

AIMSTUTORIAL

INTERMEDIATE MATHEMATICS IA STUDY MATERIAL



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(A PLACE TO LEARN)

9000687600

BLUE PRINT OF MATHEMATICS -1A

SL.NO	CHAPTER NAME	VSAQ(2M)	SAQ(4M)	LAQ(7M)	TOTAL
1	FUNCTIONS	2+2	-	7	11
2	MATHEMATICAL INDUCTION			7	7
3	MATRICES	2+2	4	7+7	22
4	ADDITION OF VECTORS	2+2	4		8
5	MULTIPLICATION OF VECTORS	2	4	7	13
6	TRIGONOMETRY RATIO'S UPTO TRANSFORMATION	2+2	4	7	15
7	TRIGONOMTRY EQUATIONS	-	4	-	4
8	INVERSE TRIGONOMETRY		4	-	4
9	HYPERBOLIC TRIGONOMETRY	2	-	-	2
10	PROPERTIES OF TRIANGLES	-	4	7	11
TOTAL MARKS		20/20	20/28	35/49	75/97

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1. If $f : A \rightarrow B$, $g : B \rightarrow C$ are two bijection, then prove that $(gof) : A \rightarrow C$ is also a bijection.

A: Given: $f: A \rightarrow B$, $g: B \rightarrow C$ are bijection.

Part 1:
To prove that $(gof) : A \rightarrow C$ is one-one.

Now $f: A \rightarrow B$, $g: B \rightarrow C$ are one-one functions.

$\Rightarrow (gof) : A \rightarrow C$ is a function.

Let $a_1, a_2 \in A \Rightarrow f(a_1), f(a_2) \in B$

and $(gof)(a_1), (gof)(a_2) \in C$.

Suppose that $(gof)(a_1) = (gof)(a_2)$

$$\Rightarrow g[f(a_1)] = g[f(a_2)]$$

$$\Rightarrow f(a_1) = f(a_2) \quad \because g \text{ is one-one}$$

$$\Rightarrow a_1 = a_2 \quad \because f \text{ is one-one}$$

$\therefore (gof) : A \rightarrow C$ is one-one.....(1)

Part 2:- To prove that $(gof) : A \rightarrow C$ is onto.

Now $f: A \rightarrow B$, $g: B \rightarrow C$ are onto functions.
 $(gof) : A \rightarrow C$ is a function.

Let $c \in C$; Since $g: B \rightarrow C$ is onto, there exists at least one element $b \in B$ such that $g(b) = c$.

Since $f: A \rightarrow B$ is also onto, there exists at least one element $a \in A$ such that $f(a) = b$

Now $(gof)(a) = g[f(a)]$

$$= g(b)$$

$$= c$$

\therefore For $c \in C$, there is an element $a \in A$ such that $(gof)(a) = c$.

so $\therefore (gof) : A \rightarrow C$ is onto.....(2)

Since $(gof) : A \rightarrow C$ is both one-one and onto

From (1) & (2)

$\therefore (gof) : A \rightarrow C$ is a bijection.

2. If $f: A \rightarrow B$, $g: B \rightarrow C$ are bijection, then prove that $(gof)^{-1} = f^{-1}g^{-1}$.

A: Given that

$f: A \rightarrow B$ is a bijection.

$g: B \rightarrow C$ is a bijection.

$f^{-1}: B \rightarrow A$ is a bijection.

$g^{-1}: C \rightarrow B$ is a bijection.

$(gof): A \rightarrow C$ is also a bijection.

$(gof)^{-1}: C \rightarrow A$ is a bijection.

$f^{-1}g^{-1}: C \rightarrow A$ is a bijection.

Thus $(gof)^{-1}$ and $(f^{-1}g^{-1})$ both the functions exist and have same domain C and same co domain A .

Let c be any element in C .

Since $g: B \rightarrow C$ is onto, there exists atleast one element $b \in B$ such that $g(b) = c$

$$\Rightarrow b = g^{-1}(c) \quad \because g \text{ is a bijection}$$

Since $f: A \rightarrow B$ is onto, there exists atleast one element $a \in A$ such that $f(a) = b$.

$$\Rightarrow a = f^{-1}(b) \quad \because f \text{ is a bijection}$$

$$\begin{aligned} \text{Consider } (gof)(a) &= g[f(a)] \\ &= g(b) \end{aligned}$$

$$\therefore (gof)(a) = c.$$

$$\Rightarrow a = (gof)^{-1}(c) \quad \because \text{gof is a bijection}$$

Consider $(f^{-1}g^{-1})(c) = f^{-1}[g^{-1}(c)]$

$$= f^{-1}(b)$$

$$= a$$

$$\therefore (f^{-1}g^{-1})(c) = (gof)^{-1}(c) \quad \forall c \in C.$$

$$\therefore (gof)^{-1} = f^{-1}g^{-1}$$



3. If $f : A \rightarrow B$, I_A and I_B are identity functions on A and B, then prove that $fol_A = I_B \circ f = f$.

A: Given that $f : A \rightarrow B$

$I_A : A \rightarrow A$ is defined by $I_A(a) = a \quad \forall a \in A$.

$I_B : B \rightarrow B$ is defined by $I_B(b) = b \quad \forall b \in B$.

Part 1:- To prove that $fol_A = f$

Now

$I_A : A \rightarrow A$,

$f : A \rightarrow B$

$\Rightarrow fol_A : A \rightarrow B$

Also $f : A \rightarrow B$

Thus fol_A and f both the functions exist and have same domain A and same co domain B.

Let $a \in A$

Since $f : A \rightarrow B$, there exists a unique element $b \in B$ such that $f(a) = b$

$$\text{Consider } (fol_A)(a) = f[I_A(a)]$$

$$= f(a)$$

$$\therefore (fol_A)(a) = f(a) \quad \forall a \in A$$

Hence $fol_A = f$(1)

Part 2:- To show that $I_B \circ f = f$

Now

$f : A \rightarrow B$,

$I_B : B \rightarrow B$

$\Rightarrow I_B \circ f : A \rightarrow B$

Also $f : A \rightarrow B$

Thus $I_B \circ f$ and f both the functions exist and have same domain A and co domain B.

$$\text{Consider } (I_B \circ f)(a) = I_B[f(a)]$$

$$= I_B(b)$$

$$= b$$

$$= f(a)$$

$$\therefore (I_B \circ f)(a) = f(a) \quad \forall a \in A$$

$$\Rightarrow I_B \circ f = f$$
.....(2)

$$\therefore \text{From (1) \& (2)} \quad fol_A = f = I_B \circ f.$$

4. If $f : A \rightarrow B$ is a bijection, then show that $fol_A = I_B$ and $f^{-1} \circ f = I_A$.

A: Given that $f : A \rightarrow B$ is a bijection

$$\Rightarrow f^{-1} : B \rightarrow A.$$

Part 1:- To show that $fol_A = I_B$

Now

$$f^{-1} : B \rightarrow A,$$

$$f : A \rightarrow B$$

$$\Rightarrow f \circ f^{-1} : B \rightarrow B.$$

$$\text{Also } I_B : B \rightarrow B$$

Thus $f \circ f^{-1}$ and I_B have same domain B and same co domain B.

Let a be any element in A.

Since $f : A \rightarrow B$, there is a unique element $b \in B$ such that $f(a) = b$

such that $f(a) = b$

$$\Rightarrow a = f^{-1}(b) \quad \because f \text{ is a bijection}$$

$$\text{Consider } (f \circ f^{-1})(b) = f[f^{-1}(b)]$$

$$= f(a)$$

$$= b$$

$$= I_B(b)$$

$$\therefore I_B : B \rightarrow B \Rightarrow I_B(b) = b$$

$$\therefore (f \circ f^{-1})(b) = I_B(b) \quad b \in B$$

$$\text{Thus } f \circ f^{-1} = I_B$$

Part 2:- To prove that $f^{-1} \circ f = I_A$

Now

$$f : A \rightarrow B,$$

$$f^{-1} : B \rightarrow A$$

$$\Rightarrow f^{-1} \circ f : A \rightarrow A$$

$$\text{Also } I_A : A \rightarrow A$$

Thus $f^{-1} \circ f$ and I_A have the same domain A and the same co domain A.

$$\text{Now } (f^{-1} \circ f)(a) = f^{-1}[f(a)]$$

$$= f^{-1}(b)$$

$$= a$$

$$= I_A(a)$$

$$\therefore I_A : A \rightarrow A \Rightarrow I_A(a) = a$$

$$(f^{-1} \circ f)(a) = I_A(a) \quad \forall a \in A$$

$$\therefore f^{-1} \circ f = I_A$$

$$\therefore f \circ f^{-1} = I_B \text{ and } f^{-1} \circ f = I_A.$$

5. If $f : A \rightarrow B$, $g : B \rightarrow A$ are two functions such that $gof = I_A$ and $fog = I_B$ then prove that $g = f^{-1}$.

A: Given that $f: A \rightarrow B$, $g: B \rightarrow A$ are two functions
Such that $gof = I_A$ and $fog = I_B$.

Part 1: - To prove that f is one-one.

