1. If $\cos\alpha + \cos\beta + \cos\gamma = \sin\alpha + \sin\beta + \sin\gamma = 0$ Then show that

(i) $\cos3\alpha + \cos3\beta + \cos3\gamma = 3\cos(\alpha+\beta+\gamma)$.

(ii) $\sin3\alpha + \sin3\beta + \sin3\gamma = 3\sin(\alpha+\beta+\gamma)$.

A: Given:

$\cos\alpha + \cos\beta + \cos\gamma$
$= \sin\alpha + \sin\beta + \sin\gamma = 0$

Let a= $\cos\alpha + i\sin\alpha$
b= $\cos\beta + i\sin\beta$
c= $\cos\gamma + i\sin\gamma$

$\therefore a+b+c = (\cos\alpha + \cos\beta + \cos\gamma)$
$+(\sin\alpha + \sin\beta + \sin\gamma) = 0 + i0$
$a+b+c = 0$:
We know that if

$a+b+c = 0$ then $a^3 + b^3 + c^3 = 3abc$

$\Rightarrow (cis\alpha)^3 + (cis\beta)^3 + (cis\gamma)^3$

$= 3(cis\alpha)(cis\beta)(cis\gamma)$

By applying De M theorem we get

$(\cos3\alpha + i\sin3\alpha) + (\cos3\beta + i\sin3\beta) + (\cos3\gamma + i\sin3\gamma)$

$= 3[\cos(\alpha+\beta+\gamma) + i\sin(\alpha+\beta+\gamma)]$.

$\Rightarrow (\cos3\alpha + \cos3\beta + \cos3\gamma) + i(\sin3\alpha + \sin3\beta + \sin3\gamma) = 3\cos(\alpha+\beta+\gamma) + i3\sin(\alpha+\beta+\gamma)$.

Now equating the real and imaginary parts on both sides we get

$\cos3\alpha + \cos3\beta + \cos3\gamma = 3\cos(\alpha+\beta+\gamma)$.
and
$\sin3\alpha + \sin3\beta + \sin3\gamma = 3\sin(\alpha+\beta+\gamma)$.

2. If $\cos\alpha + \cos\beta + \cos\gamma = \sin\alpha + \sin\beta + \sin\gamma = 0$ Then show that

$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \sin^2\alpha + \sin^2\beta + \sin^2\gamma = \frac{3}{2}$

A: Given:

$\cos\alpha + \cos\beta + \cos\gamma = \sin\alpha + \sin\beta + \sin\gamma = 0$

Let a= $cis\alpha \Rightarrow \frac{1}{a} = cis(-\alpha)$

b= $cis\beta \Rightarrow \frac{1}{b} = cis(-\beta)$

c= $cis\gamma \Rightarrow \frac{1}{c} = cis(-\gamma)$

$\therefore a+b+c = (\cos\alpha + \cos\beta + \cos\gamma)$
$+(\sin\alpha + \sin\beta + \sin\gamma) = 0 + i0$ $\{\frac{1}{a}+\frac{1}{b}+\frac{1}{c} = 0\}$
$a+b+c = 0$ S.O.B

$\Rightarrow (a+b+c)^2 = 0$

$\Rightarrow a^2 + b^2 + c^2 + 2ab + 2bc + 2ac = 0$

$\Rightarrow a^2 + b^2 + c^2 = -2(ab+bc+ac)$

$\Rightarrow a^2 + b^2 + c^2 = -2abc(\frac{1}{c}+\frac{1}{a}+\frac{1}{b})$

$\Rightarrow (cis\alpha)^2 + (cis\beta)^2 + (cis\gamma)^2 = -2abc(0)$

$\Rightarrow cis2\alpha + cis2\beta + cis2\gamma = 0$

$\Rightarrow (\cos2\alpha + \cos2\beta + \cos2\gamma)$
$+(\sin2\alpha + \sin2\beta + \sin2\gamma) = 0 + i0$

Equating the real parts

$\cos2\alpha + \cos2\beta + \cos2\gamma = 0$ .... (1)

