$

$\Rightarrow \quad \tan^{-1}\left(\frac{2\tan\theta}{1 - \tan^2\theta}\right) = \tan^{-1}(\tan 2\theta) = \tan^{-1}(\tan(2\theta - \pi)) = 2\theta - \pi = 2\tan^{-1} x - \pi$

56. c. $\sin^{-1}\left(\frac{\sqrt{r} - \sqrt{r-1}}{\sqrt{r(r+1)}}\right) = \tan^{-1}\left(\frac{\sqrt{r} - \sqrt{r-1}}{1 + \sqrt{r(r-1)}}\right)$

$\Rightarrow \quad \sum_{r=1}^{n} \sin^{-1}\left(\frac{\sqrt{r} - \sqrt{r-1}}{\sqrt{r(r+1)}}\right) = \sum_{r=1}^{n} (\tan^{-1}\sqrt{r} - \tan^{-1}\sqrt{r-1}) = \tan^{-1}\sqrt{n}$

57. b. $\sum_{r=1}^{n} \tan^{-1}\left(\frac{2^{r-1}}{1 + 2^{2r-1}}\right) = \sum_{r=1}^{n} \tan^{-1}\left(\frac{2^{r-1}}{1 + 2^r\, 2^{r-1}}\right)$

$= \sum_{r=1}^{n} \tan^{-1}\left(\frac{2^r - 2^{r-1}}{1 + 2^r\, 2^{r-1}}\right)$

$= \sum_{r=1}^{n} [\tan^{-1}(2^r) - \tan^{-1}(2^{r-1})] = \tan^{-1}(2^n) - \tan^{-1}(1) = \tan^{-1}(2^n) - \frac{\pi}{4}$

58. a. We have $\sum_{m=1}^{n} \tan^{-1}\left(\frac{2m}{m^4 + m^2 + 2}\right) = \sum_{m=1}^{n} \tan^{-1}\left(\frac{2m}{1 + (m^2 + m + 1)(m^2 - m + 1)}\right)$

$= \sum_{m=1}^{n} \tan^{-1}\left(\frac{(m^2 + m + 1) - (m^2 - m + 1)}{1 + (m^2 + m + 1)(m^2 - m + 1)}\right)$

$$= \sum_{m=1}^{n} [\tan^{-1}(m^2+m+1) - \tan^{-1}(m^2-m+1)]$$

$$= (\tan^{-1}3 - \tan^{-1}1) + (\tan^{-1}7 - \tan^{-1}3) + (\tan^{-1}13 - \tan^{-1}7) \quad + \cdots + [\tan^{-1}(n^2+n+1) - \tan^{-1}(n^2-n+1)]$$

$$= \tan^{-1}(n^2+n+1) - \tan^{-1}1 = \tan^{-1}\left(\frac{n^2+n}{2+n^2+n}\right)$$

For $n \to \infty$, sum $= \tan^{-1}(1) = \frac{\pi}{4}$

59. a. $\tan^{-1}\left(\frac{1}{1+r+r^2}\right) = \tan^{-1}\left(\frac{r+1-r}{1+r(r+1)}\right)$

$$= \tan^{-1}(r+1) - \tan^{-1}(r)$$

$$\Rightarrow \sum_{r=0}^{n} [\tan^{-1}(r+1) - \tan^{-1}(r)] = \tan^{-1}(n+1) - \tan^{-1}(0)$$

$$= \tan^{-1}(n+1)$$

$$\Rightarrow \sum_{r=0}^{\infty} \tan^{-1}\left(\frac{1}{1+r+r^2}\right) = \tan^{-1}(\infty) = \frac{\pi}{2}$$

60. b. Obviously, $x = \sin(\theta+\beta)$ and $y = \sin(\theta-\beta)$

$$\Rightarrow 1 + xy = 1 + \sin(\theta+\beta)\sin(\theta-\beta) = 1 + \sin^2\theta - \sin^2\beta = \sin^2\theta + \cos^2\beta$$

61. a. Let $\sqrt{\tan\alpha} = \tan x$, then $u = \cot^{-1}(\tan x) - \tan^{-1}(\tan x) = \frac{\pi}{2} - x - x = \frac{\pi}{2} - 2x$

$$\Rightarrow 2x = \frac{\pi}{2} - u \qquad \Rightarrow \frac{\pi}{4} - \frac{u}{2}$$

$$\Rightarrow \tan x = \tan\left(\frac{\pi}{4} - \frac{u}{2}\right)$$

$$\Rightarrow \sqrt{\tan\alpha} = \tan\left(\frac{\pi}{4} - \frac{u}{2}\right)$$

62. b. $\tan\left[\frac{\pi}{4} + \frac{1}{2}\cos^{-1}\frac{a}{b}\right] + \tan\left[\frac{\pi}{4} - \frac{1}{2}\cos^{-1}\frac{a}{b}\right]$

Let $\frac{1}{2}\cos^{-1}\frac{a}{b} = \theta \Rightarrow \cos 2\theta = \frac{a}{b}$

Thus, $\tan\left[\frac{\pi}{4} + \theta\right] + \tan\left[\frac{\pi}{4} - \theta\right] = \frac{1+\tan\theta}{1-\tan\theta} + \frac{1-\tan\theta}{1+\tan\theta}$

$$= \frac{1+\tan^2\theta + 2\tan\theta + 1 + \tan^2\theta - 2\tan\theta}{(1-\tan^2\theta)}$$

$$= \frac{2(1+\tan^2\theta)}{1-\tan^2\theta} = \frac{2}{\cos 2\theta} = \frac{2}{(a/b)} = \frac{2b}{a}$$

63. a. $2\tan^{-1}\left[\sqrt{\frac{a-b}{a+b}}\tan\frac{\theta}{2}\right] = \cos^{-1}\left[\frac{1-\left(\frac{a-b}{a+b}\right)\tan^2\frac{\theta}{2}}{1+\left(\frac{a-b}{a+b}\right)\tan^2\frac{\theta}{2}}\right]$ $\left[\because 2\tan^{-1}x = \cos^{-1}\frac{1-x^2}{1+x^2}\right]$

$$= \cos^{-1}\left[\frac{(a+b)-(a-b)\tan^2\frac{\theta}{2}}{(a+b)+(a-b)\tan^2\frac{\theta}{2}}\right]$$

$$= \cos^{-1}\left[\frac{a\left(1-\tan^2\frac{\theta}{2}\right)+b\left(1+\tan^2\frac{\theta}{2}\right)}{a\left(1+\tan^2\frac{\theta}{2}\right)+b\left(1-\tan^2\frac{\theta}{2}\right)}\right]$$

$$= \cos^{-1}\left[\frac{\frac{a\left(1-\tan^2\frac{\theta}{2}\right)}{1+\tan^2\frac{\theta}{2}}+b}{a+b\left(\frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}\right)}\right]$$

$$= \cos^{-1}\left[\frac{a\cos\theta+b}{a+b\cos\theta}\right]$$

64. d. $\cot^{-1}\left[\frac{\sqrt{1-\sin x}+\sqrt{1+\sin x}}{\sqrt{1-\sin x}-\sqrt{1+\sin x}}\right]$

$$= \cot^{-1}\left[\frac{\left(\sqrt{1-\sin x}+\sqrt{1+\sin x}\right)}{\left(\sqrt{1-\sin x}-\sqrt{1+\sin x}\right)}\,\frac{\left(\sqrt{1-\sin x}+\sqrt{1+\sin x}\right)}{\left(\sqrt{1-\sin x}+\sqrt{1+\sin x}\right)}\right]$$

$$= \cot^{-1}\left[\frac{(1-\sin x)+(1+\sin x)+2\sqrt{1-\sin^2 x}}{(1-\sin x)-(1+\sin x)}\right] = \cot^{-1}\left[\frac{2(1+\cos x)}{-2\sin x}\right]$$

$$= \cot^{-1}\left[-\frac{2\cos^2(x/2)}{2\sin(x/2)\cos(x/2)}\right] = \cot^{-1}\left(-\cot\frac{x}{2}\right) = \cot^{-1}\left[\cot\left(\pi-\frac{x}{2}\right)\right] = \pi-\frac{x}{2}$$

65. b. $\tan^{-1}\left(\dfrac{x\cos\theta}{1-x\sin\theta}\right)-\cot^{-1}\left(\dfrac{\cos\theta}{x-\sin\theta}\right)=\tan^{-1}\left(\dfrac{x\cos\theta}{1-x\sin\theta}\right)-\tan^{-1}\left(\dfrac{x-\sin\theta}{\cos\theta}\right)$

$$=\tan^{-1}\left(\frac{\dfrac{x\cos\theta}{1-x\sin\theta}-\dfrac{x-\sin\theta}{\cos\theta}}{1+\left(\dfrac{x\cos\theta}{1-x\sin\theta}\right)\left(\dfrac{x-\sin\theta}{\cos\theta}\right)}\right)$$

$$=\tan^{-1}\left(\frac{x\cos^2\theta-x+\sin\theta+x^2\sin\theta-x\sin^2\theta}{\cos\theta-x\cos\theta\sin\theta+x^2\cos\theta-x\cos\theta\sin\theta}\right)$$

$$=\tan^{-1}\left(\frac{-x\sin^2\theta+\sin\theta+x^2\sin\theta-x\sin^2\theta}{\cos\theta-2x\cos\theta\sin\theta+x^2\cos\theta}\right)$$

$$=\tan^{-1}\left(\frac{-2x\sin^2\theta+\sin\theta+x^2\sin\theta}{\cos\theta-2x\cos\theta\sin\theta+x^2\cos\theta}\right)$$

$$=\tan^{-1}\left(\frac{\sin\theta(-2x\sin\theta+1+x^2)}{\cos\theta(1-2x\sin\theta+x^2)}\right)=\tan^{-1}(\tan\theta)=\theta$$

66. d. $\tan^{-1}\left(\dfrac{\tan x}{4}\right)+\tan^{-1}\left(\dfrac{3\sin 2x}{5+3\cos 2x}\right)=\tan^{-1}\left(\dfrac{\tan x}{4}\right)+\tan^{-1}\left(\dfrac{\dfrac{6\tan x}{1+\tan^2 x}}{5+\dfrac{3(1-\tan^2 x)}{1+\tan^2 x}}\right)$

$$=\tan^{-1}\left(\frac{\tan x}{4}\right)+\tan^{-1}\left(\frac{6\tan x}{8+2\tan^2 x}\right)$$

$$=\tan^{-1}\left(\frac{\tan x}{4}\right)+\tan^{-1}\left(\frac{3\tan x}{4+\tan^2 x}\right)$$

$$=\tan^{-1}\left(\frac{\dfrac{\tan x}{4}+\dfrac{3\tan x}{4+\tan^2 x}}{1-\dfrac{3\tan^2 x}{4(4+\tan^2 x)}}\right)\quad\left[\text{as}\left|\frac{\tan x}{4}\ \frac{3\tan x}{4+\tan^2 x}\right|<1\right]$$

$$=\tan^{-1}\left(\frac{16\tan x+\tan^3 x}{16+\tan^2 x}\right)$$

$$=\tan^{-1}(\tan x)=x$$

67. a. $\cot^{-1}\left(\sqrt{\cos\alpha}\right)-\tan^{-1}\left(\sqrt{\cos\alpha}\right)=x$

$$\Rightarrow \tan^{-1}\left(\frac{1}{\sqrt{\cos\alpha}}\right) - \tan^{-1}\left(\sqrt{\cos\alpha}\right) = x$$

$$\Rightarrow \tan^{-1}\frac{\frac{1}{\sqrt{\cos\alpha}} - \sqrt{\cos\alpha}}{1 + \frac{1}{\sqrt{\cos\alpha}}\sqrt{\cos\alpha}} = x$$

$$\Rightarrow \tan^{-1}\frac{1-\cos\alpha}{2\sqrt{\cos\alpha}} = x$$

$$\Rightarrow \tan x = \frac{1-\cos\alpha}{2\sqrt{\cos\alpha}}$$

$$\Rightarrow \cot x = \frac{2\sqrt{\cos\alpha}}{1-\cos\alpha}$$

$$\Rightarrow \operatorname{cosec} x = \sqrt{1 + \frac{4\cos\alpha}{(1-\cos\alpha)^2}} = \frac{1+\cos\alpha}{1-\cos\alpha}$$

$$\Rightarrow \sin x = \frac{1-\cos\alpha}{1+\cos\alpha} = \frac{2\sin^2(\alpha/2)}{2\cos^2(\alpha/2)} = \tan^2\alpha/2$$

68. c. $\tan\left(\frac{\pi}{4}+\frac{1}{2}\cos^{-1}x\right) + \tan\left(\frac{\pi}{4}-\frac{1}{2}\cos^{-1}x\right) = \dfrac{1+\tan\left(\frac{1}{2}\cos^{-1}x\right)}{1-\tan\left(\frac{1}{2}\cos^{-1}x\right)} + \dfrac{1-\tan\left(\frac{1}{2}\cos^{-1}x\right)}{1+\tan\left(\frac{1}{2}\cos^{-1}x\right)}$

$$= \frac{\left(1+\tan\left(\frac{1}{2}\cos^{-1}x\right)\right)^2 + \left(1-\tan\left(\frac{1}{2}\cos^{-1}x\right)\right)^2}{1-\tan^2\left(\frac{1}{2}\cos^{-1}x\right)}$$

$$= 2\frac{1+\tan^2\left(\frac{1}{2}\cos^{-1}x\right)}{1-\tan^2\left(\frac{1}{2}\cos^{-1}x\right)}$$

$$= \frac{2}{\cos(\cos^{-1}x)} = \frac{2}{x}$$

69. c. We have $(\sin^{-1}x)^3 + (\cos^{-1}x)^3 = (\sin^{-1}x + \cos^{-1}x)^3 - 3\sin^{-1}x\cos^{-1}x\,(\sin^{-1}x + \cos^{-1}x)$

$$= \frac{\pi^3}{8} - 3(\sin^{-1}x\cos^{-1}x)\frac{\pi}{2}$$

$$= \frac{\pi^3}{8} - \frac{3\pi}{2}\sin^{-1}x\left(\frac{\pi}{2} - \sin^{-1}x\right)$$

$$= \frac{\pi^3}{8} - \frac{3\pi^2}{4}\sin^{-1}x + \frac{3\pi}{2}(\sin^{-1}x)^2$$

$$= \frac{\pi^3}{8} + \frac{3\pi}{2}\left[(\sin^{-1}x)^2 - \frac{\pi}{2}\sin^{-1}x\right]$$

$$= \frac{\pi^3}{8} + \frac{3\pi}{2}\left[\left(\sin^{-1}x - \frac{\pi}{4}\right)^2\right] - \frac{3\pi^3}{32}$$

$$= \frac{\pi^3}{32} + \frac{3\pi}{2}\left(\sin^{-1}x - \frac{\pi}{4}\right)^2$$

So, the least value is $\frac{\pi^3}{32}$ when $\left(\sin^{-1}x - \frac{\pi}{4}\right) = 0$.

And the greatest value occurs when $\left(\sin^{-1}x - \frac{\pi}{4}\right)^2 = \left(-\frac{\pi}{2} - \frac{\pi}{4}\right)^2 = \frac{9\pi^2}{16}$.

Therefore, the greatest value is $\frac{\pi^3}{32} + \frac{9\pi^2}{16} \times \frac{3\pi}{2} = \frac{7\pi^3}{8}$.

70. c. $f(x) = \sin^{-1}x + \tan^{-1}x + \sec^{-1}x$, clearly domain of $f(x)$ is $x = \pm 1$.

Thus, the range is $\{f(1), f(-1)\}$, i.e., $\left\{\frac{\pi}{4}, \frac{3\pi}{4}\right\}$.

71. a. $1 + x^2 \geq 2|x| \quad \Rightarrow \quad \frac{2|x|}{1+x^2} \leq 1$

$\Rightarrow \quad -1 \leq \frac{2x}{1+x^2} \leq 1 \qquad \Rightarrow \quad \tan^{-1}\left(\frac{2x}{1+x^2}\right) \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$

72. c. $[\cot^{-1}x] + [\cos^{-1}x] = 0$

As $\cos^{-1}x, \cot^{-1}x \geq 0$, $[\cot^{-1}x] = [\cos^{-1}x] = 0$

$[\cot^{-1}x] = 0 \quad \Rightarrow \quad x \in (\cot 1, \infty)$ (i)

$[\cos^{-1}x] = 0 \quad \Rightarrow \quad x \in (\cos 1, 1]$ (ii)

Hence, from Eqs. (i) and (ii), $x \in (\cot 1, 1]$.

73. d. $\frac{3\pi}{2} < 5 < \frac{5\pi}{2}$

$\Rightarrow \sin^{-1}(\sin 5) = 5 - 2\pi$

Given $\sin^{-1}(\sin 5) > x^2 - 4x$

$\Rightarrow \quad x^2 - 4x + 4 < 9 - 2\pi$

$\Rightarrow \quad (x-2)^2 < 9 - 2\pi$

$\Rightarrow \quad -\sqrt{9-2\pi} < x - 2 < \sqrt{9-2\pi}$

$\Rightarrow \quad 2 - \sqrt{9-2\pi} < x < 2 + \sqrt{9-2\pi}$

74. c. $\frac{\alpha^3}{2}\operatorname{cosec}^2\left(\frac{1}{2}\tan^{-1}\frac{\alpha}{\beta}\right) + \frac{\beta^3}{2}\sec^2\left(\frac{1}{2}\tan^{-1}\frac{\beta}{\alpha}\right)$

$$= \alpha^3 \frac{1}{1 - \cos\left(\tan^{-1}\left(\frac{\alpha}{\beta}\right)\right)} + \beta^3 \frac{1}{1 + \cos\left(\tan^{-1}\frac{\beta}{\alpha}\right)}$$

$$= \alpha^3 \frac{1}{1-\cos\left(\cos^{-1}\left(\frac{\beta}{\sqrt{\alpha^2+\beta^2}}\right)\right)} + \beta^3 \frac{1}{1+\cos\left(\cos^{-1}\frac{\alpha}{\sqrt{\alpha^2+\beta^2}}\right)}$$

$$= \alpha^3 \frac{1}{1-\frac{\beta}{\sqrt{\alpha^2+\beta^2}}} + \beta^3 \frac{1}{1+\frac{\alpha}{\sqrt{\alpha^2+\beta^2}}}$$

$$= \sqrt{\alpha^2+\beta^2}\left(\frac{\alpha^3}{\sqrt{\alpha^2+\beta^2}-\beta} + \frac{\beta^3}{\sqrt{\alpha^2+\beta^2}+\alpha}\right)$$

$$= \sqrt{\alpha^2+\beta^2}\left(\alpha^3 \frac{(\sqrt{\alpha^2+\beta^2}+\beta)}{\alpha^2} + \beta^3 \frac{(\sqrt{\alpha^2+\beta^2}-\alpha)}{\beta^2}\right)$$

$$= \sqrt{\alpha^2+\beta^2}\left[\alpha\left(\sqrt{\alpha^2+\beta^2}+\beta\right)+\beta\left(\sqrt{\alpha^2+\beta^2}-\alpha\right)\right]$$

$$= \sqrt{\alpha^2+\beta^2}(\alpha+\beta)\sqrt{\alpha^2+\beta^2}$$

$$= (\alpha+\beta)(\alpha^2+\beta^2)$$

75. d. $\lim_{|x|\to\infty} \cos(\tan^{-1}(\sin(\tan^{-1} x))) = \cos(\tan^{-1}(\sin(\tan^{-1}\infty)))$

$$= \cos(\tan^{-1}(\sin(\pi/2)))$$

$$= \cos(\tan^{-1}(1)) = \cos(\pi/4) = \frac{1}{\sqrt{2}}$$

76. d. $\sin^{-1}(-(x-1)(x-2)) + \cos^{-1}((x-3)(x-1)) = \frac{\pi}{4}$

For $x \in [1, 2]$ $\Rightarrow$ $\sin^{-1}(-(x-1)(x-2)) \in [0, \pi/2)$

and $\cos^{-1}((x-3)(x-1)) \in [\pi/2, \pi]$ $\Rightarrow$ no solution in the given range

Also, $-1 \le 3x-2-x^2 \le 1$ and $-1 \le x^2-4x+3 \le 1$ $\Rightarrow$ $2-\sqrt{2} \le x \le \frac{3+\sqrt{5}}{2}$

77. d. $2^{2\pi/\sin^{-1}x} - 2(a+2)2^{\pi/\sin^{-1}x} + 8a < 0$

$(2^{\pi/\sin^{-1}x} - 4)(2^{\pi/\sin^{-1}x} - 2a) < 0$

Now $2^{\pi/\sin^{-1}x} \in \left(0, \frac{1}{4}\right] \cup [4, \infty)$

Now for $2^{\pi/\sin^{-1}x} \in \left(0, \frac{1}{4}\right]$, we have $(2^{\pi/\sin^{-1}x} - 4) < 0$

$\Rightarrow$ $2^{\pi/\sin^{-1}x} - 2a > 0$

$\Rightarrow$ $2a < 2^{\pi/\sin^{-1}x}$ $\Rightarrow$ $2a < \frac{1}{4}$

$\Rightarrow \quad 0 \le a < \frac{1}{8}$

Similarly, for $2^{\pi/\sin^{-1}x} \in [4, \infty), a > 2$, we get

$a \in \left[0, \frac{1}{8}\right) \cup (2, \infty)$

78. b. Given that $\sin^{-1} x + \tan^{-1} x = 2k + 1$

The range of the function $\sin^{-1} x + \tan^{-1} x$ is $\left[\frac{-3\pi}{4}, \frac{3\pi}{4}\right]$ [as both functions are increasing]

Therefore, the integral values of k are -1 and 0.

79. c. From the given equation $\sin^2\theta - 2\sin\theta + 3 = 5^{\sec^2 y} + 1$, we get

$(\sin\theta - 1)^2 + 2 = 5^{\sec^2 y} + 1$

L.H.S. ≤ 6, R.H.S. ≥ 6

Possible solution is $\sin\theta = -1$ when L.H.S. = R.H.S. $\Rightarrow \cos^2\theta = 0 \Rightarrow \cos^2\theta - \sin\theta = 1$

80. d. $[\cot^{-1} x] + 2[\tan^{-1} x] = 0 \Rightarrow [\cot^{-1} x] = 0, [\tan^{-1} x] = 0$

or $[\cot^{-1} x] = 2, [\tan^{-1} x] = -1$

Now $[\cot^{-1} x] = 0 \Rightarrow x \in (\cot 1, \infty)$

$[\tan^{-1} x] = 0 \Rightarrow x \in (0, \tan 1)$

Therefore, for $[\cot^{-1} x] = [\tan^{-1} x] = 0, x \in (\cot 1, \tan 1)$

$[\cot^{-1} x] = 2 \Rightarrow x \in (\cot 3, \cot 2]$

$[\tan^{-1} x] = -1 \Rightarrow x \in [-\tan 1, 0) \quad \Rightarrow$ No such x exists.

Thus, the solution set is $(\cot 1, \tan 1)$.

81. a. Expanding, we have

$(\tan^{-1} x)^3 + (\tan^{-1} 2x)^3 + (\tan^{-1} 3x)^3 = 3\tan^{-1} x \tan^{-1} 2x \tan^{-1} 3x$

$\Rightarrow x = 0$

82. a. $2\cos^{-1} x = \cot^{-1}\left(\frac{2x^2 - 1}{2x\sqrt{1 - x^2}}\right)$

Put $x = \cos\theta$: LHS $= 2\theta$; $0 \le \theta \le \pi$ and $-1 \le x \le 1$ (i)

R.H.S. $= \cot^{-1}\left(\frac{\cos 2\theta}{2\cos\theta|\sin\theta|}\right) = \cot^{-1}(\cot 2\theta) = 2\theta$ if $0 < 2\theta < \pi$ (ii)

From Eqs. (i) and (ii), we get $0 < \theta < \pi/2$

$\therefore x \in (0, 1)$

83. d. $\frac{x}{\sqrt{1 + x^2}} = \frac{x + 1}{\sqrt{(x + 1)^2 + 1}}$

$\Rightarrow x^2 [(x+1)^2 + 1] = (x+1)^2 [(x^2+1)]$
$\Rightarrow x^2 (x+1)^2 + x^2 = x^2 (x+1)^2 + (x+1)^2$
$\Rightarrow x^2 = (x+1)^2 \Rightarrow x+1 = x$ not possible as $x \to \infty$
$\Rightarrow x+1 = -x \Rightarrow x = -1/2$ which is also not possible as for this L.H.S. < 0 but R.H.S. > 0

84. c. Let $\tan^{-1}(x) = \theta \Rightarrow x = \tan\theta \Rightarrow \cos\theta = x \Rightarrow \dfrac{1}{\sqrt{1+x^2}} = x$

$$\Rightarrow x^2(1+x^2) = 1 \Rightarrow x^2 = \frac{-1 \pm \sqrt{5}}{2} \Rightarrow x^2 = \frac{\sqrt{5}-1}{2} \Rightarrow \frac{x^2}{2} = \frac{\sqrt{5}-1}{4}$$

Now $\cos^{-1}\left(\dfrac{\sqrt{5}-1}{4}\right) = \cos^{-1}\left(\sin\dfrac{\pi}{10}\right) = \cos^{-1}\left(\cos\dfrac{2\pi}{5}\right) = \dfrac{2\pi}{5} = \dfrac{2\pi}{5}$

85. b. $\sin\cos^{-1}(\cos(\tan^{-1}x)) = p$
For $x \in R$ $\tan^{-1}x \in (-\pi/2, \pi/2)$
$\cos(\tan^{-1}x) \in (0,1]$
$\cos^{-1}\cos(\tan^{-1}x) \in [0, \pi/2)$
$\sin(\cos^{-1}(\cos(\tan^{-1}x))) \in [0, 1)$

86. a. $\sin^{-1}x - \cos^{-1}x = \sin^{1}(3x-2)$

$$\Rightarrow \frac{\pi}{2} - \cos^{-1}x - \cos^{-1}x = \frac{\pi}{2} - \cos^{-1}(3x-2)$$

$\Rightarrow 2\cos^{-1}x = \cos^{-1}(3x-2)$. Also $x \in [-1, 1]$

$\Rightarrow \cos^{-1}(2x^2-1) = \cos^{-1}(3x-2)$ and $(3x-2) \in [-1,1]$, i.e., $-1 \le 3x-2 \le 1$

$\Rightarrow 2x^2 - 1 = 3x - 2$; hence, $x \in \left[\dfrac{1}{3}, 1\right]$

$\Rightarrow 2x^2 - 3x + 1 = 0 \Rightarrow x = 1$ or $\dfrac{1}{2}$

87. c. $\sin^{-1}\sqrt{1-x^2} + \cos^{-1}x = \cot^{-1}\dfrac{\sqrt{1-x^2}}{x} - \sin^{-1}x$

or $\dfrac{\pi}{2} + \sin^{-1}\sqrt{1-x^2} = \cot^{-1}\dfrac{\sqrt{1-x^2}}{x}$

$\tan^{-1}\dfrac{\sqrt{1-x^2}}{x} + \sin^{-1}\sqrt{1-x^2} = 0$

$\Rightarrow x \in [-1, 0) \cup \{1\}$

88. c. Here $|\cos x| = \sin^{-1}(\sin x)$

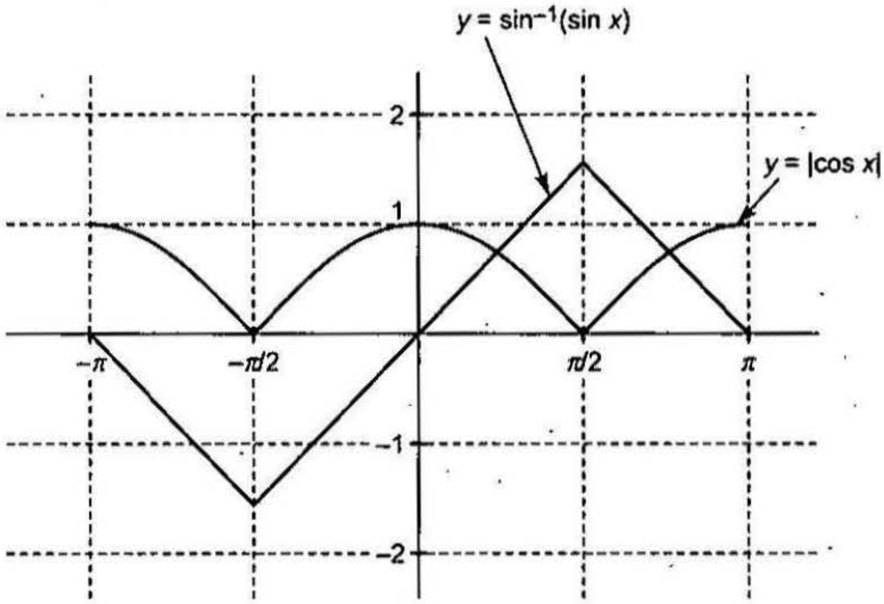


Fig. 4.29

From the graph, number of solutions is 2

89.b. $3\cos^{-1}x - \pi x - \dfrac{\pi}{2} = 0 \quad \Rightarrow \cos^{-1}x = \dfrac{\pi x}{3} + \dfrac{\pi}{6}$

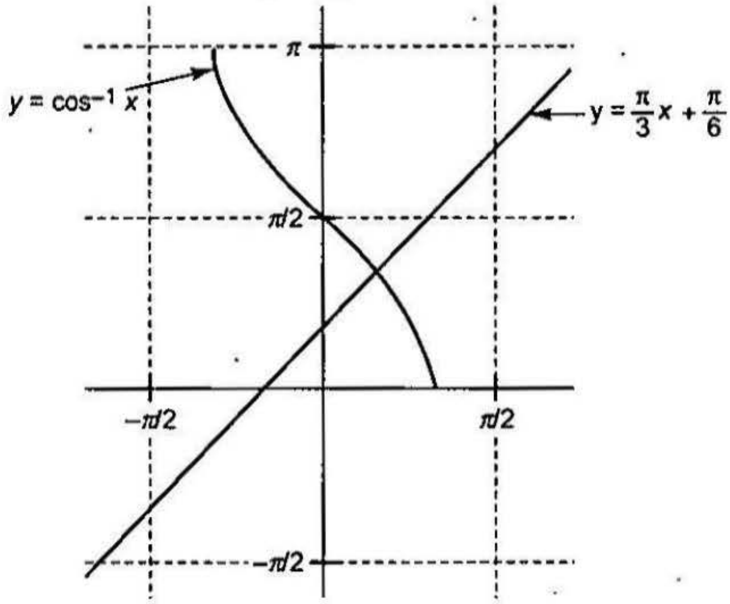


Fig. 4.30

90. b. We have

$$\left|\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)\right| < \frac{\pi}{3} \quad \Rightarrow -\frac{\pi}{3} < \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) < \frac{\pi}{3}$$

$$\Rightarrow 0 \le \cos^{-1}\frac{1-x^2}{1+x^2} < \frac{\pi}{3} \quad \Rightarrow \frac{1}{2} < \frac{1-x^2}{1+x^2} \le 1$$

$$\Rightarrow 1 + x^2 < 2(1-x^2) \le 2(1+x^2) \quad \Rightarrow 0 \le x^2 < \frac{1}{3} \Rightarrow -\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$$

91.b. $\sin^{-1}x+\sin^{-1}y=\frac{\pi}{2} \Rightarrow \sin^{-1}x=\frac{\pi}{2}-\sin^{-1}y \Rightarrow \sin^{-1}x=\sin^{-1}\sqrt{1-y^2} \Rightarrow x^2+y^2=1$

$$\Rightarrow \frac{1+x^4+y^4}{x^2-x^2y^2+y^2}=\frac{1+(x^2+y^2)^2-2x^2y^2}{1-x^2y^2}=\frac{1+1-2x^2y^2}{1-x^2y^2}=2$$

92.d. $\sin^{-1}(x^2-4x+6)+\cos^{-1}(x^2-4x+6)=\sin^{-1}((x-2)^2+2)+\cos^{-1}((x-2)^2+2)$

$(x-2)^2+2\ge 2$, for which $\sin^{-1}x$ and $\cos^{-1}x$ are not defined.

93. a. $\frac{\pi}{2}-\cos^{-1}\cos\left(\frac{2(x^2+5|x|+3)-2}{x^2+5|x|+3}\right)=\cot\cot^{-1}\left(\frac{2}{9|x|}-2\right)+\frac{\pi}{2}$

$$\frac{\pi}{2}-2+\frac{2}{x^2+5|x|+3}=\frac{2}{9|x|}-2+\frac{\pi}{2}$$

$$\Rightarrow |x|^2-4|x|+3=0$$

$$|x|=1,3 \Rightarrow x=\pm 1, \pm 3$$

94. c. $2\tan^{-1}(\operatorname{cosec}\tan^{-1}x-\tan\cot^{-1}x)=2\tan^{-1}[\operatorname{cosec}\tan^{-1}x-\tan\cot^{-1}x]$

$$=2\tan^{-1}\left[\operatorname{cosec}\left\{\operatorname{cosec}^{-1}\frac{\sqrt{1+x^2}}{x}\right\}-\tan^{-1}\left\{\tan^{-1}\left(\frac{1}{x}\right)\right\}\right]$$

$$=2\tan^{-1}\left[\sqrt{\frac{1+x^2}{x}}-\frac{1}{x}\right]=2\tan^{-1}\left[\frac{\sqrt{1+x^2}-1}{x}\right]$$

$$=2\tan^{-1}\left[\frac{\sec\theta-1}{\tan\theta}\right] \qquad [\text{putting } x=\tan\theta]$$

$$=2\tan^{-1}\left[\frac{1-\cos\theta}{\sin\theta}\right]=2\tan^{-1}\left[\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right]$$

$$=2\tan^{-1}\tan\frac{\theta}{2}=2\times\frac{\theta}{2}=\theta=\tan^{-1}x$$

Multiple Correct Answers Type

1. **b, c, d.**

$6x^2+11x+3=0$

$\Rightarrow\ (2x+3)(3x+1)=0$

$\Rightarrow\ x=-3/2, -1/3$

For $x=-3/2$, $\cos^{-1}x$ is not defined as domain of $\cos^{-1}x$ is $[-1, 1]$ and for $x=-1/3$, $\operatorname{cosec}^{-1}x$ is not defined as domain of $\operatorname{cosec}^{-1}x$ is $R-(-1, 1)$. However, $\cot^{-1}x$ is defined for both of these values as domain of $\cot^{-1}x$ is R.

2. a, b, c.

Let $\tan^{-1}(-2)=\theta \Rightarrow \tan\theta=-2 \Rightarrow \theta=(-\pi/2, 0)$

$\Rightarrow 2\theta=(-\pi, 0)$

$$\cos(-2\theta)=\cos 2\theta=\frac{1-\tan^2\theta}{1+\tan^2\theta}=\frac{-3}{5}$$

$$\Rightarrow -2\theta=\cos^{-1}\left(\frac{-3}{5}\right)=\pi-\cos^{-1}\frac{3}{5}$$

$$\Rightarrow 2\theta=-\pi+\cos^{-1}\frac{3}{5}=-\pi+\tan^{-1}\frac{4}{3}=-\pi+\cot^{-1}\frac{3}{4}=-\pi+\frac{\pi}{2}-\tan^{-1}\frac{3}{4}=-\frac{\pi}{2}-\tan^{-1}\frac{3}{4}$$

$$=-\frac{\pi}{2}+\tan^{-1}\left(-\frac{3}{4}\right)$$

3. a, b, d.

$\tan^{-1}(x-1)+\tan^{-1}(x)+\tan^{-1}(x+1)=\tan^{-1}3x$

$\Rightarrow \tan^{-1}(x-1)+\tan^{-1}(x)=\tan^{-1}3x-\tan^{-1}(x+1)$

$$\Rightarrow \tan^{-1}\left[\frac{(x-1)+x}{1-(x-1)(x)}\right]=\tan^{-1}\left[\frac{3x-(x+1)}{1+3x(x+1)}\right]$$

$$\Rightarrow \frac{2x-1}{1-x^2+x}=\frac{2x-1}{1+3x^2+3x}$$

$$\Rightarrow (1-x^2+x)(2x-1)=(1+3x^2+3x)(2x-1)$$

$$\Rightarrow x=0, \pm\frac{1}{2}$$

4. b.

We know that $\sin^{-1}x$ is defined for $x\in[-1, 1]$ and $\sec^{-1}x$ is defined for $x\in(-\infty,-1]\cup[1,\infty)$.

Hence, the given function is defined for $x\in\{-1, 1\}$.

Therefore, $f(1)=\pi/2, f(-1)=\pi/2$.

5. a, c, d.

$(\sin^{-1}x+\sin^{-1}w)(\sin^{-1}y+\sin^{-1}z)=\pi^2$

$\Rightarrow \sin^{-1}x+\sin^{-1}w=\sin^{-1}y+\sin^{-1}z=\pi$

or $\sin^{-1}x+\sin^{-1}w=\sin^{-1}y+\sin^{-1}z=-\pi$

$\Rightarrow x=y=z=w=1$ or $x=y=z=w=-1$

Hence, the maximum value of $\begin{vmatrix} x^{N_1} & y^{N_2} \\ z^{N_3} & w^{N_4}\end{vmatrix}=\begin{vmatrix} 1 & -1 \\ 1 & 1\end{vmatrix}=2$ and minimum value $\begin{vmatrix} -1 & 1 \\ 1 & 1\end{vmatrix}=-2$.

Also, there are 16 different determinants as each place value is either 1 or -1.

6. a, b, c, d.

$$\text{Since } |\tan^{-1}x|=\begin{cases}\tan^{-1}x, & \text{if } x\ge 0\\ -\tan^{-1}x, & \text{if } x<0\end{cases}$$

$\Rightarrow |\tan^{-1}x|=\tan^{-1}|x|\ \forall x\in R$

$\Rightarrow \tan|\tan^{-1}x|=\tan\tan^{-1}|x|=|x|$

Also $|\cot^{-1} x| = \cot^{-1} x;\ \forall\, x \in R$
$\Rightarrow \cot|\cot^{-1} x| = x,\ \forall\, x \in R$

$$\tan^{-1}|\tan x| = \begin{cases} x, & \text{if } \tan x > 0 \\ -x, & \text{if } \tan x < 0 \end{cases}$$

$$\sin|\sin^{-1} x| = \begin{cases} x, & x \in [0, 1] \\ -x, & x \in [-1, 0) \end{cases}$$

7. a, c.

Domain of $f(x) = \log_e \cos^{-1} x$ is $x \in [-1, 1)$
$\therefore\ [\alpha] = -1$ or 0

8. a, b, c.

(a) $\sin\left(\tan^{-1} 3 + \tan^{-1} \dfrac{1}{3}\right) = \sin\dfrac{\pi}{2} = 1$

(b) $\cos\left(\dfrac{\pi}{2} - \sin^{-1} \dfrac{3}{4}\right) = \cos\left(\cos^{-1} \dfrac{3}{4}\right) = \dfrac{3}{4}$

(c) $\sin\left(\dfrac{1}{4} \sin^{-1} \dfrac{\sqrt{63}}{8}\right)$

Let $\sin^{-1} \dfrac{\sqrt{63}}{8} = \theta$

$\Rightarrow\ \sin\theta = \dfrac{\sqrt{63}}{8} \Rightarrow \cos\theta = \dfrac{1}{8}$

We have $\cos\dfrac{\theta}{2} = \sqrt{\dfrac{1+\cos\theta}{2}} = \dfrac{3}{4}$

$\Rightarrow\ \sin\dfrac{\theta}{4} = \sqrt{\dfrac{1-\cos\dfrac{\theta}{2}}{2}} = \dfrac{1}{2\sqrt{2}}$

Now, $\log_2 \sin\left(\dfrac{1}{4} \sin^{-1} \dfrac{\sqrt{63}}{8}\right) = \log_2 \dfrac{1}{2\sqrt{2}} = -\dfrac{3}{2}$

(d) $\cos^{-1} \dfrac{\sqrt{5}}{3} = \theta \Rightarrow \cos\theta = \dfrac{\sqrt{5}}{3}$

$\therefore\ \tan\dfrac{\theta}{2} = \dfrac{3-\sqrt{5}}{2}$ which is irrational.