Let $a_1, a_2 \in A \Rightarrow f(a_1), f(a_2) \in B$

Consider $f(a_1) = f(a_2)$

$$\begin{aligned} &\Rightarrow g[f(a_1)] = g[f(a_2)] \\ &\Rightarrow (gof)(a_1) = (gof)(a_2) \\ &\Rightarrow I_A(a_1) = I_A(a_2) \quad \because \text{gof} = I_A \\ &\Rightarrow a_1 = a_2 \end{aligned}$$

Thus $f: A \rightarrow B$ is one-one.

Part 2:-To prove that f is onto.

Let $b \in B$

$\therefore g: B \rightarrow A$, there exists a unique element $a \in A$ such that $g(b) = a$.

$$\begin{aligned} \text{Now } f(a) &= f[g(b)] \\ &= (fog)(b) \\ &= I_B(b) \quad \because \text{fog} = I_B \\ &= b \end{aligned}$$

So $f: A \rightarrow B$ is onto.

Since f is one-one and onto, so f is a bijection.

$$\Rightarrow f^{-1}: B \rightarrow A$$

Also $g: B \rightarrow A$

Thus both the functions f^{-1} and g have the same domain B and same co domain A .

Part 3:- To show that $g = f^{-1}$

From previous part, $f(a) = b$

$$\Rightarrow a = f^{-1}(b)$$

Also $g(b) = a$

$$\therefore g(b) = f^{-1}(b) \quad \forall b \in B.$$

$$\therefore g = f^{-1}.$$

6. Let $A = \{1, 2, 3\}$, $B = \{a, b, c\}$, $C = \{p, q, r\}$. If $f : A \rightarrow B$, $g : B \rightarrow C$ are defined by $f = \{(1, a), (2, c), (3, b)\}$, $g = \{(a, q), (b, r), (c, p)\}$, then show that $f^{-1}g^{-1} = (gof)^{-1}$.

A: Given that

$$A = \{1, 2, 3\}, B = \{a, b, c\}, C = \{p, q, r\}$$

$f: A \rightarrow B$, $g: B \rightarrow C$ are given by

$$\begin{aligned} f &= \{(1, a), (2, c), (3, b)\} \\ \Rightarrow f^{-1} &= \{(a, 1), (b, 3), (c, 2)\} \text{ and} \\ g &= \{(a, q), (b, r), (c, p)\} \\ \Rightarrow g^{-1} &= \{(q, a), (r, b), (p, c)\} \end{aligned}$$

f	g	gof	$(gof)^{-1}$
$(1, a)$	$\rightarrow (a, q)$	$(1, q)$	$(q, 1)$
$(2, c)$	$\rightarrow (c, p)$	$(2, p)$	$(p, 2)$
$(3, b)$	$\rightarrow (b, r)$	$(3, r)$	$(r, 3) \dots (1)$

g^{-1}	f^{-1}	$g^{-1} \circ f^{-1}$
(q, a)	$\rightarrow (a, 1)$	$(q, 1)$
(r, b)	$\rightarrow (b, 3)$	$(r, 3)$
(p, c)	$\rightarrow (c, 2)$	$(p, 2) \dots (2)$

\therefore From (1) and (2) $(f^{-1}g^{-1}) = (gof)^{-1}$.

7. If $f : Q \rightarrow Q$ defined by $f(x) = 5x + 4$ for all $x \in Q$, show that f is a bijection and find f^{-1} .

A: Given that

$f : Q \rightarrow Q$ is defined by $f(x) = 5x + 4$

Part 1:- To prove that f is one-one

Let $x_1, x_2 \in Q$ (domain) and

$$f(x_1) = f(x_2)$$

$$\Rightarrow 5x_1 + 4 = 5x_2 + 4$$

$$\Rightarrow 5x_1 = 5x_2$$

$$\Rightarrow x_1 = x_2$$

$\therefore f : Q \rightarrow Q$ is one-one.

Part 2:- To prove that f is onto

Let $y \in$ the codomain Q and $x \in$ domain Q such that

$$f(x) = y$$

$$\Rightarrow 5x + 4 = y$$

$$\Rightarrow x = \frac{y - 4}{5}$$

So for every $y \in$ codomain Q , there is a preimage \in domain Q such that

$$f\left(\frac{y-4}{5}\right) = y$$

Thus $f : Q \rightarrow Q$ is onto.

Part 3:-To find $f^{-1}(x)$

Since f is both one-one, onto, so it is a bijection.

$$f(x) = y \Rightarrow x = f^{-1}(y)$$

$$5x + 4 = y \Rightarrow x = \frac{y-4}{5}$$

$$\therefore f^{-1}(x) = \frac{x-4}{5}.$$

8. If $f = \{(4, 5), (5, 6), (6, -4)\}$ and $g = \{(4, -4), (6, 5), (8, 5)\}$ then find

$$(i) f + g \quad (ii) f - g \quad (iii) 2f + 4g$$

$$(iv) f + 4 \quad (v) fg \quad (vi) \frac{f}{g}$$

$$(vii) |f| \quad (viii) \sqrt{f} \quad (ix) f^2 \quad (X) f^3.$$

Sol:

Given that $f =$

$\{(4, 5), (5, 6), (6, -4)\}$ and

$g = \{(4, -4), (6, 5), (8, 5)\}$

$$(i) f + g = \{(4, 5 + (-4)), (6, +(-4) + 5)\} \\ = \{(4, 1), (6, 1)\}$$

$$(ii) f - g = \{(4, 5 - (-4)), (6, -(-4) + 5)\} \\ = \{(4, 9), (6, -9)\}$$

$$(iii) 2f + 4g$$

$$\{(4, 10), (5, 12), (6, -8)\} + \\ \{(4, -16), (6, 20), (8, 20)\} = \{(4, -6), (6, 12)\}$$

$$(iv) f + 4 = \{(4, 5+4), (5, 6+4), (6, -4+4)\} = \{(4, 9), (5, 10), (6, 0)\}$$

$$(v) fg = \{(4, -20), (6, -20)\}$$

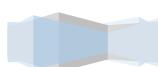
$$(vi) \frac{f}{g} = \left\{ \left(4, -\frac{5}{4}\right), \left(6, -\frac{4}{5}\right) \right\}$$

$$(vii) |f| = \{(4, 5), (5, 6), (6, 4)\}$$

$$(viii) \sqrt{f} = \{(4, \sqrt{5}), (5, \sqrt{6})\}$$

$$(ix) f^2 = \{(4, 25), (5, 36), (6, 16)\}$$

$$(X) f^3 = \{(4, 125), (5, 216), (6, -64)\}.$$



1. Using mathematical induction, Show that $1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$.

Sol: let $p(n) = 1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$.

Step-1:T0 P.T P(1) is true (put n=1)

<u>L.H.S</u>	<u>R.H.S</u>
n^2	$\frac{n(n+1)(2n+1)}{6}$
$= (1)^2$	$= \frac{1 \cdot 2 \cdot 3}{6}$
$= 1$	$= 1$

L.H.S=R.H.S \therefore p (1) is true for n=1.

$t_n = n^2$
 $t_k = k^2$
 $t_{k+1} = (k+1)^2$

Step-2: let us assume that $p(k)$ is true for $n = k$.

AIMS

$$\Rightarrow p(k) = 1^2 + 2^2 + 3^2 + \dots + k^2 = \frac{k(k+1)(2k+1)}{6}$$

Step-3: To P.T $p(k+1)$ is true {adding both sides $(k+1)$ term}

$$\begin{aligned}
 p(k+1) &= 1^2 + 2^2 + 3^2 + \dots + k^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + \frac{(k+1)^2}{1} \\
 &= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} \\
 &= \frac{(k+1)[k(2k+1) + 6(k+1)]}{6} \\
 &= \frac{(k+1)[2k^2 + k + 6k + 6]}{6} \\
 &= \frac{(k+1)[2k^2 + 7k + 6]}{6} \\
 &= \frac{(k+1)[2k^2 + 4k + 3k + 6]}{6} \\
 &= \frac{(k+1)[2k(k+2) + 3(k+2)]}{6} \\
 &= \frac{(k+1)(k+2)(2k+3)}{6} \\
 &= \frac{(k+1)(k+2)[2(k+1)+1]}{6}
 \end{aligned}$$

\therefore Thus $p(k+1)$ is true for $n = k+1$.

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by the principle of mathematical induction, $p(n)$ is true for all $n \in \mathbb{N}$.

2. Using mathematical induction, Show that $1^3 + 2^3 + 3^3 + \dots + n^3 = \frac{n^2(n+1)^2}{4}$.

Sol: let

$$p(n) = 1^3 + 2^3 + 3^3 + \dots + n^3 = \frac{n^2(n+1)^2}{4}.$$

Step-1: put $n=1$

L.H.S	R.H.S
$= n^3$	$= \frac{n^2(n+1)^2}{4}$
$= (1)^3$	$= \frac{1 \cdot 2^2}{4}$
$= 1$	$= 1$

L.H.S=R.H.S $\therefore p(1)$ is true for $n=1$.

$$t_n = n^3$$

$$t_k = k^3$$

$$t_{k+1} = (k+1)^3$$

Step-2: let us assume that $p(k)$ is true for $n = k$.

$$\Rightarrow p(k) = 1^3 + 2^3 + 3^3 + \dots + k^3 = \frac{k^2(k+1)^2}{4}$$

Step-3: adding both sides $(k+1)$ term

$$\begin{aligned}
 p(k+1) &= 1^3 + 2^3 + 3^3 + \dots + k^3 + (k+1)^3 = \frac{k^2(k+1)^2}{4} + \frac{(k+1)^3}{1} \\
 &= \frac{k^2(k+1)^2 + 4(k+1)^3}{4} \\
 &= \frac{(k+1)^2\{k^2 + 4(k+1)\}}{4} \\
 &= \frac{(k+1)^2\{k^2 + 4k + 4\}}{4} \\
 &= \frac{(k+1)^2(k+2)^2}{4}
 \end{aligned}$$

\therefore Thus $p(k+1)$ is true for $n = k+1$.

by the principle of mathematical induction, $p(n)$ is true for all $n \in \mathbb{N}$.