$\Rightarrow (2\cos^2\alpha - 1) + (2\cos^2\beta - 1) + (2\cos^2\gamma - 1) = 0$
$\Rightarrow 2(\cos^2\alpha + \cos^2\beta + \cos^2\gamma) = 3$

$\therefore \cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{3}{2}$

Also from (1) $\cos2\alpha + \cos2\beta + \cos2\gamma = 0$

$\Rightarrow (1 - 2\sin^2\alpha) + (1 - 2\sin^2\beta) + (1 - 2\sin^2\gamma) = 0$

$\Rightarrow 3 = 2(\sin^2\alpha + \sin^2\beta + \sin^2\gamma)$

$\therefore \sin^2\alpha + \sin^2\beta + \sin^2\gamma = \frac{3}{2}$

3. **If n is a positive integer, S.T**

$$(1+i)^n+(1-i)^n=2^{\frac{n+2}{2}}\cos\left(\frac{n\pi}{4}\right)$$

A: $1+i\equiv(a+ib)$⇨a=1, b=1

r=$\sqrt{a^2+b^2}$=$\sqrt{1^2+1^2}$=$\sqrt{2}$

$\theta=\tan^{-1}\left(\frac{b}{a}\right)$=$\tan^{-1}\left(\frac{1}{1}\right)$ ⇨$\theta=\frac{\pi}{4}$

$1+i=r(cos\theta+isin\theta)$

=$\sqrt{2}(cos\frac{\pi}{4}+isin\frac{\pi}{4})$

Now $(1+i)^n=\left[\sqrt{2}(cos\frac{\pi}{4}+isin\frac{\pi}{4})\right]^n$

by applying De M theorem

$(1+i)^n=2^{\frac{n}{2}}\left[cos\frac{n\pi}{4}+isin\frac{n\pi}{4}\right]$ … … (1)

And

$(1-i)^n=2^{\frac{n}{2}}\left[cos\frac{n\pi}{4}-isin\frac{n\pi}{4}\right]$ … … (2)

Adding (1) + (2)

⇨ $(1+i)^n+(1-i)^n$

$=2^{\frac{n}{2}}\left[cos\frac{n\pi}{4}+isin\frac{n\pi}{4}\right]$

$+2^{\frac{n}{2}}\left[cos\frac{n\pi}{4}+isin\frac{n\pi}{4}\right]$

⇨$(1+i)^n+(1-i)^n=2^{\frac{n}{2}}\left(cos\frac{n\pi}{4}+isin\frac{n\pi}{4}+cos\frac{n\pi}{4}-isin\frac{n\pi}{4}\right)$

⇨$(1+i)^n+(1-i)^n=2^{\frac{n}{2}}\left(2cos\frac{n\pi}{4}\right)$

⇨$(1+i)^n+(1-i)^n=2^{\frac{n}{2}}2^1cos\frac{n\pi}{4}$

$$(1+i)^n+(1-i)^n=2^{\frac{n+2}{2}}\cos\left(\frac{n\pi}{4}\right)$$

4. **If n is a positive integer, S.T**

$$(1+i)^{2n}+(1-i)^{2n}=2^{n+1}\cos\left(\frac{n\pi}{2}\right)$$

A: $1+i\equiv(a+ib)$⇨a=1, b=1

r=$\sqrt{a^2+b^2}$=$\sqrt{1^2+1^2}$=$\sqrt{2}$

$\theta=\tan^{-1}\left(\frac{b}{a}\right)$=$\tan^{-1}\left(\frac{1}{1}\right)$ ⇨$\theta=\frac{\pi}{4}$

$1+i=r(cos\theta+isin\theta)$

=$\sqrt{2}(cos\frac{\pi}{4}+isin\frac{\pi}{4})$

Now $(1+i)^n=\left[\sqrt{2}(cos\frac{\pi}{4}+isin\frac{\pi}{4})\right]^n$

by applying De M theorem

$(1+i)^n=2^{\frac{n}{2}}\left[cos\frac{n\pi}{4}+isin\frac{n\pi}{4}\right]$ …… (1)

And

$(1-i)^{2n}=2^{\frac{2n}{2}}\left[cos\frac{2n\pi}{4}-isin\frac{2n\pi}{4}\right]$ …. (2)