9. c, d.

$xy < 0 \Rightarrow\ x + \dfrac{1}{x} \geq 2, y + \dfrac{1}{y} \leq -2$

or $x + \dfrac{1}{x} \leq -2, y + \dfrac{1}{y} \geq 2$

$$x+\frac{1}{x}\geq 2\Rightarrow \quad \sec^{-1}\left(x+\frac{1}{x}\right)\in\left[\frac{\pi}{3},\frac{\pi}{2}\right)$$

$$y+\frac{1}{y}\leq -2\Rightarrow \sec^{-1}\left(y+\frac{1}{y}\right)\in\left(\frac{\pi}{2},\frac{2\pi}{3}\right] \quad \Rightarrow \quad z\in\left(\frac{5\pi}{6},\frac{7\pi}{6}\right)$$

10. a, d.

Let $f(x)=(\sin^{-1}x)^2+(\cos^{-1}x)^2$

$$=(\sin^{-1}x+\cos^{-1}x)^2-2\sin^{-1}x\cos^{-1}x$$

$$=\frac{\pi^2}{4}-2\sin^{-1}x\left[\frac{\pi}{2}-\sin^{-1}x\right]$$

$$=\frac{\pi^2}{4}-\pi\sin^{-1}x+2(\sin^{-1}x)^2$$

$$=2\left[(\sin^{-1}x)^2-\frac{\pi}{2}\sin^{-1}x+\frac{\pi^2}{8}\right]$$

$$=2\left(\sin^{-1}x-\frac{\pi}{4}\right)^2+2\left[\frac{\pi^2}{16}\right]$$

Now, $-\frac{\pi}{2}\leq\sin^{-1}x\leq\frac{\pi}{2}$

$$\Rightarrow \quad -\frac{3\pi}{4}\leq\sin^{-1}x-\frac{\pi}{4}\leq\frac{\pi}{4}$$

$$\Rightarrow \quad 0\leq\left(\sin^{-1}x-\frac{\pi}{4}\right)^2\leq\frac{9\pi^2}{16}$$

$$\Rightarrow \quad 0\leq 2\left(\sin^{-1}x-\frac{\pi}{4}\right)^2\leq\frac{9\pi^2}{8}$$

$$\Rightarrow \quad \frac{\pi^2}{8}\leq\left(\sin^{-1}x-\frac{\pi}{4}\right)^2+\frac{\pi^2}{8}\leq\frac{5\pi^2}{4}$$

11. a, d.

For the given equation $0\leq x, y\leq 1$.

Also, $\sin^{-1}x+\sin^{-1}y=\frac{\pi}{2}$

$$\Rightarrow \quad \sin^{-1}x=\cos^{-1}y=\sin^{-1}\sqrt{1-y^2}$$

$$\Rightarrow \quad x=\sqrt{1-y^2} \qquad \Rightarrow \quad x^2+y^2=1 \qquad \text{(i)}$$

Again, $\sin 2x=\cos 2y$

$$\Rightarrow \quad \cos\left(\frac{\pi}{2}-2x\right)=\cos 2y$$

$$\Rightarrow \quad \frac{\pi}{2}-2x=2n\pi\pm 2y\text{, where } n\in I$$

$$\Rightarrow \quad x\pm y=\frac{\pi}{4}-n\pi \qquad \text{(ii)}$$

From Eqs. (i) and (ii), we get

$$x=\frac{\pi}{8}+\sqrt{\frac{1}{2}-\frac{\pi^2}{64}} \text{ and } y=\sqrt{\frac{1}{2}-\frac{\pi^2}{64}}-\frac{\pi}{8}$$

12. a, c.

We have

$$\cot^{-1}\left(\frac{n^2-10n+21.6}{\pi}\right)>\frac{\pi}{6}$$

$$\Rightarrow \frac{n^2-10n+21.6}{\pi}<\cot\frac{\pi}{6} \quad \text{(as } \cot x \text{ is decreasing for } 0<x<\pi)$$

$$\Rightarrow n^2-10n+21.6<\pi\sqrt{3}$$

$$\Rightarrow n^2-10n+25+21.6-25<\pi\sqrt{3}$$

$$\Rightarrow (n-5)^2<\pi\sqrt{3}+3.4$$

$$\Rightarrow -\sqrt{\sqrt{3}\pi+3.4}<n-5<\sqrt{\sqrt{3}\pi+3.4}$$

$$\Rightarrow 5-\sqrt{\sqrt{3}\pi+3.4}<n<5+\sqrt{\sqrt{3}\pi+3.4} \quad \text{(i)}$$

Since $\sqrt{3}\pi=5.5$ nearly, $\sqrt{\sqrt{3}\pi+3.4}\sim\sqrt{8.9}\sim 2.9$

$$\Rightarrow 2.1<n<7.9$$

$$\therefore n=3,4,5,6,7 \quad \{\text{as } n\in N\}$$

13. a, b, d.

Let t_r denote the rth term of the series 3, 7, 13, 21, ... and

$$\begin{array}{rl} S= & 3+7+13+21+\cdots+t_n \\ -S= & \quad 3+7+13+\cdots+t_{n-1}+t_n \\ \hline & 0=3+4+6+8+\cdots+2n-t_n \end{array}$$

$$\Rightarrow t_n=3+4+6+\cdots+2n=1+2\times\frac{1}{2}n(n+1)=n^2+n+1$$

Let $T_r=\cot^{-1}(r^2+r+1)=\tan^{-1}\left(\frac{1}{r^2+r+1}\right)=\tan^{-1}\left(\frac{r+1-r}{1+r(r+1)}\right)=\tan^{-1}(r+1)-\tan^{-1}r$

Thus, the sum of the first n terms of the given series is

$$\sum_{r=1}^{n}\left[\tan^{-1}(r+1)-\tan^{-1}r\right]=\tan^{-1}(n+1)-\tan^{-1}(1)$$

$$=\tan^{-1}\left[\frac{n+1-n}{1+1(n+1)}\right]=\tan^{-1}\left(\frac{n}{n+2}\right)=\tan^{-1}\left(\frac{1}{1+\frac{2}{n}}\right)$$

$$\Rightarrow S_\infty=\lim_{n\to\infty}\tan^{-1}\left(\frac{1}{1+\frac{2}{n}}\right)=\frac{\pi}{4}, \quad S_{10}=\tan^{-1}\frac{10}{12}=\tan^{-1}\frac{5}{6}$$

$$S_6=\tan^{-1}\frac{3}{4}=\sin^{-1}\frac{3}{5}$$

$$S_{20}=\tan^{-1}\frac{10}{11}=\cot^{-1}1.1$$

14. b.

$f(x) = \sin^{-1}|\sin kx| + \cos^{-1}(\cos kx)$

Let $g(x) = \sin^{-1}|\sin x| + \cos^{-1}(\cos x)$

$$g(x) = \begin{cases} 2x, & 0 \le x \le \dfrac{\pi}{2} \\ \pi, & \dfrac{\pi}{2} < x \le \dfrac{3\pi}{2} \\ 4\pi - 2x, & \dfrac{3\pi}{2} < x \le 2\pi \end{cases}$$

$g(x)$ is periodic with period 2π and is constant in the continuous interval $\left[2n\pi + \dfrac{\pi}{2}, 2n\pi + \dfrac{3\pi}{2}\right]$ (where $n \in I$) and $f(x) = g(kx)$.

So, $f(x)$ is constant in the interval $\left[\dfrac{2n\pi}{k} + \dfrac{\pi}{2k}, \dfrac{2n\pi}{k} + \dfrac{3\pi}{2k}\right]$

$$\Rightarrow \frac{\pi}{4} = \frac{3\pi}{2k} - \frac{\pi}{2k} \Rightarrow \frac{\pi}{k} = \frac{\pi}{4} \Rightarrow k = 4$$

15. a, c.

Given equation is $x^2 + 2x\sin(\cos^{-1} y) + 1 = 0$. Since x is real, $D \ge 0$

$\therefore \quad 4(\sin(\cos^{-1}y))^2 - 4 \ge 0$

$\Rightarrow \quad (\sin(\cos^{-1}y))^2 \ge 1$

$\Rightarrow \quad \sin(\cos^{-1}y) = \pm 1$

$\Rightarrow \quad \cos^{-1}y = \pm\dfrac{\pi}{2} \Rightarrow y = 0$

Putting value of y in the original equation, we have $x^2 + 2x + 1 = 0 \Rightarrow x = -1$.

Hence, the equation has only one solution.

16. b, c.

$$1 \le \frac{\pi}{\cos^{-1} x} < \infty \quad \Rightarrow \quad 2 \le 2^{\frac{\pi}{\cos^{-1} x}} < \infty$$

Hence, 2 should lie between or on the roots of $t^2 - \left(a + \dfrac{1}{2}\right)t - a^2 = 0$ where $t = 2^{\pi/\cos^{-1}x}$,

$\Rightarrow \quad f(2) \le 0 \quad \Rightarrow \quad a^2 + 2a - 3 \ge 0 \Rightarrow a \in (-\infty, -3] \cup [1, \infty)$

17. a, c.

The given relation is possible when $a - \dfrac{a^2}{3} + \dfrac{a^3}{9} + \cdots = 1 + b + b^2 + \cdots$

Also, $-1 \le a - \dfrac{a^2}{3} + \dfrac{a^3}{9} + \cdots \le 1$ and $-1 \le 1 + b + b^2 + \cdots \le 1$

$\Rightarrow \quad |b| < 1 \Rightarrow |a| < 3$ and $\dfrac{a}{1 + \dfrac{a}{3}} = \dfrac{1}{1-b}$

$$= \tan^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{1-\frac{x}{y}}{1+\frac{x}{y}}\right) = \tan^{-1}\left(\frac{x}{y}\right) + \tan^{-1} 1 - \tan^{-1}\frac{x}{y} = \frac{\pi}{4}.$$

Now, in Eq. (i), putting $\frac{x}{y} = \frac{3}{4}$, we get

$$\tan^{-1}\left(\frac{3}{4}\right) + \tan^{-1}\left(\frac{1}{7}\right) = \frac{\pi}{4}$$

Hence, both the statements are correct and statement 2 is the correct explanation of statement 1.

9. a. See theory

Linked Comprehension Type

For Problems 1 – 3

1. d, 2. b, 3. c

Sol.

$\sin^{-1} x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\cos^{-1} y \in [0, \pi]$

$\sec^{-1} z \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right] \quad \Rightarrow \quad \sin^{-1}x + \cos^{-1} y + \sec^{-1} z \le \frac{\pi}{2} + \pi + \pi = \frac{5\pi}{2}$

Also, $t^2 - \sqrt{2\pi}\, t + 3\pi$

$$= t^2 - 2\sqrt{\frac{\pi}{2}}\, t + \frac{\pi}{2} - \frac{\pi}{2} + 3\pi = \left(t - \sqrt{\frac{\pi}{2}}\right)^2 + \frac{5\pi}{2} \ge \frac{5\pi}{2}$$

The given inequation exists if equality holds, i.e.,

$$\text{L.H.S.} = \text{R.H.S.} = \frac{5\pi}{2}$$

$\Rightarrow \quad x = 1, y = -1, z = -1$ and $t = \sqrt{\frac{\pi}{2}} \quad \Rightarrow \quad \cos^{-1}(\cos 5t^2) = \cos^{-1}\left(\cos\left(\frac{5\pi}{2}\right)\right) = \frac{\pi}{2}$

$\cos^{-1}(\min\{x, y, z\}) = \cos^{-1}(-1) = \pi$

For Problems 4 – 6

4. b, 5. a, 6. b

Sol.

Given $ax + b(\sec(\tan^{-1} x)) = c$ and $ay + b(\sec(\tan^{-1} y)) = c$

Let $\tan^{-1}x = \alpha$ and $\tan^{-1}y = \beta$, then the given relations are

$a \tan\alpha + b \sec\alpha = c$ and $a \tan\beta + b\sec\beta = c$

From these two relations, we can conclude that equation $a\tan\theta + b\sec\theta = c$ has roots α and β.

$a\tan\theta + b\sec\theta = c$

$\Rightarrow \quad b\sec\theta = c - a\tan\theta$

$\Rightarrow \quad b^2\sec^2\theta = c^2 - 2ac\tan\theta + a^2\tan^2\theta$

$\Rightarrow \quad b^2 + b^2\tan^2\theta = c^2 - 2ac\tan\theta + a^2\tan^2\theta$

$\Rightarrow \quad (a^2 - b^2)\tan^2\theta - 2ac\tan\theta + c^2 - b^2 = 0$

Therefore, sum of the roots, $\tan\alpha + \tan\beta = x + y = \dfrac{2ac}{a^2 - b^2}$

and the product of roots, $\tan\alpha \tan\beta = xy = \dfrac{c^2 - b^2}{a^2 - b^2}$

and $\dfrac{x+y}{1-xy} = \dfrac{\dfrac{2ac}{a^2-b^2}}{1-\dfrac{c^2-b^2}{a^2-b^2}} = \dfrac{2ac}{a^2-c^2}$

For Problems 7 – 9

7. b, 8. d, 9. c

Sol.

Let $\cos^{-1} x = a \Rightarrow a \in [0, \pi]$
and $\sin^{-1} y = b \Rightarrow b \in [-\pi/2, \pi/2]$

We have $a + b^2 = \dfrac{p\pi^2}{4}$ (i)

and $ab^2 = \dfrac{\pi^4}{16}$ (ii)

Since $b^2 \in [0, \pi^2/4]$, we get $a + b^2 \in [0, \pi + \pi^2/4]$

So, from Eq. (i) we get $0 \le \dfrac{p\pi^2}{4} \le \pi + \dfrac{\pi^2}{4}$

i.e., $0 \le p \le \dfrac{4}{\pi} + 1$

Since $p \in Z$, so $p = 0$, 1 or 2.
Substituting the value of b^2 from Eq. (i) in Eq. (ii), we get

$a\left(\dfrac{p\pi^2}{4} - a\right) = \dfrac{\pi^2}{16} \quad \Rightarrow \quad 16a^2 - 4p\pi^2 a + \pi^4 = 0$ (iii)

since $a \in R \Rightarrow D \ge 0$
i.e., $16p^2\pi^4 - 64\pi^4 \ge 0 \quad \Rightarrow \quad p^2 \ge 4 \Rightarrow p \ge 2 \quad \Rightarrow \quad p = 2$
Substituting $p = 2$ in Eq. (iii), we get
$16a^2 - 8\pi^2 a + \pi^4 = 0$

$\Rightarrow \quad (4a - \pi^2)^2 = 0 \quad \Rightarrow \quad a = \dfrac{\pi^2}{4} = \cos^{-1} x \quad \Rightarrow \quad x = \cos\dfrac{\pi^2}{4}$

From Eq. (ii), we get $\dfrac{\pi^2}{4}\ b^2 = \dfrac{\pi^4}{16} \quad \Rightarrow \quad b = \pm\dfrac{\pi}{2} = \sin^{-1} y \quad \Rightarrow \quad y = \pm 1$

For Problems 10 – 12

10. c, 11. a, 12. d

Sol.

Let $\cos^{-1} x = \theta \Rightarrow x = \cos\theta$, where $\theta \in [0, \pi]$
$\cos^{-1}(4x^3 - 3x) = \cos^{-1}(4\cos^3\theta - 3\cos\theta) = \cos^{-1}(\cos 3\theta) = \cos^{-1}(\cos\alpha)$

$\Rightarrow \quad \dfrac{3a}{a+3} = \dfrac{1}{1-b}$, there are infinitely many solutions

$\Rightarrow \quad 3a - 3ab = a + 3 \quad \Rightarrow \quad 2a - 3ab = 3$

$\Rightarrow \quad b = \dfrac{2a-3}{3a}$ and $a = \dfrac{3}{2-3b}$

18. a, b.

We know that

if $|x| \le 1$, then $2\tan^{-1} x = \sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$

if $x > 1$, $2\tan^{-1} x = \pi - \sin^{-1} \dfrac{2x}{1+x^2}$

if $x < -1$, $2\tan^{-1} x = -\pi - \sin^{-1} \dfrac{2x}{1+x^2}$

Hence, the required values are $x < -1$ or $x > 1$.

19. a, d.

Case 1: If $0 \le x \le \dfrac{1}{2}$, then

$$\cos^{-1}\left(\frac{x}{2} + \frac{1}{2}\sqrt{3-3x^2}\right) = \cos^{-1}\left(x\frac{1}{2} + \sqrt{1-x^2}\,\frac{\sqrt{3}}{2}\right) = \cos^{-1} x - \cos^{-1}\frac{1}{2}$$

Case2: If $\dfrac{1}{2} \le x \le 1$, then

$$\cos^{-1}\left(\frac{x}{2} + \frac{1}{2}\sqrt{3-3x^2}\right) = \cos^{-1}\frac{1}{2} - \cos^{-1} x$$

20. a, b, c.

a. $\cos(\tan^{-1}(\tan(4-\pi))) = \cos(4-\pi) = \cos(\pi-4) = -\cos 4 > 0$

b. $\sin(\cot^{-1}(\cot(4-\pi))) = \sin(4-\pi) = -\sin 4 > 0$ (as $\sin 4 < 0$)

c. $\tan(\cos^{-1}(\cos(2\pi-5))) = \tan(2\pi-5) = -\tan 5 > 0$ (as $\tan 5 < 0$)

d. $\cot(\sin^{-1}(\sin(\pi-4)) = \cot(\pi-4) = -\cot 4 < 0$

21. a, b. $\cos^{-1} x + \cos^{-1} y + \cos^{-1} z = \pi \Rightarrow \sin^{-1} + \sin^{-1} y + \sin^{-1} z = \pi/2$

$$\Rightarrow \cos^{-1} x + \cos^{-1} y = \cos^{-1}(-z) \Rightarrow xy - \sqrt{1-x^2}\sqrt{1-y^2} = -z \Rightarrow x^2 + y^2 + z^2 + 2xyz = 1$$

22.a, b.

a. $\cos(\tan^{-1}(\tan(4-\pi))) = \cos(4-\pi) = \cos(\pi-4) = -\cos 4 > 0$

b. $\sin(\cot^{-1}(\cot(4-\pi))) = \sin(4-\pi) = -\sin 4 > 0$ (as $\sin 4 < 0$)

c. $\tan(\cos^{-1}(\cos(2\pi-5))) = \tan(2\pi-5) = -\tan 5 > 0$ (as $\tan 5 < 0$)

d. $\cot(\sin^{-1}(\sin(\pi-4)) = \cot(\pi-4) = -\cot 4 < 0$

23.b, c, d.

$$\cos\left(-\frac{14\pi}{5}\right) = \cos\frac{14\pi}{5} = \cos\frac{4\pi}{5}$$

Hence, $\cos\dfrac{1}{2}\cos^{-1}\left(\cos\dfrac{4\pi}{5}\right) = \cos\dfrac{2\pi}{5}$

Reasoning Type

1. a. Statement 2 is correct, from which we can say $\cot^{-1}x + \cos^{-1}2x = -\pi$ is not possible. Hence, both the statements are correct and statement 2 is the correct explanation of statement 1.

2. d. Obviously, statement 2 is correct, but when $x \in [-1, 1]$ we have $\tan^{-1}x \in [-\pi/4, \pi/4]$.

It implies that $\frac{\pi}{2} + \tan^{-1}x \in [\pi/4, 3\pi/4]$.

Hence, statement 2 is true but statement 1 is false.

3. c. $\operatorname{cosec}^{-1} x > \sec^{-1}x$

$$\Rightarrow \operatorname{cosec}^{-1} x > \frac{\pi}{2} - \operatorname{cosec}^{-1} x$$

$$\Rightarrow \operatorname{cosec}^{-1} x > \frac{\pi}{4}$$

$$\Rightarrow 1 \le x < \sqrt{2} \text{ and } \left(\frac{1}{2} + \frac{1}{\sqrt{2}}\right) \in [1, \sqrt{2})$$

But statement 2 is false.

4. a. $\sin^{-1} x = \tan^{-1} \frac{x}{\sqrt{1-x^2}} > \tan^{-1} x > \tan^{-1} y$ $\left[\because x > y, \frac{x}{\sqrt{1-x^2}} > x\right]$

Therefore, statement 2 is true.

Now, $e < \pi \Rightarrow \frac{1}{\sqrt{e}} > \frac{1}{\sqrt{\pi}}$

By statement 2, we have

$$\sin^{-1}\left(\frac{1}{\sqrt{e}}\right) > \tan^{-1}\left(\frac{1}{\sqrt{e}}\right) > \tan^{-1}\left(\frac{1}{\sqrt{\pi}}\right)$$

Therefore, statement 1 is true.

5. d. $30 - 9\pi \in [0, \pi]$ is true but it is not principal value of $\cos^{-1}(\cos 30)$ as $\cos^{-1}(\cos 30) = \cos^{-1}(\cos(9\pi + (30 - 9\pi))) = \cos^{-1}(-\cos(30 - 9\pi)) = \pi - (30 - 9\pi) = 10\pi - 30$.

Hence, statement 2 is true but statement 1 is false.

6. a. $f(x) = \sin^{-1}\left(\frac{2x}{1+x^2}\right) = \pi - 2\tan^{-1}x, x \ge 1$

$$\Rightarrow f'(x) = -\frac{2}{1+x^2} \Rightarrow f'(2) = -\frac{2}{5}$$

Thus statement 1 is true, statement 2 is true and statement 2 is the correct explanation of statement 1.

7. b. We know that $\tan^{-1}x$ and $\cot^{-1}x$ have domain R, also $\tan x$ and $\cot x$ are unbounded functions. On the other hand; $\sec x$ is an unbounded function, but its range is $R - (-1, 1)$, and not R.

8. a. For $x > 0, y > 0$,

$$\tan^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y-x}{y+x}\right) \quad \text{(i)}$$

where $\alpha = 3\theta \in [0, 3\pi]$

Refer the graph of $y = \cos^{-1}(\cos\alpha)$, $\alpha \in [0, 3\pi]$.

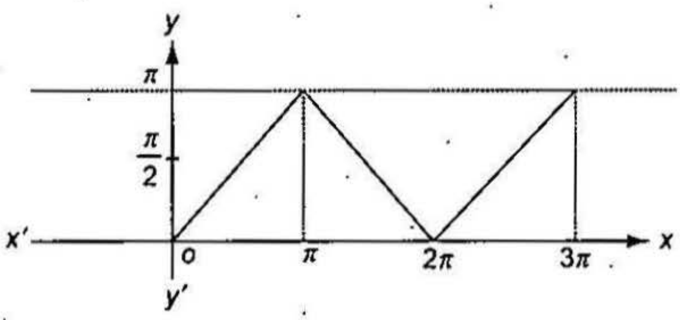


Fig. 4.31

From the graph,

$\cos^{-1}(4x^3 - 3x) = \cos^{-1}(\cos\alpha)$

$$= \begin{cases} \alpha, & 0 \le \alpha < \pi \\ 2\pi - \alpha, & \pi \le \alpha \le 2\pi \\ \alpha - 2\pi, & 2\pi < \alpha \le 3\pi \end{cases}$$

$$= \begin{cases} 3\cos^{-1}x, & 0 \le 3\cos^{-1}x < \pi \\ 2\pi - 3\cos^{-1}x, & \pi \le 3\cos^{-1}x \le 2\pi \\ 3\cos^{-1}x - 2\pi, & 2\pi < 3\cos^{-1}x \le 3\pi \end{cases}$$

$$= \begin{cases} 3\cos^{-1}x, & 0 \le \cos^{-1}x < (\pi/3) \\ 2\pi - 3\cos^{-1}x, & (\pi/3) \le \cos^{-1}x \le (2\pi/3) \\ 3\cos^{-1}x - 2\pi, & (2\pi/3) < 3\cos^{-1}x \le \pi \end{cases}$$

$$= \begin{cases} 3\cos^{-1}x, & (1/2) < x \le 1 \\ 2\pi - 3\cos^{-1}x, & (-1/2) \le x \le (1/2) \\ 3\cos^{-1}x - 2\pi, & -1 \le x < -(1/2) \end{cases}$$

$$= \begin{cases} 3\cos^{-1}x - 2\pi, & -1 \le x < (-1/2) \\ 2\pi - 3\cos^{-1}x, & (-1/2) \le x \le (1/2) \\ 3\cos^{-1}x, & (1/2) < x \le 1 \end{cases}$$

For $x \in \left[-1, -\frac{1}{2}\right)$, $\cos^{-1}(4x^3 - 3x) = 3\cos^{-1}x - 2\pi$

$\Rightarrow$ $a = -2\pi$ and $b = 3 \Rightarrow a + b\pi = \pi$

For $x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$, $\cos^{-1}(4x^3 - 3x) = 2\pi - 3\cos^{-1}x$

$\Rightarrow$ $a = 2\pi$ and $b = -3$ $\Rightarrow$ $\sin^{-1}\left(\sin\frac{a}{b}\right) = \sin^{-1}\left(\sin\frac{2\pi}{-3}\right)$

$$= \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3}$$

For $x \in \left(\frac{1}{2}, 1\right]$, $\cos^{-1}(4x^3 - 3x) = 3\cos^{-1}x \Rightarrow \quad a = 0$ and $b = 3$

$\therefore \quad \lim\limits_{y\to a} b\cos(y) = \lim\limits_{y\to 0} 3\cos(y) = 3$

Matrix-Match Type

1. a → p; b → q, s; c → r, s; d → r, s

a. $\cos^{-1}(4x^3 - 3x) = 3\cos^{-1}x$

$0 \le \cos^{-1}(4x^3 - 3x) \le \pi$

$\Rightarrow \quad 0 \le 3\cos^{-1}x \le \pi \quad \Rightarrow \quad 0 \le \cos^{-1}x \le (\pi/3) \quad \Rightarrow \quad (1/2) \le x \le 1$

b. $\sin^{-1}(3x - 4x^3) = 3\sin^{-1}x$

$(-\pi/2) \le \sin^{-1}(3x - 4x^3) \le (\pi/2)$

$\Rightarrow \quad (-\pi/2) \le 3\sin^{-1}x \le (\pi/2)$

$\Rightarrow \quad (-\pi/6) \le \sin^{-1}x \le (\pi/6)$

$\Rightarrow \quad (-1/2) \le x \le (1/2)$

c. $\cos^{-1}(4x^3 - 3x) = 3\sin^{-1}x$

$0 \le \cos^{-1}(3x - 4x^3) \le \pi$

$\Rightarrow \quad 0 \le 3\sin^{-1}x \le \pi$

$\Rightarrow \quad 0 \le \sin^{-1}x \le \pi/3$

$\Rightarrow \quad 0 \le x \le (\sqrt{3}/2)$

d. $\sin^{-1}(3x - 4x^3) = 3\cos^{-1}x$

$(-\pi/2) \le \sin^{-1}(3x - 4x^3) \le (\pi/2)$

$\Rightarrow \quad (-\pi/2) \le 3\cos^{-1}x \le (\pi/2)$

$\Rightarrow \quad (-\pi/6) \le \cos^{-1}x \le (\pi/6)$

$\Rightarrow \quad 0 \le \cos^{-1}x \le (\pi/6)$

$\Rightarrow \quad 0 \le x \le (\sqrt{3}/2)$

2. a. → q, r, s; b → q; c → r, s; d → p.

a. $(\sin^{-1}x)^2 + (\sin^{-1}y)^2 = \dfrac{\pi^2}{2}$

$\Rightarrow \quad (\sin^{-1}x)^2 = (\sin^{-1}y)^2 = \dfrac{\pi^2}{4}$

$\Rightarrow \quad \sin^{-1}x = \pm\dfrac{\pi}{2}, \sin^{-1}y = \pm\dfrac{\pi}{2}$

$\Rightarrow \quad x = \pm 1$ and $y = \pm 1$

$\Rightarrow \quad x^3 + y^3 = -2, 0, 2$

b. $(\cos^{-1}x)^2 + (\cos^{-1}y)^2 = 2\pi^2$

$\Rightarrow \quad (\cos^{-1}x)^2 = (\cos^{-1}y)^2 = \pi$

$\Rightarrow \quad x = y = -1$

$\Rightarrow \quad x^5 + y^5 = -2$

c. $(\sin^{-1} x)^2 (\cos^{-1} y)^2 = \dfrac{\pi^4}{4}$

$\Rightarrow \quad (\sin^{-1} x)^2 = \dfrac{\pi^2}{4}$ and $(\cos^{-1} y)^2 = \pi^2$

$\Rightarrow \quad (\sin^{-1} x) = \pm\dfrac{\pi}{2}$ and $(\cos^{-1} y) = \pi$

$\Rightarrow \quad x = \pm 1$ and $y = -1$

$\Rightarrow \quad -|x - y| = 0, 2$

d. $|\sin^{-1} x - \sin^{-1} y| = \pi$

$\Rightarrow \quad \sin^{-1} x = -\dfrac{\pi}{2}$ and $\sin^{-1} y = \dfrac{\pi}{2}$

or $\sin^{-1} x = \dfrac{\pi}{2}$ and $\sin^{-1} y = -\dfrac{\pi}{2}$

$\Rightarrow \quad x^y = 1^{(-1)}$ or $(-1)^1 = 1$ or -1

3. **a → p, q; b → q; c → q, r, s; d → p, r**

Refer the graphs of $y = \sin^{-1}(\sin x)$, $y = \cos^{-1}(\cos x)$, $y = \tan^{-1}(\tan x)$ and $y = \cot^{-1}(\cot x)$.

4. **a → q; b → s; c → p; d → r**

a. $\sin^{-1}\dfrac{4}{5} = \tan^{-1}\dfrac{4}{3}$

$2\tan^{-1}\dfrac{1}{3} = \tan^{-1}\dfrac{3}{4} = \cot^{-1}\dfrac{4}{3}$

and $\tan^{-1} x + \cot^{-1} x = \dfrac{\pi}{2}$

b. $\tan^{-1}\dfrac{12}{5} + \tan^{-1}\dfrac{3}{4} + \tan^{-1}\dfrac{63}{16} = \pi + \tan^{-1}\dfrac{48+15}{20-36} + \tan^{-1}\dfrac{63}{16}$

$$= \pi - \tan^{-1}\frac{63}{16} + \tan^{-1}\frac{63}{16} = \pi$$

c. $A = \tan^{-1}\dfrac{x\sqrt{3}}{2\lambda - x}$ and $B = \tan^{-1}\left(\dfrac{2x - \lambda}{\lambda\sqrt{3}}\right)$

Now, $\tan(A - B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

$$= \frac{\dfrac{x\sqrt{3}}{2\lambda - x} - \dfrac{2x - \lambda}{\lambda\sqrt{3}}}{1 + \dfrac{x\sqrt{3}}{2\lambda - x}\dfrac{2x - \lambda}{\lambda\sqrt{3}}}$$

$$= \frac{3x\lambda + (2x - \lambda)(x - 2\lambda)}{\sqrt{3}\left[\lambda(2\lambda - x) + x(2x - \lambda)\right]}$$

$$= \frac{1}{\sqrt{3}}\left[\frac{2x^2 - 2\lambda x + 2\lambda^2}{2x^2 - 2\lambda x + 2\lambda^2}\right] = \frac{1}{\sqrt{3}} = \tan 30^\circ$$

$\therefore \quad A - B = 30^\circ$

d. $\tan^{-1}\frac{1}{7}+2\tan^{-1}\frac{1}{3}=\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{2\frac{1}{3}}{1-\frac{1}{9}}=\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{3}{4}$

$$=\tan^{-1}\frac{\frac{1}{7}+\frac{3}{4}}{1-\frac{1}{7}\frac{3}{4}}=\tan^{-1}1=\pi/4$$

5. a $\rightarrow$ s; b $\rightarrow$ p; c $\rightarrow$ q; d $\rightarrow$ r

a. $f(x)=\sin^{-1}x+\cos^{-1}x+\cot^{-1}x$

$$=\frac{\pi}{2}+\cot^{-1}x, x\in[-1,1]$$

For $x\in[-1,1]$, $\cot^{-1}x\in\left[\frac{\pi}{4},\frac{3\pi}{4}\right] \Rightarrow \frac{\pi}{2}+\cot^{-1}x\in\left[\frac{3\pi}{4},\frac{5\pi}{4}\right]$

b. $f(x)=\cot^{-1}x+\tan^{-1}x+\operatorname{cosec}^{-1}x$

$$=\frac{\pi}{2}+\operatorname{cosec}^{-1}x, \text{ where } x\in(-\infty,-1]\cup[1,\infty)$$

Now $\operatorname{cosec}^{-1}x\in\left[-\frac{\pi}{2},0\right)\cup\left(0,\frac{\pi}{2}\right] \Rightarrow \frac{\pi}{2}+\operatorname{cosec}^{-1}x\in\left[0,\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\pi\right]$

c. $f(x)=\cot^{-1}x+\tan^{-1}x+\cos^{-1}x$

$$=\frac{\pi}{2}+\cos^{-1}x, \text{ where } x\in[-1,1] \Rightarrow \frac{\pi}{2}+\cos^{-1}x\in\left[\frac{\pi}{2},\frac{3\pi}{2}\right]$$

d. $\sec^{-1}x+\operatorname{cosec}^{-1}x+\sin^{-1}x$, where $x\in\{-1,1\}$

$$=\frac{\pi}{2}+\sin^{-1}x, \text{ where } x\in\{-1,1\}$$

Hence, $f(x)\in\{0,\pi\}$.

6. a $\rightarrow$ q; b $\rightarrow$ r; c $\rightarrow$ p, r; d $\rightarrow$ q, r, s

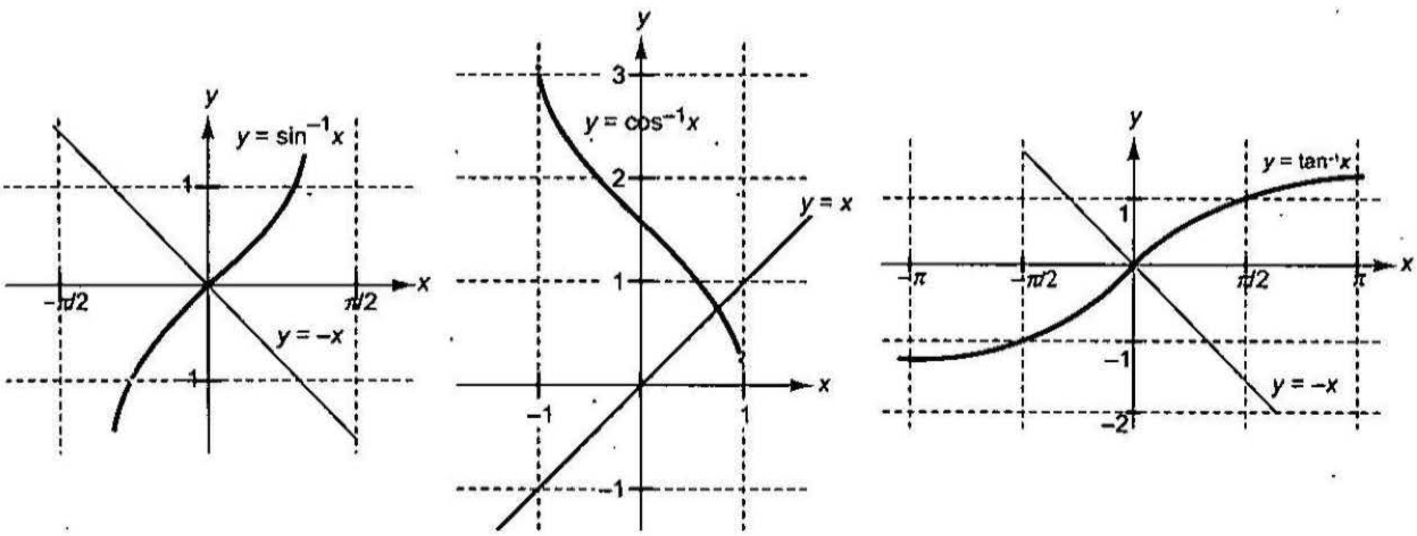


Fig. 4.32

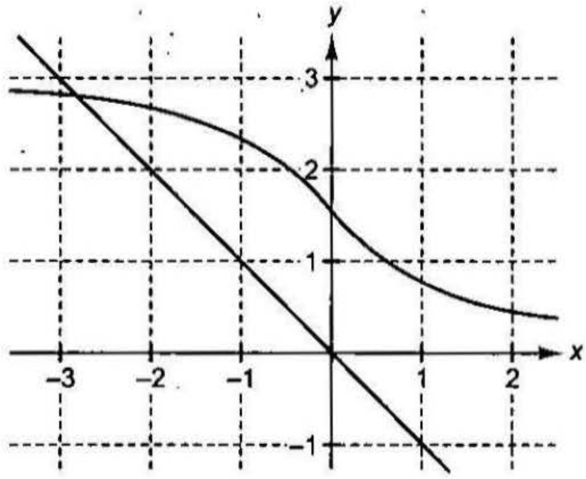


Fig. 4.33

Refer graph for solution.

Integer Type

1. (5) $(\cot^{-1}x)(\tan^{-1}x)+\left(2-\frac{\pi}{2}\right)\cot^{-1}x-3\tan^{-1}x-3\left(2-\frac{\pi}{2}\right)>0$

$\Rightarrow \cot^{-1}x>0$

$\Rightarrow (\cot^{-1}x-3)(2-\cot^{-1}x)>0$

$\Rightarrow (\cot^{-1}x-3)(\cot^{-1}x-2)<0$

$\Rightarrow 2<\cot^{-1}x<3$

$\Rightarrow \cot 3<x<\cot 2$ [as $\cot^{-1}x$ is a decreasing function]

$\Rightarrow$ Hence, $x\in(\cot 3, \cot 2)$

$\Rightarrow \cot^{-1}a+\cot^{-1}b=\cot^{-1}(\cot 3)+\cot^{-1}(\cot 2)=5$

2. (2) Since $\sin^{-1}$ is defined for $[-1, 1]$

$\therefore a=0$

$\therefore x=\sin^{-1}1+\cos^{-1}1-\tan^{-1}1=\frac{\pi}{4}$

$\Rightarrow \sec^2x=2$

3. (6) Let $\tan^{-1}u=\alpha\Rightarrow\tan\alpha=u$

$\tan^{-1}v=\beta\Rightarrow\tan\beta=v$

$\tan^{-1}w=\gamma\Rightarrow\tan\gamma=w$

$$\tan(\alpha+\beta+\gamma)=\frac{s_1-s_3}{1-s_2}=\frac{0-(-11)}{1-(-10)}=\frac{11}{11}=1$$

$\therefore \alpha+\beta+\gamma=\tan^{-1}(1)=\frac{\pi}{4}$

$\Rightarrow 3\operatorname{cosec}^2(\tan^{-1}u+\tan^{-1}v+\tan^{-1}w)=6$

4. (3) $\sin^{-1}\left(x^2-\frac{x^4}{3}+\frac{x^6}{9}-\cdots\right)+\cos^{-1}\left(x^4-\frac{x^8}{3}+\frac{x^{12}}{9}\cdots\right)=\frac{\pi}{2}$

$\Rightarrow \left(x^2-\frac{x^4}{3}+\frac{x^6}{9}\cdots\cdots\right)=\left(x^4-\frac{x^8}{3}+\frac{x^{12}}{9}\cdots\cdots\right)$

$$\Rightarrow \frac{x^2}{1+\frac{x^2}{3}} = \frac{x^4}{1+\frac{x^4}{3}}$$

$$\Rightarrow \frac{3}{3+x^2} = \frac{3x^2}{3+x^4} \text{ or } x=0$$

$$\Rightarrow 9+3x^4 = 9x^2+3x^4 \text{ or } x=0$$

$$\Rightarrow x^2=1 \Rightarrow x=0, 1 \text{ or } -1$$

Therefore, the number of values is equal to 3.