3. S.T $1.2.3 + 2.3.4 + 3.4.5 + \dots \dots \text{ up to } n \text{ terms} = \frac{n(n+1)(n+2)(n+3)}{4}$.

Sol: let

$$p(n) = 1.2.3 + 2.3.4 + \dots + n(n+1)(n+2) = \frac{n(n+1)(n+2)(n+3)}{4}.$$

Step-1: put $n=1$

L.H.S	R.H.S
$n(n+1)(n+2)$	$\frac{n(n+1)(n+2)(n+3)}{4}$
$= 1.2.3$	$= \frac{1.2.3.4}{4}$
$= 6$	$= 6$

AIMS

L.H.S=R.H.S $\therefore p(1)$ is true for $n=1$.

Step-2: let us assume that $p(k)$ is true for $n = k$.

$$\Rightarrow p(k) = 1.2.3 + 2.3.4 + \dots + k(k+1)(k+2) = \frac{k(k+1)(k+2)(k+3)}{4}$$

Step-3: adding both sides $(k+1)$ term

$$\Rightarrow P(k+1) = 1.2.3 + 2.3.4 + \dots + k(k+1)(k+2) + (k+1)(k+2)(k+3)$$

$$= \frac{k(k+1)(k+2)(k+3)}{4} + \frac{(k+1)(k+2)(k+3)}{1}$$

$$= \frac{k(k+1)(k+2)(k+3) + 4(k+1)(k+2)(k+3)}{4}$$

$$= \frac{(k+1)(k+2)(k+3)(k+4)}{4}$$

\therefore Thus $p(k+1)$ is true for $n = k + 1$.

by the principle of mathematical induction, $p(n)$ is true for all $n \in \mathbb{N}$.



4. S.T $2.3+3.4+4.5+\dots \text{ up to } n \text{ terms} = \frac{n(n^2+6n+11)}{3}$.

Sol: let $p(n) = 2.3 + 3.4 + \dots + (n+1)(n+2) = \frac{n(n^2+6n+11)}{3}$.

Step-1: put $n=1$

L.H.S	R.H.S
$(n+1)(n+2)$	$\frac{n(n^2+6n+11)}{3}$
$= 2.3$	$= \frac{1.(1+6+11)}{3}$
$= 6$	$= \frac{18}{3} = 6$

L.H.S=R.H.S $\therefore p(1)$ is true for $n=1$.

AIMS

Step-2: let us assume that $p(k)$ is true for $n = k$.

$$\Rightarrow p(k) = 2.3 + 3.4 + \dots + (k+1)(k+2) = \frac{k(k^2+6k+11)}{3}$$

Step-3: adding both sides $(k+1)$ term

$$\begin{aligned}
 P(k+1) &= 2.3 + 3.4 + \dots + (k+1)(k+2) + (k+2)(k+3) \\
 &= \frac{k(k^2+6k+11)}{3} + (k+2)(k+3) \\
 &= \frac{k(k^2+6k+11)}{3} + \frac{(k+2)(k+3)}{1} \\
 &= \frac{k(k^2+6k+11)+3(k+2)(k+3)}{3} \\
 &= \frac{k^3+6k^2+11k+3(k^2+5k+6)}{3} \\
 &= \frac{k^3+6k^2+11k+3k^2+15k+18}{3} \\
 &= \frac{k^3+9k^2+26k+18}{3} \\
 &= \frac{(k+1)(k^2+8k+18)}{3} \\
 &= \frac{(k+1)(k^2+2k+1+6k+6+11)}{3} \\
 &= \frac{(k+1)[(k+1)^2+6(k+1)+11]}{3}
 \end{aligned}$$

\therefore Thus $p(k+1)$ is true for $n = k+1$.

by the principle of mathematical induction, $p(n)$ is true for all $n \in \mathbb{N}$.

5. S.T $a + (a + d) + \dots + up\ to\ n\ terms = \frac{n[2a + (n-1)d]}{2}$.

sol: let $p(n) = a + (a + d) + \dots + a + (n-1)d = \frac{n[2a + (n-1)d]}{2}$.

Step-1: put $n=1$

L.H.S	R.H.S
$a + (n-1)d$	$\frac{n[2a + (n-1)d]}{2}$
$= a+0$	$= \frac{1.[2a+0]}{2} = \frac{2a}{2}$
$= a$	$= a$

L.H.S=R.H.S $\therefore p(1)$ is true for $n=1$.

AIMS

Step-2: let us assume that $p(k)$ is true for $n = k$.

$$\Rightarrow p(k) = a + (a + d) + \dots + a + (k-1)d = \frac{k}{2}\{2a + (k-1)d\}$$

Step-3: adding both sides $(k+1)$ term

$$\begin{aligned} \Rightarrow p(k+1) &= a + (a + d) + \dots + \{a + (k-1)d\} + (a + kd) \\ &= \frac{k}{2}\{2a + (k-1)d\} + (a + kd) \\ &= \frac{k\{2a + (k-1)d\}}{2} + \frac{\{a + kd\}}{1} \\ &= \frac{2ak + k^2d - kd + 2a + 2kd}{2} \\ &= \frac{2a(k+1) + k^2d + kd}{2} \\ &= \frac{2a(k+1) + kd(k+1)}{2} \\ &= \frac{(k+1)\{2a + kd\}}{2} \\ &= \frac{(k+1)\{2a + (k+1-1)d\}}{2} \end{aligned}$$

\therefore Thus $p(k+1)$ is true for $n = k+1$.

by the principle of mathematical induction, $p(n)$ is true for all $n \in N$.



6. S.T $a + ar + ar^2 + \dots \text{ up to } n \text{ terms} = a \frac{(r^n - 1)}{(r - 1)}$

sol: let $p(n) = a + ar + ar^2 + \dots a.r^{n-1} = a \frac{(r^n - 1)}{(r - 1)}$

Step-1: put $n=1$

<u>L.H.S</u>	<u>R.H.S</u>
$a.r^{n-1}$	$a \frac{(r^n - 1)}{(r - 1)}$
$= a.r^0$	$= a \frac{r-1}{r-1}$
$= a$	$= a$

L.H.S=R.H.S $\therefore p(1)$ is true for $n=1$.



AIMS

Step-2: let us assume that $p(k)$ is true for $n = k$.

$$\Rightarrow p(k) = a + ar + ar^2 + \dots a.r^{k-1} = a \frac{(r^k - 1)}{(r - 1)}.$$

Step-3: adding both sides $(k + 1)$ term

$$p(k + 1) = a + ar + ar^2 + \dots a.r^{k-1} + a.r^k$$

$$= a \frac{(r^k - 1)}{(r - 1)} + a.r^k$$

$$= a \left\{ \frac{(r^k - 1)}{(r - 1)} + \frac{r^k}{1} \right\}$$

$$= a \left(\frac{r^k - 1 + r^k \cdot r - r^k}{r - 1} \right)$$

$$= a \left(\frac{r^{k+1} - 1}{r - 1} \right).$$

\therefore Thus $p(k + 1)$ is true for $n = k + 1$.

by the principle of mathematical induction, $p(n)$ is true for all $n \in \mathbb{N}$.



7. S.T $1^2 + (1^2 + 2^2) + (1^2 + 2^2 + 3^2) + \dots \text{up to } n \text{ terms} = \frac{n(n+1)^2(n+2)}{12}$.

Sol: let $p(n) = 1^2 + (1^2 + 2^2) + (1^2 + 2^2 + 3^2) + \dots + \frac{n(n+1)(2n+1)}{6}$
 $= \frac{n(n+1)^2(n+2)}{12}$.

Step-1: put $n=1$

L.H.S	R.H.S
$\frac{n(n+1)(2n+1)}{6}$	$\frac{n(n+1)^2(n+2)}{12}$
$= \frac{1 \cdot 2 \cdot 3}{6}$	$= \frac{1(1+1)^2(1+2)}{12} = \frac{12}{12}$
$= 1$	$= 1$

L.H.S=R.H.S $\therefore p(1)$ is true for $n=1$.

AIMS

Step-2: let us assume that $p(k)$ is true for $n = k$.

$$\Rightarrow p(k) = 1^2 + (1^2 + 2^2) + (1^2 + 2^2 + 3^2) + \dots + \frac{k(k+1)(2k+1)}{6} = \frac{k(k+1)^2(k+2)}{12}.$$

Step-3: adding both sides $(k + 1)$ term

$$\begin{aligned}
 p(k+1) &= 1^2 + (1^2 + 2^2) + (1^2 + 2^2 + 3^2) + \dots + \frac{k(k+1)(2k+1)}{6} \\
 &+ \frac{(k+1)(k+2)(2k+3)}{6} = \frac{k(k+1)^2(k+2)}{12} + \frac{(k+1)(k+2)(2k+3)}{6} \\
 &= \frac{k(k+1)^2(k+2) + 2(k+1)(k+2)(2k+3)}{12} \\
 &= \frac{(k+1)(k+2)\{k(k+1)+2(2k+3)\}}{12} \\
 &= \frac{(k+1)(k+2)\{k^2+k+4k+6\}}{12} \\
 &= \frac{(k+1)(k+2)\{k^2+5k+6\}}{12} \\
 &= \frac{(k+1)(k+2)\{(k+2)(k+3)\}}{12} \\
 &= \frac{(k+1)(k+2)^2(k+3)}{12}.
 \end{aligned}$$

\therefore Thus $p(k + 1)$ is true for $n = k + 1$.

by the principle of mathematical induction, $p(n)$ is true for all $n \in \mathbb{N}$.