Adding (1) + (2)

⇨ $(1+i)^{2n}+(1-i)^{2n}=2^n\left[cos\frac{n\pi}{2}+isin\frac{n\pi}{2}\right]+2^n\left[cos\frac{n\pi}{2}+isin\frac{n\pi}{2}\right]$

⇨ $(1+i)^{2n}+(1-i)^{2n}=2^n\left(cos\frac{n\pi}{2}+isin\frac{n\pi}{2}+cos\frac{n\pi}{2}-isin\frac{n\pi}{2}\right)$

⇨ $(1+i)^{2n}+(1-i)^{2n}=2^n\left(2cos\frac{n\pi}{2}\right)$

⇨$(1+i)^{2n}+(1-i)^{2n}=2^n2^1cos\frac{n\pi}{2}$

$$(1+i)^{2n}+(1-i)^{2n}=2^{n+1}\cos\left(\frac{n\pi}{2}\right)$$

5. If α, β are the roots of the equation $x^2 - 2x + 4 = 0$. Then S.T $\alpha^n + \beta^n = 2^{n+1}\cos\left(\frac{n\pi}{3}\right)$.

Sol: given equation $x^2 - 2x + 4 = 0$.

$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$x = \frac{-(-2) \pm \sqrt{(-2)^2-4(1)(4)}}{2(1)}$

$x = \frac{2 \pm \sqrt{4-16}}{2} = \frac{2 \pm \sqrt{-12}}{2} = \frac{2 \pm \sqrt{i^2 4\times 3}}{2} = \frac{2 \pm 2i\sqrt{3}}{2}$

$x = 1 \pm i\sqrt{3}$

Let $\alpha = 1 + i\sqrt{3}$, $\beta = 1 - i\sqrt{3}$

Now $\alpha^n + \beta^n = (1+i\sqrt{3})^n + (1-i\sqrt{3})^n$

$1 + i\sqrt{3} \equiv (a+ib)$⇨a=1, b=1

r=$\sqrt{a^2+b^2}$=$\sqrt{1^2+\sqrt{3}^2}$=2

$\theta = \tan^{-1}\left(\frac{b}{a}\right)$=$\tan^{-1}\left(\frac{\sqrt{3}}{1}\right)$ ⇨$\theta = \frac{\pi}{3}$

$1 + i\sqrt{3} = r(\cos\theta + i\sin\theta)$

$=\left[2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^n + 2\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)^n\right]$

$= 2^n\left(\cos\frac{n\pi}{3} + i\sin\frac{n\pi}{3}\right) + 2^n\left(\cos\frac{n\pi}{3} - i\sin\frac{n\pi}{3}\right)$

$=2^n\left[\cos\frac{n\pi}{3} + i\sin\frac{n\pi}{3} + \cos\frac{n\pi}{3} - i\sin\frac{n\pi}{3}\right]$

$=2^n\left[\cos\frac{n\pi}{3} + \cos\frac{n\pi}{3}\right]$

$=2^n.2\left[\cos\frac{n\pi}{3}\right]$

$\therefore\ \alpha^n + \beta^n = 2^{n+1}\cos\left(\frac{n\pi}{3}\right)$.

6. Prove that $(1+\cos\theta + i\sin\theta)^n + (1+\cos\theta - i\sin\theta)^n$

$$= 2^{n+1}\cos^n\left(\frac{\theta}{2}\right)\cos\left(\frac{n\theta}{2}\right)$$

A: Given

$(1+\cos\theta + i\sin\theta)^n + (1+\cos\theta - i\sin\theta)^n$

$= \left[2\cos^2\left(\frac{\theta}{2}\right) + i2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)\right]^n + \left(2\cos^2\left(\frac{\theta}{2}\right) - i2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)\right)^n$

By applying De M theorem we get

$=2^n\cos^n\left(\frac{\theta}{2}\right)\left[\cos\left(\frac{n\theta}{2}\right) + i\sin\left(\frac{n\theta}{2}\right)\right]$