5. (7) $f(x)=\sqrt{3\cos^{-1}(4x)-\pi}$ is defined

If $\cos^{-1}4x \ge \frac{\pi}{3} \Rightarrow 4x \le \frac{1}{2} \Rightarrow x \le \frac{1}{8}$ (i)

Also, $-1 \le 4x \le 1 \Rightarrow \frac{-1}{4} \le x \le \frac{1}{4}$ (ii)

Therefore, from Eqs. (i) and (ii), we have domain: $x \in \left[\frac{-1}{4}, \frac{1}{8}\right]$

$\Rightarrow 4a+64b=7$

6. (9) $1+\sin(\cos^{-1}x)+\sin^2(\cos^{-1}x)+\cdots\infty=2$

$$\Rightarrow \frac{1}{1-\sin(\cos^{-1}x)}=2$$

$$\Rightarrow \frac{1}{2}=1-\sin(\cos^{-1}x)$$

$$\Rightarrow \sin(\cos^{-1}x)=\frac{1}{2} \Rightarrow \cos^{-1}x=\frac{\pi}{6}$$

$$\Rightarrow x=\frac{\sqrt{3}}{2} \Rightarrow 12x^2=9$$

7. (9) $\tan^{-1}\left(x+\frac{3}{x}\right)-\tan^{-1}\left(x-\frac{3}{x}\right)=\tan^{-1}\frac{6}{x}$

$$\Rightarrow \tan^{-1}\left(\frac{\left(x+\frac{3}{x}\right)-\left(x-\frac{3}{x}\right)}{1+\left(x+\frac{3}{x}\right)\left(x-\frac{3}{x}\right)}\right)=\tan^{-1}\frac{6}{x}$$

$$\Rightarrow x^2-\frac{9}{x^2}=0 \Rightarrow x^4=9$$

8. (4) $f(x)=\sin^{-1}x+2\tan^{-1}x+(x+2)^2-3$

Domain of $f(x)$ is $[-1, 1]$.

Also $f(x)$ is an increasing function in the domain

$$\therefore p = f_{\min.}(x) = f(-1) = -\frac{\pi}{2} + 2\left(\frac{-\pi}{4}\right) + 1 - 3 = -\pi - 2$$

and $q = f_{\max.}(x) = f(1) = \frac{\pi}{2} + 2\left(\frac{\pi}{4}\right) + 9 - 6 = \pi + 6.$

Therefore, the range of $f(x)$ is $[-\pi - 2, \pi + 6]$.

Hence, $(p + q) = 4$.

9. (6) $T_n = \tan^{-1}\left(\frac{n+1-1}{1+(n+1)1}\right)$

$$= \tan^{-1}(n+1) - \tan^{-1}(n)$$

Hence, $S_n = \tan^{-1}(n+1) - \tan^{-1} 1$

$$= \tan^{-1}\left(\frac{n+1-1}{1+(n+1)\cdot 1}\right) = \left(\tan^{-1}\frac{n}{n+2}\right) = \frac{1}{2}\cos^{-1}\left(\frac{24}{145}\right)$$

$$\Rightarrow 2\left(\tan^{-1}\frac{n}{n+2}\right) = \cos^{-1}\left(\frac{24}{145}\right)$$

$$\Rightarrow \cos^{-1}\left(\frac{2(n+1)}{n^2+2n+2}\right) = \cos^{-1}\left(\frac{24}{145}\right)$$

$$\Rightarrow \left(\frac{2(n+1)}{n^2+2n+2}\right) = \left(\frac{24}{145}\right)$$

$$\Rightarrow 12(n+1)^2 - 145(n+1) + 12 = 0$$

$$\Rightarrow ((n+1) - 12)(12(n+1) - 1) = 0$$

$$\Rightarrow n + 1 = 12 \Rightarrow n = 11$$

10. (1) We have $g(x) = \sin^{-1}(\cos x) = \frac{\pi}{2} - \cos^{-1}(\cos x)$

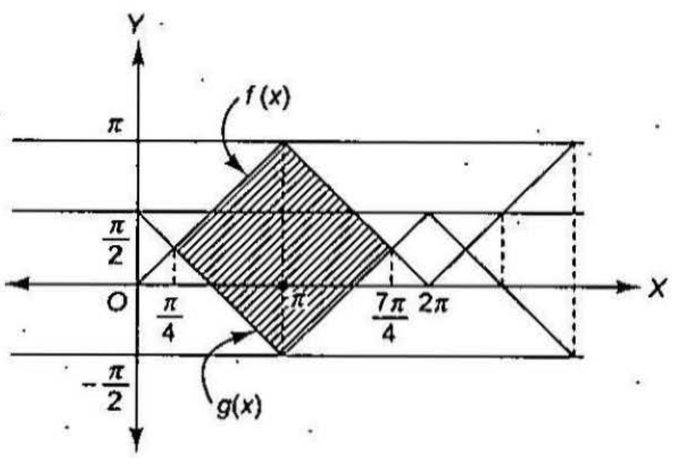


Fig. 4.34

Both the curves bound the regions of same area

in $\left[\frac{\pi}{4}, \frac{7\pi}{4}\right], \left[\frac{9\pi}{4}, \frac{15\pi}{4}\right]$ and so on

Therefore, the required area = area of shaded square $= \frac{9\pi^2}{8} = \frac{a\pi^2}{b}$

$\therefore\ a = 9$ and $b = 8 \Rightarrow a - b = 1$

11. (3) We must have $x(x+3) \geq 0$

$\Rightarrow x \geq 0$ or $x \leq -3$ (i)

Also, $-1 \leq x^2 + 3x + 1 \leq 1$

$\Rightarrow x(x+3) \leq 0 \Rightarrow -3 \leq x \leq 0$ (ii)

From Eqs. (i) and (ii), we get $x = \{0, -3\}$

Hence, required sum is 3.

12. (1) Given expression is defined only for $x = 1$ and -1

$\therefore f(1) = 1$ and $f(-1) = (1+\pi)(1+\pi) = (1+\pi)^2$

Hence, the least value is 1.

13. (3) $\cos^{-1}(x) + \cos^{-1}(2x) + \cos^{-1}(3x) = \pi$

$\Rightarrow \cos^{-1}(2x) + \cos^{-1}(3x) = \pi - \cos^{-1}(x) = \cos^{-1}(-x)$

$\Rightarrow \cos^{-1}[(2x)(3x) - \sqrt{1-4x^2}\sqrt{1-9x^2}\,] = \cos^{-1}(-x)$

$\Rightarrow 6x^2 - \sqrt{1-4x^2}\sqrt{1-9x^2} = -x$

$\Rightarrow (6x^2 + x)^2 = (1-4x^2)(1-9x^2)$

$\Rightarrow x^2 + 12x^3 = 1 - 13x^2$

$\Rightarrow 12x^3 + 14x^2 - 1 = 0$

$\Rightarrow a = 12;\ b = 14; c = 0$

$\Rightarrow b - a - c = 14 - 12 + 1 = 3$

14. (1) $\tan^{-1}(3x) + \tan^{-1}(5x) = \tan^{-1}(7x) + \tan^{-1}(2x)$

$\Rightarrow \tan^{-1}(3x) - \tan^{-1}(2x) = \tan^{-1}(7x) - \tan^{-1}(5x)$

$\Rightarrow \tan^{-1}\left(\frac{3x-2x}{1+6x^2}\right) = \tan^{-1}\left(\frac{7x-5x}{1+35x^2}\right)$

$\Rightarrow \frac{x}{1+6x^2} = \frac{2x}{1+35x^2}$

$\Rightarrow x = 0$ or $1 + 35x^2 = 2 + 12x^2$

$\Rightarrow x = 0$ or $x = \frac{1}{\sqrt{23}}$ or $-\frac{1}{\sqrt{23}}$

Archives

Subjective

1. $$\cos(2\cos^{-1}x+\sin^{-1}x)=\cos\left(\cos^{-1}x+\frac{\pi}{2}\right)$$

$$=-\sin(\cos^{-1}x)=-\sin\left(\sin^{-1}\sqrt{1-x^2}\right)=-\sqrt{1-x^2}$$

At $x=\frac{1}{5}$, value $=-\sqrt{1-\frac{1}{25}}=-\sqrt{\frac{24}{25}}=-\frac{2\sqrt{6}}{5}$

2. L.H.S. $=\cos(\tan^{-1}(\sin(\cot^{-1}x)))$

$$=\cos\left(\tan^{-1}\left(\sin\left(\sin^{-1}\frac{1}{\sqrt{1+x^2}}\right)\right)\right)\text{ if } x>0$$

and $\cos\left(\tan^{-1}\left(\sin\left(\pi-\sin^{-1}\frac{1}{\sqrt{1+x^2}}\right)\right)\right)$ if $x<0$

In each case, L.H.S. $=\cos\left(\tan^{-1}\frac{1}{\sqrt{1+x^2}}\right)=\cos\left(\cos^{-1}\sqrt{\frac{1+x^2}{2+x^2}}\right)=\sqrt{\frac{x^2+1}{x^2+2}}$

Objective

Fill in the blanks

1. $$\theta=\tan^{-1}\sqrt{\frac{a(a+b+c)}{bc}}+\tan^{-1}\sqrt{\frac{b(a+b+c)}{ca}}+\tan^{-1}\sqrt{\frac{c(a+b+c)}{ab}}$$

$$\sqrt{\frac{a(a+b+c)}{bc}}\sqrt{\frac{b(a+b+c)}{ca}}=\frac{a+b+c}{c}=1+\frac{b}{c}+\frac{a}{c}>1$$

$$\Rightarrow\quad\theta=\pi+\tan^{-1}\frac{\sqrt{\frac{a(a+b+c)}{bc}}+\sqrt{\frac{b(a+b+c)}{ca}}}{1-\sqrt{\frac{a(a+b+c)}{bc}}+\sqrt{\frac{b(a+b+c)}{ca}}}+\tan^{-1}\sqrt{\frac{c(a+b+c)}{ab}}$$

$$=\pi+\tan^{-1}\frac{\sqrt{\frac{a+b+c}{c}}\left(\sqrt{\frac{a}{b}}+\sqrt{\frac{b}{a}}\right)}{1-\frac{a+b+c}{c}}+\tan^{-1}\sqrt{\frac{c(a+b+c)}{ab}}$$

$$=\pi+\tan^{-1}\left(-\sqrt{\frac{c(a+b+c)}{ab}}\right)+\tan^{-1}\sqrt{\frac{c(a+b+c)}{ab}}=\pi\quad\Rightarrow\quad\tan\theta=0$$

2. $$\tan\left(2\tan^{-1}\frac{1}{5}-\frac{\pi}{4}\right)=\tan\left(\tan^{-1}\left(\frac{2/5}{1-(1/5)^2}\right)-\tan^{-1}(1)\right)$$

$$= \tan\left(\tan^{-1}\left(\frac{5}{12}\right) - \tan^{-1}(1)\right)$$

$$= \tan\left(\tan^{-1}\left(\frac{(5/12)-1}{1+(5/12)}\right)\right)$$

$$= \tan(\tan^{-1}(-7/17)) = -7/17$$

3. We have,

$$A = 2\tan^{-1}(2\sqrt{2}-1)$$

$$= 2\tan^{-1}(2\times 1.414 - 1)$$

$$= 2\tan^{-1}(1.828) > 2\tan^{-1}\sqrt{3} = 2\pi/3 \quad \Rightarrow \quad A > (2\pi/3) \qquad \text{(i)}$$

Also, $B = 3\sin^{-1}(1/3) + \sin^{-1}(3/5) = \sin^{-1}\left[3\times\frac{1}{3} - 4\times\frac{1}{27}\right] + \sin^{-1}(3/5)$

$$= \sin^{-1}\left(\frac{23}{27}\right) + \sin^{-1}(0.6) = \sin^{-1}(0.852) + \sin^{-1}(0.6) < \sin^{-1}(\sqrt{3}/2) + \sin^{-1}(\sqrt{3}/2) = 2\pi/3$$

$$\Rightarrow \quad B < (2\pi/3) \qquad \text{(ii)}$$

From Eqs. (i) and (ii), we conclude $A > B$.

Multiple choice questions with one correct answer

1. d. $\tan\left(\cos^{-1}\left(\frac{4}{5}\right) + \tan^{-1}\left(\frac{2}{3}\right)\right) = \tan\left(\tan^{-1}\frac{3}{4} + \tan^{-1}\frac{2}{3}\right) = \tan\left(\tan^{-1}\left(\frac{(3/4)+(2/3)}{1-(3/4)\times(2/3)}\right)\right)$

$$= \frac{17}{12}\times\frac{12}{6} = \frac{17}{6}$$

2. e. The principal value of $\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$ = principal value of $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \pi/3$

3. d. $\tan\left(\cos^{-1}\frac{1}{5\sqrt{2}} - \sin^{-1}\frac{4}{\sqrt{17}}\right) = \tan[\tan^{-1}7 - \tan^{-1}4] = \tan\left(\tan^{-1}\left(\frac{3}{29}\right)\right) = \frac{3}{29}$

4. c. $\tan^{-1}\sqrt{[x(x+1)]} = (\pi/2) - \sin^{-1}\sqrt{(x^2+x+1)} = \cos^{-1}\sqrt{x^2+x+1} = \tan^{-1}\frac{\sqrt{-x^2-x}}{\sqrt{x^2+x+1}}$

$$\Rightarrow \quad \sqrt{x(x+1)} = \frac{\sqrt{-x^2-x}}{\sqrt{x^2+x+1}} \Rightarrow \quad x = 0, -1 \text{ are the only real solutions.}$$

5. b. $\sin^{-1}\left(x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots\right) + \cos^{-1}\left(x^2 - \frac{x^4}{2} + \frac{x^6}{4} + \cdots\right) = \frac{\pi}{2}$

$$\Rightarrow \quad \cos^{-1}\left(x^2 - \frac{x^4}{2} + \frac{x^6}{4}\cdots\right) = \frac{\pi}{2} - \sin^{-1}\left(x - \frac{x^2}{2} + \frac{x^3}{4}\cdots\right)$$

$$= \cos^{-1}\left(x - \frac{x^2}{2} + \frac{x^3}{4}\cdots\right)$$

$\Rightarrow \quad x^2-\frac{x^4}{2}+\frac{x^6}{4}\cdots=x-\frac{x^2}{2}+\frac{x^3}{4}\cdots$

We have G. P. of infinite terms on both sides.

$\therefore \quad \frac{x}{1-\left(-\frac{x}{2}\right)}=\frac{x^2}{1-\left(\frac{-x^2}{2}\right)} \quad \Rightarrow \quad \frac{2x}{2+x}=\frac{2x^2}{2+x^2} \quad \Rightarrow \quad 2x+x^3=2x^2+x^3$

$\Rightarrow \quad x(x-1)=0 \quad \Rightarrow \quad x=0, 1$ but $0<|x|<\sqrt{2} \quad \Rightarrow \quad x=1$

6. a. For $f(x)=\sqrt{\sin^{-1}(2x)+\frac{\pi}{6}}$ to be defined and real $\sin^{-1} 2x+(\pi/6)\geq 0$

$\Rightarrow \quad \sin^{-1} 2x \geq -\frac{\pi}{6}$ (i)

But we know that $-\pi/2\leq \sin^{-1} 2x \leq \pi/2$ (ii)

Combining Eqs. (i) and (ii), we get

$-\pi/6 \leq \sin^{-1} 2x \leq \pi/2$

$\Rightarrow \quad \sin(-\pi/6)\leq 2x \leq \sin(\pi/2) \quad \Rightarrow \quad -1/2\leq 2x\leq 1$

$\Rightarrow \quad -1/4\leq x\leq 1/2$

$\therefore \quad D_f=\left[-\frac{1}{4},\frac{1}{2}\right]$

7. d. $\sin\left[\cot^{-1}(x+1)\right]=\sin\left(\sin^{-1}\frac{1}{\sqrt{x^2+2x+2}}\right)=\frac{1}{\sqrt{x^2+2x+2}}$

$\cos\left(\tan^{-1}x\right)=\cos\left(\cos^{-1}\frac{1}{\sqrt{1+x^2}}\right)=\frac{1}{\sqrt{1+x^2}}$

Thus, $\frac{1}{\sqrt{x^2+2x+2}}=\frac{1}{\sqrt{1+x^2}} \quad \Rightarrow \quad x^2+2x+2=1+x^2 \Rightarrow x=-\frac{1}{2}$

Match the following type

1. a → p; b → q; c → p; d → s

$\sin^{-1}(ax)+\cos^{-1}y+\cos^{-1}(bxy)=\frac{\pi}{2}$

$\Rightarrow \quad \cos^{-1}y+\cos^{-1}(bxy)=\frac{\pi}{2}-\sin^{-1}(ax)=\cos^{-1}(ax)$

Let $\cos^{-1}y=\alpha$, $\cos^{-1}(bxy)=\beta$, $\cos^{-1}(ax)=\gamma$

$\Rightarrow \quad y=\cos\alpha,\ bxy=\cos\beta,\ ax=\cos\gamma$

Therefore, we get $\alpha+\beta=\gamma$. $\quad \Rightarrow \quad \cos(\gamma-\alpha)=bxy \quad \Rightarrow \quad \cos\gamma\cos\alpha+\sin\gamma\sin\alpha=bxy$

$\Rightarrow \quad (a-b)\,xy=-\sin\alpha\sin\gamma \quad \Rightarrow \quad (a-b)^2x^2y^2=\sin^2\alpha\sin^2\gamma=(1-\cos^2\alpha)(1-\cos^2\gamma)$

$$\Rightarrow \quad (a-b)^2 x^2y^2 = (1-a^2x^2)(1-y^2) \tag{i}$$

a. For $a = 1$, $b = 0$, Eq. (i) reduces to

$x^2 y^2 = (1-x^2)(1-y^2) \Rightarrow x^2 + y^2 = 1$

b. For $a = 1$, $b = 1$, Eq. (i) becomes $(1-x^2)(1-y^2) = 0$

$\Rightarrow \quad (x^2-1)(y^2-1) = 0$

c. For $a = 1$, $b = 2$, Eq. (i) reduces to

$x^2y^2 = (1-x^2)(1-y^2)$

$\Rightarrow \quad x^2 + y^2 = 1$

d. For $a = 2$, $b = 2$, Eq. (i) reduces to $0 = (1-4x^2)(1-y^2)$

$\Rightarrow \quad (4x^2-1)(y^2-1) = 0$

CHAPTER

5

Solutions and Properties of Triangle

- Standard Symbols
- Sine Rule
- Cosine Rule
- Projection Formula
- Half-Angle Formulae
- Area of Triangle
- Escribed Circles of A Triangle And Their Radii
- Miscellaneous Topics
- Solution of Triangles (Ambiguous Cases)

STANDARD SYMBOLS

The following symbols in relation to ΔABC are universally adopted.

- $m\angle BAC = A$
  $m\angle ABC = B$
  $m\angle BCA = C$
  $A + B + C = \pi$
  $AB = c, BC = a, CA = b$
- Semi-perimeter of the triangle, $s = \dfrac{a+b+c}{2}$
  So, $a + b + c = 2s$

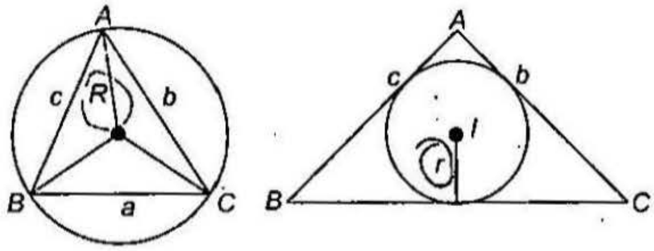


Fig. 5.1

- The radius of the circumcircle of the triangle, i.e., circumradius = R
- The radius of the incircle of the triangle, i.e., inradius = r
- Area of the triangle = Δ

SINE RULE

The sine rule is $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = 2R$

We shall prove here that $\dfrac{a}{\sin A} = 2R$.

Case I:

$0 < A < \dfrac{\pi}{2}$

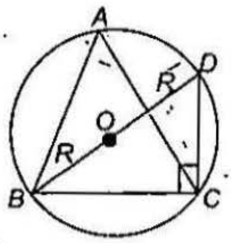


Fig. 5.2

Suppose O is the circumcentre of ΔABC.

$\overline{BO}$ intersects the circumcircle at D.

Here, $BD = 2OB = 2R$ and $\angle D = \angle BDC \cong \angle A$ [angles in the same segment] (i)

Now in ΔBCD, $m\angle BCD = \dfrac{\pi}{2}$ [angle in a semicircle]

$\Rightarrow \sin D = \dfrac{BC}{BD} = \dfrac{a}{2R}$

$\Rightarrow \quad \sin A = \dfrac{a}{2R}$ [using Eq. (i)]

$\Rightarrow \quad \dfrac{a}{\sin A} = 2R$

Case II:

ΔABC is right angled and $A = \dfrac{\pi}{2}$

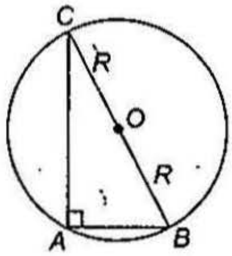


Fig. 5.3

$\overline{BC}$ is a diameter of the circumcircle.

$\therefore \quad BC = 2R$

Now, $a = BC = 2R = 2R\sin\dfrac{\pi}{2} = 2R\sin A$

$\therefore \quad \dfrac{a}{\sin A} = 2R$

Case III:

$\dfrac{\pi}{2} < A < \pi$

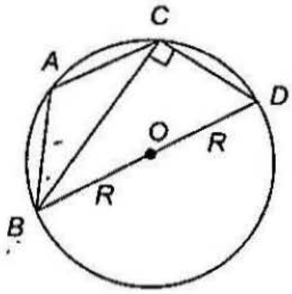


Fig. 5.4

As $\angle A$ is obtuse, so A is on the minor arc BC.
Now, take any point D on the major arc BC.

Here, $m\angle BDC = \pi - A < \dfrac{\pi}{2}$ $\quad \left[\dfrac{\pi}{2} < A < \pi\right]$ (ii)

Now in ΔBCD, $\sin(\angle BDC) = \dfrac{BC}{BD}$

$\Rightarrow \quad \sin(\pi - A) = \dfrac{a}{2R}$ [using Eq. (ii)]

$\Rightarrow \quad \sin A = \dfrac{a}{2R}$

$\Rightarrow \quad a = 2R\sin A$

Thus, in each case, $\frac{a}{\sin A} = 2R$.

Similarly, we can prove that $\frac{b}{\sin B} = 2R$ and $\frac{c}{\sin C} = 2R$.

$$\therefore \quad \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

Nepier's Formula

i. $\tan\left(\frac{B-C}{2}\right) = \frac{b-c}{b+c}\cot\frac{A}{2}$

ii. $\tan\left(\frac{C-A}{2}\right) = \frac{c-a}{c+a}\cot\frac{B}{2}$

iii. $\tan\left(\frac{A-B}{2}\right) = \frac{a-b}{a+b}\cot\frac{C}{2}$

Proof:

i. From the sine rule, we have $\frac{b}{\sin B} = \frac{c}{\sin C}$

$$\Rightarrow \quad \frac{\sin B}{\sin C} = \frac{b}{c}$$

$$\Rightarrow \quad \frac{\sin B - \sin C}{\sin B + \sin C} = \frac{b-c}{b+c}$$

$$\Rightarrow \quad \frac{2\cos\left(\frac{B+C}{2}\right)\sin\left(\frac{B-C}{2}\right)}{2\sin\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right)} = \frac{b-c}{b+c}$$

$$\Rightarrow \quad \cot\left(\frac{B+C}{2}\right)\tan\left(\frac{B-C}{2}\right) = \frac{b-c}{b+c}$$

$$\Rightarrow \quad \tan\frac{A}{2}\tan\left(\frac{B-C}{2}\right) = \frac{b-c}{b+c}$$

$$\Rightarrow \quad \frac{\tan\left(\frac{B-C}{2}\right)}{\cot\frac{A}{2}} = \frac{b-c}{b+c}$$

$$\Rightarrow \quad \tan\left(\frac{B-C}{2}\right) = \frac{b-c}{b+c}\cot\frac{A}{2}$$

Similarly, we can prove the other formulae.

Note:

These formulae are also known as tangent rules. These are useful in calculating the remaining parts of a triangle when two sides and included angle are given.

Example 5.1 **The perimeter of a triangle ABC is 6 times the arithmetic mean of the sines of its angles. If the side a is 1, then find angle A.**

Sol.

Given that $a+b+c=6\times\dfrac{\sin A+\sin B+\sin C}{3}$

$\Rightarrow\ 2R(\sin A+\sin B+\sin C)=2(\sin A+\sin B+\sin C)\quad \Rightarrow R=1$

Now, $\dfrac{a}{\sin A}=2R$

$\Rightarrow\ \sin A=\dfrac{1}{2}\ (\because a=1)\quad \Rightarrow A=30°$

Example 5.2 **If $\dfrac{\cos A}{a}=\dfrac{\cos B}{b}=\dfrac{\cos C}{c}$ and the side $a=2$, then find the area of the triangle.**

Sol. $\dfrac{\cos A}{a}=\dfrac{\cos B}{b}=\dfrac{\cos C}{c}$

$\Rightarrow\ \dfrac{\cos A}{2R\sin A}=\dfrac{\cos B}{2R\sin B}=\dfrac{\cos C}{2R\sin C}$

$\Rightarrow\ \tan A=\tan B=\tan C$

$\Rightarrow\ \Delta$ is equilateral

$\Rightarrow\ \text{Area }\Delta=\dfrac{\sqrt{3}}{4}a^2=\sqrt{3}\ (\text{as } a=2)$

Example 5.3 **If $A=75°$, $B=45°$, then prove that $b+c\sqrt{2}=2a$.**

Sol. $A=75°, B=45°\Rightarrow C=60°$

$\dfrac{a}{\sin A}=\dfrac{b}{\sin B}=\dfrac{c}{\sin C}=2R$

$\Rightarrow\ \dfrac{a}{\sin 75°}=\dfrac{b}{\sin 45°}=\dfrac{c}{\sin 60°}=2R$

$\Rightarrow b+c\sqrt{2}=\dfrac{\sin 45°}{\sin 75°}a+\sqrt{2}\dfrac{\sin 60°}{\sin 75°}a$

$$=\frac{\frac{1}{\sqrt{2}}}{\frac{\sqrt{3}+1}{2\sqrt{2}}}a+\sqrt{2}\frac{\frac{\sqrt{3}}{2}}{\frac{\sqrt{3}+1}{2\sqrt{2}}}a=\frac{2}{\sqrt{3}+1}a+\frac{2\sqrt{3}a}{\sqrt{3}+1}=2a$$

Example 5.4 **If the base angles of a triangle are $22\frac{1}{2}°$ and $112\frac{1}{2}°$, then prove that the altitude of the triangle is equal to $\frac{1}{2}$ of its base.**

Sol.

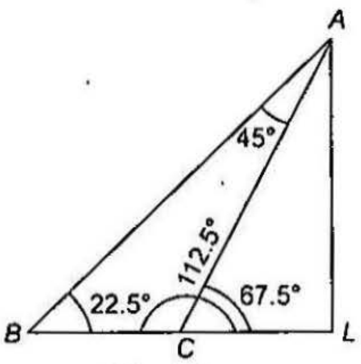


Fig. 5.5

In ΔABC, $\dfrac{BC}{\sin 45°} = \dfrac{AC}{\sin 22\frac{1}{2}°}$ (i)

In ΔALC, $\dfrac{AL}{AC} = \sin 67\frac{1}{2}°$

$\therefore \quad AL = AC \cos 22\frac{1}{2}°$

$\Rightarrow \quad AL = \dfrac{BC}{\sin 45°} \sin 22\frac{1}{2}° \cos 22\frac{1}{2}°$ [using Eq. (i)]

$= \dfrac{1}{2} BC \dfrac{\sin 45°}{\sin 45°}$

$\Rightarrow \quad AL = \dfrac{1}{2} BC = \dfrac{1}{2}$ of base.

Example 5.5 **Prove that** $\dfrac{\mathbf{1 + \cos(A-B)\cos C}}{\mathbf{1 + \cos(A-C)\cos B}} = \dfrac{\mathbf{a^2 + b^2}}{\mathbf{a^2 + c^2}}$.

Sol.

$$\frac{1 + \cos(A-B)\cos C}{1 + \cos(A-C)\cos B} = \frac{1 - \cos(A-B)\cos(A+B)}{1 - \cos(A-C)\cos(A+C)}$$

$$= \frac{1 - (\cos^2 A - \sin^2 B)}{1 - (\cos^2 A - \sin^2 C)} = \frac{\sin^2 A + \sin^2 B}{\sin^2 A + \sin^2 C}$$

$$= \frac{a^2 + b^2}{a^2 + c^2} \text{ [using } a = 2R \sin A, \text{ etc.]}$$

Example 5.6 **Prove that** $\dfrac{\mathbf{a^2 \sin(B-C)}}{\mathbf{\sin B + \sin C}} + \dfrac{\mathbf{b^2 \sin(C-A)}}{\mathbf{\sin C + \sin A}} + \dfrac{\mathbf{c^2 \sin(A-B)}}{\mathbf{\sin A + \sin B}} = \mathbf{0}$.

Sol.

$$\frac{a^2 \sin(B-C)}{\sin B + \sin C} = \frac{4R^2 \sin^2 A \sin(B-C)}{\sin B + \sin C}$$

$$= \frac{4R^2 \sin A \sin(B+C) \sin(B-C)}{\sin B + \sin C}$$

$$= \frac{4R^2 \sin A (\sin^2 B - \sin^2 C)}{\sin B + \sin C}$$

$$= 4R^2 \sin A (\sin B - \sin C)$$

Similarly,

$$\frac{b^2 \sin(C-A)}{\sin C + \sin A} = 4R^2 \sin B (\sin C - \sin A)$$

and $\dfrac{c^2 \sin(A-B)}{\sin A + \sin B} = 4R^2 \sin C(\sin A - \sin B)$

Adding, we get

$$\frac{a^2 \sin(B-C)}{\sin B + \sin C} + \frac{b^2 \sin(C-A)}{\sin C + \sin A} + \frac{c^2 \sin(A-B)}{\sin A + \sin B} = 0$$

Example 5.7 **In any triangle, if $\dfrac{a^2 - b^2}{a^2 + b^2} = \dfrac{\sin(A-B)}{\sin(A+B)}$, then prove that the triangle is either right angled or isosceles.**

Sol. $\dfrac{a^2 - b^2}{a^2 + b^2} = \dfrac{\sin(A-B)}{\sin(A+B)}$

$\Rightarrow \quad \dfrac{4R^2 \sin^2 A - 4R^2 \sin^2 B}{4R^2 \sin^2 A + 4R^2 \sin^2 B} = \dfrac{\sin(A-B)}{\sin(A+B)}$

$\Rightarrow \quad \dfrac{\sin(A+B)\sin(A-B)}{\sin^2 A + \sin^2 B} = \dfrac{\sin(A-B)}{\sin(A+B)}$

$\Rightarrow \quad \sin(A-B) = 0$ or $\dfrac{\sin(\pi - C)}{\sin^2 A + \sin^2 B} = \dfrac{1}{\sin(\pi - C)}$

$\Rightarrow \quad A = B$ or $\sin^2 C = \sin^2 A + \sin^2 B$

$\Rightarrow \quad A = B$ or $c^2 = a^2 + b^2$ [from the sine rule]

Therefore, the triangle is isosceles or right angled.

Example 5.8 **$ABCD$ is a trapezium such that $AB \parallel CD$ and CB is perpendicular to them. If $\angle ADB = \theta$, $BC = p$ and $CD = q$, show that $AB = \dfrac{(p^2 + q^2)\sin\theta}{p\cos\theta + q\sin\theta}$.**

Sol.

Let $\angle ABD = \angle BDC = \alpha$

$\therefore \quad \angle BAD = 180° - (\theta + \alpha)$

By the sine formula, in ΔABD, we have

$$\frac{AB}{\sin\theta} = \frac{BD}{\sin(180° - (\theta + \alpha))}$$

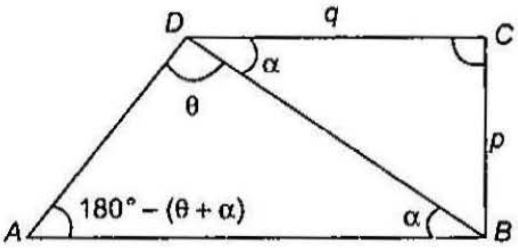


Fig. 5.6

$$\therefore \quad AB = \frac{BD \sin\theta}{\sin(\theta + \alpha)} = \frac{BD \sin\theta}{\sin\theta \cos\alpha + \cos\theta \sin\alpha} \quad \text{(i)}$$

In ΔBCD, $\sin\alpha = p/BD$
and $\cos\alpha = q/BD$. Also $BD^2 = p^2 + q^2$
Therefore, from Eq. (i), we have

$$AB = \frac{BD \sin\theta}{\sin\theta\,(q/BD) + \cos\theta\,(p/BD)} = \frac{BD^2 \sin\theta}{q\sin\theta + p\cos\theta} = \frac{(p^2+q^2)\sin\theta}{p\cos\theta + q\sin\theta}$$

Example 5.9 **In a triangle ABC, $\angle A = 60°$ and $b : c = \sqrt{3}+1 : 2$, then find the value of $(\angle B - \angle C)$.**

Sol.

$$\frac{b}{c} = \frac{\sqrt{3}+1}{2} \Rightarrow \frac{b-c}{b+c} = \frac{\sqrt{3}+1-2}{\sqrt{3}+1+2} = \frac{\sqrt{3}-1}{(\sqrt{3}+1)}\frac{1}{\sqrt{3}}$$

Now using $\tan\frac{B-C}{2} = \frac{b-c}{b+c}\cot\frac{A}{2}$, we get $\frac{\sqrt{3}-1}{(\sqrt{3}+1)}\frac{\sqrt{3}}{\sqrt{3}} = 2-\sqrt{3} \Rightarrow \frac{B-C}{2} = 15°$

$\therefore B - C = 30°$

Concept Application Exercise 5.1

1. Prove that $\dfrac{c}{a+b} = \dfrac{1-\tan\frac{1}{2}A\tan\frac{1}{2}B}{1+\tan\frac{1}{2}A\tan\frac{1}{2}B}$.
2. If the angles of triangle ABC are in the ratio 3:5:4, then find the value of $a + b + c\sqrt{2}$.
3. Prove that $a\cos A + b\cos B + c\cos C = 4R\sin A\sin B\sin C$.
4. Find the value of $\dfrac{a^2+b^2+c^2}{R^2}$ in any right-angled triangle.
5. In triangle ABC, if $\cos^2 A + \cos^2 B - \cos^2 C = 1$, then identify the type of the triangle.
6. If angles A, B and C of a triangle ABC are in A.P. and if $\dfrac{b}{c} = \dfrac{\sqrt{3}}{\sqrt{2}}$, then find angle A.
7. Prove that $b^2\cos 2A - a^2\cos 2B = b^2 - a^2$.

COSINE RULE

In a ΔABC, we have

$$\cos A = \frac{b^2+c^2-a^2}{2bc},$$

$$\cos B = \frac{c^2+a^2-b^2}{2ca}$$

and $$\cos C = \frac{a^2+b^2-c^2}{2ab}$$

We shall prove that $\cos A = \dfrac{b^2 + c^2 - a^2}{2ab}$

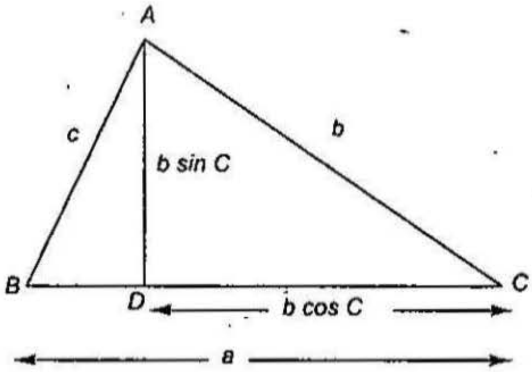


Fig. 5.7

From Fig. 5.7, $BD = a - b\cos C$. Now, in triangle ABD,

$AB^2 = AD^2 + BD^2$

$$\Rightarrow \quad c^2 = (b \sin C)^2 + (a - b\cos C)^2$$
$$= b^2 \sin^2 C + a^2 + b^2 \cos^2 C - 2ab \cos C$$
$$= (b^2 \sin^2 C + b^2 \cos^2 C) + a^2 - 2ab \cos C$$
$$= b^2 + a^2 - 2ab \cos C$$
$$\Rightarrow \quad \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

Note:

- *The above proof will not change even if $\angle A$ is a right angle or an obtuse angle.*
- *If the lengths of the three sides of a triangle are known, we can find all the angles by using cosine rule because this rule gives us* $\cos A$, $\cos B$ *and* $\cos C$. *We know that A, B and C are in $(0, \pi)$ and the* cosine *function is one-one in* $[0, \pi]$, So, *A, B and C are precisely determined. Similarly, if two sides (say b and c) and the included angle A are given, the* cosine *rule* $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$ *will give us a and then knowing a, b and c we can find B and C by the cosine rule.*

Example 5.10 **In ΔABC, prove that $(a-b)^2 \cos^2 \dfrac{C}{2} + (a+b)^2 \sin^2 \dfrac{C}{2} = c^2$.**

Sol. L.H.S.

$$= (a^2 + b^2 - 2ab)\cos^2 \frac{C}{2} + (a^2 + b^2 + 2\,ab)\sin^2 \frac{C}{2}$$
$$= a^2 + b^2 + 2ab\left(\sin^2 \frac{C}{2} - \cos^2 \frac{C}{2}\right)$$
$$= a^2 + b^2 - 2ab \cos C$$
$$= a^2 + b^2 - (a^2 + b^2 - c^2) = c^2$$

Example 5.11 **In ΔABC, if $(a + b + c)(a - b + c) = 3ac$, then find $\angle B$.**

Sol.

$(a+c)^2 - b^2 = 3ac \Rightarrow a^2 + c^2 - b^2 = ac$

But $\cos B = \dfrac{a^2 + c^2 - b^2}{2ac} = \dfrac{1}{2} \Rightarrow B = \dfrac{\pi}{3}$

Example 5.12 If $a = \sqrt{3}, b = \dfrac{1}{2}\left(\sqrt{6} + \sqrt{2}\right)$ and $c = \sqrt{2}$, then find ÐA.

Sol. $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc} = \dfrac{\frac{1}{4}\left(8 + 4\sqrt{3}\right) + 2 - 3}{\sqrt{12} + \sqrt{4}} = \dfrac{1+\sqrt{3}}{2(1+\sqrt{3})} = \dfrac{1}{2} \quad \Rightarrow A = \dfrac{\pi}{3}$

Example 5.13 If the angles A, B, C of a triangle are in A.P. and sides a, b, c are in G.P., then prove that a^2, b^2, c^2 are in A.P.

Sol.

Given, $2B = A + C \Rightarrow 3B = \pi \Rightarrow B = \pi/3$ (i)

Also a, b, c in G.P. $\Rightarrow b^2 = ac$ (ii)

Now, $\cos B = \cos 60° = \dfrac{1}{2} = \dfrac{c^2 + a^2 - b^2}{2ca}$

$\Rightarrow \quad ca = c^2 + a^2 - b^2$

$\Rightarrow \quad 2b^2 = c^2 + a^2$ [by using Eq. (ii)]

$\Rightarrow \quad a^2, b^2, c^2$ are in A.P.

Example 5.14 If in a triangle ABC, $\angle C = 60°$, then prove that $\dfrac{1}{a+c} + \dfrac{1}{b+c} = \dfrac{3}{a+b+c}$.