8. S.T $\frac{1^3}{1} + \frac{1^3+2^3}{1+3} + \frac{1^3+2^3+3^3}{1+3+5} + \dots \text{ up to } n \text{ terms} = \frac{n}{24}(2n^2 + 9n + 13).$

sol: let $p(n) = \frac{1^3}{1} + \frac{1^3+2^3}{1+3} + \frac{1^3+2^3+3^3}{1+3+5} + \dots + \frac{(n+1)^2}{4} = \frac{n}{24}(2n^2 + 9n + 13).$

Step-1: put $n=1$

L.H.S	R.H.S
$\frac{(n+1)^2}{4}$	$\frac{n}{24}(2n^2 + 9n + 13).$
$= \frac{4}{4} = 1$	$= \frac{(2+9+13)}{24} = \frac{24}{24} = 1$
$= 1$	$= 1$

L.H.S=R.H.S $\therefore p(1)$ is true for $n=1.$

AIMS

Step-2: let us assume that $p(k)$ is true for $n = k.$

$$\Rightarrow p(n) = \frac{1^3}{1} + \frac{1^3+2^3}{1+3} + \frac{1^3+2^3+3^3}{1+3+5} + \dots + \frac{(k+1)^2}{4} = \frac{k}{24}(2k^2 + 9k + 13).$$

Step-3: adding both sides $(k + 1)$ term

$$\begin{aligned} p(n) &= \frac{1^3}{1} + \frac{1^3+2^3}{1+3} + \frac{1^3+2^3+3^3}{1+3+5} + \dots + \frac{(k+1)^2}{4} + \frac{(k+2)^2}{4} \\ &= \frac{k}{24}(2k^2 + 9k + 13) + \frac{(k+2)^2}{4} \\ &= \frac{2k^3 + 9k^2 + 13k + 6(k^2 + 4k + 4)}{24} \\ &= \frac{(2k^3 + 9k^2 + 13k + 6k^2 + 24k + 24)}{24} \\ &= \left(\frac{2k^3 + 15k^2 + 37k + 24}{24} \right) \\ &= \frac{(k+1)(2k^2 + 13k + 24)}{24} \\ &= \frac{(k+1)(2k^2 + 4k + 2 + 9k + 9 + 13)}{24} \\ &= \frac{(k+1)\{2(k+1)^2 + 9(k+1) + 13\}}{24} \end{aligned}$$

$\therefore P(k+1)$ is true

Hence, by the principle of mathematical induction, the given statement is true for all $n \in N.$



9. Show that $\frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + \dots \text{ upto } n \text{ terms} = \frac{n}{3n+1}$.

Sol: let

$$p(n) = \frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + \dots + \frac{1}{(3n-2)(3n+1)} = \frac{n}{3n+1}.$$

Step-1: put n=1

L.H.S	R.H.S
$\frac{1}{(3n-2)(3n+1)}$ $= \frac{1}{(3-2)(3+1)}$ $= \frac{1}{1.4}$	$\frac{n}{3n+1}$ $\frac{1}{3+1}$ $= \frac{1}{1.4}$

L.H.S=R.H.S $\therefore p(1)$ is true for n=1.



AIMS

Step-2: let us assume that $p(k)$ is true for $n = k$.

$$\Rightarrow p(k) = \frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + \dots + \frac{1}{(3k-2)(3k+1)} = \frac{k}{3k+1}.$$

Step-3: adding both sides $(k+1)$ term

$$\begin{aligned} p(k+1) &= \frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + \dots + \frac{1}{(3k-2)(3k+1)} + \frac{1}{(3k+1)(3k+4)} \\ &= \frac{k}{3k+1} + \frac{1}{(3k+1)(3k+4)} \\ &= \frac{1}{(3k+1)} \left\{ \frac{k}{1} + \frac{1}{3k+4} \right\} \\ &= \frac{1}{(3k+1)} \left\{ \frac{3k^2+4k+1}{3k+4} \right\} \\ &= \frac{1}{(3k+1)} \left\{ \frac{3k^2+3k+1+k+1}{3k+4} \right\} \\ &= \frac{1}{(3k+1)} \left\{ \frac{3k(k+1)+1(k+1)}{3k+4} \right\} \\ &= \frac{1}{(3k+1)} \left\{ \frac{(3k+1)(k+1)}{3k+4} \right\} \\ &= \frac{(k+1)}{(3k+4)} \\ &= \frac{(k+1)}{\{3(k+1)+1\}} \end{aligned}$$

$\therefore P(k+1)$ is true

Hence, by the principle of mathematical induction, the given statement is true for all $n \in N$.



10. Show that $49^n + 16n - 1$ is divisible by 64 for all + ve integers n.

Sol: let $p(n)$ = be the statement

Step-1: put $n=1$

$$\begin{aligned} P(1) &= 49^1 + 16(1) - 1 = 49 + 16 - 1 \\ &= 64 \\ &= 64(1) \end{aligned}$$

Thus $p(1)$ is divisible by 64.

Step-2: let us assume that $p(k)$ is true for $n = k$.

$$\Rightarrow p(k) = 49^k + 16k - 1 = 64m, \text{ for some } m \in N$$

AIMS

$$\Rightarrow 49^k = (64m - 16k + 1) \dots (1)$$

Step-3: if $n=k+1$

$$\Rightarrow p(k+1) = 49^{k+1} + 16(k+1) - 1$$

$$= 49^k \cdot 49 + 16k + 16 - 1$$

$$= 49\{64m - 16k + 1\} + 16k + 15$$

$$= 49 \cdot 64k - 16k \cdot 49 + 49 + 16k + 15$$

$$= 49 \cdot 64k - 16k \cdot 48 + 64$$

$$= 49 \cdot 64k - 16k \cdot 12 \times 4 + 64$$

$$= 49 \cdot 64k - 12k \cdot 64 \times 4 + 64$$

$$= 64\{49m - 12k + 1\}$$

∴ thus the statement is divisible by 64

Hence, by the principle of mathematical induction,
 $49^n + 16n - 1$ is divisible by 64 for all $n \in N$.



11. Show that $3 \cdot 5^{2n+1} + 2^{3n+1}$ is divisible by 17 for all + ve integers n.

Sol: let $p(n)$ = be the statement

Step-1: put n=1

$$\begin{aligned} P(1) &= 3 \cdot 5^{2n+1} + 2^{3n+1} = 3 \cdot 5^3 + 2^4 \\ &= 3(125) + 16 \\ &= 375 + 16 \\ &= 391 = 17 \times 23 \end{aligned}$$

Thus $p(1)$ is divisible by 17.

Step-2: let us assume that $p(k)$ is true for $n = k$.

$\Rightarrow p(k) = 3 \cdot 5^{2n+1} + 2^{3n+1} = 17m$, for some $m \in N$

$$\Rightarrow 3 \cdot 5^{2n+1} = (17m - 2^{3n+1}) \dots\dots\dots (1)$$

Step-3: if $n=k+1$

$$\Rightarrow p(k+1) = 3 \cdot 5^{2n+1} + 2^{3n+1}$$

$$= 3 \cdot 5^{2k+1} \cdot 5^2 + 2^{3k+1} \cdot 2^3$$

$$= 25(17m - 2^{3n+1}) + 2^{3k+1} \cdot 8$$

$$= 25 \cdot 17m - 25 \cdot 2^{3n+1} + 8 \cdot 2^{3n+1}$$

$$= 25 \cdot 17m - 17 \cdot 2^{3n+1}$$

$$= 17(25m - 2^{3n+1})$$

\therefore thus the statement is divisible by 17

Hence, by the principle of mathematical induction,
 $3 \cdot 5^{2n+1} + 2^{3n+1}$ is divisible by 17 for all $n \in N$.



12. S.T $2 \cdot 4^{2n+1} + 3^{3n+1}$ is divisible by 11, for all $n \in N$.

Sol: Sol: let $p(n) =$ be the statement

Step-1: put $n=1$

$$\begin{aligned} P(1) &= 2 \cdot 4^{2n+1} + 3^{3n+1} = 2 \cdot 4^3 + 3^4 \\ &= 2(64) + 81 \\ &= 128 + 81 \\ &= 201 = 11 \times 19 \end{aligned}$$

Thus $p(1)$ is divisible by 11.