$+2^n\cos^n\left(\frac{\theta}{2}\right)\left[\cos\left(\frac{n\theta}{2}\right) - i\sin\left(\frac{n\theta}{2}\right)\right]$

$=2^n\cos^n\left(\frac{\theta}{2}\right)\left[\cos\left(\frac{n\theta}{2}\right) + i\sin\left(\frac{n\theta}{2}\right) + \cos\left(\frac{n\theta}{2}\right) - i\sin\left(\frac{n\theta}{2}\right)\right]$

$=2^n\cos^n\left(\frac{\theta}{2}\right).2\cos\left(\frac{n\theta}{2}\right)$

$= 2^{n+1}\cos^n\left(\frac{\theta}{2}\right)\cos\left(\frac{n\theta}{2}\right)$.

7. **If n is a positive integer, show that** $(p+iq)^{1/n}+(p-iq)^{i/n}=2(p^2+q^2)^{1/2n}\cos\left[\frac{1}{n}\tan^{-1}\left(\frac{q}{p}\right)\right]$

Sol: $(p+iq)^{1/n}+(p-iq)^{i/n}$

$$=\left[\sqrt{p^2+q^2}\left\{\frac{p}{\sqrt{p^2+q^2}}+i\frac{q}{\sqrt{p^2+q^2}}\right\}\right]^{1/n}+\left[\sqrt{p^2+q^2}\left\{\frac{p}{\sqrt{p^2+q^2}}-i\frac{q}{\sqrt{p^2+q^2}}\right\}\right]^{\frac{1}{n}}$$

By applying De M theorem for an rational index

$$=(p^2+q^2)^{1/2n}\left[\{cos\alpha+isin\alpha\}^{1/n}+\{cos\alpha-isin\alpha\}^{1/n}\right]$$

where $cos\alpha=\frac{p}{\sqrt{p^2+q^2}}$,

$sin\alpha=\frac{q}{\sqrt{p^2+q^2}}$

⇨ $\frac{q}{p}=\frac{sin\alpha}{cos\alpha}=tan\alpha$

⇨ $\alpha=\tan^{-1}\left(\frac{q}{p}\right)$

$$=(p^2+q^2)^{1/2n}\left[\cos\left(\frac{\alpha}{n}\right)+isin\left(\frac{\alpha}{n}\right)+\cos\left(\frac{\alpha}{n}\right)-isin\left(\frac{\alpha}{n}\right)\right]$$

$$=(p^2+q^2)^{1/2n}\left[2\cos\left(\frac{\alpha}{n}\right)\right]$$

$$=(p^2+q^2)^{1/2n}\left[2\cos\left\{\frac{1}{n}\tan^{-1}\left(\frac{q}{p}\right)\right\}\right]$$

$$\therefore\ (p+iq)^{\frac{1}{n}}+(p-iq)^{\frac{i}{n}}=2(p^2+q^2)^{1/2n}\cos\left[\frac{1}{n}\tan^{-1}\left(\frac{q}{p}\right)\right]$$

8. **Show that** $\left[\frac{1+\sin\left(\frac{\pi}{8}\right)+i\cos\left(\frac{\pi}{8}\right)}{1+\sin\left(\frac{\pi}{8}\right)-i\cos\left(\frac{\pi}{8}\right)}\right]^{8/3}=-1$

Sol:

consider $\frac{1+\sin\left(\frac{\pi}{8}\right)+i\cos\left(\frac{\pi}{8}\right)}{1+\sin\left(\frac{\pi}{8}\right)-i\cos\left(\frac{\pi}{8}\right)}$

$$=\frac{1+\cos\left(\frac{\pi}{2}-\frac{\pi}{8}\right)+i\sin\left(\frac{\pi}{2}-\frac{\pi}{8}\right)}{1+\cos\left(\frac{\pi}{2}-\frac{\pi}{8}\right)-i\sin\left(\frac{\pi}{2}-\frac{\pi}{8}\right)}$$

$$=\frac{1+\cos(\theta)+i\sin(\theta)}{1+\cos(\theta)-i\sin(\theta)}\qquad\left[\frac{\pi}{2}-\frac{\pi}{8}=\frac{8\pi-\pi}{8}=\frac{3\pi}{8}=\theta\ \ let\right]$$