Sol.

By the cosine formula, we have

$c^2 = a^2 + b^2 - 2ab\cos C \quad \Rightarrow \quad c^2 = a^2 + b^2 - 2ab\cos 60° = a^2 + b^2 - ab$ (i)

Now, $\dfrac{1}{a+c} + \dfrac{1}{b+c} - \dfrac{3}{a+b+c}$

$= \left[\dfrac{(b+c)(a+b+c) + (a+c)(a+b+c) - 3(a+c)(b+c)}{(a+b)(b+c)(a+b+c)}\right]$

$= \dfrac{(a^2 + b^2 - ab) - c^2}{(a+b)(b+c)(a+b+c)} = 0$ [from Eq. (i)]

$\Rightarrow \quad \dfrac{1}{a+c} + \dfrac{1}{b+c} = \dfrac{3}{a+b+c}$

Example 5.15 In a triangle, if the angles A, B and C are in A.P, show that $2\cos\dfrac{1}{2}(A-C) = \dfrac{a+c}{\sqrt{a^2 - ac + c^2}}$.

Sol.

Since angles A, B and C are in A.P.

$\therefore \quad A + C = 2B$

But, $A + B + C = 180° \Rightarrow 3B = 180° \Rightarrow B = 60°$

Now, $\cos B = \dfrac{1}{2} = \dfrac{a^2 + c^2 - b^2}{2ac}$

$\Rightarrow \quad a^2 + c^2 - b^2 = ac$

$\Rightarrow \quad a^2 - ac + c^2 = b^2$

$\Rightarrow \quad \dfrac{a+c}{\sqrt{a^2 - ac + c^2}} = \dfrac{a+c}{b} = \dfrac{2R(\sin A + \sin C)}{2R\sin B}$

$$= \frac{2\sin\left(\frac{A+C}{2}\right)\cos\left(\frac{A-C}{2}\right)}{\sin B} = \frac{2\sin 60°}{\sin 60°}\cos\left(\frac{A-C}{2}\right) = 2\cos\left(\frac{A-C}{2}\right)$$

Example 5.16 **The sides of a triangle are x^2+x+1, $2x+1$ and x^2-1, prove that the greatest angle is 120°.**

Sol.

Let $a = x^2+x+1$, $b = 2x+1$ and $c = x^2-1$.

First of all, we have to decide which side is the greatest. We know that in a triangle, the length of each side is greater than zero. Therefore, we have $b = 2x+1 > 0$ and $c = x^2 - 1 > 0$.

$\Rightarrow \quad x > -1/2$ and $x^2 > 1$

$\Rightarrow \quad x > -1/2$ and $x < -1$ or $x > 1 \Rightarrow x > 1$

$a = x^2 + x + 1 = (x+1/2)^2 + (3/4)$ is always positive.

Thus, all sides a, b and c are positive when $x > 1$.

Now, $x > 1 \Rightarrow x^2 > x$

$\Rightarrow \quad x^2+x+1 > x+x+1$

$\Rightarrow \quad x^2+x+1 > 2x+1 \Rightarrow a > b$

Also, when $x > 1$,

$x^2+x+1 > x^2-1 \Rightarrow a > c$

Thus, $a = x^2+x+1$ is the greatest side and the angle A opposite to this side is the greatest angle .

$$\therefore \quad \cos A = \frac{b^2+c^2-a^2}{2bc}$$

$$= \frac{(2x+1)^2 + (x^2-1)^2 - (x^2+x+1)^2}{2(2x+1)(x^2-1)}$$

$$= \frac{-2x^3 - x^2 + 2x + 1}{2(2x^3 + x^2 - 2x - 1)} = -\frac{1}{2}$$

$$= \cos 120°$$

$$\Rightarrow \quad A = 120°$$

Example 5.17 **Triangle ABC has $BC = 1$ and $AC = 2$. Find the maximum possible value of angle A**

Sol.

Using cosine rule, we have

$$\cos\theta = \frac{x^2+4-1}{4x}$$

$$= \frac{x^2+3}{4x}$$

$$= \frac{1}{4}\left[x + \frac{3}{x}\right]$$

$$= \frac{1}{4}\left[\left(\sqrt{x} - \sqrt{\frac{3}{x}}\right)^2 + 2\sqrt{3}\right]$$

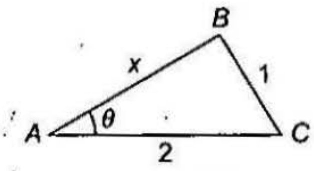


Fig. 5.8

Hence, $\cos\theta$ is minimum if $x = \sqrt{3}$.

Therefore, the minimum value of $\cos\theta = 2 \times \frac{\sqrt{3}}{4} = \frac{\sqrt{3}}{2}$, and the maximum value of $\theta = \frac{\pi}{6}$

Example 5.18 **Let a, b and c be the three sides of a triangle, then prove that the equation $b^2x^2 + (b^2 + c^2 - a^2)x + c^2 = 0$ has imaginary roots.**

Sol.

$b^2x^2 + (b^2 + c^2 - a^2)x + c^2 = 0$
Let $f(x) = b^2x^2 + (2bc\cos A)x + c^2 = 0$
Also in ΔABC, where $A \in (0, \pi)$ in a triangle, we find $\cos A \in (-1, 1)$
$\Rightarrow 2bc\cos A \in (-2bc, 2bc)$
$\Rightarrow D = (2bc\cos A)^2 - 4b^2c^2 = 4b^2c^2(\cos^2 A - 1) < 0$
Hence, the roots are imaginary.

Example 5.19 **Let $a \le b \le c$ be the lengths of the sides of a triangle. If $a^2 + b^2 < c^2$, then prove that Δ is obtuse angled.**

Sol.

$a^2 + b^2 < c^2$
$\Rightarrow a^2 + b^2 < a^2 + b^2 - 2ab\cos C$
$\Rightarrow \cos C < 0$
$\Rightarrow C$ is obtuse

Concept Application Exercise 5.2

1. If the sides of a triangle are a, b and $\sqrt{a^2 + ab + b^2}$, then find the greatest angle.
2. Prove that $(a-b)^2\cos^2\frac{C}{2} + (a+b)^2\sin^2\frac{C}{2} = c^2$.
3. If the line segment joining the points $A(a, b)$ and $B(c, d)$ subtends an angle θ at the origin, then prove that $\cos\theta = \frac{ac+bd}{\sqrt{(a^2+b^2)(c^2+d^2)}}$.
4. Prove that in ΔABC, a^2, b^2, c^2 are in A.P. if and only if $\cot A$, $\cot B$, $\cot C$ are in A.P.
5. If $x, y > 0$, then prove that the triangle whose sides are given by $3x + 4y$, $4x + 3y$ and $5x + 5y$ units is obtuse angled.

6. In ΔABC, angle A is $120°$, $BC + CA = 20$ and $AB + BC = 21$. Find the length of the side BC.
7. In ΔABC, $AB = 1$, $BC = 1$ and $AC = 1/\sqrt{2}$. In ΔMNP, $MN = 1$, $NP = 1$ and $\angle MNP = 2\angle ABC$. Find the side MP.

PROJECTION FORMULA

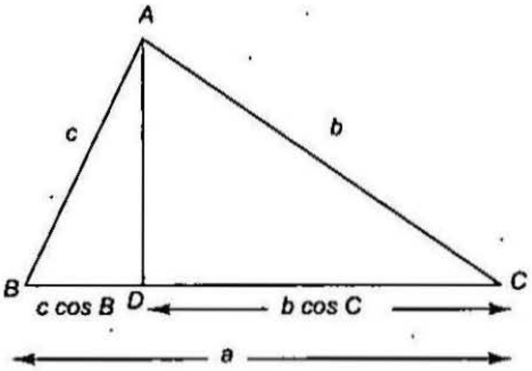


Fig. 5.9

Projection of AB on $BC = BD = c \cos B$
Projection of AC on $BC = CD = b \cos C$
Now, $BC = a = BD + DC = c \cos B + b \cos C$
Similarly, other formulae follow.

Example 5.20 **Prove that $a(b \cos C - c \cos B) = b^2 - c^2$.**

Sol. $a(b \cos C - c \cos B) = (b \cos C + c \cos B)(b \cos C - c \cos B)$
$= b^2 \cos^2 C - c^2 \cos^2 B$
$= b^2 (1 - \sin^2 C) - c^2 (1 - \sin^2 B)$
$= b^2 - c^2 - (b^2 \sin^2 C - c^2 \sin^2 B)$
$= b^2 - c^2$ [as by the sine rule $b \sin C = c \sin B$]

Example 5.21 **If in a triangle $a \cos^2 \frac{C}{2} + c\cos^2 \frac{A}{2} = \frac{3b}{2}$, then find the relation between the sides of the triangle.**

Sol. $a \cos^2 \frac{C}{2} + c\cos^2 \frac{A}{2} = \frac{3b}{2}$
$\Rightarrow$ $a(1 + \cos C) + c(1 + \cos A) = 3b$
$\Rightarrow$ $a + c + (a \cos C + c \cos A) = 3b$
$\Rightarrow$ $a + c + b = 3b$ [by the projection formula]
$\Rightarrow$ $a + c = 2b$
$\Rightarrow$ a, b, c are in A.P.

Example 5.22 **Prove that $(b + c) \cos A + (c + a) \cos B + (a + b) \cos C = 2s$.**

Sol. $(b + c) \cos A + (c + a) \cos B + (a + b) \cos C$
$= (b \cos A + a \cos B) + (c \cos A + a \cos C) + (b \cos C + c \cos B) = c + b + a = 2s$

Concept Application Exercise 5.3

1. In ΔABC, prove that $c\cos(A-\alpha)+a\cos(C+\alpha)=b\cos\alpha$.
2. Prove that $\dfrac{\cos C+\cos A}{c+a}+\dfrac{\cos B}{b}=\dfrac{1}{b}$.
3. Prove that $a(b^2+c^2)\cos A+b(c^2+a^2)\cos B+c(a^2+b^2)\cos C=3abc$.

HALF-ANGLE FORMULAE

1. i. $\sin\dfrac{A}{2}=\sqrt{\dfrac{(s-b)(s-c)}{bc}}$

   ii. $\sin\dfrac{B}{2}=\sqrt{\dfrac{(s-c)(s-a)}{ca}}$

   iii. $\sin\dfrac{C}{2}=\sqrt{\dfrac{(s-a)(s-b)}{ab}}$

Proof:

i. $$\sin^2\frac{A}{2}=\frac{1-\cos A}{2}$$
$$=\frac{1}{2}\left[1-\frac{b^2+c^2-a^2}{2bc}\right]$$
$$=\frac{1}{2}\left[\frac{2bc-b^2-c^2+a^2}{2bc}\right]$$
$$=\frac{1}{2}\left[\frac{a^2-(b-c)^2}{2bc}\right]$$
$$=\frac{(a-b+c)(a+b-c)}{4bc}$$
$$=\frac{(a+b+c-2b)(a+b+c-2c)}{4bc}$$
$$=\frac{(2s-2b)(2s-2c)}{4bc}\qquad [\because a+b+c=2s]$$
$$\Rightarrow \sin^2\frac{A}{2}=\frac{(s-b)(s-c)}{bc}$$

As $0<\dfrac{A}{2}<\dfrac{\pi}{2}$, so $\sin\dfrac{A}{2}>0$.

Hence, $\sin\dfrac{A}{2}=\sqrt{\dfrac{(s-b)(s-c)}{bc}}$

The other formulae can be proved similarly.

2. i. $\cos\dfrac{A}{2}=\sqrt{\dfrac{s(s-a)}{bc}}$

   ii. $\cos\dfrac{B}{2}=\sqrt{\dfrac{s(s-b)}{ca}}$

iii. $\cos\frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}$

Proof:

i. $\cos^2\frac{A}{2} = \frac{1+\cos A}{2}$

$= \frac{1}{2}\left[1 + \frac{b^2+c^2-a^2}{2bc}\right]$ [using cosine rule]

$= \frac{1}{2}\left[\frac{2bc+b^2+c^2-a^2}{2bc}\right]$

$= \frac{1}{2}\left[\frac{(b+c)^2-a^2}{2bc}\right]$

$= \frac{1}{2}\left[\frac{(b+c+a)(b+c-a)}{2bc}\right]$

$= \frac{(b+c+a)(a+b+c-2a)}{4bc}$

$= \frac{2s(2s-2a)}{4bc}$ [$\because a+b+c=2s$]

$\therefore \cos^2\frac{A}{2} = \frac{s(s-a)}{bc}$

As $0 < \frac{A}{2} < \frac{\pi}{2}$ so $\cos\frac{A}{2} > 0$

Hence, $\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$.

The other formulae can be proved in the same way.

3. From the above formulae, we can prove

i. $\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$

ii. $\tan\frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}}$

iii. $\tan\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$

Example 5.23 **If $\cos\frac{A}{2} = \sqrt{\frac{b+c}{2c}}$, then prove that $a^2+b^2=c^2$.**

Sol. $\cos\frac{A}{2} = \sqrt{\frac{b+c}{2c}} \Rightarrow \frac{s(s-a)}{bc} = \frac{b+c}{2c}$ [squaring]

$\Rightarrow 2s(2s-2a) = 2b(b+c)$

$\Rightarrow (b+c+a)(b+c-a) = 2b^2+2bc$

$\Rightarrow (b+c)^2 - a^2 = 2b^2+2bc$

$\Rightarrow c^2 = a^2 + b^2$

Example 5.24 **If the cotangents of half the angles of a triangle are in A.P., then prove that the sides are in A.P.**

Sol. $\cot\frac{A}{2}=\sqrt{\frac{s(s-a)}{(s-b)(s-c)}}=\frac{s(s-a)}{\Delta}$

Similarly, $\cot\frac{B}{2}=\frac{s(s-b)}{\Delta}$ and $\cot\frac{C}{2}=\frac{s(s-c)}{\Delta}$.

$\cos\frac{A}{2}$, $\cot\frac{B}{2}$ and $\cot\frac{C}{2}$ are in A.P.

$\Rightarrow$ $\frac{s(s-a)}{\Delta}$, $\frac{s(s-b)}{\Delta}$ and $\frac{s(s-c)}{\Delta}$ are in A.P.

$\Rightarrow$ $s-a$, $s-b$ and $s-c$ are in A.P.

$\Rightarrow$ a, b and c are in A.P.

Concept Application Exercise 5.4

1. If $b+c=3a$, then find the value of cot $\cot\frac{B}{2}\cot\frac{C}{2}$.
2. Prove that $bc\cos^2\frac{A}{2}+ca\cos^2\frac{B}{2}+ab\cos^2\frac{C}{2}=s^2$.
3. If in ΔABC, $\tan\frac{A}{2}=\frac{5}{6}$, $\tan\frac{C}{2}=\frac{2}{5}$, then prove that a, b and c are in A.P.

AREA OF TRIANGLE

Different formulae for area of triangle are as follows.

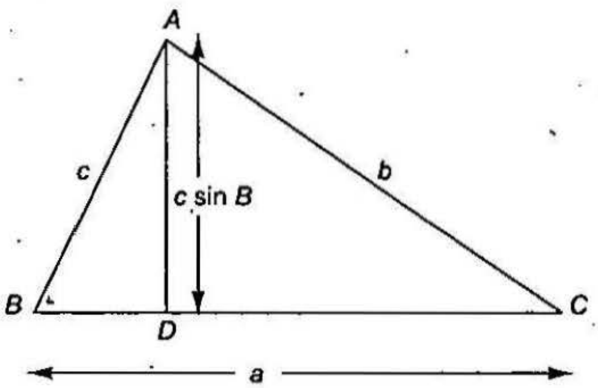


Fig. 5.10

From Fig. 5.10, area of triangle ABC is

$\Delta=\frac{1}{2}AD\times BC=\frac{1}{2}c\sin B\times a=\frac{1}{2}ac\sin B$

Similarly, we can prove that $\Delta=\frac{1}{2}ab\sin C=\frac{1}{2}bc\sin A$

Also, by the sine rule, $\sin A = \dfrac{a}{2R}$

$$\Rightarrow \quad \Delta = \frac{abc}{4R} = \frac{(2R\sin A)(2R\sin B)(2R\sin C)}{4R}$$

$$= 2R^2 \sin A \sin B \sin C$$

Also, $\Delta = \dfrac{1}{2} ac \sin B$

$$= ac \sin\frac{B}{2} \cos\frac{B}{2}$$

$$= ac \sqrt{\frac{(s-c)(s-a)}{ca}} \sqrt{\frac{s(s-b)}{ca}}$$

$$= \sqrt{s(s-a)(s-b)(s-c)}$$

Example 5.25 **If in triangle ABC, $\Delta^2 = a^2 - (b-c)^2$, then find the value of $\tan A$.**

Sol. $\Delta^2 = (a+b-c)(a-b+c)$

$$\Rightarrow \quad \Delta^2 = [2(s-b)\, 2(s-c)]^2$$

$$\Rightarrow \quad s(s-a)(s-b)(s-c) = 16(s-b)^2 (s-c)^2$$

$$\Rightarrow \quad \frac{(s-b)(s-c)}{s(s-a)} = \frac{1}{16}$$

$$\Rightarrow \quad \tan\frac{A}{2} = \frac{1}{4}$$

$$\Rightarrow \quad \tan^2 A = \frac{2\tan(A/2)}{1-\tan^2(A/2)} = \frac{2.(1/4)}{1-(1/16)} = \frac{8}{15}$$

Example 5.26 **Prove that $a^2 \sin 2B + b^2 \sin 2A = 4\Delta$.**

Sol. $a^2 \sin 2B + b^2 \sin 2A = 4R^2 [\sin^2 A\,(2 \sin B \cos B) + \sin^2 B\,(2 \sin A \cos A)]$ [using sine rule]

$$= 8R^2 \sin A \sin B (\sin A \cos B + \sin B \cos A)$$

$$= 8R^2 \sin A \sin B \sin(A+B)$$

$$= 8R^2 \sin A \sin B \sin C = 4\Delta$$

Example 5.27 **Prove that $\dfrac{(a+b+c)(b+c-a)(c+a-b)(a+b-c)}{4b^2c^2} = \sin^2 A$.**

Sol. $$\frac{(a+b+c)(b+c-a)(c+a-b)(a+b-c)}{4b^2c^2} = \frac{2s\,2(s-a)\;2(s-b)\;2(s-c)}{4b^2c^2}$$

$$= \frac{4\Delta^2}{b^2c^2}$$

$$= \frac{4}{b^2c^2}\left(\frac{1}{2}bc\sin A\right)^2 = \sin^2 A$$

Example 5.28 **If the sides of a triangle are 17, 25 and 28, then find the greatest length of the altitude.**

Sol. We know from geometry that the greatest altitude is perpendicular to the shortest side.
Let $a = 17, b = 25$ and $c = 28$.

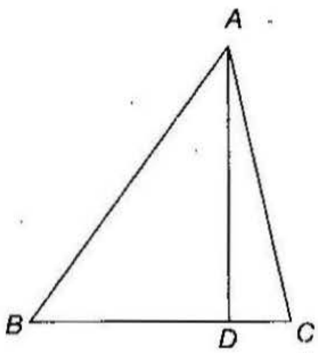


Fig. 5.11

Now, $\Delta = \frac{1}{2} AD \times BC \quad \Rightarrow AD = \frac{2\Delta}{17}$

where $\Delta^2 = s(s-a)(s-b)(s-c) = 210^2$

$$\Rightarrow \quad AD = \frac{420}{17}$$

Example 5.29 **In equilateral triangle ABC with interior point D, if the perpendicular distances from D to the sides of 4, 5 and 6, respectively, are given, then find the area of ΔABC.**

Sol.

Area of triangle is $\Delta = \frac{a\times 4 + a\times 5 + a\times 6}{2}$

$$\Rightarrow \frac{a(4+5+6)}{2} = \frac{3\sqrt{3}}{4}a^2$$

$$\Rightarrow \frac{15}{2} = \frac{\sqrt{3}\,a}{4}$$

$$\Rightarrow a = \frac{30}{\sqrt{3}} = 10\sqrt{3}$$

$$\Rightarrow \Delta = \frac{\sqrt{3}}{4}\times 100\times 3 = 75\sqrt{3}$$

Example 5.30 If area of a triangle is 2 sq. units, then find the value of the product of the arithmetic mean of the lengths of the sides of a triangle and harmonic mean of the lengths of the altitudes of the triangle.

Sol.

$$ah_1 = bh_2 = ch_3 = 2\Delta$$

$$\Rightarrow \frac{1}{h_1}+\frac{1}{h_2}+\frac{1}{h_3} = \frac{a+b+c}{2\Delta}$$

$$\Rightarrow \frac{a+b+c}{3} \times \frac{3}{\frac{1}{h_1}+\frac{1}{h_2}+\frac{1}{h_3}} = 2\Delta = 4$$

Example 5.31 A triangle has sides 6, 7 and 8. The line through its incentre parallel to the shortest side is drawn to meet the other two sides at P and Q. Then find the length of the segment PQ

Sol.

$$\Delta = r \times s$$

$$\therefore \frac{21 \times r}{2} = \frac{6 \times h}{2} = 3h$$

$$\Rightarrow \frac{r}{h} = \frac{2}{7}$$

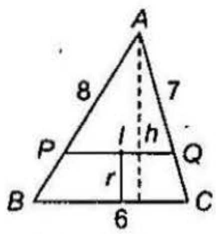


Fig. 5.12

Now APQ and ABC are similiar

$$\Rightarrow \frac{h-r}{h} = \frac{PQ}{6}$$

$$\Rightarrow 1 - \frac{r}{h} = \frac{PQ}{6}$$

$$\Rightarrow 1 - \frac{2}{7} = \frac{PQ}{6}$$

$$\Rightarrow \frac{5}{7} = \frac{PQ}{6} \Rightarrow PQ = \frac{30}{7}$$

Concept Application Exercise 5.5

1. If $c^2 = a^2 + b^2$, then prove that $4s(s-a)(s-b)(s-c) = a^2b^2$.
2. If the sides of a triangle are in the ratio 3:7:8, then find $R : r$.
3. In triangle ABC, if $a = 2$ and $bc = 9$, then prove that $R = 9/2\Delta$.
4. The area of triangle ABC is equal to $(a^2 + b^2 - c^2)$, where a, b and c are the side of the triangle. Find the value of $\tan C$.
5. Let the lengths of the altitudes drawn from the vertices of ΔABC to the opposite sides are 2, 2 and 3. If the area of ΔABC is Δ, then find the area of triangle.

Different Circles and Centres Connected With Triangle

Circumcircle and Circumcentre (O)

The circle passing through the angular point of ΔABC is called its circumcircle. The centre of this circle is the point of intersection of the perpendicular bisectors of the sides and is called the circumcentre. Its radius is denoted by R.

- Circumcentre of an acute-angled triangle lies inside the triangle.

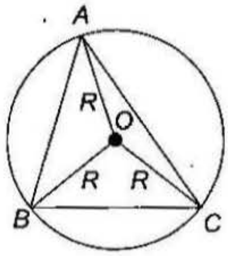


Fig. 5.13

- Circumcentre of an obtuse-angled triangle lies outside the triangle.

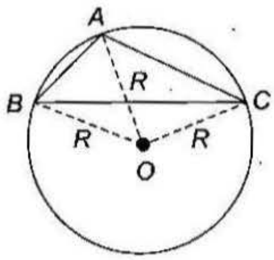


Fig. 5.14

- Circumcentre of a right-angled triangle is the mid-point of the hypotenuse.

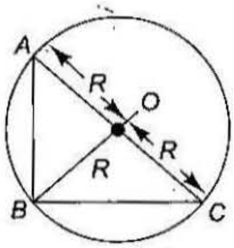


Fig. 5.15

- Distance of the circumcentre from the sides can be calculated as follows.

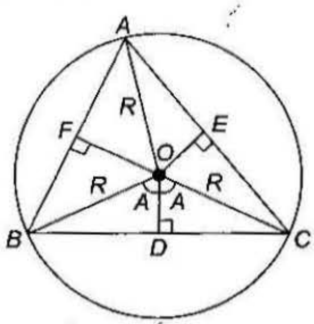


Fig. 5.16

At the circumcentre, the perpendicular bisectors of the sides are concurrent.
Also, $\angle BOC = 2\angle BAC = 2A$. Triangles BOD and COD are congruent.
Hence, $\angle BOD = A$.

Now, in ΔBOD, $\cos A = \dfrac{OD}{OB} = \dfrac{OD}{R} \quad \Rightarrow OD = R\cos A$

Similarly, $OE = R\cos B$ and $OF = R\cos C$.

In-Circle and In-Centre (I)

Point of intersection of the internal bisectors of a triangle is called the in-center of the triangle. Also, it is the centre of the circle touching all the three sides internally. In-centre always lies inside the triangle.

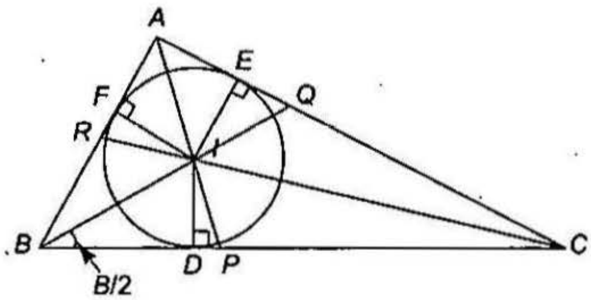


Fig. 5.17

- Points $AFIE$, $BDIF$ and $CEID$ are concyclic
- Internal bisector AP divides side BC in ratio $AB:AC$

or $\dfrac{BP}{PC} = \dfrac{AB}{AC} = \dfrac{c}{b} \quad \Rightarrow BP = ck, CP = bk$

But $BP + CP = a$

$\Rightarrow \quad ck + bk = a$

$\Rightarrow \quad k = \dfrac{a}{b+c}$

$\Rightarrow \quad BP = \dfrac{ac}{b+c}$ and $CP = \dfrac{ab}{b+c}$

Similarly, $AQ = \dfrac{cb}{a+c}$, $CQ = \dfrac{ab}{a+c}$

and $AR = \dfrac{bc}{a+b}$, $BR = \dfrac{ac}{a+b}$

- Area of the triangle in terms of r

$$\Delta_{ABC} = \Delta_{IBC} + \Delta_{IAC} + \Delta_{IAB} = \frac{1}{2}ra + \frac{1}{2}rb + \frac{1}{2}rc = \frac{1}{2}r(a+b+c) = rs$$

- $r = (s-a)\tan\dfrac{A}{2} = (s-b)\tan\dfrac{B}{2} = (s-c)\tan\dfrac{C}{2}$

Proof: $(s-a)\tan\dfrac{A}{2} = (s-a)\sqrt{\dfrac{(s-b)(s-c)}{s(s-a)}}$

$$= \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$$

$$= \frac{\sqrt{s(s-a)(s-b)(s-c)}}{s}$$

$$= \frac{\Delta}{s} = r$$

Similarly, the results $r = (s-b)\tan\dfrac{B}{2}$ and $r = (s-c)\tan\dfrac{C}{2}$ can be proved.

- $r = 4R \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

Proof: $4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = 4R\sqrt{\frac{(s-b)(s-c)}{bc}}\sqrt{\frac{(s-c)(s-a)}{ca}}\sqrt{\frac{(s-a)(s-b)}{ab}}$

$$= 4R\frac{(s-a)(s-b)(s-c)}{abc}$$

$$= \frac{4R}{abc}\frac{s(s-a)(s-b)(s-c)}{s}$$

$$= \frac{1}{\Delta}\frac{\Delta^2}{s}$$

$$\left[\text{where } \Delta = \frac{abc}{4R} \text{ and } \Delta = \sqrt{s(s-a)(s-b)(s-c)}\right] = \frac{\Delta}{s} = r$$

- **Distance of the in-centre from the vertex**

In ΔIDB, $\sin\frac{B}{2} = \frac{ID}{BI} = \frac{r}{BI} \quad \Rightarrow BI = \frac{r}{\sin\frac{B}{2}}$

Similarly, $AI = \frac{r}{\sin\frac{A}{2}}$ and $CI = \frac{r}{\sin\frac{C}{2}}$

- **Length of angle bisector AP**

Area of ΔABP + Area of ΔACP = Area of ΔABC

$\Rightarrow (1/2)\,AB\,AP\sin(A/2) + (1/2)\,AC\,AP\sin(A/2) = (1/2)AB\,AC\sin A$

$\Rightarrow (1/2)(c+b)\,AP\sin(A/2) = (1/2)[cb\,2\sin(A/2)\cos(A/2)]$

$\Rightarrow AP = \left(\frac{2bc}{b+c}\right)\cos(A/2)$

Similarly, length of angle bisector through point B and C is $BQ = \left(\frac{2ac}{a+c}\right)\cos(B/2)$,

$CR = \left(\frac{2ab}{a+b}\right)\cos(C/2)$

Orthocentre

Orthocentre (H) is the point of intersection of the altitudes of a triangle.

- Orthocentre (H) of an acute-angled triangle lies inside the triangle.
Here, H is the orthocentre of ΔABC.

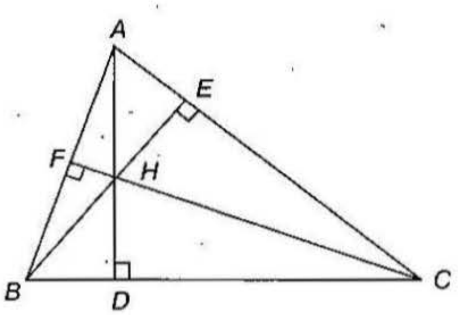


Fig. 5.18

- Orthocentre (H) of an obtuse-angled triangle lies outside the triangle. Here, H is orthocenter of ΔABC.

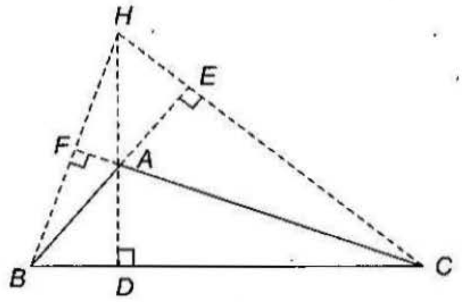


Fig. 5.19

- Orthocentre (H) of a right-angled triangle ABC lies at the right angle itself. In Fig. 5.20, the orthocentre H coincides with the right angle B.

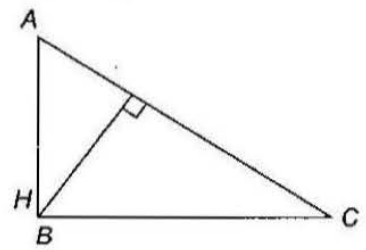


Fig. 5.20

- Image of orthocentre (H) in any side of a triangle lies on the circumcircle.

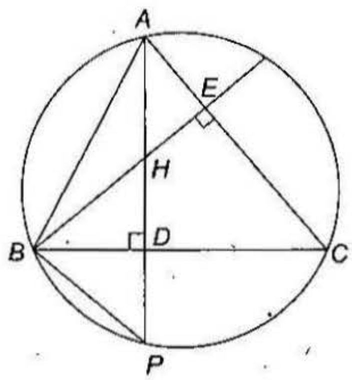


Fig. 5.21

$\angle HBD = \angle EBC = (\pi/2) - C \quad \Rightarrow \angle BHD = C$

Also, $\angle BPD = \angle BPA = \angle BCA = C$

Thus, ΔBPD and ΔBHD are congruent.

This implies $HD = DP \quad \Rightarrow P$ is image in H in BC.

- Distance of the orthocentre from vertices and sides of a triangle can be calculated as follows.

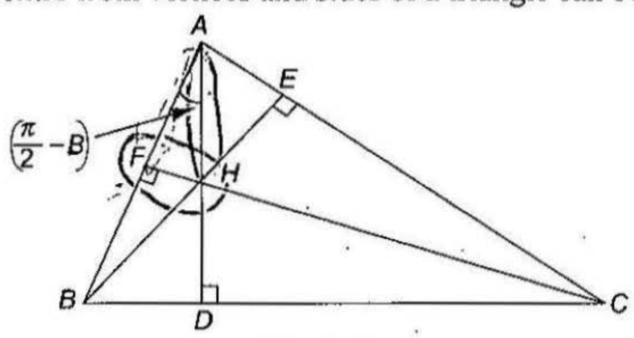


Fig. 5.22

In ΔADB, $\angle BAD = \frac{\pi}{2} - B$

In ΔAFC, $AF = b\cos A$ [projection of AC on AB]

In ΔAFH, $\cos\left(\frac{\pi}{2} - B\right) = \frac{AF}{AH} = \frac{b\cos A}{AH}$

$$\Rightarrow \quad AH = \frac{b\cos A}{\sin B} = 2R\cos A$$

Similarly, $BH = 2R\cos B$ and $CH = 2R\cos C$

In ΔAFH, $\tan\left(\frac{\pi}{2} - B\right) = \frac{FH}{AF} = \frac{FH}{b\cos A}$

$$\Rightarrow \quad FH = \frac{b\cos A\cos B}{\sin B} = 2R\cos A\cos B$$

Similarly, $EH = 2R\cos A\cos C$ and $HD = 2R\cos B\cos C$

Pedal Triangle

The triangle formed by the feet of the altitudes on the sides of a triangle is called a pedal triangle.

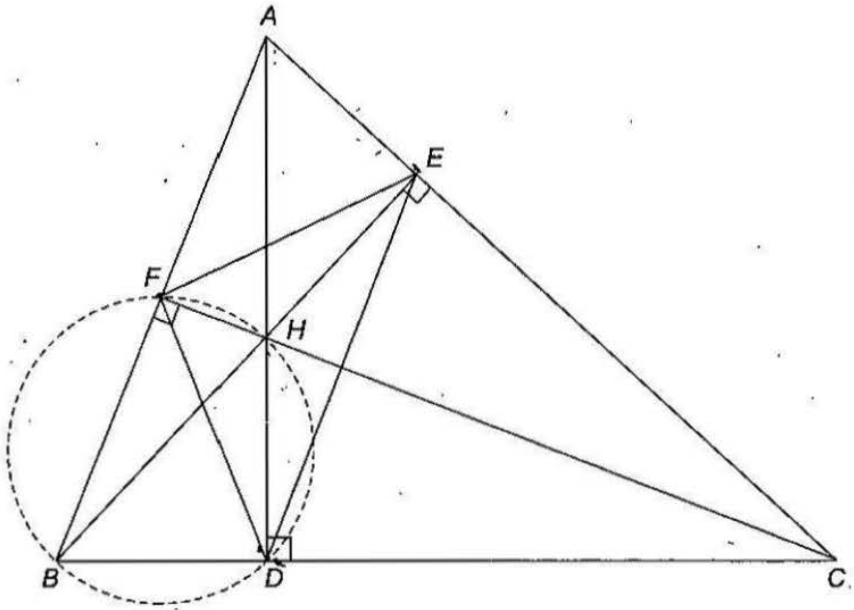


Fig. 5.23

- **In an acute-angled triangle, orthocentre of ΔABC is the in-centre of the pedal triangle DEF.**

Proof:

Points F, H, D and B are concyclic. $\Rightarrow \angle FDH = \angle FBH = \angle ABE = \frac{\pi}{2} - A$

Similarly, points D, H, E and C are concyclic $\Rightarrow \angle HDE = \angle HCE = \angle ACF = \frac{\pi}{2} - A$.

Thus, $\angle FDH = \angle HDE \Rightarrow AD$ is angle bisector of $\angle FDE$. Hence, altitudes of ΔABC are internal angle bisectors of the pedal triangle. Thus, orthocentre of ΔABC is the in-centre of the pedal triangle DEF.

- **Sides of pedal triangle in acute-angled triangle**

In ΔAFE, $AF = b\cos A$, $AE = c\cos A$

By cosine rule, $EF^2 = AE^2 + AF^2 - 2\,AE\,AF\cos(\angle EAF)$

$$\Rightarrow \quad EF^2 = b^2\cos^2 A + c^2\cos^2 A - 2\,bc\cos^3 A$$
$$= \cos^2 A\,(b^2 + c^2 - 2\,bc\cos A) = \cos^2 A\,(a^2)$$
$$= a\cos A$$

- **Circumradius of pedal triangle**

Let circumradius be R'

$$\Rightarrow \quad 2R' = \frac{EF}{\sin(\angle EDF)} = \frac{a\cos A}{\sin(\pi - 2A)} = \frac{a\cos A}{2\sin A\cos A} = \frac{a}{2\sin A} = R$$

$$\Rightarrow \quad R' = R/2$$

Centroid of Triangle

In $\Delta\,ABC$, the mid-points of the sides BC, CA and AB are D, E and F, respectively. The lines, AD, BE and CF are called medians of the triangle ABC, the points of concurrency of three medians is called the centroid. Generally, it is represented by G.

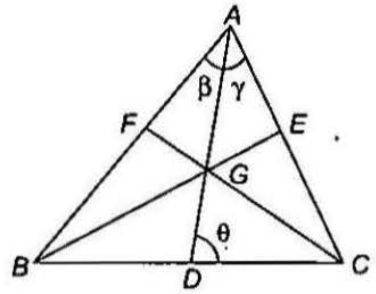


Fig. 5.24

Also, $AG = \frac{2}{3}AD$, $BG = \frac{2}{3}BE$ and $CG = \frac{2}{3}CF$.

- Length of medians and the angles that the medians make with sides

From Fig. 5.24, we have

$$AD^2 = AC^2 + CD^2 - 2AC \times CD \times \cos C$$
$$= b^2 + \frac{a^2}{4} - ab\cos C$$
$$= b^2 + \frac{a^2}{4} - ab\left(\frac{b^2 + a^2 - c^2}{2ab}\right)$$
$$= \frac{2b^2 + 2c^2 - a^2}{4}$$

$$\Rightarrow \quad AD = \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}$$

Similarly, $BE = \frac{1}{2}\sqrt{2c^2 + 2a^2 - b^2}$

and $CF = \frac{1}{2}\sqrt{2a^2 + 2b^2 - c^2}$

- Apollonius theorem

$AB^2 + AC^2 = 2(AD^2 + BD^2)$

Proof:

$$2(AD^2+BD^2)=2\left[\frac{1}{4}(2b^2+2c^2-a^2)+\frac{a^2}{4}\right]=b^2+c^2=AB^2+AC^2$$

- Centroid (G) of a triangle is situated on the line joining its circumcentre (O) and orthocenter (H) and divide this line in the ratio 1:2.

Proof:

Let AL be a perpendicular from A on BC, then H lies on AL. If OD is perpendicular from O on BC, then D is mid-point of BC.

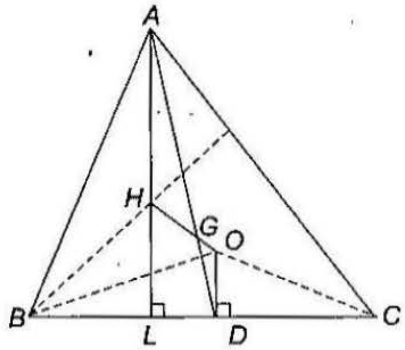


Fig. 5.25

Therefore, AD is a median of ΔABC. Let the line HO meet the median AD at G. Now, we shall prove that G is the centroid of the ΔABC. Obviously, ΔOGD and ΔHGA are similar triangles.

$$\therefore \frac{OG}{HG}=\frac{GD}{GA}=\frac{OD}{HA}=\frac{R\cos A}{2R\cos A}=\frac{1}{2}$$

$\therefore GD=\frac{1}{2}GA \Rightarrow G$ is centroid of ΔABC and $OG:HG=1:2$.