Step-2: let us assume that $p(k)$ is true for $n = k$.

$$\Rightarrow p(k) = 2 \cdot 4^{2k+1} + 3^{3k+1} = 11m, \text{ for some } m \in N$$

$$\Rightarrow 2 \cdot 4^{2k+1} = (11m - 2^{3k+1}) \dots\dots (1)$$

Step-3: if $n=k+1$

$$\Rightarrow p(k+1) = 2 \cdot 4^{2(k+1)+1} + 3^{3(k+1)+1}$$

$$= 2 \cdot 4^{2k+1} \cdot 4^2 + 3^{3k+1} \cdot 3^3$$

$$= 16(11m - 2^{3k+1}) + 3^{3k+1} \cdot 27$$

$$= 16 \cdot 11m - 16 \cdot 3^{3n+1} + 27 \cdot 3^{3n+1}$$

$$= 16 \cdot 11m + 11 \cdot 3^{3n+1}$$

$$= 11(26m - 3^{3n+1})$$

∴ thus the statement is divisible by 11

Hence, by the principle of mathematical induction,
 $2 \cdot 4^{2n+1} + 3^{3n+1}$ is divisible by 11 for all $n \in N$.



13. $2+3.2+3.2^2+\dots \text{ upto } n \text{ terms} = n.2^n$

Sol: let $p(n) = 2+3.2+3.2^2+\dots+(n+1)2^{n-1} = n.2^n$.

Step-1: put $n=1$

L.H.S	R.H.S
2	$n.2^n$
$=2$	$=1.2$
$=2$	$=2$

L.H.S=R.H.S $\therefore p(1)$ is true for $n=1$.

AIMS

Step-2: let us assume that $p(k)$ is true for $n=k$.

$$\Rightarrow p(k) = 2+3.2+3.2^2+\dots+(k+1)2^{k-1} = k.2^k.$$

Step-3: adding both sides $(k+1)$ term

$$\begin{aligned}
 \Rightarrow p(k+1) &= 2+3.2+3.2^2+\dots+(k+1)2^{k-1}+(k+2)2^k \\
 &= k.2^k + (k+2)2^k \\
 &= 2^k(k+k+2) \\
 &= 2^k(2k+2) \\
 &= 2^k \cdot 2(k+1) \\
 &= 2^{k+1} \cdot (k+1)
 \end{aligned}$$

\therefore Thus $p(k+1)$ is true for $n=k+1$.
by the principle of mathematical induction, $p(n)$ is true for all $n \in \mathbb{N}$.



$$1. \text{ S.T} \begin{vmatrix} \mathbf{b} + \mathbf{c} & \mathbf{c} + \mathbf{a} & \mathbf{a} + \mathbf{b} \\ \mathbf{c} + \mathbf{a} & \mathbf{a} + \mathbf{b} & \mathbf{b} + \mathbf{c} \\ \mathbf{a} + \mathbf{b} & \mathbf{b} + \mathbf{c} & \mathbf{c} + \mathbf{a} \end{vmatrix} = 2 \begin{vmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \\ \mathbf{b} & \mathbf{c} & \mathbf{a} \\ \mathbf{c} & \mathbf{a} & \mathbf{b} \end{vmatrix}$$

Sol: L.H.S $\begin{vmatrix} b + c & c + a & a + b \\ c + a & a + b & b + c \\ a + b & b + c & c + a \end{vmatrix}$

$$R_1 \rightarrow R_1 + R_2 + R_3$$

$$= \begin{vmatrix} 2(a+b+c) & 2(a+b+c) & 2(a+b+c) \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}$$

$$= 2 \begin{vmatrix} (a+b+c) & (a+b+c) & (a+b+c) \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}$$

$$R_2 \rightarrow R_2 - R_1 : R_3 \rightarrow R_3 - R_1$$

$$= 2 \begin{vmatrix} (a+b+c) & (a+b+c) & (a+b+c) \\ -b & -c & -a \\ -c & -a & -b \end{vmatrix}$$

$$R_1 \rightarrow R_1 + R_2 + R_3$$

$$= 2 \begin{vmatrix} a & b & c \\ -b & -c & -a \\ -c & -a & -b \end{vmatrix}$$

$$= 2(-)(-) \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

$$= 2 \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \text{ R.H.S}$$

$$\begin{aligned}
 2. \quad & \left| \begin{array}{ccc} a & b & c \\ b & c & a \\ c & a & b \end{array} \right|^2 = \left| \begin{array}{ccc} 2bc - a^2 & c^2 & b^2 \\ c^2 & 2ac - b^2 & a^2 \\ b^2 & a^2 & 2ab - c^2 \end{array} \right| \\
 & = (a^2 + b^2 + c^2 - 3abc)^2 \\
 L.H.S \quad & \left| \begin{array}{ccc} a & b & c \\ b & c & a \\ c & a & b \end{array} \right|^2 = \left| \begin{array}{ccc} a & b & c \\ b & c & a \\ c & a & b \end{array} \right| \left| \begin{array}{ccc} a & b & c \\ b & c & a \\ c & a & b \end{array} \right| (R_2 \leftrightarrow R_3) \\
 & = \left| \begin{array}{ccc} a & b & c \\ b & c & a \\ c & a & b \end{array} \right| (-) \left| \begin{array}{ccc} a & b & c \\ c & a & b \\ b & c & a \end{array} \right| \\
 & = \left| \begin{array}{ccc} a & b & c \\ b & c & a \\ c & a & b \end{array} \right| \left| \begin{array}{ccc} -a & -b & -c \\ c & a & b \\ b & c & a \end{array} \right| \\
 & = \left| \begin{array}{ccc} -a^2 + bc + bc & -ab + ab + c^2 & -ac + b^2 + ac \\ -ab + c^2 + ab & -b^2 + ac + ac & -bc + bc + a^2 \\ -ac + ac + b^2 & -bc + a^2 + bc & -c^2 + ab + ab \end{array} \right| \\
 & = \left| \begin{array}{ccc} 2bc - a^2 & c^2 & b^2 \\ c^2 & 2ac - b^2 & a^2 \\ b^2 & a^2 & 2ab - c^2 \end{array} \right| \dots (1) \\
 \text{now} \quad & \left| \begin{array}{ccc} a & b & c \\ b & c & a \\ c & a & b \end{array} \right|^2 \\
 & = [a(bc - a^2) - b(b^2 - ac) + c(ab - c^2)]^2 \\
 & = (abc - a^3 - b^3 + abc + abc - c^3)^2 \\
 & = (3abc - a^3 - b^3 - c^3)^2 \\
 & = (a^3 + b^3 + c^3 - 3abc)^2 \dots \dots (2)
 \end{aligned}$$

from (1)&(2)

$$L.H.S = R.H.S$$

3. If $\begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix} = 0$ and $\begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} \neq 0$,
show that $abc = -1$.

$$Sol: Given \begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} + \begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} + abc \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = 0 \quad \{C_1 \leftrightarrow C_2\}$$

$$\Rightarrow \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} - abc \begin{vmatrix} a & 1 & a^2 \\ b & 1 & b^2 \\ c & 1 & c^2 \end{vmatrix} = 0 \quad \{C_2 \leftrightarrow C_3\}$$

$$\Rightarrow \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} + abc \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} (1 + abc) = 0 \quad \left[\because \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} \neq 0 \right]$$

$$\therefore (1 + abc) = 0$$

$$\Rightarrow abc = -1.$$

$$4. S.T \begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix} \\ = 2(a+b+c)^3$$

$$Sol: L.H.S = \begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix}$$

$$C_1 \rightarrow C_1 + C_2 + C_3$$

$$= \begin{vmatrix} 2(a+b+c) & a & b \\ 2(a+b+c) & b+c+2a & b \\ 2(a+b+c) & a & c+a+2b \end{vmatrix}$$

$$= 2(a+b+c) \begin{vmatrix} 1 & a & b \\ 1 & b+c+2a & b \\ 1 & a & c+a+2b \end{vmatrix}$$

$$R_2 \rightarrow R_2 - R_1: R_3 \rightarrow R_3 - R_1$$

$$= 2(a+b+c) \begin{vmatrix} 1 & a & b \\ 0 & (a+b+c) & 0 \\ 0 & 0 & (a+b+c) \end{vmatrix}$$

$$= 2(a+b+c)^1 \begin{vmatrix} a+b+c & 0 \\ 0 & a+b+c \end{vmatrix}$$

$$= 2(a+b+c)^1(a+b+c)^2$$

$$= 2(a+b+c)^3$$

$$5. S.T \begin{vmatrix} a-b-c & 2a & 2b \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

$$= (a+b+c)^3$$

$$Sol: L.H.S = \begin{vmatrix} a-b-c & 2a & 2b \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

$$R_1 \rightarrow R_1 + R_2 + R_3$$

$$= \begin{vmatrix} (a+b+c) & (a+b+c) & (a+b+c) \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

$$= (a+b+c) \begin{vmatrix} 1 & 1 & 1 \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