$$=\frac{2\cos^2\left(\frac{\theta}{2}\right)+i2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}{2\cos^2\left(\frac{\theta}{2}\right)-i2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}$$

$$=\frac{\cos\left(\frac{\theta}{2}\right)\left[\cos\left(\frac{\theta}{2}\right)+i\sin\left(\frac{\theta}{2}\right)\right]}{\cos\left(\frac{\theta}{2}\right)\left[\cos\left(\frac{\theta}{2}\right)-i\sin\left(\frac{\theta}{2}\right)\right]}$$

$$=\frac{\left[\cos\left(\frac{\theta}{2}\right)+i\sin\left(\frac{\theta}{2}\right)\right]}{\left[\cos\left(\frac{\theta}{2}\right)-i\sin\left(\frac{\theta}{2}\right)\right]}$$

$$=\frac{cis\left(\frac{\theta}{2}\right)}{cis\left(-\frac{\theta}{2}\right)}$$

$$=\mathbf{cis}\left(\frac{\theta}{2}+\frac{\theta}{2}\right)$$

$$=\mathbf{cis}(\theta)$$

$$\left[\frac{1+\sin\left(\frac{\pi}{8}\right)+i\cos\left(\frac{\pi}{8}\right)}{1+\sin\left(\frac{\pi}{8}\right)-i\cos\left(\frac{\pi}{8}\right)}\right]^{8/3}=\left[\mathbf{cis}\left(\frac{3\pi}{8}\right)\right]^{8/3}$$

By applying De M theorem for an integral index

$$=cis\left(\frac{3\pi}{8}\times\frac{8}{3}\right)=cis(\pi)=-1$$

9. If n is an integer and Z =cisθ, then show that $\frac{Z^{2n}-1}{Z^{2n}+1} = i\tan n\theta$.

So: $\frac{Z^{2n}-1}{Z^{2n}+1} = \frac{(\cos\theta+i\sin\theta)^{2n}-1}{(\cos\theta+i\sin\theta)^{2n}+1}$

$= \frac{\cos(2n\theta)+i\sin(2n\theta)-1}{\cos(2n\theta)+i\sin(2n\theta)+1}$

By applying De M theorem for an integral index

$= \frac{i\sin(2n\theta)-[1-\cos(2n\theta)]}{i\sin(2n\theta)+[1+\cos(2n\theta)]}$ $\quad [\boldsymbol{i^2 = -1}]$

$= \frac{i\,2\sin n\theta\cos n\theta+i^2[2sin^2 n\theta]}{i2\sin n\theta\cos n\theta+[2cos^2 n\theta]}$

$= \frac{i\,2\sin n\theta[\cos n\theta+i\sin n\theta]}{2\cos n\theta[\cos n\theta+i\sin n\theta]}$

$= \frac{i\sin n\theta}{\cos n\theta}$

$= i\tan n\theta$.

10. Solve the equation $x^9 - x^5 + x^4 - 1 = 0$.

Sol:

Given equation $x^9 - x^5 + x^4 - 1 = 0$

⇨ $x^5(x^4 - 1) + 1(x^4 - 1) = 0$

⇨ $(x^4 - 1)(x^5 + 1) = 0$

Now $(x^5 + 1) = 0$

$x^5 = -1$

$x^5 = \cos\pi + \sin\pi$

$x^5 = cis\pi$

$x^5 = cis(2k\pi + \pi), k \in Z$

$\therefore x = [cis(2k+1)\pi]^{1/5}$

$= cis(2k+1)(\frac{\pi}{5})$ where k=0, 1, 2, 3, 4

x= $cis(\frac{\pi}{5})$, $cis(\frac{3\pi}{5})$, $cis(\frac{5\pi}{5})$, $cis(\frac{7\pi}{5})$, $cis(\frac{9\pi}{5})$

Also $x^4 - 1 = 0$

$(x^2 + 1)(x^2 - 1) = 0$

$x^2 = -1 \;||\; x^2 = 1$

$x^2 = i^2 \;||\; x^2 = 1$

$x = \pm i, \pm 1$