Example 5.32 ***ABC* is an acute-angled triangle with circumcentre '*O*' orthocentre *H*. If *AO* = *AH*, then find the angle *A*.**

Sol. $OA=HA$

$R=2R\cos A$

$\Rightarrow \cos A=\frac{1}{2} \quad \Rightarrow \quad A=\frac{\pi}{3}$

Example 5.33 **If x, y and z are the distances of incentre from the vertices of the triangle *ABC*, respectively, then prove that $\frac{abc}{xyz}=\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$.**

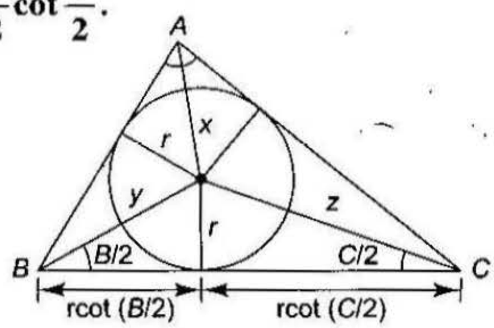


Fig. 5.26

Sol. $x=r\operatorname{cosec}\frac{A}{2}$ and $a=r\left(\cot\frac{B}{2}+\cot\frac{C}{2}\right)$

$$\Rightarrow \frac{a}{x}=\left(\cot\frac{B}{2}+\cot\frac{C}{2}\right)\sin\frac{A}{2}=\frac{\sin\frac{A}{2}\cos\frac{A}{2}}{\sin\frac{B}{2}\sin\frac{C}{2}}$$

$$\Rightarrow \frac{abc}{xyz}=\frac{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}=\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$$

Example 5.34 Let ABC be a triangle with $\angle BAC = 2\pi/3$ and $AB = x$ such that $(AB)(AC) = 1$. If x varies, then find the longest possible length of the angle bisector AD

Sol. $AD = y = \dfrac{2bc}{b+c}\cos\dfrac{A}{2} = \dfrac{bx}{b+x}$ (as $c = x$)

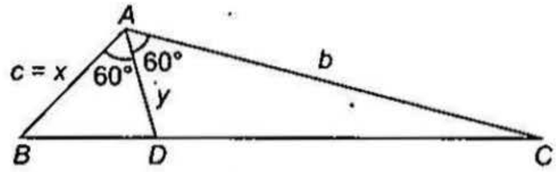


Fig. 5.27

But $bx = 1 \Rightarrow b = \dfrac{1}{x}$

$\therefore y = \dfrac{x}{1+x^2} = \dfrac{1}{x+\dfrac{1}{x}}$

$\Rightarrow y_{\max} = \dfrac{1}{2}$ since the minimum value of the denominator is 2 if $x > 0$.

Example 5.35 Let ABC be an acute triangle whose orthocentre is at H. If altitude from A is produced to meet the circumcircle of triangle ABC at D, then prove $HD = 4R\cos B\cos C$

Sol. ΔBHN and ΔBDN are congruent.

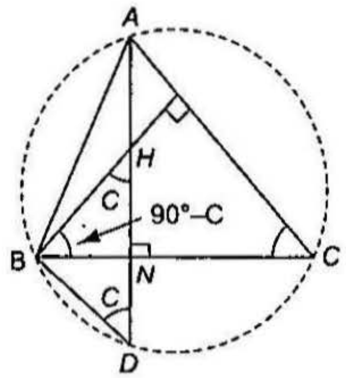


Fig. 5.28

$\therefore HN = ND = 2R\cos B\cos C$

$\therefore HD = 4R\cos B\cos C$

Example 5.36 In an acute-angled triangle ABC, point D, E and F are the feet of the perpendiculars from A, B and C onto BC, AC and AB, respectively. H is orthocentre. If $\sin A = 3/5$ and $BC = 39$, then find the length of AH.

Sol. Given $\sin A = 3/5 \Rightarrow \cos A = 4/5$

Also $a = 39$

$\therefore \dfrac{a}{\sin A} = 2R$

$\Rightarrow \dfrac{39\times 5}{3} = 2R$

$\Rightarrow 2R = 65$

$\Rightarrow AH = 2R\cos A = 65\cdot\dfrac{4}{5} = 52$

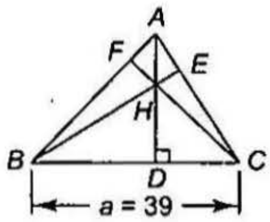


Fig. 5.29

Example 5.37 In triangle ABC, CD is the bisector of the angle C. If $\cos\frac{C}{2}=\frac{1}{3}$ and $CD=6$, then find the value of $\left(\frac{1}{a}+\frac{1}{b}\right)$.

Sol. $\Delta=\Delta_1+\Delta_2$

$$\Rightarrow \frac{1}{2}ab\sin C=\frac{1}{2}6b\sin\frac{C}{2}+\frac{1}{2}6a\sin\frac{C}{2}$$

$$\Rightarrow ab\sin\frac{C}{2}\cos\frac{C}{2}=\frac{1}{2}6b\sin\frac{C}{2}+\frac{1}{2}6a\sin\frac{C}{2}$$

$$\Rightarrow \frac{1}{a}+\frac{1}{b}=\frac{1}{9}$$

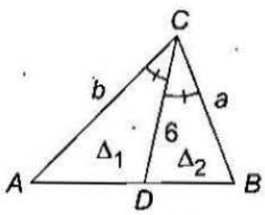


Fig. 5.30

Example 5.38 Let f, g and h be the lengths of the perpendiculars from the circumcentre of ΔABC on the sides a, b and c, respectively, then prove that $\frac{a}{f}+\frac{b}{g}+\frac{c}{h}=\frac{1}{4}\frac{abc}{fgh}$.

Sol. Distance of circumcentre from to side BC is $R\cos A=f$
Similarly, $g=R\cos B$, $h=R\cos C$

$$\Rightarrow \frac{a}{f}+\frac{b}{g}+\frac{c}{h}=\frac{2R\sin A}{R\cos A}+\frac{2R\sin B}{R\cos B}+\frac{2R\sin C}{R\cos C}=2(\tan A+\tan B+\tan C)$$

Also, $\frac{a}{f}\frac{b}{g}\frac{c}{h}=8\tan A\tan B\tan C$

But in triangle, $\tan A+\tan B+\tan C=\tan A\tan B\tan C$

$$\Rightarrow \frac{a}{f}+\frac{b}{g}+\frac{c}{h}=\frac{1}{4}\frac{abc}{fgh}$$

Example 5.39 If the incircle of ΔABC touches its sides, respectively, at L, M and N and if x, y, z are the circumradii of the triangles MIN, NIL, and LIM where I is the incenter, then prove that $xyz=\frac{1}{2}Rr^2$.

Sol. In Fig. 5.31, $ANIM$ is a cyclic quadrilateral.
Also, AI is the diameter of circumcircle MNI.
Let $AI=2x$

$$\Rightarrow \operatorname{cosec}\frac{A}{2}=\frac{2x}{r}$$

$$\Rightarrow x=\frac{r}{2\sin\frac{A}{2}},\ y=\frac{r}{2\sin\frac{B}{2}},\ z=\frac{r}{2\sin\frac{C}{2}}$$

$$\Rightarrow xyz=\frac{r^3}{8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}=\frac{r^3}{2\frac{r}{R}}=\frac{r^2R}{2}$$

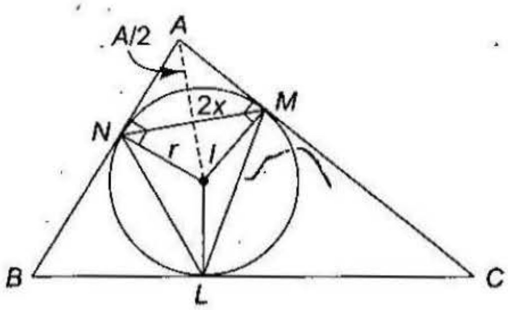


Fig. 5.31

ESCRIBED CIRCLES OF A TRIANGLE AND THEIR RADII

The circle which touches the side BC and two sides AB and AC produced of triangle ABC is called the escribed circle opposite to the angle A. Its radius is denoted by r_1. Similarly, r_2 and r_3 denote the radii of the escribed circles opposite to the angles B and C, respectively. The centres of the escribed circles are called the ex-centres. The centre of the escribed circle opposite to the angle A is the point of intersection of the external bisectors of angles B and C. The internal bisector of angle A also passes through the same point. The centre is generally denoted by I_1.

In any ΔABC, we have

i. $r_1 = \dfrac{\Delta}{s-a}$, $r_2 = \dfrac{\Delta}{s-b}$, $r_3 = \dfrac{\Delta}{s-c}$

ii. $r_1 = s \tan \dfrac{A}{2}$, $r_2 = s \tan \dfrac{B}{2}$, $r_3 = s \tan \dfrac{C}{2}$

iii. $r_1 = 4R \sin \dfrac{A}{2} \cos \dfrac{B}{2} \cos \dfrac{C}{2}$

$r_2 = 4R \cos \dfrac{A}{2} \sin \dfrac{B}{2} \cos \dfrac{C}{2}$

$r_3 = 4R \cos \dfrac{A}{2} \cos \dfrac{B}{2} \sin \dfrac{C}{2}$

Proof:

Let I_1 be the point of intersection of external bisectors of angles B and C of ΔABC. Suppose the circle touches the side BC at D and sides AB and AC produced at F and E, respectively. Clearly, $I_1D = I_1E = I_1F = r_1$.

i. Area of ΔABC = area of ΔI_1AC + Area of ΔI_1AB – Area of ΔI_1BC

$$\Rightarrow \quad \Delta = \frac{1}{2} r_1 b + \frac{1}{2} r_1 c - \frac{1}{2} r_1 a$$

$$= \frac{1}{2} r_1 (b + c - a)$$

$$= \frac{r_1}{2} (2s - 2a)$$

$$\Rightarrow \quad r_1 = \frac{\Delta}{s-a}$$

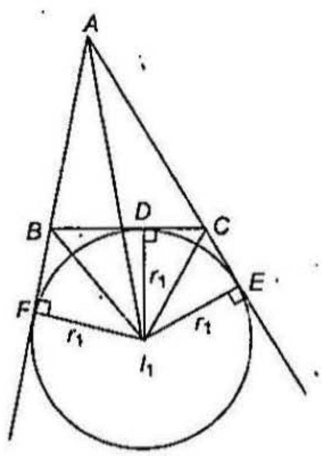


Fig. 5.32

Similarly, it can be shown that $r_2 = \dfrac{\Delta}{s-b}$ and $r_3 = \dfrac{\Delta}{s-c}$

ii. $s \tan \dfrac{A}{2} = s\sqrt{\dfrac{(s-b)(s-c)}{s(s-a)}} = \dfrac{\sqrt{s(s-b)(s-c)(s-a)}}{s-a} = \dfrac{\Delta}{s-a} = r_1$

Similarly, $r_2 = s \tan \dfrac{B}{2}$ and $r_3 = s \tan \dfrac{C}{2}$.

iii. $4R \sin \dfrac{A}{2} \cos \dfrac{B}{2} \cos \dfrac{C}{2} = 4R\sqrt{\dfrac{(s-b)(s-c)}{bc}}\sqrt{\dfrac{s(s-b)}{ac}}\sqrt{\dfrac{s(s-c)}{ab}}$

$$= 4R\,\frac{s(s-a)(s-b)(s-c)}{abc(s-a)}$$

$$= \frac{4R}{abc}\frac{\Delta^2}{(s-a)} = \frac{\Delta}{s-a}$$

$$= r_1$$

Similarly, we can prove for r_2 and r_3.

Example 5.40 **Prove that $r_1 + r_2 + r_3 - r = 4R$.**

Sol. $r_1 + r_2 + r_3 - r = \dfrac{\Delta}{s-a} + \dfrac{\Delta}{s-b} + \dfrac{\Delta}{s-c} - \dfrac{\Delta}{s} = \Delta\left[\dfrac{s-b+s-a}{(s-a)(s-b)} + \dfrac{(s-s+c)}{s(s-c)}\right]$

$$= \Delta\left[\frac{c}{(s-a)(s-b)} + \frac{c}{s(s-c)}\right]$$

$$= \Delta c\left[\frac{s(s-c)+(s-a)(s-b)}{s(s-a)(s-b)(s-c)}\right]$$

$$= \frac{\Delta c}{\Delta^2}\,[2s^2 - s(a+b+c) + ab]$$

$$= (c/\Delta)\,[2s^2 - s(2s) + ab] = 4\,(abc/4\Delta) = 4R$$

Example 5.41 **Prove that $\cos A + \cos B + \cos C = 1 + r/R$.**

Sol. $\cos A + \cos B + \cos C = 1 + 4 \sin (A/2) \sin (B/2) \sin (C/2)$

$$= 1 + [4R \sin (A/2) \sin (B/2) \sin (C/2)]/R = 1 + r/R$$

Example 5.42 **Prove that $\dfrac{a\cos A + b\cos B + c\cos C}{a+b+c} = \dfrac{r}{R}$.**

Sol. We have,

$a \cos A + b \cos B + c \cos C = R(2 \sin A \cos A + 2 \sin B \cos B + 2 \sin A \sin C)$

$$= R(\sin 2A + \sin 2B + \sin 2C)$$

$$= 4R \sin A \sin B \sin C$$

and $a+b+c=2R(\sin A+\sin B+\sin C)=8R4\cos(A/2)\cos(B/2)\cos(C/2)$

$$\Rightarrow \frac{a\cos A+b\cos B+c\cos C}{a+b+c}$$

$$=\frac{4R\sin A\sin B\sin C}{8R\cos A/2\ \cos B/2\ \cos C/2}$$

$=[4R\sin(A/2)\sin(B/2)\sin(C/2)]/R=r/R$

Example 5.43 **If in a triangle $r_1=r_2+r_3+r$, prove that the triangle is right angled.**

Sol. We have $r_1=r_2+r_3+r$

$\Rightarrow\ r_1-r=r_2+r_3$

$$\Rightarrow\ \frac{\Delta}{s-a}-\frac{\Delta}{s}=\frac{\Delta}{s-b}+\frac{\Delta}{s-c}$$

$$\Rightarrow\ \frac{\Delta a}{s(s-a)}=\frac{\Delta(2s-b-c)}{(s-b)(s-c)}=\frac{\Delta a}{(s-b)(s-c)}$$

$\Rightarrow\ s(s-a)=(s-b)(s-c)$

$\Rightarrow\ s^2-sa=s^2-(b+c)s+bc$

$\Rightarrow\ 2s(b+c-a)=2bc$

$\Rightarrow\ (a+b+c)(b+c-a)=2bc$

$\Rightarrow\ (b+c)^2-a^2=2bc$

$\Rightarrow\ b^2+c^2=a^2$

Hence, the triangle is right angled.

Example 5.44 **Prove that $\frac{r_1+r_2}{1+\cos C}=2R$.**

Sol. $$r_1+r_2=4R\left(\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}+\sin\frac{B}{2}\cos\frac{A}{2}\cos\frac{C}{2}\right)$$

$$=4R\left(\cos\frac{C}{2}\left(\sin\frac{A}{2}\cos\frac{B}{2}+\sin\frac{B}{2}\cos\frac{A}{2}\right)\right)$$

$$=4R\left(\cos^2\frac{C}{2}\right)=2R(1+\cos C)$$

$$\Rightarrow\ \frac{r_1+r_2}{1+\cos C}=2R$$

Concept Application Exercise 5.6

1. In ΔABC, if $r_1<r_2<r_3$, then find the order of lengths of the sides.
2. Find the radius of the in-circle of a triangle where sides are 18, 24 and 30 cm.
3. If in ΔABC, $(a-b)(s-c)=(b-c)(s-a)$, prove that r_1, r_2, r_3 are in A.P.
4. In triangle ABC, $\angle A=\frac{\pi}{2}$, prove that $r+2R=\frac{1}{2}(b+c+a)$.

5. Prove that $\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}=\frac{1}{r}$.

6. Prove that $r_1r_2+r_2r_3+r_3r_1=\frac{1}{4}(a+b+c)^2$.

7. In any triangle ABC, find the least value of $\frac{r_1+r_2+r_3}{r}$.

Geometry Relating to Ex-centres

Consider an acute-angled ΔABC.

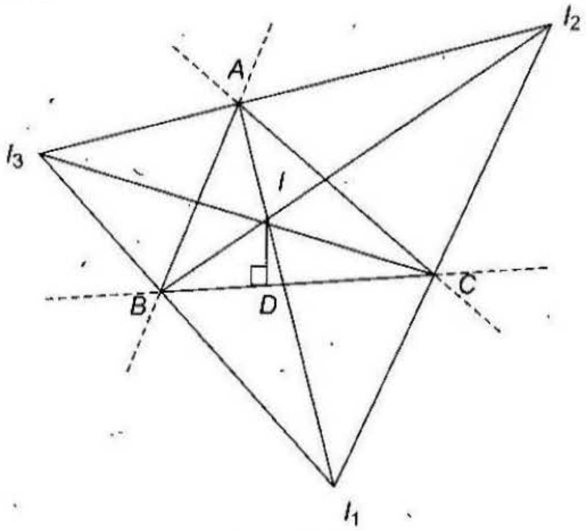


Fig. 5.33

At I_1, external bisectors of $\angle B$ and $\angle C$ and internal bisector of $\angle A$ are concurrent.

Also $\angle IBC = B/2$ and $\angle CBI_1 = \frac{\pi}{2}-\frac{B}{2}$ $\Rightarrow \angle IBI_1 = \pi/2$ or $BI \perp I_1I_3$

Similarly, $CI \perp I_1I_2$ and $AI \perp I_2I_3$.

Thus, the in-centre of triangle ABC is orthocentre of $\Delta I_1I_2I_3$ and ABC is the pedal triangle of $\Delta I_1I_2I_3$.

Distance Between In-centre and Ex-centre

In ΔIDB, $BI = \frac{ID}{\sin \angle IBD} = \frac{r}{\sin(B/2)}$

Also in ΔIBI_1, $II_1 = \frac{BI}{\cos \angle BII_1} = \frac{r}{\sin(B/2)\cos\left(\frac{\pi}{2}-\frac{C}{2}\right)} = \frac{r}{\sin(B/2)\sin(C/2)}$

Similarly, $II_2 = \frac{r}{\sin(A/2)\sin(C/2)}$ and $II_3 = \frac{r}{\sin(A/2)\sin(B/2)}$

Distance Between Ex-centres

Let us find I_1I_2. Points B, I, C and I_1 are concyclic.

Hence, $\angle II_1C = \angle IBC = B/2$.

Similarly, points A, I, C and I_2 are concyclic. So, $\angle II_2C = \angle IAC = A/2$.

Then in ΔII_1I_2, $\angle I_1II_2 = \pi - \frac{A+B}{2}$.

Now in $\Delta\, I_1 I_2$ from the sine rule, we get

$$\frac{I_1 I_2}{\sin\left(\pi - \frac{A+B}{2}\right)} = \frac{I I_1}{\sin\left(\frac{A}{2}\right)}$$

$$\Rightarrow \quad \frac{I_1 I_2}{\cos\left(\frac{C}{2}\right)} = \frac{\dfrac{r}{\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)}}{\sin\left(\frac{A}{2}\right)}$$

$$\Rightarrow \quad I_1 I_2 = \frac{r\cos\left(\frac{C}{2}\right)}{\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)} = 4R\cos\left(\frac{C}{2}\right)$$

Similarly, $I_2 I_3 = 4R\cos\left(\frac{A}{2}\right)$ and $I_1 I_3 = 4R\cos\left(\frac{B}{2}\right)$.

MISCELLANEOUS TOPICS

m–n Theorem

Let D be a point on the side BC of a $\Delta\, ABC$ such that $BD{:}DC = m{:}n$ and $\angle ADC = \theta$, $\angle BAD = \alpha$ and $\angle DAC = \beta$. Then

i. $(m+n)\cot\theta = m\cot\alpha - n\cot\beta$
ii. $(m+n)\cot\theta = n\cot B - m\cot C$

Proof:

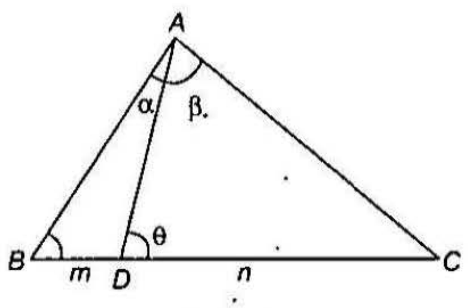


Fig. 5.34

i. Given $\dfrac{BD}{DC} = \dfrac{m}{n}$ and $\angle ADC = \theta$

$\angle ADB = (180° - \theta)$, $\angle BAD = \alpha$ and $\angle DAC = \beta$

$\angle ABD = 180° - (\alpha + 180° - \theta) = \theta - \alpha = B$

From ΔABD, $\dfrac{BD}{\sin\alpha} = \dfrac{AD}{\sin(\theta - \alpha)}$ (i)

From ΔADC, $\dfrac{DC}{\sin\beta}=\dfrac{AD}{\sin(\theta+\beta)}$ (ii)

Dividing Eq. (i) by Eq. (ii), we get

$$\frac{BD\sin\beta}{DC\sin\alpha}=\frac{\sin(\theta+\beta)}{\sin(\theta-\alpha)} \quad \text{(iii)}$$

$$\Rightarrow \quad \frac{m}{n}\frac{\sin\beta}{\sin\alpha}=\frac{\sin\theta\cos\beta+\cos\theta\sin\beta}{\sin\theta\cos\alpha-\cos\theta\sin\alpha}$$

$\Rightarrow \quad m\sin\beta(\sin\theta\cos\alpha-\cos\theta\sin\alpha)=n\sin\alpha(\sin\theta\cos\beta+\cos\theta\sin\beta)$

$\Rightarrow \quad m\cot\alpha-m\cot\theta=n\cot\beta+n\cot\theta$ [dividing both sides by $\sin\alpha\sin\beta\sin\theta$]

$\Rightarrow \quad (m+n)\cot\theta=m\cot\alpha-n\cot\beta$ (iv)

ii. We have $\angle CAD=180°-(\theta+C)$

$\angle ABC=B, \angle ACD=C, \angle BAD=(\theta-B)$

Putting these values in Eq. (iii), we get

$m\sin(\theta+C)\sin B=n\sin C\sin(\theta-B)$

$\Rightarrow \quad m(\sin\theta\cos C+\cos\theta\sin C)\sin B=n\sin C(\sin\theta\cos B-\cos\theta\sin B)$

Dividing both sides by $\sin\theta\sin B\sin C$, we get

$m(\cot C+\cot\theta)=n(\cot B-\cot\theta)$

$\therefore \quad (m+n)\cot\theta=n\cot B-m\cot C$

Example 5.45 **If the median AD of triangle ABC makes an angle $\pi/4$ with the side BC, then find the value of $|\cot B-\cot C|$.**

Sol.

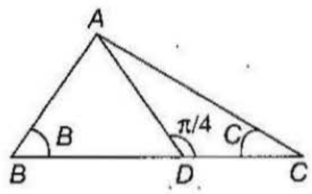


Fig. 5.35

By m–n theorem,

$(BD+DC)\cot(\pi/4)=DC\cot B-BD\cot C \quad \Rightarrow |\cot B-\cot C|=2$

Inequality

In Chapter 2, we have proved that $\cos A+\cos B+\cos C\le\dfrac{3}{2}$. (i)

Also in ΔABC, $\cos A+\cos B+\cos C=1+4\sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2}$

$$\Rightarrow \quad \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\le\frac{1}{8} \quad \text{(ii)}$$

In ΔABC, $r=4R\sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2}\Rightarrow R\ge 2r$ [using Eq. (ii)]

Example 5.46 **Prove that $a\cos A+b\cos B+c\cos C\le s$.**

Sol. $a\cos A+b\cos B+c\cos C=R(2\sin A\cos A+2\sin B\cos B+2\sin C\cos C)$

$$= R(\sin 2A + \sin 2B + \sin 2C)$$

$$= 4R \sin A \sin B \sin C = \frac{2}{R}\left(2R^2 \sin A \sin B \sin C\right)$$

$$= \frac{2}{R}\Delta = 2\frac{rs}{R} \le s \qquad [\because R \ge 2r]$$

Example 5.47 **In triangle ABC, prove that the maximum value of $\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}$ is $\frac{R}{2s}$.**

Sol. For triangle ABC, we have

$$\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}\sqrt{\frac{(s-a)(s-c)}{s(s-b)}}\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$$

$$= \frac{\sqrt{s(s-a)(s-b)(s-c)}}{s^2}$$

$$= \frac{\Delta}{s^2} = \frac{r}{s} \le \frac{R}{2s}$$

Area of Quadrilateral

$ABCD$ is any quadrilateral where $AB = a$, $BC = b$, $CD = c$, $AD = d$ and $\angle DPA = \alpha$. Let us denote the area of the quadrilateral by S, then ΔDAC = area of ΔAPD + area of ΔDPC.

$$= \frac{1}{2} DP \times AP \times \sin\alpha + \frac{1}{2} DP \times PC \times \sin(\pi - \alpha)$$

$$= \frac{1}{2} DP(AP + PC)\sin\alpha$$

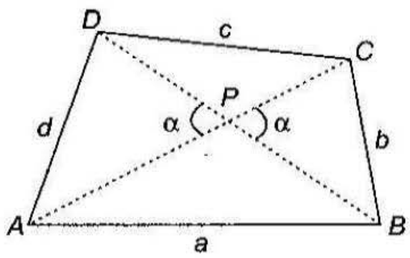


Fig. 5.36

Area of $\Delta DAC = \frac{1}{2} DP \times AC \times \sin\alpha$ (i)

Similarly, area of $\Delta ABC = \frac{1}{2} BP \times AC\sin\alpha$ (ii)

$\therefore$ S = area of ΔDAC + area of ΔABC

$$= \frac{1}{2} DP \times AC\sin\alpha + \frac{1}{2} BP \times AC\sin\alpha \qquad \text{[using (ii) and (iii)]}$$

$$= \frac{1}{2}(DP + BP)AC\sin\alpha \qquad \Rightarrow S = \frac{1}{2} BD \times AC\sin\alpha \qquad \text{(iii)}$$

Therefore, area of quadrilateral $= \frac{1}{2}$ (product of the diagonals) $\times$ (sine of included angle).

Cyclic Quadrilateral

A cyclic quadrilateral is a quadrilateral which can be circumscribed by a circle.

Note:

- *Sum of the opposite angles of a cyclic quadrilateral is* 180°.
- *In a cyclic quadrilateral, sum of the products of the opposite sides is equal to the product of the diagonals. This is known as Ptolemy's theorem.*
- *If sum of the opposite sides of a quadrilateral is equal, then and only then a circle can be inscribed in the quadrilateral.*

Regular Polygon

A regular polygon is a polygon which has equal sides as well as equal angles. In any polygon of n sides, sum of its internal angles is $(n-2)\pi$, then in regular polygon each angle is $\frac{(n-2)\pi}{n}$.

Note:

- *In the regular polygon, the circumcentre and the in-centre are the same.*

Radii of the inscribed and the circumscribed circles and area of a regular polygon of n sides with each side a.

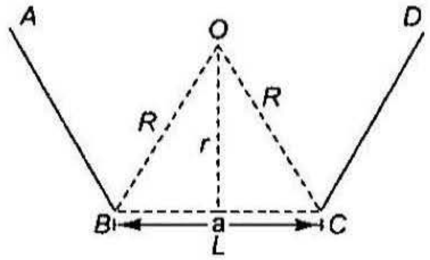


Fig. 5.37

Let AB, BC and CD be three successive sides of the polygon and O be the centre of both the incircle and the circumcircle of the polygon.

$$\angle BOC = \frac{2\pi}{n} \qquad \Rightarrow \angle BOL = \frac{1}{2}\left(\frac{2\pi}{n}\right) = \frac{\pi}{n}$$

$$a = BC = 2\,BL = 2R\sin\angle BOL = 2R\sin\frac{\pi}{n} \qquad \Rightarrow R = \frac{a}{2}\operatorname{cosec}\frac{\pi}{n}$$

Again, $a = 2\,BL = 2\,OL\tan\angle BOL = 2r\tan\frac{\pi}{n} \qquad \Rightarrow r = \frac{a}{2}\cot\frac{\pi}{n}$

Now, the area of the regular polygon $= n$ times the area of the $\Delta OBC = n\left(\frac{1}{2}OL \times BC\right)$

$$= n\frac{1}{2}\left(\frac{a}{2}\cot\frac{\pi}{n}\right)a$$

$$= \frac{na^2}{4}\cot\frac{\pi}{n} \quad \text{[in terms of side of polygon]} \qquad \text{(i)}$$

Now, $a = 2r\tan\frac{\pi}{n} \Rightarrow \Delta = nr^2\tan\left(\frac{\pi}{n}\right)$ [from Eq. (i)]

Also, $a = 2R\sin\frac{\pi}{n} \Rightarrow \Delta = \frac{nR^2}{2}\sin\left(\frac{2\pi}{n}\right)$ [from Eq. (i)]

Example 5.48 Find the sum of the radii of the circles, which are, respectively, inscribed and circumscribed about a polygon of n sides, whose side length is a.

Sol. Radius of the circumscribed circle $= R = \frac{a}{2} \operatorname{cosec} \frac{\pi}{n}$

and radius of the inscribed circle $= r = \frac{1}{2} a \cot(\pi/n)$

$$\Rightarrow \quad R + r = \frac{a}{2\sin(\pi/n)} + \frac{a\cos(\pi/n)}{2\sin(\pi/n)} = \frac{a[1+\cos(\pi/n)]}{2\times 2\sin(\pi/2n)\cos(\pi/2n)} = \frac{1}{2} a \cot\left(\frac{\pi}{2n}\right)$$

Example 5.49 If the area of the circle is A_1 and the area of the regular pentagon inscribed in the circle is A_2, then find the ratio A_1/A_2.

Sol.

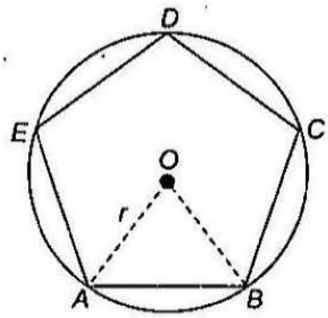


Fig. 5.38

In ΔOAB, $OA = OB = r$ and $\angle AOB = \frac{360°}{5} = 72°$

Therefore, area of $\Delta AOB = \frac{1}{2} r \times r \sin 72° = \frac{1}{2} r^2 \cos 18°$

Area of pentagon $(A_2) = 5$ (area of ΔAOB) $= 5\left(\frac{1}{2} r^2 \cos 18°\right)$ (i)

Also, area of the circle $(A_1) = \pi r^2$ (ii)

Hence, $\frac{A_1}{A_2} = \frac{\pi r^2}{\frac{5}{2} r^2 \cos 18°} = \frac{2\pi}{5} \sec\left(\frac{\pi}{10}\right)$ [from Eqs. (i) and (ii)]

Example 5.50 Prove that the area of a regular polygon of $2n$ sides inscribed in a circle is the geometric mean of the areas of the inscribed and circumscribed polygons of n sides.

Sol. Let a be the radius of the circle.
Then,

S_1 = Area of regular polygon of n sides inscribed in the circle $= \frac{1}{2} na^2 \sin(2\pi/n)$

S_2 = Area of regular polygon of n sides circumscribing the circle $= na^2 \tan(\pi/n)$

S_3 = Area of regular polygon of $2n$ sides inscribed in the circle $= na^2 \sin(\pi/n)$

[replacing n by $2n$ is S_1]

Therefore, geometric mean of S_1 and $S_2 = \sqrt{(S_1 S_2)} = na^2 \sin(\pi/n) = S_3$

SOLUTION OF TRIANGLES (AMBIGUOUS CASES)

The three sides a, b, c and the three angles A, B, C are called the elements of the triangle ABC. When any three of these six elements (except all the three angles) of a triangle are given, the triangle is known completely; that is, the other three elements can be expressed in terms of the given elements and can be evaluated. This process is called the solution of triangles.

- If the three sides a, b and c are given, angle A is obtained from $\tan\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$ or $\cos A=\frac{b^2+c^2-a^2}{2bc}$. B and C can be obtained in a similar way.
- If two sides b and c and the included angle A are given, then $\tan\frac{B-C}{2}=\frac{b-c}{b+c}\cot\frac{A}{2}$ gives $\frac{B-C}{2}$. Also, $\frac{B+C}{2}=90°-\frac{A}{2}$, so that B and C can be evaluated. The third side is given by $a=b\frac{\sin A}{\sin B}$ or $a^2=b^2+c^2-2bc\cos A$.
- If two sides b and c and the angle B (opposite to side b) are given, then $\sin C=\frac{c}{b}\sin B$, $A=180°-(B+C)$ and $a=\frac{b\sin A}{\sin B}$ give the remaining elements.

Case I:
$b < c\sin B$
We draw the side c and angle B. Now, it is obvious from Fig. 5.39 that there is no triangle possible.

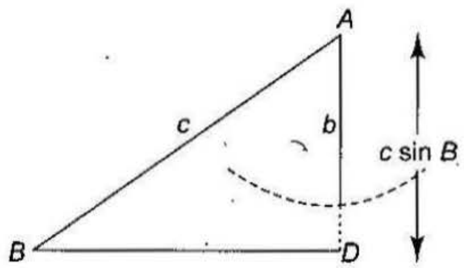


Fig. 5.39

Case II:
$b = c\sin B$ and B is an acute angle, then there is only one triangle possible.

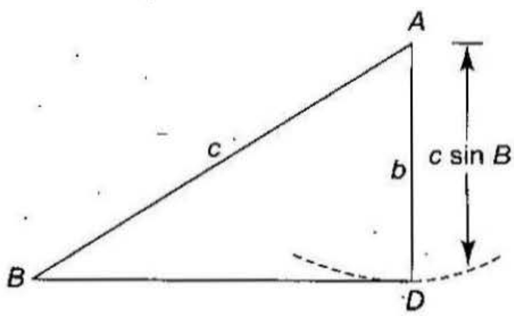


Fig. 5.40

Case III:

$b > c \sin B$, $b < c$ and B is an acute angle, then there are two values of angle C. Hence, two triangles are possible.

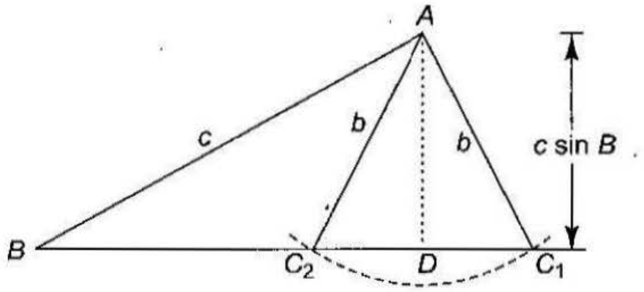


Fig. 5.41

Case IV:

$b > c \sin B$, $c < b$ and B is an acute angle, then there is only one triangle possible.

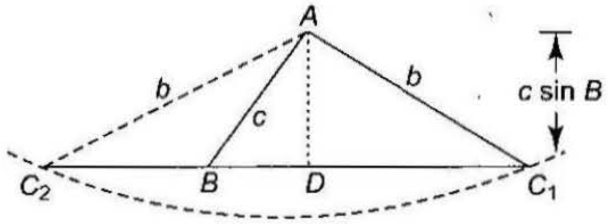


Fig. 5.42

Case V:

$b > c \sin B$, $c > b$ and B is an obtuse angle. For any choice of point C, b will be greater than c which is a contradiction as $c > b$ (given). So, there is no triangle possible.

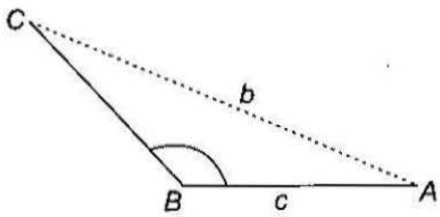


Fig. 5.43

Case VI:

$b > c \sin B$, $c < b$ and B is an obtuse angle. We can see that the circle with A as centre and b as radius will cut the line only in one point. So, only one triangle is possible.

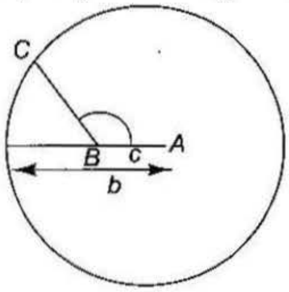


Fig. 5.44

Case VII:

$b > c$ and $B = 90°$

Again the circle with A as centre and b as radius will cut the line only in one point. So, only one triangle is possible.

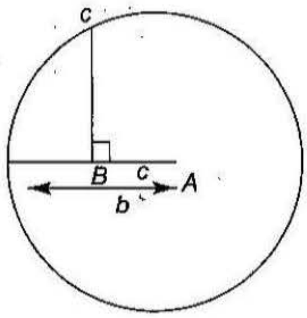


Fig. 5.45

Case VIII:

$b \le c$ and $B = 90°$

The circle with A as centre and b as radius will not cut the line in any point. So, no triangle is possible.

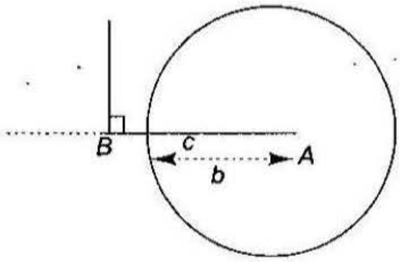


Fig. 5.46

Alternative method:

By applying cosine rule, we have $\cos B = \dfrac{a^2 + c^2 - b^2}{2ac}$

$\Rightarrow \quad a^2 - (2c \cos B)\, a + (c^2 - b^2) = 0$

$\Rightarrow \quad a = c \cos B \pm \sqrt{(c\cos B)^2 - (c^2 - b^2)}$

$\quad = c \cos B \pm \sqrt{b^2 - (c \sin B)^2}$

This equation leads to the following cases:

Case I: If $b < c \sin B$, no such triangle is possible.

Case II: Let $b = c \sin B$. There are further following two cases:

a. B is an obtuse angle $\Rightarrow \cos B$ is negative. There exists no such triangle.

b. B is an acute angle $\Rightarrow \cos B$ is positive. There exists only one such triangle.

Case III: Let $b > c \sin B$. There are further following two cases:

a. B is an acute angle $\Rightarrow \cos B$ is positive. In this case, two values of a will exist if and only if $c \cos B > \sqrt{b^2 - (c \sin B)^2}$ or $c > b \Rightarrow$ two such triangles are possible. If $c < b$, only one such triangle is possible.

b. B is an obtuse angle $\Rightarrow \cos B$ is negative. In this case, triangle will exist if and only if $\sqrt{b^2 - (c \sin B)^2} > c|\cos B| \Rightarrow b > c$. So, in this case, only one such triangle is possible. If $b < c$ there exists no such triangle.

Note:

- *If one side a and angles B and C are given, then* $A = 180° - (B + C)$ *and* $b = \dfrac{a \sin B}{\sin A}$, $c = \dfrac{a \sin C}{\sin A}$.
- *If the three angles A, B and C are given, we can only find the ratios of the sides a, b and c by using the sine rule (since there are infinite number of similar triangles possible).*

Example 5.51 **If $b = 3$, $c = 4$ and $B = \pi/3$, then find the number of triangles that can be constructed.**

Sol. We have,

$$\frac{\sin B}{b} = \frac{\sin C}{c} \quad \Rightarrow \frac{\sin(\pi/3)}{3} = \frac{\sin C}{4} \quad \Rightarrow \sin C = \frac{2}{\sqrt{3}} > 1 \text{ which is not possible.}$$

Hence, no triangle is possible.

Example 5.52 **If $A = 30°$, $a = 7$ and $b = 8$ in ΔABC, then find the number of triangles that can be constructed.**

Sol. We have $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} \quad \Rightarrow \sin B = \dfrac{b \sin A}{a} = \dfrac{8 \sin 30°}{7} = 4/7$

Thus, we have, $b > a > b \sin A$.

Hence, angle B has two values given by $\sin B = 4/7$.