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$$C_2 \rightarrow C_2 - C_1: C_3 \rightarrow C_3 - C_1$$

$$= (a+b+c) \begin{vmatrix} 1 & 0 & 0 \\ 2b & -(a+b+c) & 0 \\ 2c & 0 & -(a+b+c) \end{vmatrix}$$

$$= (a+b+c)^1 \begin{vmatrix} -(a+b+c) & 0 \\ 0 & -(a+b+c) \end{vmatrix}$$

$$= (a+b+c)^1 (a+b+c)^2$$

$$= (a+b+c)^3 \quad R.H.S$$

$$6. S.T \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(ab+bc+ca).$$

$$Sol: L.H.S \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$$

$$R_1 \rightarrow R_1 - R_2: R_2 \rightarrow R_2 - R_3$$

$$= \begin{vmatrix} 0 & a^2 - b^2 & a^3 - b^3 \\ 0 & b^2 - c^2 & b^3 - c^3 \\ 1 & c^2 & c^3 \end{vmatrix} = \begin{vmatrix} 0 & (a-b)(a+b) & (a-b)(a^2 + ab + b^2) \\ 0 & (b-c)(b+c) & (b-c)(b^2 + bc + c^2) \\ 1 & c^2 & c^3 \end{vmatrix}$$

$$= (a-b)(b-c) \begin{vmatrix} 0 & (a+b) & (a^2 + ab + b^2) \\ 0 & (b+c) & (b^2 + bc + c^2) \\ 1 & c^2 & c^3 \end{vmatrix}$$

$$R_2 \rightarrow R_2 - R_1:$$

$$= (a-b)(b-c) \begin{vmatrix} 0 & a+b & a^2 + ab + b^2 \\ 0 & (c-a) & b^2 + bc + c^2 - a^2 - ab - b^2 \\ 1 & c^2 & c^3 \end{vmatrix}$$

$$b^2 + bc + c^2 - a^2 - ab - b^2$$

$$= bc + c^2 - a^2 - ab$$

$$\Rightarrow b(c-a) + (c-a)(c+a) \Rightarrow (c-a)(a+b+c)$$

$$= (a-b)(b-c) \begin{vmatrix} 0 & a+b & a^2 + ab + b^2 \\ 0 & (c-a) & (c-a)(a+b+c) \\ 1 & c^2 & c^3 \end{vmatrix}$$

$$= (a-b)(b-c)(c-a) \begin{vmatrix} 0 & a+b & a^2 + ab + b^2 \\ 0 & 1 & a+b+c \\ 1 & c^2 & c^3 \end{vmatrix} Expanding\ along\ C_1$$

$$= (a-b)(b-c)(c-a)1[(a+b)(a+b+c) - a^2 - ab - b^2]$$

$$= (a-b)(b-c)(c-a)\{a^2 + ab + ac + ab + b^2 + bc - a^2 - ab - b^2\}$$

$$= (a-b)(b-c)(c-a)(ab + bc + ca).$$

$$7. S.T \begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} = abc(a-b)(b-c)(c-a)$$

$$Sol: \begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix}$$

$$= abc \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$$

$$C_1 \rightarrow C_1 - C_2; C_2 \rightarrow C_2 - C_3$$

$$= abc \begin{vmatrix} 0 & 0 & 1 \\ a-b & b-c & c \\ a^2-b^2 & b^2-c^2 & c^2 \end{vmatrix}$$

$$= abc \begin{vmatrix} 0 & 0 & 1 \\ a-b & b-c & c \\ (a-b)(a+b) & (b-c)(b+c) & c^2 \end{vmatrix}$$

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$$= abc(a-b)(b-c) \begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & c \\ (a+b) & (b+c) & c^2 \end{vmatrix}$$

Expanding along R₁

$$= abc(a-b)(b-c)1\{b+c-a-b\}$$

$$= abc(a-b)(b-c)(c-a)$$

8. find x if $\begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ x-4 & 2x-9 & 3x-16 \\ x-8 & 2x-27 & 3x-64 \end{vmatrix} = 0.$

sol: $\begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ x-4 & 2x-9 & 3x-16 \\ x-8 & 2x-27 & 3x-64 \end{vmatrix} = 0$

$$R_2 \rightarrow R_2 - R_1: R_3 \rightarrow R_3 - R_1$$

$$\Rightarrow \begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ -2 & -6 & -12 \\ -6 & -24 & -60 \end{vmatrix} = 0.$$

$$\Rightarrow \begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ 1 & 3 & 6 \\ 1 & 4 & 10 \end{vmatrix} = 0.$$

$$\Rightarrow (x-2)[30-24] - (2x-3)[10-6] + (3x-4)[4-3] = 0$$

$$\Rightarrow (x-2)[6] - (2x-3)[4] + (3x-4)[1] = 0$$

$$\Rightarrow 6x - 12 - 8x + 12 + 3x - 4 = 0$$

$$\Rightarrow x - 4 = 0$$

$$\therefore x = 4.$$

9. If $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ is a non-singular matrix,

then prove that A is invertible and $A^{-1} = \frac{\text{adj}A}{\det A}$.

$$\text{Sol: } A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \text{ and } \text{adj}A = \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix}$$

$$\text{Now } A \cdot \text{adj}A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \cdot \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix}$$

$$= \begin{bmatrix} a_1A_1 + b_1B_1 + c_1C_1 & a_1A_2 + b_1B_2 + c_1C_2 & a_1A_3 + b_1B_3 + c_1C_3 \\ a_2A_1 + b_2B_1 + c_2C_1 & a_2A_2 + b_2B_2 + c_2C_2 & a_2A_3 + b_2B_3 + c_2C_3 \\ a_3A_1 + b_3B_1 + c_3C_1 & a_3A_2 + b_3B_2 + c_3C_2 & a_3A_3 + b_3B_3 + c_3C_3 \end{bmatrix}$$

$$= \begin{bmatrix} \det A & 0 & 0 \\ 0 & \det A & 0 \\ 0 & 0 & \det A \end{bmatrix}$$

$$= \det A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \det A \cdot I$$

$$\therefore A \cdot \frac{\text{adj}A}{\det A} = I$$

Similarly we can prove that $\frac{\text{adj}A}{\det A} = I$

$$\therefore A^{-1} = \frac{\text{adj}A}{\det A}$$

10. If $A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & 4 \\ -2 & 2 & 1 \end{bmatrix}$, then find $(A')^{-1}$.

$$\text{Sol: } A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & 4 \\ -2 & 2 & 1 \end{bmatrix}$$

$$\Rightarrow A' = \begin{bmatrix} 1 & 0 & -2 \\ -2 & -1 & 2 \\ 3 & 4 & 1 \end{bmatrix}$$

$$\begin{aligned} |A'| &= 1 \begin{vmatrix} -1 & 2 \\ 4 & 1 \end{vmatrix} + 0 \begin{vmatrix} -2 & 2 \\ 3 & 1 \end{vmatrix} + 3 \begin{vmatrix} -2 & -1 \\ 3 & 4 \end{vmatrix} \\ &= 1(-1 - 8) + 0(-2 - 6) + 3(-8 + 3) \\ &= -9 + 0 + 10 = 1 \end{aligned}$$

$$\text{Adj}(A') = \begin{bmatrix} -1 & 2 & -2 & -1 \\ 4 & 1 & 3 & 4 \\ 0 & -2 & 1 & 0 \\ -1 & 2 & -2 & -1 \end{bmatrix}$$

$$\begin{aligned} &= \begin{bmatrix} (-1 - 8) & (6 + 2) & (-8 + 3) \\ (-8 + 0) & (1 + 6) & (0 - 4) \\ (0 - 2) & (4 - 2) & (-1 - 0) \end{bmatrix} \\ &= \begin{bmatrix} -9 & 8 & -5 \\ -8 & 7 & -4 \\ -2 & 2 & -1 \end{bmatrix} \end{aligned}$$

$$= \begin{bmatrix} -9 & -8 & -2 \\ 8 & 7 & 2 \\ -5 & -4 & -1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A'|} \text{adj}(A') = \frac{1}{1} \begin{bmatrix} -9 & -8 & -2 \\ 8 & 7 & 2 \\ -5 & -4 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -9 & -8 & -2 \\ 8 & 7 & 2 \\ -5 & -4 & -1 \end{bmatrix}$$

11. Solve the following equations by using Cramer's rule method.

$$\text{Sol: let } A = \begin{bmatrix} 3 & 4 & 5 \\ 2 & -1 & 8 \\ 5 & -2 & 7 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 18 \\ 13 \\ 20 \end{bmatrix}$$

$$\begin{aligned}\Delta &= \begin{vmatrix} 3 & 4 & 5 \\ 2 & -1 & 8 \\ 5 & -2 & 7 \end{vmatrix} = 3 \begin{vmatrix} -1 & 8 \\ -2 & 7 \end{vmatrix} - 4 \begin{vmatrix} 2 & 8 \\ 5 & 7 \end{vmatrix} + 5 \begin{vmatrix} 2 & -1 \\ 5 & -2 \end{vmatrix} \\ &= 3(-7 + 16) - 4(14 - 40) + 5(-4 + 5) \\ &= 3(9) - 4(-26) + 5(1) \\ &= 27 + 10 + 5 \\ &= 136 \neq 0 \text{ cramer's rule applicable}\end{aligned}$$

$$\begin{aligned}\Delta_1 &= \begin{vmatrix} 18 & 4 & 5 \\ 13 & -1 & 8 \\ 20 & -2 & 7 \end{vmatrix} = 18 \begin{vmatrix} -1 & 8 \\ -2 & 7 \end{vmatrix} - 4 \begin{vmatrix} 13 & 8 \\ 20 & 7 \end{vmatrix} + 5 \begin{vmatrix} 13 & -1 \\ 20 & -2 \end{vmatrix} \\ &= 18(-7 + 16) - 4(91 - 160) + 5(-26 + 20) \\ &= 18(9) - 4(-69) + 5(-6) \\ &= 162 + 276 - 30 \\ &= 408 \\ \Delta_2 &= \begin{vmatrix} 3 & 18 & 5 \\ 2 & 13 & 8 \\ 5 & 20 & 7 \end{vmatrix} = 3 \begin{vmatrix} 13 & 8 \\ 20 & 7 \end{vmatrix} - 18 \begin{vmatrix} 2 & 8 \\ 5 & 7 \end{vmatrix} + 5 \begin{vmatrix} 2 & 13 \\ 5 & 20 \end{vmatrix} \\ &= 3(91 - 160) - 18(14 - 40) + 5(40 - 65) \\ &= 3(-69) - 18(-26) + 5(-25) \\ &= -207 + 468 - 125 \\ &= 136\end{aligned}$$

$$\begin{aligned}\Delta_3 &= \begin{vmatrix} 3 & 4 & 18 \\ 2 & -1 & 13 \\ 5 & -2 & 20 \end{vmatrix} = 3 \begin{vmatrix} -1 & 13 \\ -2 & 20 \end{vmatrix} - 4 \begin{vmatrix} 2 & 13 \\ 5 & 20 \end{vmatrix} + 18 \begin{vmatrix} 2 & -1 \\ 5 & -2 \end{vmatrix} \\ &= 3(-20 + 26) - 4(40 - 65) + 5(-4 + 5) \\ &= 3(6) - 4(-25) + 5(1) \\ &= 18 + 100 + 18 \\ &= 136\end{aligned}$$

$$\Rightarrow x = \frac{\Delta_1}{\Delta} = \frac{408}{136}, y = \frac{\Delta_2}{\Delta} = \frac{136}{136}, z = \frac{\Delta_3}{\Delta} = \frac{136}{136}$$

$$\therefore x = 3, y = 1 \text{ and } z = 1.$$

12. Sol: let $A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 9 \\ 6 \\ 2 \end{bmatrix}$

$$\Delta = \begin{vmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix} = 2 \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} + 1 \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} + 3 \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}$$

$$\begin{aligned} &= 2(1+1) + 1(1-1) + 3(-1-1) \\ &= 2(2) + 1(0) + 3(-2) \\ &= 4 - 6 = -2 \neq \text{cramer's rule applicable.} \end{aligned}$$

$$\Delta_1 = \begin{vmatrix} 9 & -1 & 3 \\ 6 & 1 & 1 \\ 2 & -1 & 1 \end{vmatrix} = 6 \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} + 1 \begin{vmatrix} 6 & 1 \\ 2 & 1 \end{vmatrix} + 3 \begin{vmatrix} 6 & 1 \\ 2 & -1 \end{vmatrix}$$

$$\begin{aligned} &= 9(1+1) + 1(6-2) + 3(-6-2) \\ &= 9(2) + 1(4) + 3(-8) \\ &= 18 + 4 - 24 = 22 - 24 = -2 \end{aligned}$$

$$\Delta_2 = \begin{vmatrix} 2 & 9 & 3 \\ 1 & 6 & 1 \\ 1 & 2 & 1 \end{vmatrix} = 2 \begin{vmatrix} 6 & 1 \\ 2 & 1 \end{vmatrix} - 9 \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} + 3 \begin{vmatrix} 1 & 6 \\ 1 & 2 \end{vmatrix}$$

$$\begin{aligned} &= 2(6-2) - 9(1-1) + 3(2-6) \\ &= 2(4) + 1(0) + 3(-24) \\ &= 8 - 12 = -4 \end{aligned}$$

$$\Delta_3 = \begin{vmatrix} 2 & -1 & 9 \\ 1 & 1 & 6 \\ 1 & -1 & 2 \end{vmatrix} = 2 \begin{vmatrix} 1 & 6 \\ -1 & 2 \end{vmatrix} + 1 \begin{vmatrix} 1 & 6 \\ 1 & 2 \end{vmatrix} + 9 \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}$$

$$\begin{aligned} &= 2(2+6) + 1(2-6) + 9(-1-1) \\ &= 2(8) + 1(-4) + 9(-2) \\ &= 16 - 4 - 18 \\ &= 16 - 22 = -6 \\ \Rightarrow x &= \frac{\Delta_1}{\Delta} = \frac{-2}{-2} = 1, y = \frac{\Delta_2}{\Delta} = \frac{-4}{-2} = 2, z = \frac{\Delta_3}{\Delta} = \frac{-6}{-2} = 3 \end{aligned}$$

$$\therefore x = 1, y = 2 \text{ and } z = 3.$$

$$13. A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & 7 \\ 2 & 1 & -1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 9 \\ 52 \\ 0 \end{bmatrix}$$

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 5 & 7 \\ 2 & 1 & -1 \end{vmatrix} = 1 \begin{vmatrix} 5 & 7 \\ 1 & -1 \end{vmatrix} - 1 \begin{vmatrix} 2 & 7 \\ 2 & -1 \end{vmatrix} + 1 \begin{vmatrix} 2 & 5 \\ 2 & 1 \end{vmatrix}$$

$$\begin{aligned} &= 1(-5 - 7) - 1(-2 - 14) + 1(2 - 10) \\ &= 1(-12) - 1(-16) + 1(-8) \\ &= -12 + 16 - 8 = -4 \neq \text{cramer's rule applicable.} \end{aligned}$$

$$\Delta_1 = \begin{vmatrix} 9 & 1 & 1 \\ 52 & 5 & 7 \\ 0 & 1 & -1 \end{vmatrix} = 9 \begin{vmatrix} 5 & 7 \\ 1 & -1 \end{vmatrix} - 1 \begin{vmatrix} 52 & 7 \\ 0 & -1 \end{vmatrix} + 1 \begin{vmatrix} 52 & 5 \\ 0 & 1 \end{vmatrix}$$

$$\begin{aligned} &= 9(-5 - 7) - 1(-52 - 0) + 1(52 - 0) \\ &= 9(-12) - 1(-52) + 1(52) \\ &= -108 + 52 + 52 \\ &= -108 + 104 = -4 \end{aligned}$$

$$\Delta_2 = \begin{vmatrix} 1 & 9 & 1 \\ 2 & 52 & 7 \\ 2 & 0 & -1 \end{vmatrix} = 1 \begin{vmatrix} 52 & 7 \\ 0 & -1 \end{vmatrix} - 9 \begin{vmatrix} 2 & 7 \\ 2 & -1 \end{vmatrix} + 1 \begin{vmatrix} 2 & 52 \\ 0 & 1 \end{vmatrix}$$

$$\begin{aligned} &= 1(-52 - 0) - 9(-2 - 14) + 1(0 - 104) \\ &= 1(-52) - 9(-16) + 1(-104) \\ &= -52 + 144 - 104 \\ &= -156 + 144 = -12 \end{aligned}$$

$$\Delta_3 = \begin{vmatrix} 1 & 1 & 9 \\ 2 & 5 & 52 \\ 2 & 1 & 0 \end{vmatrix} = 1 \begin{vmatrix} 5 & 52 \\ 1 & 0 \end{vmatrix} - 1 \begin{vmatrix} 2 & 52 \\ 2 & 0 \end{vmatrix} + 9 \begin{vmatrix} 2 & 5 \\ 2 & 1 \end{vmatrix}$$

$$\begin{aligned} &= 1(0 - 52) - 1(0 - 104) + 9(2 - 10) \\ &= 1(-52) - 1(-104) + 9(-8) \\ &= -52 + 104 - 72 \\ &= -124 + 104 = -20 \\ \Rightarrow x &= \frac{\Delta_1}{\Delta} = \frac{-4}{-4} = 1, y = \frac{\Delta_2}{\Delta} = \frac{-12}{-4} = 3, z = \frac{\Delta_3}{\Delta} = \frac{-20}{-4} = 5 \end{aligned}$$

$$\therefore x = 1, y = 3 \text{ and } z = 5.$$

$$14. A = \begin{bmatrix} 2 & -1 & 3 \\ -1 & 2 & 1 \\ 3 & 1 & -4 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 8 \\ 4 \\ 0 \end{bmatrix}$$

$$\begin{aligned}\Delta &= \begin{vmatrix} 2 & -1 & 3 \\ -1 & 2 & 1 \\ 3 & 1 & -4 \end{vmatrix} = 2 \begin{vmatrix} 2 & 1 \\ 1 & -4 \end{vmatrix} + 1 \begin{vmatrix} -1 & 1 \\ 3 & -4 \end{vmatrix} + 3 \begin{vmatrix} -1 & 2 \\ 3 & 1 \end{vmatrix} \\ &= 2(-8 - 1) + 1(4 - 3) + 3(-1 - 6) \\ &= 2(-9) + 1(1) + 3(-7) \\ &= -18 + 1 - 21 \\ &= -38\end{aligned}$$

$$\begin{aligned}\Delta_1 &= \begin{vmatrix} 8 & -1 & 3 \\ 4 & 2 & 1 \\ 0 & 1 & -4 \end{vmatrix} = 8 \begin{vmatrix} 2 & 1 \\ 1 & -4 \end{vmatrix} + 1 \begin{vmatrix} 4 & 1 \\ 0 & -4 \end{vmatrix} + 3 \begin{vmatrix} 4 & 2 \\ 0 & 1 \end{vmatrix} \\ &= 8(-8 - 1) + 1(-16 - 0) + 3(4 - 0) \\ &= 8(-9) + 1(-16) + 3(4) \\ &= -72 - 16 + 12 = -76\end{aligned}$$