Example 5.53 **If in triangle ABC, $a = (1 + \sqrt{3})$ cm, $b = 2$ cm and $\angle C = 60°$, then find the other two angles and the third side.**

Sol. From $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$, we have

$$\frac{1}{2} = \frac{\left(1+\sqrt{3}\right)^2 + 4 - c^2}{2\left(1+\sqrt{3}\right)2}$$

$$\Rightarrow \quad 2 + 2\sqrt{3} = 1 + 3 + 2\sqrt{3} + 4 - c^2$$

$$\Rightarrow \quad c^2 = 6 \Rightarrow c = \sqrt{6} \text{ cm}$$

Also, $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c} \quad \Rightarrow \dfrac{\sin A}{1+\sqrt{3}} = \dfrac{\sin B}{2} = \dfrac{\sqrt{3}/2}{\sqrt{6}} \Rightarrow \sin B = \dfrac{1}{\sqrt{2}}$

$$\Rightarrow \quad B = 45° \Rightarrow A = 180° - (60° + 45°) = 75°$$

Example 5.54 **In ΔABC, the sides b, c and the angle B are given such that a has two values a_1 and a_2. Then prove that $|a_1 - a_2| = 2\sqrt{b^2 - c^2 \sin^2 B}$.**

Sol. $\cos B = \dfrac{c^2 + a^2 - b^2}{2ca}$

$$\Rightarrow \quad a^2 - 2c \cos B\, a + c^2 - b^2 = 0$$

$$\Rightarrow \quad a_1 + a_2 = 2c \cos B,\ a_1 a_2 = c^2 - b^2$$

$$\Rightarrow \quad (a_1 - a_2)^2 = (a_1 + a_2)^2 - 4a_1 a_2$$

$$= 4c^2 \cos^2 B - 4(c^2 - b^2) = 4b^2 - 4c^2 \sin^2 B = 4(b^2 - c^2 \sin^2 B)$$

$$\Rightarrow \quad |a_1 - a_2| = 2\sqrt{b^2 - c^2 \sin^2 B}$$

Example 5.55 In ΔABC, a, c and A are given and b_1, b_2 are two values of the third side b such that $b_2 = 2b_1$. Then prove that $\sin A = \sqrt{\frac{9a^2 - c^2}{8c^2}}$.

Sol. We have $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$ $\Rightarrow b^2 - 2bc\cos A + (c^2 - a^2) = 0$

It is given that b_1 and b_2 are the roots of this equation.

Therefore, $b_1 + b_2 = 2c\cos A$ and $b_1 b_2 = c^2 - a^2$

$\Rightarrow$ $3b_1 = 2c\cos A, 2b_1^2 = c^2 - a^2$ $(\because b_2 = 2b_1 \text{ given})$

$\Rightarrow$ $2\left(\frac{2c}{3}\cos A\right)^2 = c^2 - a^2$

$\Rightarrow$ $8c^2(1 - \sin^2 A) = 9c^2 - 9c^2$

$\Rightarrow$ $\sin A = \sqrt{\frac{9a^2 - c^2}{8c^2}}$

EXERCISES

Subjective Type

Solutions on page 5.63

1. O is the circumcentre of ΔABC and R_1, R_2, R_3 are, respectively, the radii of the circumcircles of the triangles OBC, OCA and OAB. Prove that $\frac{a}{R_1} + \frac{b}{R_2} + \frac{c}{R_3} = \frac{abc}{R^3}$.
2. In triangle ABC, D is on AC such that $AD = BC$, $BD = DC$, $\angle DBC = 2x$ and $\angle BAD = 3x$, all angles are in degrees, then find the value of x.
3. If in ΔABC, the distances of the vertices from the orthocentre are x, y and z, then prove that $\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = \frac{abc}{xyz}$.
4. In ΔABC, a semicircle is inscribed, which lies on the side c. If x is the length of the angle bisector through angle C, then prove that the radius of the semicircle is $x \sin (C/2)$.
5. Prove that the distance between the circumcentre and the orthocentre of triangle ABC is $R\sqrt{1 - 8\cos A \cos B \cos C}$.
6. Prove that the distance between the circumcentre and the incentre of triangle ABC is $\sqrt{R^2 - 2Rr}$.
7. The two adjacent sides of a cyclic quadrilateral are 2 and 5 and the angle between them is 60°. If the area of the quadrilateral is $4\sqrt{3}$, find the remaining two sides.
8. If p and q are perpendiculars from the angular points A and B of the ΔABC drawn to any line through the vertex C, then prove that $a^2b^2 \sin^2 C = a^2p^2 + b^2q^2 - 2abpq \cos C$.
9. If I is the incentre of ΔABC and R_1, R_2 and R_3 are, respectively, the radii of the circumcircles of the triangles IBC, ICA and IAB, then prove that $R_1 R_2 R_3 = 2rR^2$.
10. In a circle of radius r, chords of lengths a and b cm subtend angles θ and 3θ, respectively at the centre. Show that $r = a\sqrt{\frac{a}{3a - b}}$ cm.

11. If in triangle ABC, the median AD and the perpendicular AE from the vertex A to the side BC divide the angle A into three equal parts, show that $\cos\frac{A}{3}\sin^2\frac{A}{3}=\frac{3a^2}{32bc}$.

12. Perpendiculars are drawn from the angles A, B and C of an acute-angled triangle on the opposite sides, and produced to meet the circumscribing circle. If these produced parts are α, β, γ, respectively, then show that $\frac{a}{\alpha}+\frac{b}{\beta}+\frac{c}{\gamma}=2(\tan A+\tan B+\tan C)$.

13. Show that the line joining the incentre to the circumcentre of triangle ABC is inclined to the side BC at an angle $\tan^{-1}\left(\frac{\cos B+\cos C-1}{\sin C-\sin B}\right)$.

14. If the tangents of the angles of a triangle are in A.P., prove that the squares of the sides are in the ratio of $x^2(x^2+9):(3+x^2)^2:9(1+x^2)$ where x is the tangent of the least or the greatest angle.

15. In ABC, right angled at C, if $\tan A=\sqrt{\frac{\sqrt{5}-1}{2}}$, show that the sides a, b and c are in G.P.

Objective Type

Solutions on page 5.72

Each question has four choices a, b, c, and d, out of which *only one* answer is correct.

1. If in ΔABC, $\sin A\cos B=\frac{\sqrt{2}-1}{\sqrt{2}}$ and $\sin B\cos A=\frac{1}{\sqrt{2}}$, then the triangle is

   a. equilateral b. isosceles c. right angled d. righ-angled isosceles

2. ABC is an equilateral triangle of side 4 cm. If R, r and h are the circumradius, inradius and altitude, respectively, then $\frac{R+r}{h}$ is equal to

   a. 4 b. 2 c. 1 d. 3

3. In ΔABC, if $\frac{\sin A}{c\sin B}+\frac{\sin B}{c}+\frac{\sin C}{b}=\frac{c}{ab}+\frac{b}{ac}+\frac{a}{bc}$, then the value of angle A is

   a. 120° b. 90° c. 60° d. 30°

4. A piece of paper is in the shape of a square of side 1 m long. It is cut at the four corners to make a regular polygon of eight sides (octagon). The area of the polygon is

   a. $2(\sqrt{2}-1)\,\text{m}^2$ b. $(\sqrt{2}-1)\,\text{m}^2$ c. $\frac{1}{\sqrt{2}}\,\text{m}^2$ d. none of these

5. If A, B and C are angles of a triangle such that angle A is obtuse, then $\tan B\tan C$ will be less than

   a. $\frac{1}{\sqrt{3}}$ b. $\frac{\sqrt{3}}{2}$ c. 1 d. none of these

6. In ΔABC, $\angle B=\pi/3$. The range of values of x, where $x=\sin A\sin C$, is the interval

   a. $\left[-\frac{1}{4},\frac{3}{4}\right]$ b. $\left(0,\frac{3}{4}\right)$ c. $\left(0,\frac{3}{4}\right]$ d. $\left[\frac{1}{4},\frac{3}{4}\right]$

7. If in ΔABC, AC is double of AB, then the value of $\cot \frac{A}{2} \cot \frac{B-C}{2}$ is equal to

a. $\frac{1}{3}$ b. $-\frac{1}{3}$ c. 3 d. $\frac{1}{2}$

8. In a right-angled isosceles triangle, the ratio of the circumradius and inradius is

a. $2(\sqrt{2}+1):1$ b. $(\sqrt{2}+1):1$ c. $2:1$ d. $\sqrt{2}:1$

9. In triangle ABC, $a = 5$, $b = 3$ and $c = 7$, the value of $3 \cos C + 7 \cos B$ is equal to

a. 5 b. 10 c. 7 d. 3

10. If in triangle ABC, $\angle B = 90°$, then $\tan^2(A/2)$ is

a. $\frac{b-c}{b+c}$ b. $\frac{b+c}{b-c}$ c. $\frac{2bc}{b-c}$ d. none of these

11. In ΔABC, if $b^2 + c^2 = 2a^2$, then value of $\frac{\cot A}{\cot B + \cot C}$ is

a. $\frac{1}{2}$ b. $\frac{3}{2}$ c. $\frac{5}{2}$ d. $\frac{5}{3}$

12. If $\sin\theta$ and $-\cos\theta$ are the roots of the equation $ax^2 - bx - c = 0$, where a, b and c are the sides of a triangle ABC, then $\cos B$ is equal to

a. $1-\frac{c}{2a}$ b. $1-\frac{c}{a}$ c. $1+\frac{c}{2a}$ d. $1+\frac{c}{3a}$

13. In ΔABC, $a^2 + b^2 + c^2 = ac + ab\sqrt{3}$, then the triangle is

a. equilateral b. isosceles c. right angled d. none of these

14. In ΔABC, $(a+b+c)(b+c-a) = kbc$ if

a. $k<0$ b. $k>0$ c. $0<k<4$ d. $k>4$

15. If one side of a triangle is double the other, and the angles on opposite sides differ by 60°, then the triangle is

a. equilateral b. obtuse angled c. right angled d. acute angled

16. In triangle ABC, if $r_1 = 2r_2 = 3r_3$, then $a:b$ is equal to

a. $\frac{5}{4}$ b. $\frac{4}{5}$ c. $\frac{7}{4}$ d. $\frac{4}{7}$

17. If in a triangle, $\left(1-\frac{r_1}{r_2}\right)\left(1-\frac{r_1}{r_3}\right) = 2$, then the triangle is

a. right angled b. isosceles c. equilateral d. none of these

18. In an equilateral triangle, the inradius, circumradius and one of the ex-radii are in the ratio

a. 2:3:5 b. 1:2:3 c. 1:3:7 d. 3:7:9

19. In any ΔABC, if $\cot\frac{A}{2}, \cot\frac{B}{2}, \cot\frac{C}{2}$ are in A.P., then a, b, c are in

a. A.P. b. G.P. c. H.P. d. none of these

20. In ΔABC, if $A = 30°$, $b = 2$, $c = \sqrt{3}+1$, then $\frac{C-B}{2}$ is equal to

a. 15° b. 30° c. 45° d. none of these

21. In triangle ABC, if $a:b:c = 7:8:9$, then $\cos A : \cos B$ is equal to

a. $\frac{11}{63}$ b. $\frac{22}{63}$ c. $\frac{2}{9}$ d. none of these

22. In triangle ABC, if $\cos A + \cos B + \cos C = \frac{7}{4}$, then $\frac{R}{r}$ is equal to

a. $\frac{3}{4}$ b. $\frac{4}{3}$ c. $\frac{2}{3}$ d. $\frac{3}{2}$

23. In ΔABC, $\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2}$ is equal to

a. $\frac{\Delta}{r^2}$ b. $\frac{(a+b+c)^2}{abc}2R$ c. $\frac{\Delta}{r}$ d. $\frac{\Delta}{Rr}$

24. In triangle ABC, $a^2 + c^2 = 2002b^2$, then $\frac{\cot A + \cot C}{\cot B}$ is equal to

a. $\frac{1}{2001}$ b. $\frac{2}{2001}$ c. $\frac{3}{2001}$ d. $\frac{4}{2001}$

25. If the hypotenuse of a right-angled triangle is four times the length of the perpendicular drawn from the opposite vertex to it, then the difference of the two acute angles will be

a. 60° B. 15° c. 75° d. 30°

26. In ΔABC, if $\sin^2\frac{A}{2}$, $\sin^2\frac{B}{2}$ and $\sin^2\frac{C}{2}$ are in H.P., then a, b and c will be in

a. A.P. b. GP. c. H.P. d. none of these

27. Given $b = 2$, $c = \sqrt{3}$, $\angle A = 30°$, then inradius of ΔABC is

a. $\frac{\sqrt{3}-1}{2}$ b. $\frac{\sqrt{3}+1}{2}$ c. $\frac{\sqrt{3}-1}{4}$ d. none of these

28. If P is a point on the altitude AD of the triangle ABC such that $\angle CBP = B/3$, then AP is equal to

a. $2a\sin\frac{C}{3}$ b. $2b\sin\frac{C}{3}$ c. $2c\sin\frac{B}{3}$ d. $2c\sin\frac{C}{3}$

29. In triangle ABC, $\angle A = \pi/2$, then $\tan(C/2)$ is equal to

a. $\frac{a-c}{2b}$ b. $\frac{a-b}{2c}$ c. $\frac{a-c}{b}$ d. $\frac{a-b}{c}$

30. With usual notations, in triangle ABC, $a\cos(B-C) + b\cos(C-A) + c\cos(A-B)$ is equal to

a. $\frac{abc}{R^2}$ b. $\frac{abc}{4R^2}$ c. $\frac{4abc}{R^2}$ d. $\frac{abc}{2R^2}$

31. In ΔABC, $\sin A + \sin B + \sin C = 1 + \sqrt{2}$ and $\cos A + \cos + \cos C = \sqrt{2}$ if the triangle is

a. equilateral b. isosceles c. right angled d. right-angled isosceles

32. If $\frac{r}{r_1} = \frac{r_2}{r_3}$, then

a. $A = 90°$ b. $B = 90°$ c. $C = 90°$ d. none of these

33. If in a ΔABC, $\cos 3A + \cos 3B + \cos 3C = 1$, then one angle must be exactly equal to

a. 90° b. 45° c. 120° d. none of these

34. If $\cos B\cos C + \sin B\sin C\sin^2 A = 1$, then triangle ABC is

a. isosceles and right angled

b. equilateral

c. isosceles whose equal angles are greater than $\pi/4$
d. none

35. In triangle ABC, internal angle bisector $\angle A$ makes an angle θ with side BC. The value of $\sin\theta$ is equal to
a. $\left|\sin\left(\frac{B-C}{2}\right)\right|$ c. $\left|\sin\left(\frac{B}{2}-C\right)\right|$ c. $\cos\left(\frac{B-C}{2}\right)$ d. $\cos\left(\frac{B}{2}-C\right)$

36. In an acute angled triangle ABC, $r+r_1=r_2+r_3$ and $\angle B>\frac{\pi}{3}$, then
a. $b+2c<2a<2b+2c$
b. $b+4c<4a<2b+4c$
c. $b+4c<4a<4b+4c$
d. $b+3c<3a<3b+3c$

37. In triangle ABC, $\angle A=30°$, $BC=2+\sqrt{5}$, then the distance of the vertex A from the orthocentre of the triangle is
a. 1 b. $(2+\sqrt{5})\sqrt{3}$ c. $\frac{\sqrt{3}+1}{2\sqrt{2}}$ d. $\frac{1}{2}$

38. If I is the incentre of a triangle ABC, then the ratio $IA:IB:IC$ is equal to
a. $\operatorname{cosec}\frac{A}{2}:\text{coses}\frac{B}{2}:\operatorname{cosec}\frac{C}{2}$
b. $\sin\frac{A}{2}:\sin\frac{B}{2}:\sin\frac{C}{2}$
c. $\sec\frac{A}{2}:\sec\frac{B}{2}:\sec\frac{C}{2}$
d. none of these

39. In ΔABC, the bisector of the angle A meets the side BC at D and the circumscribed circle at E, then DE equals
a. $\frac{a^2\sec\frac{A}{2}}{2(b+c)}$ b. $\frac{a^2\sin\frac{A}{2}}{2(b+c)}$ c. $\frac{a^2\cos\frac{A}{2}}{2(b+c)}$ d. $\frac{a^2\operatorname{cosec}\frac{A}{2}}{2(b+c)}$

40. If D is the mid-point of the side BC of triangle ABC and AD is perpendicular to AC, then
a. $3b^2=a^2-c^2$ b. $3a^2=b^2-3c^2$ c. $b^2=a^2-c^2$ d. $a^2+b^2=5c^2$

41. Two medians drawn from the acute angles of a right-angled triangle intersect at an angle $\pi/6$. If the length of the hypotenuse of the triangle is 3 units, then the area of the triangle (in sq. units) is
a. $\sqrt{3}$ b. 3 c. $\sqrt{2}$ d. 9

42. For triangle ABC, $R=5/2$ and $r=1$. Let I be the incentre of the triangle and D, E and F be the feet of the perpendiculars from I to BC, CA and AB, respectively. The value of $\frac{ID\times IE\times IF}{IA\times IB\times IC}$ is equal to
a. $\frac{5}{2}$ b. $\frac{5}{4}$ c. $\frac{1}{10}$ d. $\frac{1}{5}$

43. If the median of ΔABC through A is perpendicular to AB, then
a. $\tan A+\tan B=0$ b. $2\tan A+\tan B=0$ c. $\tan A+2\tan B=0$ d. none of these

44. In ΔABC, the median AD divides $\angle BAC$ such that $\angle BAD:\angle CAD=2:1$. Then $\cos(A/3)$ is equal to
a. $\frac{\sin B}{2\sin C}$ b. $\frac{\sin C}{2\sin B}$ c. $\frac{2\sin B}{\sin C}$ d. none of these

45. If in ΔABC, $b=3$ cm, $c=4$ cm and the length of the perpendicular from A to the side BC is 2 cm, then the number of solutions of the triangle is
a. 1 b. 0 c. 3 d. 2

46. In in triangle ABC, $\sum \sin\frac{A}{2}=\frac{6}{5}$ and $\sum II_1=9$ (where I_1, I_2 and I_3 are ex-centres and I is in-centre, then circumradius R is equal to

a. $\frac{15}{8}$ b. $\frac{15}{4}$ c. $\frac{15}{2}$ d. $\frac{4}{12}$

47. In triangle ABC, medians AD and CE are drawn. If $AD=5$, $\angle DAC=\pi/8$ and $\angle ACE=\pi/4$, then the area of the triangle ABC is equal to

a. $\frac{25}{9}$ b. $\frac{25}{3}$ c. $\frac{25}{18}$ d. $\frac{10}{3}$

48. In triangle ABC, if $\tan(A/2)=5/6$ and $\tan(B/2)=20/37$, the sides a, b and c are in

a. A.P. b. G.P. c. H.P. d. none of these

49. If H is the orthocentre of a acute-angled triangle ABC whose circumcircle is $x^2+y^2=16$, then circumdiameter of the triangle HBC is

a. 1 b. 2 c. 4 d. 8

50. In triangle ABC, $a=5$, $b=4$ and $c=3$. G is the centroid of the triangle. Circumradius of triangle GAB is equal to

a. $2\sqrt{13}$ b. $\frac{5}{12}\sqrt{13}$ c. $\frac{5}{3}\sqrt{13}$ d. $\frac{3}{2}\sqrt{13}$

51. In triangle ABC, line joining circumcentre and incentre is parallel to side AC, then $\cos A+\cos C$ is equal to

a. -1 b. 1 c. -2 d. 2

52. In triangle ABC, line joining the circumcentre and orthocentre is parallel to side AC, then the value of $\tan A\tan C$ is equal to

a. $\sqrt{3}$ b. 3 c. $3\sqrt{3}$ d. none of these

53. If in ΔABC, $8R^2=a^2+b^2+c^2$, then the triangle ABC is

a. right angled b. isosceles c. equilateral d. none of these

54. In triangle ABC, $\frac{a}{b}=\frac{2}{3}$ and $\sec^2 A=\frac{8}{5}$. Then the number of triangles satisfying these conditions is

a. 0 b. 1 c. 2 d. 3

55. We are given b, c and $\sin B$ such that B is acute and $b<c\sin B$. Then

a. no triangle is possible b. one triangle is possible
c. two triangles are possible d. a right-angled triangle is possible

56. If a, b and A are given in a triangle and c_1, c_2 are the possible values of the third side, then $c_1^2+c_2^2-2c_1c_2\cos A=$ is equal to

a. $4a^2\sin 2A$ b. $4a^2\sin^2 A$ c. $4a^2\cos 2A$ d. $4a^2\cos^2 A$

57. In ΔABC, a, b, A are given and c_1, c_2 are two values of the third side c. The sum of the areas of the two triangles with sides a, b, c_1 and a, b, c_2 is

a. $(1/2)b^2\sin 2A$ b. $(1/2)a^2\sin 2A$ c. $b^2\sin 2A$ d. none of these

58. The area of the circle and the area of a regular polygon of n sides and of perimeter equal to that of the circle are in the ratio of

a. $\tan\left(\frac{\pi}{n}\right):\frac{\pi}{n}$ b. $\cos\left(\frac{\pi}{n}\right):\frac{\pi}{n}$ c. $\sin\frac{\pi}{n}:\frac{\pi}{n}$ d. $\cot\left(\frac{\pi}{n}\right):\frac{\pi}{n}$

59. The ratio of the area of a regular polygon of n sides inscribed in a circle to that of the polygon of same number of sides circumscribing the same circle is 3:4. Then the value of n is

a. 6 b. 4 c. 8 d. 12

60. In triangle ABC, if P, Q, R divides sides BC, AC and AB, respectively, in the raito $k : 1$ (in order). If the ratio $\left(\frac{\text{area } PQR}{\text{area } ABC}\right)$ is $\frac{1}{3}$, then k is equal to

a. 1/3 b. 2 c. 3 d. none of these

61. The side of triangle ABC are in A.P. (order being a, b, c) and satisfy $\frac{2!}{1!9!}+\frac{2!}{3!7!}+\frac{1}{5!5!}=\frac{8^a}{(2b)!}$, then the value of $\cos A + \cos B$ is

a. $\frac{12}{7}$ b. $\frac{13}{7}$ c. $\frac{11}{7}$ d. $\frac{10}{7}$

62. Let ABC be a triangle with $\angle B = 90°$. Let AD be the bisector of $\angle A$ with D on BC. Supoose $AC = 6$ cm and the area of the triangle ADC is 10 cm^2. Then the length of BD in cm is equal to

a. $\frac{3}{5}$ b. $\frac{3}{10}$ c. $\frac{5}{3}$ d. $\frac{10}{3}$

63. In any triangle ABC, $\frac{a^2+b^2+c^2}{R^2}$ has the maximum value of

a. 3 b. 6 c. 9 d. none of these

64. If a, b and c are the sides of a triangle, then the minimum value of $\frac{2a}{b+c-a}+\frac{2b}{c+a-b}+\frac{2c}{a+b-c}$ is

a. 3 c. 9 c. 6 d. 1

65. In any triangle, the minimum value of $r_1r_2r_3/r^3$ is equal to

a. 1 b. 9 c. 27 d. none of these

66. In a convex quadrilateral $ABCD$, $AB = a$, $BC = b$, $CD = c$ and $DA = d$. This quadrilateral is such that a circle can be inscribed in it and a circle can be also circumscribed about it, then $\tan^2(A/2)$ is equal to

a. $\frac{ad}{bc}$ b. $\frac{ab}{cd}$ c. $\frac{cd}{ab}$ d. $\frac{bc}{ad}$

67. In triangle ABC, $\sin A$, $\sin B$ and $\sin C$ are in A.P, then

a. the altitudes are in H.P. b. the altitudes are in A.P.
c. the altitudes are in G.P. d. none of these

68. In triangle ABC, $\angle C = 2\pi/3$ and CD is the internal angle bisector of $\angle C$, meeting the side AB at D. Length CD is equal to

a. $\frac{ab}{2(a+b)}$ b. $\frac{2ab}{a+b}$ c. $\frac{2ab}{\sqrt{3}(a+b)}$ d. $\frac{ab}{a+b}$

69. In $\Delta\, ABC$, let R = circumradius, r = inradius, if r is the distance between the circumcentre and the incentre, then ratio R/r is equal to

a. $\sqrt{2}-1$ b. $\sqrt{3}-1$ c. $\sqrt{2}+1$ d. $\sqrt{3}+1$

70. In the given figure, AB is the diameter of the circle, centered at 'O'. If $\angle\, COA = 60°$, $AB = 2r$, $AC = d$ and $CD = l$, then l is equal to

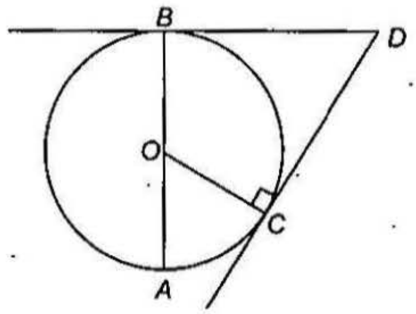


Fig. 5.47

a. $d\sqrt{3}$
b. $d/\sqrt{3}$
c. $3d$
d. $\sqrt{3}d/2$

71. In triangle ABC, base BC and area of triangle Δ are fixed. Locus of the centroid of triangle ABC is a straight line that is
   a. parallel to side BC
   b. right bisector of side BC
   c. right angle of BC
   d. inclined at an angle $\sin^{-1}\left(\frac{\sqrt{\Delta}}{BC}\right)$ to side BC

72. Let AD be a median of the ΔABC. If AE and AF are medians of the triangle ABD and ADC, respectively, and $AD = m_1$, $AE = m_2$, $AF = m_3$, then $a^2/8$ is equal to
   a. $m_2^2 + m_3^2 - 2m_1^2$
   b. $m_1^2 + m_2^2 - 2m_3^2$
   c. $m_1^2 + m_3^2 - 2m_2^2$
   d. none of these

73. A variable triangle ABC is circumscribed about a fixed circle of unit radius. Side BC always touches the circle at D and has fixed direction. If B and C vary in such a way that $(BD)(CD) = 2$, then locus of vertex A will be a straight line
   a. parallel to side BC
   b. perpendicular to side BC
   c. making an angle $(\pi/6)$ with BC
   d. making an angle $\sin^{-1}(2/3)$ with BC

74. In triangle ABC, $\angle A = \pi/3$ and its incircle is of unit radius. If the radius of the circle touching the sides AB, AC internally and incircle externally is x, then the value of x is
   a. 1/2
   b. 1/4
   c. 1/3
   d. none of these

75. The radii r_1, r_2, r_3 of the escribed circles of the triangle ABC are in H.P. If the area of the triangle is 24 cm^2 and its perimeter is 24 cm, then the length of its largest side is
   a. 10
   b. 9
   c. 8
   d. none of these

76. If 'O' is the circumcentre of ΔABC and R_1, R_2 and R_3 are the radii of the circumcircles of triangles OBC, OCA and OAB, respectively, then $\frac{a}{R_1} + \frac{b}{R_2} + \frac{c}{R_3}$ has the value equal to
   a. $\frac{abc}{2R^3}$
   b. $\frac{R^3}{abc}$
   c. $\frac{4\Delta}{R^2}$
   d. $\frac{\Delta}{4R^2}$

77. In triangle ABC, If $A - B = 120°$ and $R = 8r$ where R and r have their usual meaning, then $\cos C$ equals
   a. 3/4
   b. 2/3
   c. 5/6
   d. 7/8

78. In triangle ABC, $\angle A = 60°$, $\angle B = 40°$ and $\angle C = 80°$. If P is the centre of the circumcircle of triangle ABC with radius unity, then the radius of the circumcircle of triangle BPC is

a. 1 b. $\sqrt{3}$ c. 2 d. $\sqrt{3}/2$

79. Let area of triangle ABC is $\left(\sqrt{3}-1\right)/2$, $b = 2$ and $c = \left(\sqrt{3}-1\right)$ and $\angle A$ is acute. The measure of the angle C is

a. 15° b. 30° c. 60° d. 75°

80. In triangle ABC, $R(b+c) = a\sqrt{bc}$ where R is the circumradius of the triangle. Then the triangle is

a. isosceles but not right b. right but not isosceles
c. right isosceles d. equilateral

81. In triangle ABC, $\angle ABC = 120°$, $AB = 3$ and $BC = 4$. If perpendicular constructed to the side AB at A and to the side BC at C meets at D, then CD is equal to

a. 3 b. $\dfrac{8\sqrt{3}}{3}$ c. 5 d. $\dfrac{10\sqrt{3}}{3}$

Multiple Correct Answers Type

Solutions on page 5.95

Each question has four choices a, b, c, and d, out of which one or more answers are correct.

1. If the tangents of the angles A and B of triangle ABC satisfy the equation $abx^2 - c^2x + ab = 0$, then

a. $\tan A = a/b$ b. $\tan B = b/a$
c. $\cos C = 0$ d. $\sin^2 A + \sin^2 B + \sin^2 C = 2$

2. In a triangle, the angles are in A.P. and the lengths of the two larger sides are 10 and 9, respectively, then the length of the third side can be

a. $5+\sqrt{6}$ b. 0.7 c. $5-\sqrt{6}$ d. none of these

3. In triangle ABC if $2a^2b^2 + 2b^2c^2 = a^4 + b^4 + c^4$, then angle B is equal to

a. 45° b. 135° c. 120° d. 60°

4. If in triangle ABC, a, b, c and angle A are given and $c\sin A < a < c$, then

a. $b_1 + b_2 = 2c\cos A$ b. $b_1 + b_2 = c\cos A$ c. $b_1b_2 = c^2 - a^2$ d. $b_1b_2 = c^2 + a^2$

5. There exists triangle ABC satisfying

a. $\tan A + \tan B + \tan C = 0$

b. $\dfrac{\sin A}{2} = \dfrac{\sin B}{3} = \dfrac{\sin C}{7}$

c. $(a+b)^2 = c^2 + ab$ and $\sqrt{2}\,(\sin A + \cos A) = \sqrt{3}$

d. $\sin A + \sin B = \dfrac{\sqrt{3}+1}{2}$, $\cos A \cos B = \dfrac{\sqrt{3}}{4} = \sin A \sin B$

6. CF is the internal bisector of angle C of ΔABC, then CF is equal to

a. $\dfrac{2ab}{a+b}\cos\dfrac{C}{2}$ b. $\dfrac{a+b}{2ab}\cos\dfrac{C}{2}$ c. $\dfrac{b\sin A}{\sin\left(B+\dfrac{C}{2}\right)}$ d. none of these

7. The sides of ΔABC satisfy the equation $2a^2 + 4b^2 + c^2 = 4ab + 2ac$. Then

a. the triangle is isosceles b. the triangle is obtuse
c. $B = \cos^{-1}(7/8)$ d. $A = \cos^{-1}(1/4)$

8. Let ABC be an isosceles triangle with base BC. If 'r' is the radius of the circle inscribed in ΔABC and r_1 is the radius of the circle escribed opposite to the angle A, then the product $r_1 r$ can be equal to

a. $R^2 \sin^2 A$ b. $R^2 \sin^2 2B$ c. $\frac{1}{2}a^2$ d. $\frac{a^2}{4}$

where R is the radius of the circumcircle of the ΔABC.

9. If in a triangle, $\sin^4 A + \sin^4 B + \sin^4 C = \sin^2 B \sin^2 C + 2\sin^2 C \sin^2 A + 2\sin^2 A \sin^2 B$, then its angle A is equal to

a. 30° b. 120° c. 150° d. 60°

10. The area of a regular polygon of n sides is (where r is inradius, R is circumradius and a is side of the triangle)

a. $\frac{nR^2}{2}\sin\left(\frac{2\pi}{n}\right)$ b. $nr^2 \tan\left(\frac{\pi}{n}\right)$ c. $\frac{na^2}{4}\cot\frac{\pi}{n}$ d. $nR^2 \tan\left(\frac{\pi}{n}\right)$

11. In acute-angled triangle ABC, AD is the altitude. Circle drawn with AD as its diameter cuts the AB and AC at P and Q, respectively. Length PQ is equal to

a. $\frac{\Delta}{2R}$ b. $\frac{abc}{4R^2}$ c. $2R\sin A \sin B \sin C$ d. $\frac{\Delta}{R}$

12. If A is the area and $2s$ is the sum of the sides of a triangle, then

a. $A \le \frac{s^2}{4}$ b. $A \le \frac{s^2}{3\sqrt{3}}$ c. $A < \frac{s^2}{\sqrt{3}}$ d. none of these

13. If the angles of a triangle are 30° and 45°, and the included side is $(\sqrt{3}+1)$ cm, then

a. area of the triangle is $\frac{1}{2}(\sqrt{3}+1)$ sq. units

b. area of the triangle is $\frac{1}{2}(\sqrt{3}-1)$ sq. units

c. ratio of greater side to smaller side is $\frac{\sqrt{3}+1}{\sqrt{2}}$

d. ratio of greater side to smaller side is $\frac{1}{4\sqrt{3}}$

14. Sides of ΔABC are in A.P. If $a < \min\{b, c\}$, then $\cos A$ may be equal to

a. $\frac{4b-3c}{2b}$ b. $\frac{3c-4b}{2c}$ c. $\frac{4c-3b}{2b}$ d. $\frac{4c-3b}{2c}$

15. Lengths of the tangents from A, B and C to the incircle are in A.P., then

a. r_1, r_2, r_3 are in H.P. b. r_1, r_2, r_3 are in A.P. c. a, b, c are in A.P. d. $\cos A = \frac{4c-3b}{2b}$

16. If sides of triangle ABC are a, b and c such that $2b = a + c$, then

a. $\frac{b}{c} > \frac{2}{3}$ b. $\frac{b}{c} > \frac{1}{3}$ c. $\frac{b}{c} < 2$ d. $\frac{b}{c} < \frac{3}{2}$

17. In ΔABC, if $\cos\dfrac{A}{2}=\sqrt{\dfrac{b+c}{2c}}$, then

a. area of triangle is $\dfrac{1}{2}ab$
b. circumradius is equal to $\dfrac{1}{2}c$
c. area of triangle is $\dfrac{1}{2}bc$
d. circumradius is equal to $\dfrac{1}{2}a$

18. If the sides of a right-angled triangle are in G.P., then the cosines of the acute angle of the triangle are

a. $\dfrac{\sqrt{5}-1}{2}$
b. $\dfrac{\sqrt{5}+1}{2}$
c. $\sqrt{\dfrac{\sqrt{5}-1}{2}}$
d. $\dfrac{\sqrt{\sqrt{5}+1}}{2}$

Reasoning Type

Solutions on page 5.102

Each question has four choices a, b, c and d, out of which *only one* is correct. Each question contains STATEMENT 1 and STATEMENT 2.

a. Both the statements are TRUE and STATEMENT 2 is the correct explanation of STATEMENT 1
b. Both the statements are TRUE but STATEMENT 2 is NOT the correct explanation of STATEMENT 1
c. STATEMENT 1 is TRUE and STATEMENT 2 is FALSE
d. STATEMENT 1 is FALSE and STATEMENT 2 is TRUE

1. **Statement 1:** If side BC and ratio of r_2 and r_3 of an acute-angled triangle is given, then the locus of A is a hyperbola.
Statement 2: If base of a triangle is given and difference of two variable sides is constant (less than the base), then locus of variable vertex is a hyperbola.
2. **Statement 1:** In any ΔABC, the maximum value of $r_1+r_2+r_3=9R/2$.
Statement 2: In any ΔABC, $R\geq 2r$.
3. In acute-angled ΔABC, $a>b>c$
Statement 1: $r_1>r_2>r_3$.
Statement 2: $\cos A<\cos B<\cos C$.
4. **Statement 1:** The incentre of the triangle formed by the feet of altitudes from the vertices of triangle ABC to the opposite sides is the orthocentre of the triangle ABC.
Statement 2: The incentre of triangle ABC is orthocentre of the triangle $I_1I_2I_3$, where I_1, I_2, I_3 are excentres of triangle ABC.
5. **Statement 1:** If I is incentre of ΔABC and I_1 excentre opposite to A and P is the intersection of II_1 and BC, then $IP\cdot I_1P=BP\cdot PC$.
Statement 2: In ΔABC, I is incentre and I_1 is excentre opposite to A, then IBI_1C must be square.
6. **Statement 1:** If the quadrilateral Q_1 formed by joining mid-points of sides of another quadrilateral Q_2 is cyclic, then Q_1 is necessarily a rectangle.
Statement 2: The quadrilateral Q_1 formed by joining mid-points of sides of another quadrilateral Q_2 is always a parallelogram.
7. Let l_1, l_2, l_3 be the lengths of the internal bisectors of angles A, B, C of ΔABC, respectively.

Statement 1: $\dfrac{\cos\frac{A}{2}}{l_1}+\dfrac{\cos\frac{B}{2}}{l_2}+\dfrac{\cos\frac{C}{2}}{l_3}=2\left(\dfrac{l_1}{a}+\dfrac{l_2}{b}+\dfrac{l_3}{c}\right)$

Statement 2: $l_1^2=bc\left[1-\left(\dfrac{a}{b+c}\right)^2\right], l_2^2=ca\left[1-\left(\dfrac{b}{c+a}\right)^2\right], l_3^2=ab\left[1-\left(\dfrac{c}{a+b}\right)^2\right]$

8. **Statement 1:** In ΔABC, the centroid (G) divides line joining orthocenter (H) and circumcenter in ratio 2:1.
Statement 2: The centroid (G) divides the median AD in ratio 2:1.
9. **Statement 1:** Circumradius of $\Delta I_1I_2I_3$ is $2R$.
Statement 2: Circumradius of the triangle formed by feet of altitudes of ΔABC is $R/2$.
10. **Statement 1:** If the incircle of the triangle ABC passes through its circumcentre, then

$$\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}=\sqrt{2}.$$

Statement 2: Distance between the circumcentre and incentre is $\sqrt{R^2-2rR}$.
11. **Statement 1:** In triangle ABC, D is a point on the side AB such that $CD^2 = AD\cdot DB$, then the greatest value of $\sin A\sin B$ is $\sin^2(C/2)$.
Statement 2: Greatest value of $\sin A\sin B$ occurs when CD is the angle bisector of angle C.
12. **Statement 1:** If a, b, c are the sides of a triangle, then the minimum value of

$$\frac{2a}{b+c-a}+\frac{2b}{c+a-b}+\frac{2c}{a+b-c}\text{ is } 9.$$

Statement 2: A.M. ≥ G.M. ≥ H.M.
13. **Statement 1:** If $C=45°$, $B=60°$, then the line joining A and circumcentre (O) divides BC in ratio $2:\sqrt{3}$.
Statement 2: Line joining A and circumcenter (O) divides BC in ratio $\dfrac{\sin 2C}{\sin 2B}$.
14. **Statement 1:** If $a=3$, $b=7$, $c=8$, and internal angle bisector AI meets BC at D (where I is incentre), then $AI/ID = 11/2$.
Statement 2: Internal angle bisector of angle A divides the side BC in ratio AB/AC.

Linked Comprehension Type

Solutions on page 5.106

Based upon each paragraph, three multiple choice questions have to be answered. Each question has four choices a, b, c, and d, out of which *only one* is correct.

For Problems 1–3

Given that $\Delta=6$, $r_1=2$, $r_2=3$, $r_3=6$.

1. Circumradius R is equal to
   a. 2.5 b. 3.5 c. 1.5 d. none of these
2. Inradius is equal to
   a. 2 b. 1 c. 1.5 d. 2.5
3. Difference between the greatest and the least angle is
   a. $\cos^{-1}\frac{4}{5}$ b. $\tan^{-1}\frac{3}{4}$ c. $\cos^{-1}\frac{3}{5}$ d. none of these

For Problems 4–6

Let $a=6$, $b=3$ and $\cos(A-B)=\dfrac{4}{5}$.

4. Area of the triangle is equal to
   a. 9 b. 12 c. 11 d. 10
5. Angle C is equal to

a. $\frac{3\pi}{4}$ b. $\frac{\pi}{4}$ c. $\frac{\pi}{2}$ d. none of these

6. Value of sin A is equal to

a. $\frac{1}{2\sqrt{5}}$ b. $\frac{1}{\sqrt{3}}$ c. $\frac{1}{\sqrt{5}}$ d. $\frac{2}{\sqrt{5}}$

For Problems 7–9

p_1, p_2, p_3 are altitudes of triangle ABC from the vertices A, B, C and Δ is the area of the triangle.

7. The value of $p_1^{-2} + p_2^{-2} + p_3^{-2}$ is equal to

a. $\frac{a+b+c}{\Delta}$ b. $\frac{a^2+b^2+c^2}{4\Delta^2}$ c. $\frac{a^2+b^2+c^2}{\Delta^2}$ d. none of these

8. The value of $\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}$ is equal to

a. $\frac{2}{r}$ b. $\frac{2s}{\Delta}$ c. $\frac{8R}{abc}$ d. none of these

9. The value of $\frac{\cos A}{p_1}+\frac{\cos B}{p_2}+\frac{\cos C}{p_3}$ is equal to

a. $\frac{1}{R}$ b. $\frac{a^2+b^2+c^2}{2R}$ c. $\frac{\Delta}{2R}$ d. none of these

For Problems 10–12

Let O be a point inside a ΔABC such that $\angle OAB = \angle OBC = \angle OCA = \theta$.

10. $\cot A + \cot B + \cot C$ is equal to

a. $\tan^2\theta$ b. $\cot^2\theta$ c. $\tan\theta$ d. $\cot\theta$

11. $\operatorname{cosec}^2 A + \operatorname{cosec}^2 B + \operatorname{cosec}^2 C$ is equal to

a. $\cot^2\theta$ b. $\operatorname{cosec}^2\theta$ c. $\tan^2\theta$ d. $\sec^2\theta$

12. Area of ΔABC is equal to

a. $\left(\frac{a^2+b^2+c^2}{4}\right)\tan\theta$ b. $\left(\frac{a^2+b^2+c^2}{4}\right)\cot\theta$ c. $\left(\frac{a^2+b^2+c^2}{2}\right)\tan\theta$ d. $\left(\frac{a^2+b^2+c^2}{2}\right)\cot\theta$

For Problems 13–15

Let D, E and F be the feet of altitudes from the vertices of acute-angled triangle ABC to the sides BC, AC and AB, respectively. Triangle DEF is defined as the pedal triangle of triangle ABC. (R and r are circumradius and inradius of triangle ABC, respectively.)

13. Consider the following statements:

i. orthocentre of the triangle ABC is incentre of the triangle DEF

ii. A, B, C are excentres of triangle DEF

a. only (i) is true b. only (ii) is true

c. both (i) and (ii) are true d. both (i) and (ii) are false

14. Circumradius of a pedal triangle of triangle ABC is

a. $R/2$ b. $r/2$ c. $R/4$ d. $r/4$

15. If x, y, z are the sides of a pedal triangle, then $x+y+z$ is equal to

a. $\Delta R/2$ b. $\Delta/2R$ c. ΔR d. none of these

For Problems 16 – 18

Incircle of ΔABC touches the sides BC, AC and AB at D, E and F, respectively. Then answer the following questions.

16. $\angle DEF$ is equal to

a. $\dfrac{\pi - B}{2}$ b. $\pi - 2B$ c. $A - C$ d. none of these

17. Area of ΔDEF is

a. $2r^2 \sin(2A)\sin(2B)\sin(2C)$ b. $2r^2 \cos\dfrac{A}{2}\cos\dfrac{B}{2}\cos\dfrac{C}{2}$

c. $2r^2 \sin(A-B)\sin(B-C)\sin(C-A)$ d. none of these

18. The length of side EF is

a. $r\sin\dfrac{A}{2}$ b. $2r\sin\dfrac{A}{2}$ c. $r\cos\dfrac{A}{2}$ d. $2r\cos\dfrac{A}{2}$

For Problems 19 – 21

Internal bisectors of ΔABC meet the circumcircle at points D, E and F,

19. The length of side EF is

a. $2R\cos\left(\dfrac{A}{2}\right)$ b. $2R\sin\left(\dfrac{A}{2}\right)$ c. $R\cos\left(\dfrac{A}{2}\right)$ d. $2R\cos\left(\dfrac{B}{2}\right)\cos\left(\dfrac{C}{2}\right)$

20. Area of ΔDEF is

a. $2R^2\cos^2\left(\dfrac{A}{2}\right)\cos^2\left(\dfrac{B}{2}\right)\cos^2\left(\dfrac{C}{2}\right)$ b. $2R^2\sin\left(\dfrac{A}{2}\right)\sin\left(\dfrac{B}{2}\right)\sin\left(\dfrac{C}{2}\right)$

c. $2R^2\sin^2\left(\dfrac{A}{2}\right)\sin^2\left(\dfrac{B}{2}\right)\sin^2\left(\dfrac{C}{2}\right)$ d. $2R^2\cos\left(\dfrac{A}{2}\right)\cos\left(\dfrac{B}{2}\right)\cos\left(\dfrac{C}{2}\right)$

21. Ratio of area of triangle ABC and triangle DEF is

a. ≥ 1 b. ≤ 1 c. $\geq 1/2$ d. $\leq 1/2$

Matrix-Match Type

Solutions on page 5.111

Each question contains statements given in two columns which have to be matched.
Statements (A, B, C, D) in column I have to be matched with statements (p, q, r, s) in column II. If the correct match are a-p, a-s, b-q, b-r, c-p, c-q and d-s, then the correctly bubbled 4 × 4 matrix should be as follows:

	p	q	r	s
a	p	q	r	s
b	p	q	r	s
c	p	q	r	s
d	p	q	r	s

1.

Column I	Column II
a. $b > c\sin B$, $b < c$ and B is an acute angle	p. 0
b. $b > c\sin B$, $c < b$, and B is an acute angle	q. 2
c. $b > c\sin B$, $c < b$ and B is an obtuse angle	r. data insufficient
d. $b > c\sin B$, $c > b$ and B is an obtuse angle	s. 1

2. In acute-angled triangle ABC

Column I	Column II
a. $\cos A$, $\cos B$, $\cos C$ are in A.P.	**p.** Distances of orthocentre from vertices of triangle are in A.P.
b. $\sin (A/2)$, $\sin (B/2)$, $\sin (C/2)$ are in A.P.	**q.** Distances of orthocentre from sides of triangle are in H.P.
c. Distances of circumcentre from the vertices of the triangle ABC are in A.P.	**r.** Distances of incentre from vertices of triangle are in H.P.
d. Circumradii of triangles OBC, OAC and OAB are in H.P. (where O is cicumcentre of triangle ABC)	**s.** Distances of incentre from excentres of triangle are in A.P.

3.

Column I	Column II
a. If the sines of the angles A and B of a triangle ABC satisfy the equation $c^2x^2 - c(a+b)x + ab = 0$, the triangle can be	**p.** right angled
b. If one angle of a triangle is 30° and the lengths of the sides adjacent to it are 40 and $40\sqrt{3}$, the triangle can be	**q.** isosceles
c. If two angles of a triangle ABC satisfy the equation $81^{\sin^2 x} + 81^{\cos^2 x} = 30$, then the triangle can be ($x \in (0, \pi/2)$)	**r.** equilateral
d. In triangle ABC, $\cos A \cos B + \sin A \sin B \sin C = 1$, then the triangle can be	**s.** obtuse angled

4. Let O be the circumcentre, H be the orthocentre, I be the incentre and I_1, I_2, I_3 be the excentres of acute-angled ΔABC

Column I	Column II
a. Angle subtended by OI at vertex A	**p.** $\lvert B-C \rvert$
b. Angle subtended by HI at vertex A	**q.** $\dfrac{\lvert B-C \rvert}{2}$
c. Angle subtended by OH at vertex A	**r.** $\dfrac{B+C}{2}$
d. Angle subtended by I_2I_3 at I_1	**s.** $\dfrac{B}{2} - C$

5.

Column I (Condition)	Column II (Type of ΔABC)
a. $\cot\frac{A}{2}=\frac{b+c}{a}$	p. always right angled
b. $a\tan A+b\tan B=(a+b)\tan\left(\frac{A+B}{2}\right)$	q. always isosceles
c. $a\cos A=b\cos B$	r. may be right angled
d. $\cos A=\frac{\sin B}{2\sin C}$	s. may be right-angled isosceles

Integer Type

Solutions on page 5.115

1. Suppose α, β, γ and δ are the interior angles of regular pentagon, hexagon, decagon and dodecagon, respectively, then the value of $|\cos\alpha\sec\beta\cos\gamma\operatorname{cosec}\delta|$ is ________.
2. Let $ABCDEFGHIJKL$ be a regular dodecagon. Then the value of $\frac{AB}{AF}+\frac{AF}{AB}$ is equal to ________.
3. Two equilateral triangles are constructed from a line segment of length L. If M and m are the maximum and minimum value of the sum of the areas of two plane figures, then the value of M/m is ________.
4. In ΔABC, if $r=1$, $R=3$ and $s=5$, then the value of $\frac{a^2+b^2+c^2}{3}$ is ________.
5. Consider a ΔABC in which the sides are $a=(n+1)$, $b=(n+2)$, $c=n$ with $\tan C=4/3$, then the value of $\Delta/12$ is ________.
6. In ΔAEX, T is the midpoint of XE, and P is the midpoint of ET. If ΔAPE is equilateral of side length equal to unity, then the value of $[(AX)^2/2]$ is (where [.] represents greatest integer function) ________.
7. In ΔABC, the incircle touches the sides BC, CA and AB, respectively, at D, E and F. If the radius of the incircle is 4 units and BD, CE and AF are consecutive integers, then the value of $s/3$, where s is a semi-perimeter of triangle, is ________.
8. The altitudes from the angular points A, B and C on the opposite sides BC, CA and AB of ΔABC are 210, 195 and 182, respectively. Then the value of $a/30$ is (where $a=BC$) ________.
9. In ΔABC, if $\angle C=3\angle A$, $BC=27$ and $AB=48$. Then the value of $AC/7$ is ________.
10. The area of a right triangle is 6864 square units. If the ratio of its legs is 143 : 24, then the value of $[r/4]$, where $[\cdot]$ represents the greatest integer function, is ________.
11. In ΔABC, if $\cos A+\sin A-\frac{2}{\cos B+\sin B}=0$, then the value of $\left(\frac{a+b}{c}\right)^4$ is ________.
12. In ΔABC, $\angle C=2\angle A$ and $AC=2\,BC$, then the value of $\frac{a^2+b^2+c^2}{R^2}$ (where R is circum-radius of triangle) is ________.
13. A circle inscribed in a triangle ABC touches the side AB at D such that $AD=5$ and $BD=3$. If $\angle A=60°$, then the value of $[BC/3]$ (where $[\cdot]$ represents greatest integer function) is ________.

14. The sides of triangle ABC satisfy the relations $a+b-c=2$ and $2ab-c^2=4$, then square of the area of triangle is ________.

15. The lengths of the tangents drawn from the vertices A, B and C to the incircle of ΔABC are 5, 3 and 2, respectively. If the lengths of the parts of tangents within the triangle which are drawn parallel to the sides BC, CA and AB of the triangle to the incircle are α, β and γ, respectively, then the value of $[\alpha+\beta+\gamma]$ (where $[\cdot]$ represents greatest integer funciton) is ________.

16. If a, b and c represent the lengths of sides of a triangle, then the possible integral value of $\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}$ is ________.

17. In triangle ABC, $\sin A\sin B+\sin B\sin C+\sin C\sin A=9/4$ and $a=2$, then the value of $\sqrt{3}\Delta$, where Δ is area of triangle, is ________.

18. In ΔABC, $AB=52$, $BC=56$, $CA=60$. Let D be the foot of the altitude from A, and E be the intersection of the internal angle bisector of $\angle BAC$ with BC. Find the length DE is ________.

Archives

Solutions on page 5.121

Subjective

1. ABC is a triangle. D is the middle point of BC. If AD is perpendicular to AC, then prove that $\cos A\cos C=\frac{2(c^2-a^2)}{3ac}$. **(IIT-JEE, 1980)**

2. ABC is a triangle with $AB=AC$. D is any point on the side BC. E and F are points on the sides AB and AC, respectively, such that DE is parallel to AC and DF is parallel to AB. Prove that $DF+FA+AE+ED=AB+AC$. **(IIT-JEE, 1980)**

3. Let the angles A, B and C of triangle ABC be in A.P. and let $b:c$ be $\sqrt{3}:\sqrt{2}$. Find the angle A. **(IIT-JEE, 1981)**

4. The exradii r_1, r_2 and r_3 of ΔABC are in H.P. Show that its sides a, b and c are in A.P. **(IIT-JEE, 1983)**

5. For triangle ABC, it is given that $\cos A+\cos B+\cos C=\frac{3}{2}$. Prove that the triangle is equilateral. **(IIT-JEE, 1984)**

6. With usual notation, if in triangle ABC,
$\frac{b+c}{11}=\frac{c+a}{12}=\frac{a+b}{13}$ then prove that $\frac{\cos A}{7}=\frac{\cos B}{19}=\frac{\cos C}{25}$. **(IIT-JEE, 1984)**

7. In triangle ABC, the median to the side BC is of length $\frac{1}{\sqrt{11-6\sqrt{3}}}$ and it divides the angle A into angles 30° and 45°. Find the length of the side BC. **(IIT-JEE, 1985)**

8. If in triangle ABC, $\cos A\cos B+\sin A\sin B\sin C=1$. Show that $a:b:c=1:1:\sqrt{2}$. **(IIT-JEE, 1986)**

9. The sides of a triangle are three consecutive natural numbers and its largest angle is twice the smallest one. Determine the sides of the triangle. **(IIT-JEE, 1997)**

10. In a triangle of base a, the ratio of the other two sides is $r\ (<1)$. Show that the altitude of the triangle is less than or equal to $\frac{ar}{1-r^2}$. **(IIT-JEE, 1997)**

11. Three circles touch one another externally. The tangents at their points of contact meet at a point whose distance from a point of contanct is 4. Find the ratio of the product of the radii to the sum of the radii of the circles. **(IIT-JEE, 1992)**

12. Consider the following statements concerning triangle ABC
    i. The sides a, b and c and area (Δ) are rational
    ii. a, $\tan\frac{B}{2}$ and $\tan\frac{C}{2}$ are rational
    iii. a, $\sin A$, $\sin B$ and $\sin C$ are rational
    Prove that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i). **(IIT-JEE, 1994)**

13. Let ABC be a triangle with incentre I and inradius r. Let D, E and F be the feet of the perpendiculars from I to the sides BC, CA and AB, respectively. If r_1, r_2 and r_3 are the radii of circles inscribed in the quadrilaterals $AFIE$, $BDIF$ and $CEID$, respectively, prove that

$$\frac{r_1}{r-r_1}+\frac{r_2}{r-r_2}+\frac{r_3}{r-r_3}=\frac{r_1r_2r_3}{(r-r_1)(r-r_2)(r-r_3)}.$$ **(IIT-JEE, 2000)**

14. If Δ is the area of a triangle with side lengths a, b and c, then show that $\Delta \le \frac{1}{4}\sqrt{(a+b+c)\,abc}$. Also show that the equality occurs in the above inequality if and only if $a=b=c$. **(IIT-JEE, 2001)**

15. If I_n is the area of n-sided regular polygon inscribed in a circle of unit radius and O_n be the area of the polygon circumscribing the given circle, prove that $I_n=\frac{O_n}{2}\left(1+\sqrt{1-\left(\frac{2I_n}{n}\right)^2}\right)$. **(IIT-JEE, 2003)**

16. Two parallel chords of a circle of radius 2 are at a distance $\sqrt{3}+1$ apart. If the chord subtend angles $\frac{\pi}{k}$ and $\frac{2\pi}{k}$ at the centre, where $k>0$, then find the value of $[k]$.
    (*Note:*) $[k]$ denotes the largest integer less than or equal to k) **(IIT-JEE, 2010)**

17. Consider a triangle ABC and let a, b and c denote the lengths of the sides opposite to vertices A, B and C, respectively. Suppose $a=6$, $b=10$ and the area of triangle is $15\sqrt{3}$. If $\angle ACB$ is obtuse and if r denotes the radius of the incircle of the triangle, then find the value of r^2. **(IIT-JEE, 2010)**

Objective

Fill in the blanks

1. In ΔABC, $\angle A=90°$ and AD is an altitude. Complete the relation $\frac{BD}{DA}=\frac{AB}{(\cdots)}$. **(IIT-JEE, 1980)**

2. ABC is a triangle, P is a point on AB and Q is a point on AC such that $\angle AQP=\angle ABC$. Complete the relation $\frac{\text{area of }\Delta APQ}{\text{area of }\Delta ABC}=\frac{(\ )}{AC^2}$. **(IIT-JEE, 1980)**

3. ABC is a triangle with $\angle B$ greater than $\angle C$. D and E are points on BC such that AD is perpendicular to BC and AE is the bisector of angle A. Complete the relation $\angle DAE=\frac{1}{2}[(\cdots)-\angle C]$. **(IIT-JEE, 1980)**

4. The set of all real numbers a such that a^2+2a, $2a+3$ and a^2+3a+8 are the sides of a triangle is _____.
(IIT-JEE, 1985)

5. In triangle ABC, if $\cot A$, $\cot B$, $\cot C$ are in $A.P.$, then a^2, b^2, c^2 are in _____ progression.
(IIT-JEE, 1985)

6. A polygon of nine sides, each side of length 2, is inscribed in a circle. The radius of the circle is _____.
(IIT-JEE, 1987)

7. If the angles of a triangle are 30° and 45° and the included side is $(\sqrt{3}+1)$ cm, then the area of the triangle is _____.
(IIT-JEE, 1988)

8. If in triangle ABC, $\frac{2\cos A}{a}+\frac{\cos B}{b}+\frac{2\cos C}{c}=\frac{a}{c}+\frac{b}{ca}$, then the value of the angle A is _____ degrees.
(IIT-JEE, 1993)

9. In triangle ABC, AD is the altitude from A. Given $b>c$, $\angle C=23°$ and $AD=\frac{abc}{b^2-c^2}$, then $\angle B=$ _____.
(IIT-JEE, 1994)

10. A circle is inscribed in an equilateral triangle of side a. The area of any square inscribed in this circle is _____.
(IIT-JEE, 1994)

11. In triangle ABC, $a:b:c=4:5:6$. The ratio of the radius of the circumcircle to that of the incircle is _____.
(IIT-JEE, 1996)

Multiple choice questions with one correct answer

1. In triangle ABC, angle A is greater than angle B. If the measures of angles A and B satisfy the equation $3\sin x-4\sin^3 x-k=0$, $0<k<1$, then the measure of angle C is

a. $\frac{\pi}{3}$ b. $\frac{\pi}{2}$ c. $\frac{2\pi}{3}$ d. $\frac{5\pi}{6}$

(IIT-JEE, 1990)

2. If the lengths of the sides of triangle are 3, 5 and 7, then the largest angle of the triangle is

a. $\frac{\pi}{2}$ b. $\frac{5\pi}{6}$ c. $\frac{2\pi}{3}$ d. $\frac{3\pi}{4}$

(IIT-JEE, 1994)

3. In triangle ABC, $\angle B=\pi/3$ and $\angle C=\pi/4$. Let D divide BC internally in the ratio 1:3. Then $\frac{\sin \angle BAD}{\sin \angle CAD}$ equals

a. $\frac{1}{\sqrt{6}}$ b. $\frac{1}{3}$ c. $\frac{1}{\sqrt{3}}$ d. $\sqrt{\frac{2}{3}}$

(IIT-JEE, 1995)

4. In triangle ABC, $2ac\sin\left(\frac{1}{2}(A-B+C)\right)$ is equal to

a. $a^2+b^2-c^2$ b. $c^2+a^2-b^2$ c. $b^2-c^2-a^2$ d. $c^2-a^2-b^2$

(IIT-JEE, 2000)

5. In triangle ABC, let $\angle C=\pi/2$. If r is the inradius and R is circumradius of the triangle, then $2(r+R)$ is equal to

a. $a+b$ b. $b+c$ c. $c+a$ d. $a+b+c$

(IIT-JEE, 2000)

6. Which of the following pieces of data does NOT uniquely determine an acute-angled triangle ABC (R being the radius of the circumcircle)?

a. a, $\sin A$, $\sin B$ b. a, b, c c. a, $\sin B$, R d. a, $\sin A$, R

(IIT-JEE, 2002)

7. If the angles of a triangle are in the ratio 4:1:1, then the ratio of the longest side to the perimeter is

a. $\sqrt{3}:(2+\sqrt{3})$ b. 1:6 c. $1:2+\sqrt{3}$ d. 2:3

(IIT-JEE, 2003)

8. The side of a triangle are in the ratio $1:\sqrt{3}:2$, then the angles of the triangle are in the ratio

a. 1:3:5 b. 2:3:4 c. 3:2:1 d. 1:2:3

(IIT-JEE, 2004)

9. In an equilateral triangle, three coins of radii 1 unit each are kept so that they touch each other and also the sides of the triangle. Area of the triangle is

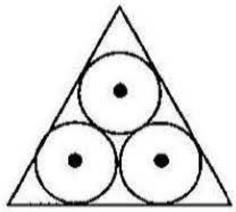


Fig. 5.48

a. $4+2\sqrt{3}$ b. $6+4\sqrt{3}$ c. $12+\frac{7\sqrt{3}}{4}$ d. $3+\frac{7\sqrt{3}}{4}$

(IIT-JEE, 2005)

10. In triangle ABC, a, b, c are the lengths of its sides and A, B, C are the angles of triangle ABC. The correct relation is given by

a. $(b-c)\sin\left(\frac{B-C}{2}\right)=a\cos\frac{A}{2}$ b. $(b-c)\cos\left(\frac{A}{2}\right)=a\sin\frac{B-C}{2}$

c. $(b+c)\sin\left(\frac{B+C}{2}\right)=a\cos\frac{A}{2}$ d. $(b-c)\cos\left(\frac{A}{2}\right)=2a\sin\frac{B+C}{2}$

(IIT-JEE, 2005)

11. One angle of an isosceles Δ is 120° and radius of its incircle $=\sqrt{3}$. Then the area of the triangle in sq. units is

a. $7+12\sqrt{3}$ b. $12-7\sqrt{3}$ c. $12+7\sqrt{3}$ d. 4π

(IIT-JEE, 2006)

12. Let $ABCD$ be a quadrilateral with area 18, with side AB parallel to the side CD and $AB = 2\,CD$. Let AD be perpendicular to AB and CD. If a circle is drawn inside the quadrilateral $ABCD$ touching all the sides, then its radius is

a. 3 b. 2 c. $\frac{3}{2}$ d. 1

(IIT-JEE, 2007)

13. Let ABC be a triangle such that $\angle ACB = \pi/6$ and let a, b and c denote the lengths of the side opposite to A, B and C, respectively. The value(s) of x for which $a = x^2 + x + 1$, $b = x^2 - 1$ and $c = 2x + 1$ is (are)

a. $-(2+\sqrt{3})$ b. $1+\sqrt{3}$ c. $2+\sqrt{3}$ d. $4\sqrt{3}$

(IIT-JEE, 2010)

Multiple choice questions with one or more than one correct answers

1. There exists a triangle ABC satisfying the conditions

a. $b \sin A = a, A < \pi/2$ b. $b \sin A > a, A > \pi/2$
c. $b \sin A > a, A < \pi/2$ d. $b \sin A < a, A < \pi/2, b > a$
e. $b \sin A < a, A > \pi/2, b = a$ **(IIT-JEE, 1986)**

2. In a triangle, the lengths of the two larger sides are 10 and 9, respectively. If the angles are in A.P., then the length of the third side can be

a. $5 - \sqrt{6}$ b. $3\sqrt{3}$ c. 5 d. $5 + \sqrt{6}$

(IIT-JEE, 1987)

3. If in a triangle PQR, $\sin P$, $\sin Q$, $\sin R$ are in A.P., then

a. the altitudes are in A.P. b. the altitudes are in H.P.
c. the medians are in G.P. d. the medians are in A.P. **(IIT-JEE, 1988)**

4. Let $A_0A_1A_2A_3A_4A_5$ be a regular hexagon inscribed in a circle of unit radius. Then the product of the lengths of the line segments A_0A_1, A_0A_2 and A_0A_4 is

a. $\frac{3}{4}$ b. $3\sqrt{3}$ c. 3 d. $\frac{3\sqrt{3}}{2}$

(IIT-JEE, 1998)

5. In ΔABC, internal angle bisector of $\angle A$ meets side BC in D. $DE \perp AD$ meets AC in E and AB in F. Then

a. AE is $H.M.$ of b and c b. $AD = \frac{2bc}{b+c} \cos \frac{A}{2}$ c. $EF = \frac{4bc}{b+c} \sin \frac{A}{2}$ d. $\Delta\, AEF$ is isosceles

(IIT-JEE, 2006)

6. A straight line through the vertex P of a triangle PQR intersects the side QR at the point S and the circumcircle of the triangle PQR at the point T. If S is not the centre of the circumcircle, then

a. $\frac{1}{PS} + \frac{1}{ST} < \frac{2}{\sqrt{QS \times SR}}$ b. $\frac{1}{PS} + \frac{1}{ST} > \frac{2}{\sqrt{QS \times SR}}$

c. $\frac{1}{PS} + \frac{1}{ST} < \frac{4}{QR}$ d. $\frac{1}{PS} + \frac{1}{ST} > \frac{4}{QR}$ **(IIT-JEE, 2008)**

7. In a triangle ABC with fixed base BC, the vertex A moves such that

$$\cos B + \cos C = 4 \sin^2 \frac{A}{2}$$

If a, b and c denote the lengths of the sides of the trangle opposite to the angles A, B and C, respectively, then

a. $b + c = 4a$ b. $b + c = 2a$
c. locus of point A is an ellipse d. locus of point A is a pair of straight lines

(IIT-JEE, 2009)

ANSWERS AND SOLUTIONS

Subjective Type

1. If O is the circumcentre of ΔABC, then
$OA = OB = OC = R$
Let R_1, R_2 and R_3 be circumradii of ΔOBC, ΔOCA and ΔOAB, respectively.

In ΔOBC, $2R_1 = \dfrac{a}{\sin 2A} \quad \Rightarrow \quad \dfrac{a}{R_1} = 2\sin 2A$

Similarly, $\dfrac{a}{R_2} = 2\sin 2B$ and $\dfrac{a}{R_3} = 2\sin 2C$

$$\Rightarrow \quad \frac{a}{R_1}+\frac{b}{R_2}+\frac{c}{R_3} = 2(\sin 2A+\sin 2B+\sin 2C) = 8\sin A\sin B\sin C = 8\frac{a}{2R}\frac{b}{2R}\frac{c}{2R} = \frac{abc}{R^3}$$

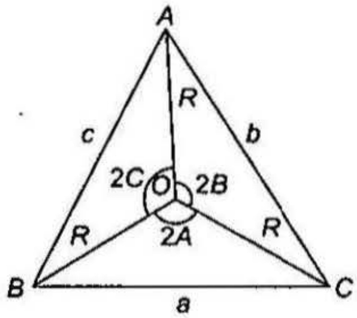


Fig. 5.49

2. In ΔABC,
$$\frac{AC}{\sin 5x} = \frac{BC}{\sin 3x}$$
$$\Rightarrow \frac{a+p}{\sin 5x} = \frac{a}{\sin 3x} \qquad \text{(i)}$$

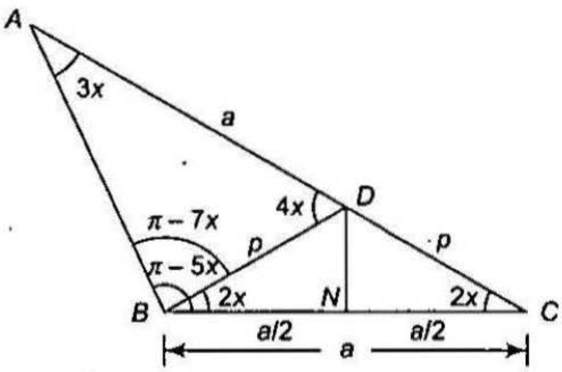


Fig. 5.50

In ΔBDN, $\cos 2x = \dfrac{a}{2p}$

$\Rightarrow a = 2p\cos 2x$

From Eq.(i), $\dfrac{2p\cos 2x + p}{\sin 5x} = \dfrac{2p\cos 2x}{\sin 3x}$

$\Rightarrow 2\sin 3x\cos 2x + \sin 3x = 2\sin 5x\cos 2x$

$\Rightarrow \sin 5x + \sin x + \sin 3x = \sin 7x + \sin 3x$
$\Rightarrow \sin 7x - \sin 5x = \sin x$
$\Rightarrow 2 \cos 6x \sin x = \sin x$

$$\Rightarrow \cos 6x = \frac{1}{2}$$

$\Rightarrow x = 10°$

3. We know that distance of orthocentre (H) from vertex (A) is $2R \cos A$
or $x = 2R \cos A, y = 2R \cos B, z = 2R \cos C$

$$\Rightarrow \quad \frac{a}{x}+\frac{b}{y}+\frac{c}{z} = \frac{2R\sin A}{2R\cos A}+\frac{2R\sin B}{2R\cos B}+\frac{2R\sin C}{2R\cos C} = \tan A+\tan B+\tan C = \tan A\tan B\tan C$$

Also, $$\frac{abc}{xyz} = \frac{(2R\sin A)(2R\sin B)(2R\sin C)}{(2R\cos A)(2R\cos B)(2R\cos C)} = \tan A\tan B\tan C$$

4.

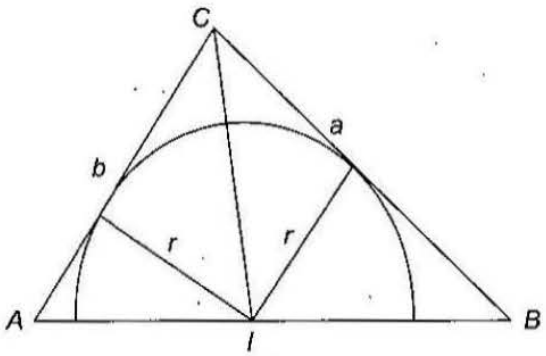


Fig. 5.51

From Fig. 5.51,
$(1/2)\, ra + (1/2)\, rb = (1/2)\, a\, b \sin C$
$\Rightarrow r(a+b) = 2\Delta$

$$\Rightarrow r = \frac{2\Delta}{a+b} = \frac{2\,abc}{4R(2R\sin A + 2R\sin B)} = \frac{abc}{4R^2(\sin A + \sin B)} \quad \text{(i)}$$

Also $x = \dfrac{2ab}{a+b}\cos\dfrac{C}{2}$ [length of angle bisector]

$$\text{From Eq. (i), } r = \frac{2\times\frac{1}{2}ab\sin C}{a+b}$$

$$= \frac{2ab\sin\frac{C}{2}\cos\frac{C}{2}}{a+b}$$

$$= \frac{2\,ab\cos\frac{C}{2}}{a+b}\cdot\sin\frac{C}{2}$$

$$= x\sin\frac{C}{2}$$

5. Let O and H be the circumcentre and the orthocentre, respectively.
If OF is the perpendicular to AB, we have
$\angle OAF = 90° - \angle AOF = 90° - C$
Also, $\angle HAL = 90° - C$
Hence, $\angle OAH = A - \angle OAF - \angle HAL$
$= A - 2(90° - C)$
$= A + 2C - 180°$
$= A + 2C - (A + B + C) = C - B$
Also, $OA = R$, and $HA = 2R\cos A$

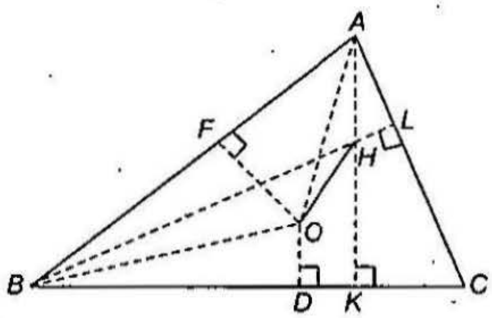


Fig. 5.52

Now in ΔAOH,
$OH^2 = OA^2 + HA^2 - 2OA\ HA\cos(\angle OAH)$
$= R^2 + 4R^2\cos^2 A - 4R^2\cos A\cos(C - B)$
$= R^2 + 4R^2\cos A\,[\cos A - \cos(C - B)]$
$= R^2 - 4R^2\cos A\,[\cos(B + C) + \cos(C - B)]$
$= R^2 - 8R^2\cos A\cos B\cos C$

Hence, $OH = R\sqrt{1 - 8\cos A\cos B\cos C}$

6. Let O be the circumcentre and OF be the perpendicular to AB.
Let I be the incentre and IE be the perpendicular to AC.
Then $\angle OAF = 90° - C$.

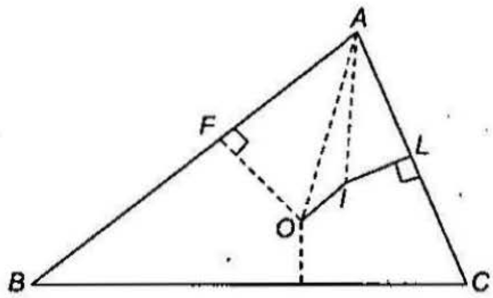


Fig. 5.53

$$\Rightarrow \angle OAI = \angle IAF - \angle OAF = \frac{A}{2} - (90° - C) = \frac{A}{2} + C - \frac{A + B + C}{2} = \frac{C - B}{2}$$

Also, $AI = \dfrac{IE}{\sin\frac{A}{2}} = \dfrac{r}{\sin\frac{A}{2}} = 4R\sin\dfrac{B}{2}\sin\dfrac{C}{2}$

Hence in ΔOAI, $OI^2 = OA^2 + AI^2 - 2OA\,AI\cos\angle OAI$

$$\Rightarrow \quad = R^2 + 16R^2\sin^2\frac{B}{2}\sin^2\frac{C}{2} - 8R^2\sin\frac{B}{2}\sin\frac{C}{2}\cos\frac{C-B}{2}$$

$$\Rightarrow \quad \frac{OI^2}{R^2} = 1 + 16\sin^2\frac{B}{2}\sin^2\frac{C}{2} - 8\sin\frac{B}{2}\sin\frac{C}{2}\left(\cos\frac{B}{2}\cos\frac{C}{2} + \sin\frac{B}{2}\sin\frac{C}{2}\right)$$

$$= 1 - 8\sin\frac{B}{2}\sin\frac{C}{2}\left(\cos\frac{B}{2}\cos\frac{C}{2} - \sin\frac{B}{2}\sin\frac{C}{2}\right)$$

$$= 1 - 8\sin\frac{B}{2}\sin\frac{C}{2}\cos\frac{B+C}{2}$$

$$= 1 - 8\sin\frac{B}{2}\sin\frac{C}{2}\sin\frac{A}{2}$$

Therefore, $OI = R\sqrt{1 - 8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}} = \sqrt{R^2 - 2Rr}$

7. Let $ABCD$ be the cyclic quadrilateral in which $AB = 2$ and $BC = 5$, $\angle ABC = 60°$

$\angle ADC = 180° - 60° = 120°$.

Area of cyclic quadrilateral $ABCD$ = Area of ΔABC + Area of ΔACD

$$\Rightarrow \quad 4\sqrt{3} = \frac{1}{2}AB\,BC\sin 60° + \frac{1}{2}CD\,DA\sin 120°$$

$$= \frac{1}{2}2\times 5\times(\sqrt{3}/2) + \frac{1}{2}xy(\sqrt{3}/2), \text{ where } CD = x, AD = y$$

$\therefore \quad xy = 6$ (i)

From ΔABC, we get

$AC^2 = AB^2 + BC^2 - 2AB\,BC\cos 60° = 4 + 25 - 20\,(1/2) = 19$ (ii)

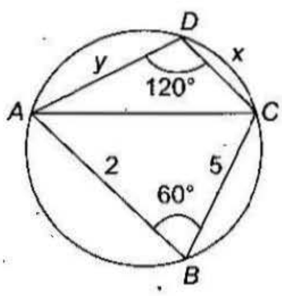


Fig. 5.54

Also from ΔACD,

$AC^2 = CD^2 + DA^2 - 2CD\,DA\cos 120° = x^2 + y^2 + xy = x^2 + y^2 + 6$ [using Eq. (i)] (iii)

Now from Eqs. (ii) and (iii), we have

$x^2 + y^2 + 6 = 19$ or $x^2 + y^2 = 13$ (iv)

Solving Eqs. (ii) and (iv), we get

$x^4 - 13x^2 + 36 = 0 \quad \Rightarrow \quad x^2 = 4, 9 \quad \Rightarrow \quad x = 2, 3 \Rightarrow y = 3, 2$

Hence, the other two sides of the cyclic quadrilateral are 3 and 2.

8. Let $\angle ACE = \alpha$. Clearly, from Fig. 5.55, we get

$$\frac{p}{AC} = \sin\alpha, \frac{q}{BC} = \sin(\alpha + C)$$

$$\Rightarrow \quad \frac{p}{b} = \sin\alpha, \frac{q}{a} = \sin\alpha\cos C + \cos\alpha\sin C$$

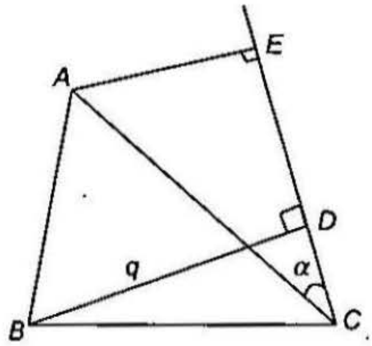


Fig. 5.55

$$\Rightarrow \quad \frac{q}{a} = \frac{p}{b}\cos C + \cos\alpha\sin C$$

$$\Rightarrow \quad \left(\frac{q}{a} - \frac{p}{b}\cos C\right)^2 = \cos^2\alpha\sin^2 C = \left(1 - \frac{p^2}{b^2}\right)\left(1 - \cos^2 C\right)$$

$$\Rightarrow \frac{q^2}{a^2} + \frac{p^2}{b^2}\cos^2 C - \frac{2pq}{ab}\cos C = 1 - \frac{p^2}{b^2} - \left(1 - \frac{p^2}{b^2}\right)\cos^2 C$$

$$\Rightarrow \quad \frac{q^2}{a^2} + \frac{p^2}{b^2} - \frac{2pq}{ab}\cos C = \sin^2 C$$

$$\Rightarrow \quad a^2p^2 + b^2q^2 - 2\,abpq\cos C = a^2b^2\sin^2 C$$

9. Let I be the in-centre of the ΔABC.

In ΔIBC, $\angle BIC = \pi - \dfrac{B+C}{2} = \pi - \dfrac{\pi - A}{2} = \dfrac{\pi + A}{2}$

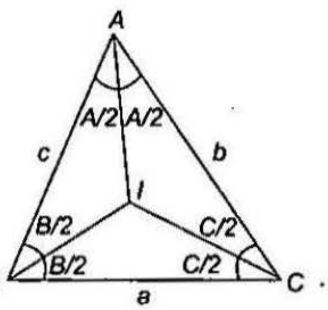


Fig. 5.56

Now, radius of circumcircle of ΔIBC, by sine rule is

$$R_1 = \frac{BC}{2\sin(\angle BIC)} = \frac{a}{2\sin\left(\frac{\pi + A}{2}\right)} = \frac{2R\sin A}{2\cos\frac{A}{2}} = 2R\sin\frac{A}{2}$$

Similarly, radius of circumcircle of ΔICA and ΔIAB are given by

$$R_2 = 2R\sin\frac{B}{2} \text{ and } R_3 = 2R\sin\frac{C}{2} \quad \Rightarrow \quad R_1R_2R_3 = 8R^3\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = 2rR^2$$

10.

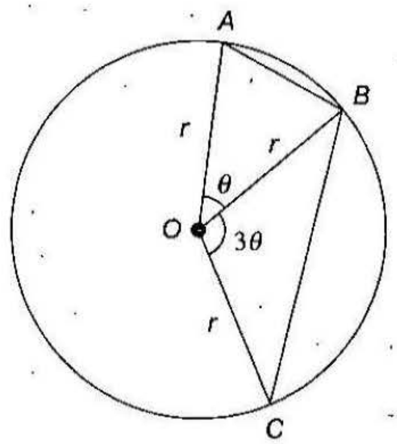


Fig. 5.57

Applying cosine rule in ΔOAB, we get

$$\cos\theta = \frac{2r^2 - a^2}{2r^2} \quad \Rightarrow \quad a^2 = 2r^2(1 - \cos\theta) \quad \Rightarrow \quad a = 2r\sin\frac{\theta}{2}$$

Applying cosine rule in ΔOBC, we get

$$\cos 3\theta = \frac{2r^2 - b^2}{2r^2}$$

$$\Rightarrow \quad b = 2r\sin\left(\frac{3\theta}{2}\right)$$

$$= 2r\left[3\sin\frac{\theta}{2} - 4\sin^3\frac{\theta}{2}\right]$$

$$= 2r\left[\frac{3a}{2r} - \frac{4a^3}{8r^3}\right]$$

$$= 3a - \frac{a^3}{r^2}$$

$$\Rightarrow \quad r^2 = \frac{a^3}{3a - b}$$

$$\Rightarrow \quad r = a\sqrt{\frac{a}{3a - b}} \text{ cm}$$

11. In ΔAEB, $\dfrac{AE}{\sin\left(90° - \dfrac{A}{3}\right)} = \dfrac{BE}{\sin\dfrac{A}{3}} = c$ (1)

In ΔAED, $\dfrac{AE}{\sin\left(90° - \dfrac{A}{3}\right)} = \dfrac{ED}{\sin\dfrac{A}{3}}$ (2)

Now $BE = ED = a/4$

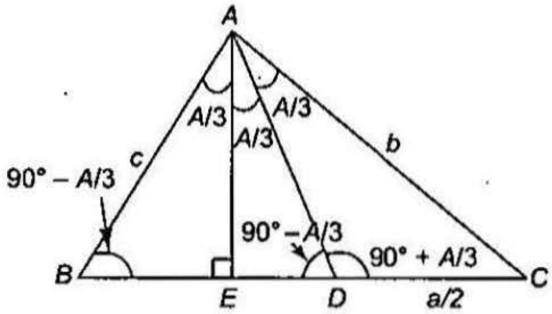


Fig. 5.58

$$\Rightarrow \cos\frac{A}{3} = \frac{AE}{c} \text{ and } \frac{AE}{\cos\frac{A}{3}} = \frac{a/4}{\sin\frac{A}{3}} \Rightarrow \sin\frac{A}{3} = \frac{a}{4c}$$

In ΔAEC, $\sin\dfrac{2A}{3} = \dfrac{EC}{b}$ or $2\sin\dfrac{A}{3}\cos\dfrac{A}{3} = \dfrac{\frac{a}{2}+\frac{a}{4}}{b} = \dfrac{3a}{4b}$

i.e., $\cos\dfrac{A}{3} = \dfrac{3c}{2b}$

Now, L.H.S. $= \cos\dfrac{A}{2}\sin^2\dfrac{A}{3} = \dfrac{3c}{2b} \times \dfrac{a^2}{16c^2} = \dfrac{3a^2}{32bc} =$ R.H.S.

12. Let AD be the perpendicular from A on BC. When AD is produced, it meets the circumscribing circle at E. From question, $DE = \alpha$.

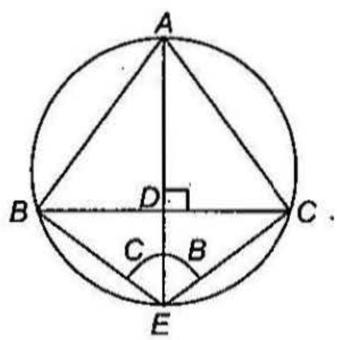


Fig. 5.59

Since angles in the same segment are equal, we have

$\angle AEB = \angle ACB = \angle C$, and $\angle AEC = \angle ABC = \angle B$

From the right-angled triangle BDE,

$\tan C = \dfrac{BD}{DE}$ (i)

From the right-angled triangle CDE,

$$\tan B = \frac{CD}{DE} \quad \text{(ii)}$$

Adding Eqs. (i) and (ii), we get

$$\tan B + \tan C = \frac{BD + CD}{DE} = \frac{BC}{DE} = \frac{a}{\alpha} \quad \text{(iii)}$$

Similarly, $\tan C + \tan A = \dfrac{b}{\beta}$ (iv)

and $\tan A + \tan B = \dfrac{c}{\gamma}$ (v)

Adding Eqs. (iii), (iv) and (v), we get

$$\frac{a}{\alpha} + \frac{b}{\beta} + \frac{c}{\gamma} = 2(\tan A + \tan B + \tan C)$$

13.

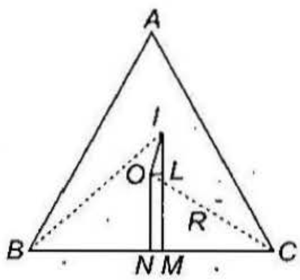


Fig. 5.60

Let I be the incentre and O be the circumcentre of the triangle ABC.
Let OL be parallel to BC. Let $\angle IOL = \theta$, $IM = r$, $OC = R$, $\angle NOC = A$

$$\Rightarrow \quad \tan\theta = \frac{IL}{OL} = \frac{IM - LM}{BM - BN}$$

$$= \frac{IM - ON}{BM - NC}$$

$$= \frac{r - R\cos A}{r\cot\frac{B}{2} - R\sin A}$$

$$= \frac{4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} - R\cos A}{4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.\cot\frac{B}{2} - R\sin A}$$

$$= \frac{\cos A + \cos B + \cos C - 1 - \cos A}{\sin A + \sin C - \sin B - \sin A}$$

$$= \frac{\cos B + \cos C - 1}{\sin C - \sin B}$$

$$\Rightarrow \quad \theta = \tan^{-1}\left[\frac{\cos B + \cos C - 1}{\sin C - \sin B}\right]$$

14. Let $\tan A = x$

$\tan A + \tan C = 2\tan B \quad \Rightarrow \quad \tan C = 2\tan B - x$

Also, $\tan A + \tan B + \tan C = \tan A \tan B \tan C$ $\quad [\because A+B+C=\pi]$

$\Rightarrow \quad x + \tan B + 2\tan B - x = x\tan B\,(2\tan B - x)$

$\Rightarrow \quad 3 = x\,(2\tan B - x)$

$$\Rightarrow \quad \frac{3+x^2}{2x} = \tan B$$

$$\Rightarrow \quad \tan C = \frac{3+x^2}{x} - x = \frac{3}{x}$$

Now,

$a^2:b^2:c^2 = \sin^2 A:\sin^2 B:\sin^2 C$

$$= \frac{\tan^2 A}{1+\tan^2 A}:\frac{\tan^2 B}{1+\tan^2 B}:\frac{\tan^2 C}{1+\tan^2 C}$$

$$= \frac{x^2}{1+x^2}:\frac{\left(\frac{3+x^2}{2x}\right)^2}{1+\left(\frac{3+x^2}{2x}\right)^2}:\frac{\left(\frac{3}{x}\right)^2}{1+\left(\frac{3}{x}\right)^2}$$

$$= \frac{x^2}{1+x^2}:\frac{(3+x^2)^2}{(x^2+9)\,(x^2+1)}:\frac{9}{x^2+9}$$

$$= x^2(x^2+9):(3+x^2)^2:9\,(1+x^2)$$

15. $A + B = 90°$

$\Rightarrow \quad \tan A = \cot B$

$$\Rightarrow \quad \frac{\sqrt{5}-1}{2} = \cot^2 B$$

$$\Rightarrow \quad \frac{\cos^2 B}{\sqrt{5}-1} = \frac{\sin^2 B}{2} = \frac{1}{\sqrt{5}+1}$$

$$\Rightarrow \quad \frac{b^2}{ac} = \frac{4R^2\sin^2 B}{4R^2\sin A\sin C} = \frac{\sin^2 B}{\sin A} = \sin^2 B\sqrt{1+\cot^2 A} \quad [\because \angle C = 90°]$$

$$= \left(\frac{2}{\sqrt{5}+1}\right)\sqrt{1+\frac{2}{\sqrt{5}-1}} \quad \left[\because \tan A = \sqrt{\frac{\sqrt{5}-1}{2}}\right]$$

$$= \frac{2}{\sqrt{5}+1}\sqrt{\frac{\sqrt{5}+1}{\sqrt{5}-1}} = \frac{2}{\sqrt{5}+1}\sqrt{\frac{(\sqrt{5}+1)^2}{4}} = 1$$

$\Rightarrow \quad b^2 = ac \Rightarrow a, b, c$ are in G.P.

Objective Type

1. c. Adding, $\sin(A+B)=1$

and subtracting, $\sin(A-B)=\dfrac{\sqrt{2}-2}{\sqrt{2}}=1-\sqrt{2}\neq 0$

$\therefore\ A+B=90°, A\neq B$

2. c.

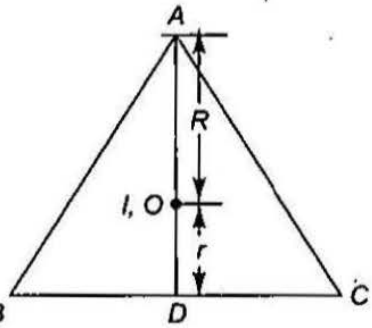


Fig. 5.61

In equilateral triangle, circumcentre (O) and incentre (I) coincide.

Also from the diagram $R+r=h \quad\Rightarrow\quad \dfrac{R+r}{h}=1$

3. b. $\dfrac{\sin A}{c\sin B}+\dfrac{\sin B}{c}+\dfrac{\sin C}{b}=\dfrac{c}{ab}+\dfrac{b}{ac}+\dfrac{a}{bc}$

$\Rightarrow \dfrac{a}{bc}+\dfrac{\sin B}{c}+\dfrac{\sin C}{b}=\dfrac{c}{ab}+\dfrac{b}{ac}+\dfrac{a}{bc}$

$\Rightarrow \dfrac{\sin B}{c}+\dfrac{\sin C}{b}=\dfrac{c}{ab}+\dfrac{b}{ac}$

$\Rightarrow \dfrac{b\sin B+c\sin C}{bc}=\dfrac{c^2+b^2}{abc}$

$\Rightarrow a=\dfrac{b^2+c^2}{b\sin B+c\sin C}=\dfrac{b(2R\sin B)+c(2R\sin C)}{b\sin B+c\sin C}$

$\Rightarrow a=2R$

$\Rightarrow \angle A=\pi/2$

4. a.

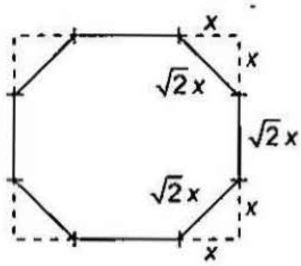


Fig. 5.62

Clearly, $x+\sqrt{2}x+x=1$. So, $x=\dfrac{1}{2+\sqrt{2}}$

The required area $=\left(1^2-4\times\dfrac{1}{2}x^2\right)\ \text{m}^2$

$$= \left\{1 - 2\ \frac{1}{(2+\sqrt{2})^2}\right\} m^2$$

$$= \left\{1 - \frac{1}{(\sqrt{2}+1)^2}\right\} m^2$$

$$= \left[1-(\sqrt{2}-1)^2\right] m^2 = 2(\sqrt{2}-1)\, m^2$$

5. c. As A is an obtuse angle,

$90° < A < 180°$

$\Rightarrow \quad 90° < 180-(B+C) < 180°$

$\Rightarrow \quad 0 < B+C < 90°$

$\Rightarrow \quad B+C < 90° \Rightarrow B < 90° - C$

$\therefore \quad \tan B < \tan(90° - C)$

$\tan B < \cot C \Rightarrow \tan B \tan C < 1$

6. d. $x = \frac{1}{2}[\cos(A-C) - \cos(A+C)] = \frac{1}{2}\left[\cos(A-C) + \frac{1}{2}\right]$

But $0 \le \cos(A-C) \le 1 \quad \Rightarrow \quad \frac{1}{2}\left(0+\frac{1}{2}\right) \le x \le \frac{1}{2}\left(1+\frac{1}{2}\right)$

7. c. Here $b = 2c$.

Now, $\tan\frac{B-C}{2} = \frac{b-c}{b+c}\cot\frac{A}{2}$

$\cot\frac{A}{2}\ \cot\frac{B-C}{2} = \frac{b+c}{b-c} = \frac{3c}{c} = 3$

8. b.

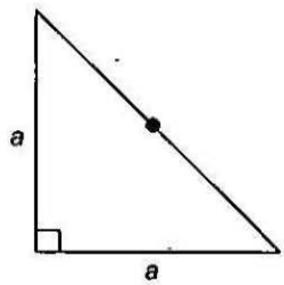


Fig. 5.63

Here, $R = \frac{\sqrt{2}a}{2} = \frac{a}{\sqrt{2}}$

$r = \frac{\Delta}{s} = \frac{\frac{1}{2}a^2}{\frac{1}{2}\left(a+a+\sqrt{2}a\right)} = \frac{a}{2+\sqrt{2}}$

$\therefore \quad \frac{R}{r} = \frac{a}{\sqrt{2}} \times \frac{2+\sqrt{2}}{a} = \sqrt{2}+1$

9. a. $\cos C = \dfrac{a^2+b^2-c^2}{2ab} = \dfrac{25+9-49}{2\times5\times3} = -\dfrac{1}{2}$

$$\cos B = \frac{a^2+c^2-b^2}{2ac} = \frac{25+49-9}{2\times5\times7} = \frac{13}{14}$$

$$\Rightarrow \quad 3\cos C + 7\cos B = -\frac{3}{2}+\frac{13}{2} = 5$$

10. a. $\angle B = 90° \Rightarrow \cos A = \dfrac{c}{b}$

$$\Rightarrow \quad \frac{1-\tan^2\frac{A}{2}}{1+\tan^2\frac{A}{2}} = \frac{c}{b}$$

$$\Rightarrow \quad \frac{\left(1+\tan^2\frac{A}{2}\right)-\left(1-\tan^2\frac{A}{2}\right)}{\left(1+\tan^2\frac{A}{2}\right)+\left(1-\tan^2\frac{A}{2}\right)} = \frac{(b-c)}{(b+c)}$$

$$\Rightarrow \quad \tan^2\frac{A}{2} = \frac{b-c}{b+c}$$

11. a. $\dfrac{\cot A}{\cot B + \cot C} = \dfrac{\dfrac{R\left(b^2+c^2-a^2\right)}{abc}}{\dfrac{R\left(a^2+c^2-b^2\right)}{abc}+\dfrac{R\left(a^2+b^2-c^2\right)}{abc}}$

$$= \frac{b^2+c^2-a^2}{2a^2} = \frac{2a^2-a^2}{2a^2} = \frac{1}{2}$$

12. c. Here, $\sin\theta - \cos\theta = \dfrac{b}{a}$ and $\sin\theta\cos\theta = \dfrac{c}{a}$

$$\Rightarrow \quad 1-2\sin\theta\cos\theta = \frac{b^2}{a^2}$$

$$\Rightarrow \quad 1-\frac{2c}{a} = \frac{b^2}{a^2}$$

$$\Rightarrow \quad a^2-b^2 = 2ac$$

Hence, $\cos B = \dfrac{a^2+c^2-b^2}{2ac} = \dfrac{2ac+c^2}{2ac} = 1+\dfrac{c}{2a}$

13. c. We have $a^2+b^2+c^2-ac-ab\sqrt{3} = 0$

$$\Rightarrow \quad \frac{a^2}{4} - ac + c^2 + \frac{3a^2}{4} + b^2 - ab\sqrt{3} = 0$$

$$\Rightarrow \quad \left[\frac{a}{2}-c\right]^2 + \left[\frac{\sqrt{3}a}{2}-b\right]^2 = 0$$

$\Rightarrow \quad a = 2c$ and $2b = \sqrt{3}a \quad \Rightarrow \quad b^2 + c^2 = a^2$

Hence, the triangle is right angled.

14. c. $(a+b+c)(b+c-a) = kbc$

$\Rightarrow \quad (b+c)^2 - a^2 = kbc$

$\Rightarrow \quad b^2 + c^2 - a^2 = (k-2)\,bc$

$\Rightarrow \quad 2bc\cos A = (k-2)\,bc$

$\Rightarrow \quad \cos A = \dfrac{k-2}{2}$

Now, A being the angle of a triangle,

$-1 < \cos A < 1 \quad \Rightarrow \quad -2 < k-2 < 2$

$\Rightarrow \quad 0 < k < 4$

15. c. $a = 2b$ and $A - B = 60°$

We know that $\tan\dfrac{A-B}{2} = \dfrac{a-b}{a+b}\cot\dfrac{C}{2}$

$\Rightarrow \quad \tan 30° = \dfrac{1}{3}\cot\dfrac{C}{2} \quad \Rightarrow \quad \tan\dfrac{C}{2} = \dfrac{1}{\sqrt{3}}$

$\Rightarrow \quad C = 60°$

Hence, $A + B = 120° \quad \Rightarrow \quad 2A = 180° \quad \Rightarrow \quad A = 90°, B = 30°, C = 60°$

16. a. $r_1 = 2r_2 = 3r_3$

$\Rightarrow \quad \dfrac{\Delta}{s-a} = 2\dfrac{\Delta}{s-b} = 3\dfrac{\Delta}{s-c}$

$\Rightarrow \quad \dfrac{1}{s-a} = \dfrac{2}{s-b} = \dfrac{3}{s-c} = k$ (say)

$\Rightarrow \quad s-a = \dfrac{1}{k}, s-b = \dfrac{2}{k}$ and $s-c = \dfrac{3}{k}$

Adding, we get $3s - (a+b+c) = \dfrac{6}{k} \quad \Rightarrow \quad s = \dfrac{6}{k} \quad \Rightarrow \quad a = \dfrac{5}{k}$ and $b = \dfrac{4}{k} \quad \Rightarrow \quad \dfrac{a}{b} = \dfrac{5}{4}$

17. a. We have $\left(1 - \dfrac{s-b}{s-a}\right)\left(1 - \dfrac{s-c}{s-a}\right) = 2$

$\Rightarrow \quad 2(b-a)(c-a) = 4(s-a)^2$

$\Rightarrow \quad 2(bc - ac - ab + a^2) = (2s - 2a)^2$

$\Rightarrow \quad 2(bc - ac - ab + a^2) = (b + c - a)^2$

$\Rightarrow \quad a^2 = b^2 + c^2$

Hence, triangle is right angled.

18. b. We have $\Delta = \dfrac{\sqrt{3}}{4}a^2, s = \dfrac{3a}{2}$

$\therefore \quad r = \dfrac{\Delta}{s} = \dfrac{a}{2\sqrt{3}}, R = \dfrac{abc}{4\Delta} = \dfrac{a^3}{\sqrt{3}a^2} = \dfrac{a}{\sqrt{3}}$

and $r_1 = \dfrac{\Delta}{s-a} = \dfrac{\sqrt{3}/4a^2}{a/2} = \dfrac{\sqrt{3}}{2}a$

Hence, $r : R : r_1 = \dfrac{a}{2\sqrt{3}} : \dfrac{a}{\sqrt{3}} : \dfrac{\sqrt{3}}{2}a = 1 : 2 : 3$

19. a. $\cot\frac{A}{2}, \cot\frac{B}{2}, \cot\frac{C}{2}$ are in A.P.

$\Rightarrow \quad 2\cot\frac{B}{2} = \cot\frac{A}{2} + \cot\frac{C}{2}$

$\Rightarrow \quad 2\sqrt{\frac{s(s-b)}{(s-a)(s-c)}} = \sqrt{\frac{s(s-a)}{(s-b)(s-c)}} + \sqrt{\frac{s(s-c)}{(s-a)(s-b)}}$

$\Rightarrow \quad 2(s-b) = s-a+s-c$

$\Rightarrow \quad 2b = a+c$

$\Rightarrow \quad a, b, c$ are in A.P.

20. b. We have $\tan\frac{C-B}{2} = \frac{c-b}{c+b}\cot\frac{A}{2}$

$\therefore \quad \tan\left(\frac{C-B}{2}\right) = \frac{\sqrt{3}+1-2}{\sqrt{3}+1+2}\cot 15^\circ$

$= \frac{\sqrt{3}-1}{\sqrt{3}+3}\,\frac{1}{\tan(45-30^\circ)}$

$= \frac{\sqrt{3}-1}{\sqrt{3}+3}\,\frac{\sqrt{3}+1}{\sqrt{3}-1} = \frac{1}{\sqrt{3}}$

$= \tan 30^\circ$

$\Rightarrow \frac{C-B}{2} = 30^\circ$

21. d. $\cos A = \frac{b^2+c^2-a^2}{2bc} = \frac{64+81-49}{2\times 8\times 9} = \frac{145-49}{144} = \frac{96}{144}$

$\cos B = \frac{a^2+c^2-b^2}{2ac} = \frac{49+81-64}{2\times 7\times 9} = \frac{66}{126} = \frac{11}{26}$

22. b. $\cos A + \cos B + \cos C = \frac{7}{4}$

$\Rightarrow \quad 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = \frac{7}{4}$

$\Rightarrow \quad 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = \frac{3}{4}$ $\quad [\because r = 4R\sin(A/2)\sin(B/2)\sin(C/2)]$

$\Rightarrow \quad \frac{r}{R} = \frac{3}{4} \Rightarrow \frac{R}{r} = \frac{4}{3}$

23. a. $\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2} = \frac{s(s-a)}{\Delta} + \frac{s(s-b)}{\Delta} + \frac{s(s-c)}{\Delta}$

$= \frac{s}{\Delta}[3s-(a+b+c)] = \frac{s^2}{\Delta} = \frac{\left(\frac{\Delta}{r}\right)^2}{\Delta} = \frac{\Delta}{r^2}$

24. b. $\dfrac{\cot A + \cot C}{\cot B} = \dfrac{\sin(A+C)\sin B}{\sin A \sin C \cos B}$

$$= \frac{\sin^2 B}{\sin A \sin C \cos B}$$

$$= \frac{4R^2 b^2}{4R^2 ac \cos B}$$

$$= \frac{2b^2}{2ac\cos B} = \frac{2b^2}{a^2+c^2-b^2}$$

$$= \frac{2b^2}{2002b^2-b^2} = \frac{2}{2001}$$

25. a.

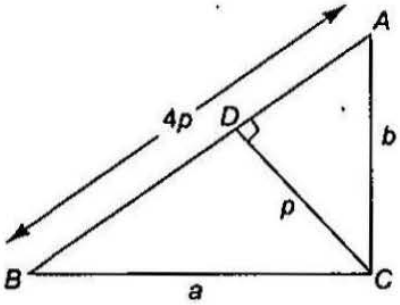


Fig. 5.64

$\Delta = \dfrac{1}{2}ab = \dfrac{1}{2}p\,4p \quad \Rightarrow \quad ab = 4p^2$

Also, $a^2+b^2=c^2=16p^2$

$\therefore \quad (a-b)^2 = a^2+b^2-2ab = 8p^2$

Also, $(a+b)^2 = a^2+b^2+2ab = 24p^2$

$$\tan\frac{A-B}{2} = \frac{a-b}{a+b}\cot\frac{C}{2} = \frac{1}{\sqrt{3}}\,1$$

$$\Rightarrow \quad \frac{A-B}{2} = 30° \quad \Rightarrow \quad A-B=60°$$

26. c. $\dfrac{1}{\sin^2\dfrac{A}{2}}, \dfrac{1}{\sin^2\dfrac{B}{2}}, \dfrac{1}{\sin^2\dfrac{C}{2}}$ are in A.P.

$$\Rightarrow \quad \frac{1}{\sin^2\dfrac{C}{2}} - \frac{1}{\sin^2\dfrac{B}{2}} = \frac{1}{\sin^2\dfrac{B}{2}} - \frac{1}{\sin^2\dfrac{A}{2}}$$

$$\Rightarrow \quad \frac{ab}{(s-a)(s-b)} - \frac{ac}{(s-a)(s-c)} = \frac{ac}{(s-a)(s-c)} - \frac{bc}{(s-b)(s-c)}$$

$$\Rightarrow \quad \left(\frac{a}{s-a}\right)\left(\frac{b(s-c)-c(s-b)}{(s-b)(s-c)}\right)=\left(\frac{c}{s-c}\right)\left(\frac{a(s-b)-b(s-a)}{(s-a)(s-b)}\right)$$

$$\Rightarrow \quad ab-ac=ac-bc$$

$$\Rightarrow \quad ab+bc=2ac$$

$$\Rightarrow \quad \frac{1}{c}+\frac{1}{a}=\frac{2}{b}, \text{ i.e., } a, b, c \text{ are in H.P.}$$

27. a. $b=2, c=\sqrt{3}, \angle A=30°$

$$a=\sqrt{b^2+c^2-2bc\cos A}=\sqrt{4+3-2\times 2\sqrt{3}\,\frac{\sqrt{3}}{2}}=1$$

$$\Rightarrow \quad r=(s-a)\tan\frac{A}{2}=\frac{b+c-a}{2}\tan\frac{A}{2}=\frac{\sqrt{3}+1}{2}\tan 15°=\frac{\sqrt{3}+1}{2}\,\frac{\sqrt{3}-1}{\sqrt{3}+1}=\frac{\sqrt{3}-1}{2}$$

28. c.

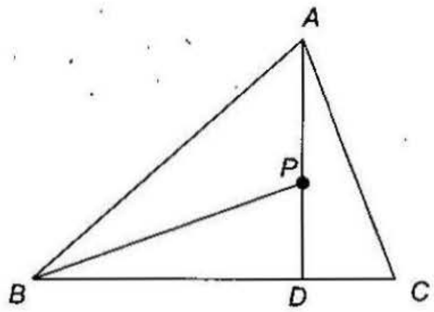


Fig. 5.65

$\angle BPA=90°+(B/3), \angle ABP=2B/3$

In ΔABP,

$$\frac{AP}{\sin(2B/3)}=\frac{c}{\sin[90°+(B/3)]}=\frac{c}{\cos(B/3)}$$ [by sine rule]

$$\Rightarrow \quad AP=\frac{c\sin(2B/3)}{\cos(B/3)}=\frac{2c\sin(B/3)\cos(B/3)}{\cos(B/3)}$$

$$=2c\sin(B/3)$$

29. d. $\angle A=\frac{\pi}{2} \Rightarrow a^2=b^2+c^2$

$$\sin C=\frac{c}{a}=\frac{2\tan\frac{C}{2}}{1+\tan^2\frac{C}{2}}$$

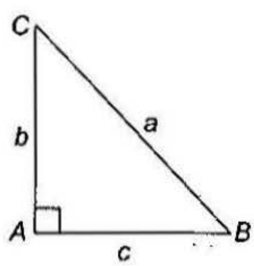


Fig. 5.66

$$\Rightarrow \quad c\tan^2\frac{C}{2} - 2a\tan\frac{C}{2} + c = 0$$

$$\Rightarrow \quad \tan\frac{C}{2} = \frac{2a \pm \sqrt{4a^2 - 4c^2}}{2c} = \frac{2a \pm 2b}{2c} = \frac{a \pm b}{c}$$

$$= \frac{a-b}{c}$$

Because if $\tan\frac{C}{2} = \frac{a+b}{c}$, then $\tan\frac{C}{2} > 1 \quad \Rightarrow \quad \frac{C}{2} > \frac{\pi}{4} \Rightarrow C > \frac{\pi}{2}$ which is not possible.

30. a. $a\cos(B-C) + b\cos(C-A) + c\cos(A-B)$

$= 2R\sin A\cos(B-C) + 2R\sin B\cos(C-A) + 2R\sin C\cos(A-B)$

$= 2R\sin(B+C)\cos(B-C) = R[\sin 2B + \sin 2C]$

$= R[\sin 2B + \sin 2C + \sin 2C + \sin 2A + \sin 2A + \sin 2B]$

$= 2R(\sin 2A + \sin 2B + \sin 2C)$

$= 8R\sin A\sin B\sin C$

$$= 8R\frac{a}{2R}\frac{b}{2R}\frac{c}{2R} = \frac{abc}{R^2}$$

31. d. If the triangle is equilateral

$$\sin A + \sin B + \sin C = \frac{3\sqrt{3}}{2}$$

If the triangle is isosceles, let $A = 30°$, $B = 30°$, $C = 120°$.

Then, $\sin A + \sin B + \sin C = 1 + \frac{\sqrt{3}}{2}$

If the triangle is right angled, let $A = 90°$, $B = 30°$, $C = 60°$.

Then, $\sin A + \sin B + \sin C = \frac{3+\sqrt{3}}{2}$

If the triangle is right-angled isosceles, then one of the angles is 90° and the remaining two are 45° each, so that

$\sin A + \sin B + \sin C = 1 + \sqrt{2}$

and $\cos A + \cos B + \cos C = \sqrt{2}$

32. c. $\frac{r}{r_1} = \frac{r_2}{r_3}$

$$\Rightarrow \quad r r_3 = r_1 r_2$$

$$\Rightarrow \quad \frac{\Delta}{s}\frac{\Delta}{s-c} = \frac{\Delta}{s-a}\frac{\Delta}{s-b}$$

$$\Rightarrow \quad \frac{(s-a)(s-b)}{s(s-c)} = 1$$

$$\Rightarrow \quad \tan^2\frac{C}{2}=1 \Rightarrow \tan\frac{C}{2}=1$$

$$\Rightarrow \quad \frac{C}{2}=45° \Rightarrow C=90°$$

33. c. Since $\cos 3A+\cos 3B+\cos 3C=1$

$$\Rightarrow \quad 4\sin\frac{3A}{2}\sin\frac{3B}{2}\sin\frac{3C}{2}=0$$

Either $\frac{3A}{2}=180°$ or $\frac{3B}{2}=180°$ or $\frac{3C}{2}=180°$

Either $A=120°$ or $B=120°$ or $C=120°$

34. a. Given, $\cos B\cos C+\sin B\sin C\sin^2 A=1$ (i)

Now, we know that $\sin^2 A\le 1$ (ii)

Also, $\sin B$ and $\sin C$ are positive.

$\Rightarrow \quad \sin B\sin C\sin^2 A\le \sin B\sin C$ (iii)

$\Rightarrow \quad 1-\cos B\cos C\le \sin B\sin C,$ [by using Eq. (i)]

$\Rightarrow \quad \cos(B-C)\ge 1 \quad \Rightarrow \quad \cos(B-C)=1 \quad \Rightarrow \quad B-C=0 \quad \Rightarrow \quad B=C$

Also, $\sin^2 A=1$, i.e., $A=\pi/2$. Hence, the triangle is right-angled isosceles.

35. c.

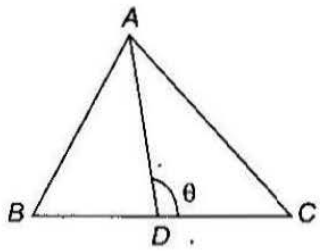


Fig. 5.67

$$\theta=\pi-\left(C+\frac{A}{2}\right) \quad \Rightarrow \quad \sin\theta=\sin\left(C+\frac{A}{2}\right)=\sin\left(C+\frac{\pi}{2}-\frac{B+C}{2}\right)=\cos\left(\frac{B-C}{2}\right)$$

36. d. $r-r_2=r_3-r_1$

$$\Rightarrow \quad \frac{\Delta}{s}-\frac{\Delta}{s-b}=\frac{\Delta}{s-c}-\frac{\Delta}{s-a}$$

$$\Rightarrow \quad \frac{-b}{s(s-b)}=\frac{c-a}{(s-a)(s-c)}$$

$$\Rightarrow \quad \frac{(s-a)(s-c)}{s(s-b)}=\frac{a-c}{b}$$

$$\Rightarrow \quad \tan^2\frac{B}{2}=\frac{a-c}{b}$$

But $\frac{B}{2}\in\left(\frac{\pi}{6},\frac{\pi}{4}\right)$

$$\Rightarrow \quad \tan^2\frac{B}{2}\in\left(\frac{1}{3},1\right)$$

$\Rightarrow \quad \frac{1}{3} < \frac{a-c}{b} < 1$

$\Rightarrow \quad b < 3a - 3c < 3b$

$\Rightarrow \quad b + 3c < 3a < 3b + 3c$

37. b.

$$R = \frac{a}{2\sin A} = \frac{2+\sqrt{5}}{2\sin 30^\circ} = \frac{2+\sqrt{5}}{2\times\frac{1}{2}} = \left(2+\sqrt{5}\right)$$

Now, $AH = 2R\cos A = 2(2+\sqrt{5})\cos 30^\circ = (2+\sqrt{5})\sqrt{3}$

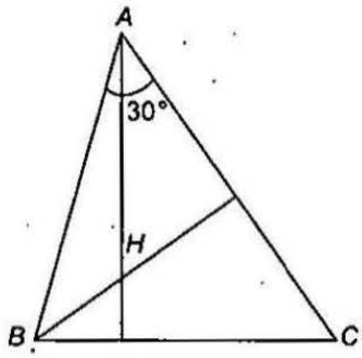


Fig. 5.68

38. a. We know that $IA = \frac{r}{\sin\frac{A}{2}}$

$\Rightarrow \quad IA : IB : IC = \operatorname{cosec}\frac{A}{2} : \operatorname{cosec}\frac{B}{2} : \operatorname{cosec}\frac{C}{2}$

39. a. Using the property of angle bisector, we have $\frac{BD}{DC} = \frac{AB}{AC} = \frac{c}{b}$

$\Rightarrow BD + DC = ck + bk = a$

$\Rightarrow k = \frac{a}{b+c}$

Also $x\,y = b\,c\,k^2$ (property of circle)

$$\Rightarrow x = \left(\frac{2bc\cos\frac{A}{2}}{b+c}\right)\frac{bc\,a^2}{(b+c)^2}$$

$$= \frac{a^2\sec\frac{A}{2}}{2(b+c)}$$

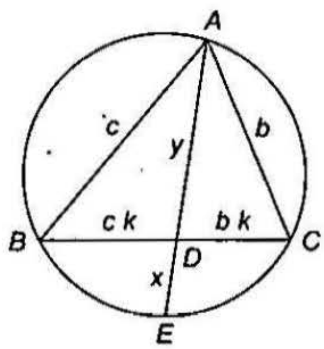


Fig. 5.69

40. a. From the right angled ΔCAD, we have

$$\cos C = \frac{b}{a/2} \Rightarrow \frac{2b}{a} = \frac{a^2+b^2-c^2}{2ab}$$

$\Rightarrow a^2 + b^2 - c^2 = 4b^2 \Rightarrow a^2 - c^2 = 3b^2$

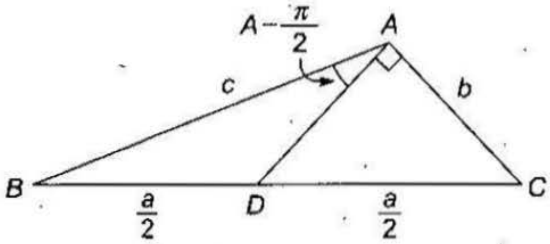


Fig. 5.70

41. a.

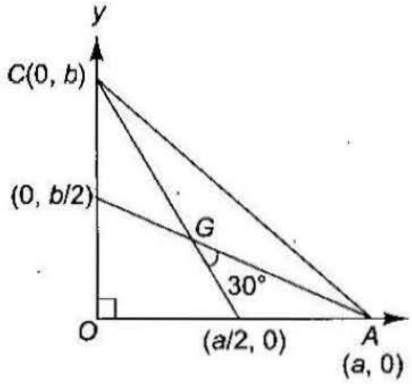


Fig. 5.71

Slope of $GC = m_1 = \frac{-2b}{a}$, slope of $AG = m_2 = \frac{-b}{2a}$

$$\tan 30° = \tan\left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \frac{\frac{3b}{2a}}{1+\frac{b^2}{a^2}} \text{ and } a^2 + b^2 = 9$$

$$\Rightarrow \frac{1}{\sqrt{3}} = \frac{3ba}{2(a^2+b^2)} \qquad \Rightarrow \frac{1}{2}ab = \left(\frac{a^2+b^2}{3\sqrt{3}}\right) = \frac{9}{3\sqrt{3}} = \sqrt{3}$$

42. c.

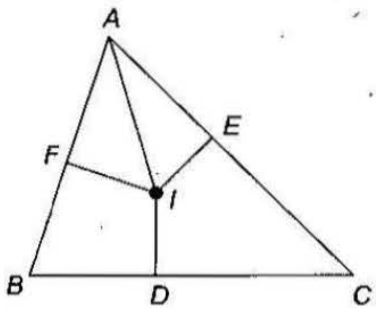


Fig. 5.72

In triangles AIF and AIE,

$$\frac{IF}{\sin(A/2)} = AI = \frac{IE}{\sin(A/2)} \qquad \Rightarrow AI^2 = \frac{IE \cdot IF}{\sin^2(A/2)}$$

$$\Rightarrow \frac{ID \cdot IE \cdot IF}{IA \cdot IB \cdot IC} = \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = \frac{r}{4R} = \frac{1}{10}$$

43. c. We have $BD = DC$ and $\angle DAB = 90°$. Draw CN perpendicular to BA produced, then in ΔBCN, we have

$DA = \frac{1}{2}CN$ and $AB = AN$

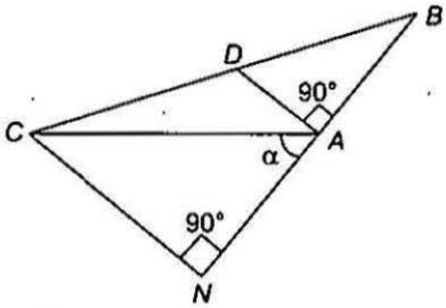


Fig. 5.73

Let $\angle CAN = \alpha$

$\because \tan A = \tan(\pi - \alpha) = -\tan\alpha = -\frac{CN}{NA} = -2\frac{AD}{AB} = -2\tan B \quad \Rightarrow \quad \tan A + 2\tan B = 0$

44. a.

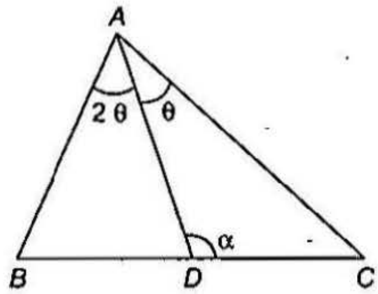


Fig. 5.74

Let $\frac{A}{3} = \angle CAD = \theta$

Now, by m–n theorem,

$(1 + 1)\cot\alpha = 1\cot 2\theta - 1\cot\theta \quad \Rightarrow \quad 2\cot(B + 2\theta) = \cot 2\theta - \cot\theta$

$\Rightarrow \cot(B + 2\theta) + \cot\theta = \cot 2\theta - \cot(B + 2\theta)$

$$\Rightarrow \frac{\sin(B + 3\theta)}{\sin(B + 2\theta)\ \sin\theta} = \frac{\sin B}{\sin(B + 2\theta)\ \sin 2\theta}$$

$$\Rightarrow \frac{\sin(B + A)}{\sin\theta} = \frac{\sin B}{\sin 2\theta}$$

$$\Rightarrow \sin C = \frac{\sin B}{2\cos\theta}$$

$$\Rightarrow \cos\frac{A}{3} = \frac{\sin B}{2\sin C}$$