$$\begin{aligned}\Delta_2 &= \begin{vmatrix} 2 & 8 & 3 \\ -1 & 4 & 1 \\ 3 & 0 & -4 \end{vmatrix} = 2 \begin{vmatrix} 4 & 1 \\ 0 & -4 \end{vmatrix} - 8 \begin{vmatrix} -1 & 1 \\ 3 & -4 \end{vmatrix} + 3 \begin{vmatrix} -1 & 4 \\ 3 & 0 \end{vmatrix} \\ &= 2(-16 - 0) - 8(4 - 3) + 3(0 - 12) \\ &= 2(-16) - 8(1) + 3(-12) \\ &= -32 - 8 - 36 = -76\end{aligned}$$

$$\begin{aligned}\Delta_3 &= \begin{vmatrix} 2 & -1 & 8 \\ -1 & 2 & 4 \\ 3 & 1 & 0 \end{vmatrix} = 2 \begin{vmatrix} 2 & 4 \\ 1 & 0 \end{vmatrix} + 1 \begin{vmatrix} -1 & 4 \\ 3 & 0 \end{vmatrix} + 8 \begin{vmatrix} -1 & 2 \\ 3 & 1 \end{vmatrix} \\ &= 2(0 - 4) + 1(0 - 12) + 8(-1 - 6) \\ &= 2(-4) + 1(-12) + 8(-7) \\ &= -8 - 12 - 56 = 76\end{aligned}$$

$$\Rightarrow x = \frac{\Delta_1}{\Delta} = \frac{-76}{-38} = 2, y = \frac{\Delta_2}{\Delta} = \frac{-76}{-38} = 2, z = \frac{\Delta_3}{\Delta} = \frac{-76}{-38} = 2$$

$$\therefore x = 2, y = 2 \text{ and } z = 2.$$

Matrix inversion method:

$$15. A = \begin{bmatrix} 3 & 4 & 5 \\ 2 & -1 & 8 \\ 5 & -2 & 7 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 18 \\ 13 \\ 20 \end{bmatrix}$$

$$AX = B \Rightarrow X = A^{-1}B$$

$$\text{And } A^{-1} = \frac{\text{adj}A}{\det A} \quad \Rightarrow \quad \text{adj}A = \begin{bmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{bmatrix} = \begin{pmatrix} -1 & 8 & 2 & -1 \\ -2 & 7 & 5 & -2 \\ 4 & 5 & 3 & 4 \\ -1 & 8 & 2 & -1 \end{pmatrix}$$

$$\text{cofactor matrix} = \begin{bmatrix} (-7 + 16) & (40 - 14) & (-4 + 5) \\ (-10 - 28) & (21 - 25) & (20 + 6) \\ (32 + 5) & (10 - 24) & (-3 - 8) \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 26 & 1 \\ -38 & -4 & 26 \\ 37 & -14 & -11 \end{bmatrix}$$

$$\text{adj}A = [\text{cofactor}]^T = \begin{bmatrix} 9 & -38 & 37 \\ 26 & -4 & -14 \\ 1 & 26 & -11 \end{bmatrix}$$

$$\det A = a_1A_1 + b_1B_1 + c_1C_1$$

$$2(9) - 4(-26) + 5(1)$$

$$= 136$$

$$A^{-1} = \frac{\text{adj}A}{\det A} = \frac{1}{136} \begin{bmatrix} 9 & -38 & 37 \\ 26 & -4 & -14 \\ 1 & 26 & -11 \end{bmatrix}$$

$$\Rightarrow X = A^{-1}B$$

$$= \frac{1}{136} \begin{bmatrix} 9 & -38 & 37 \\ 26 & -4 & -14 \\ 1 & 26 & -11 \end{bmatrix} \begin{bmatrix} 18 \\ 13 \\ 20 \end{bmatrix}$$

$$= \frac{1}{136} \begin{bmatrix} 162 - 494 + 740 \\ 468 - 52 - 280 \\ 18 + 338 + 220 \end{bmatrix}$$

$$= \frac{1}{136} \begin{bmatrix} 408 \\ 136 \\ 136 \end{bmatrix} \therefore x = 3, y = 1 \text{ and } z = 1.$$

$$(b). Sol: let A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} and B = \begin{bmatrix} 9 \\ 6 \\ 2 \end{bmatrix}$$

$$AX = B \Rightarrow X = A^{-1} \cdot B$$

$$And A^{-1} = \frac{\text{adj}A}{\det A}$$

$$\begin{aligned} \det A &= \begin{vmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix} = 2 \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} + 1 \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} + 3 \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} \\ &= 2(1+1) + 1(1-1) + 3(-1-1) \\ &= 2(2) + 1(0) + 3(-2) \\ &= 4 - 6 = -2 \neq 0 \quad A^{-1} \text{ exists.} \end{aligned}$$

$$\text{adj}A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \quad \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 3 & 2 & -1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

$$\text{cofactor matrix} = \begin{bmatrix} (1+1) & (1-1) & (-1-1) \\ (-3+1) & (2-3) & (-1+2) \\ (-1-3) & (3-2) & (2+1) \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & -2 \\ -2 & -1 & 1 \\ -4 & 1 & 3 \end{bmatrix}$$

$$\text{adj}A = [\text{cofactor}]^T = \begin{bmatrix} 2 & -2 & -4 \\ 0 & -1 & 1 \\ -2 & 1 & 3 \end{bmatrix}$$

$$A^{-1} = \frac{\text{adj}A}{\det A} = \frac{1}{-2} \begin{bmatrix} 2 & -2 & -4 \\ 0 & -1 & 1 \\ -2 & 1 & 3 \end{bmatrix}$$

$$\Rightarrow X = A^{-1} \cdot B$$

$$= \frac{1}{-2} \begin{bmatrix} 2 & -2 & -4 \\ 0 & -1 & 1 \\ -2 & 1 & 3 \end{bmatrix} \begin{bmatrix} 9 \\ 6 \\ 2 \end{bmatrix}$$

$$= \frac{1}{-2} \begin{bmatrix} 18 - 12 - 8 \\ 0 - 6 + 2 \\ -18 + 6 + 6 \end{bmatrix} = \frac{1}{-2} \begin{bmatrix} -2 \\ -4 \\ -6 \end{bmatrix}$$

$$\therefore x = 1, y = 2 \text{ and } z = 3$$

$$16. A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & 7 \\ 2 & 1 & -1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 9 \\ 52 \\ 0 \end{bmatrix}$$

$$\begin{aligned} \det A &= \begin{vmatrix} 1 & 1 & 1 \\ 2 & 5 & 7 \\ 2 & 1 & -1 \end{vmatrix} = 1 \begin{vmatrix} 5 & 7 \\ 1 & -1 \end{vmatrix} - 1 \begin{vmatrix} 2 & 7 \\ 2 & -1 \end{vmatrix} + 1 \begin{vmatrix} 2 & 5 \\ 2 & 1 \end{vmatrix} \\ &= 1(-5 - 7) - 1(-2 - 14) + 1(2 - 10) \\ &= 1(-12) - 1(-16) + 1(-8) \\ &= -12 + 16 - 8 = -4 \neq 0. A^{-1} \text{ exists.} \end{aligned}$$

$$Adj A = \begin{bmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{bmatrix} \quad \begin{pmatrix} 5 & 7 & 2 & 5 \\ 1 & -1 & 2 & 1 \\ 1 & 1 & 1 & 1 \\ 5 & 7 & 2 & 5 \end{pmatrix}$$

$$\text{cofactor matrix} = \begin{bmatrix} (-5 - 7) & (14 + 2) & (2 - 10) \\ (1 + 1) & (-1 - 2) & (2 - 1) \\ (7 - 5) & (2 - 7) & (5 - 2) \end{bmatrix}$$

$$= \begin{bmatrix} -12 & 16 & -8 \\ 2 & -3 & 1 \\ 2 & -5 & 3 \end{bmatrix}$$

$$Adj A = [\text{cofactor}]^T = \begin{bmatrix} -12 & 2 & 2 \\ 16 & -3 & -5 \\ -8 & 1 & 3 \end{bmatrix}$$

$$A^{-1} = \frac{adj A}{\det A} = \frac{1}{-4} \begin{bmatrix} -12 & 2 & 2 \\ 16 & -3 & -5 \\ -8 & 1 & 3 \end{bmatrix}$$

$$\Rightarrow X = A^{-1} \cdot B$$

$$= \frac{1}{-4} \begin{bmatrix} -12 & 2 & 2 \\ 16 & -3 & -5 \\ -8 & 1 & 3 \end{bmatrix} \begin{bmatrix} 9 \\ 52 \\ 0 \end{bmatrix}$$

$$= \frac{1}{-2} \begin{bmatrix} -108 + 104 + 0 \\ 144 - 156 + 0 \\ -72 + 52 + 0 \end{bmatrix}$$

$$= \frac{1}{-2} \begin{bmatrix} -4 \\ -12 \\ -20 \end{bmatrix} \therefore x = 1, y = 3 \text{ and } z = 5.$$

$$17. A = \begin{bmatrix} 2 & -1 & 3 \\ -1 & 2 & 1 \\ 3 & 1 & -4 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 8 \\ 4 \\ 0 \end{bmatrix}$$

$$\begin{aligned} \det A &= \begin{vmatrix} 2 & -1 & 3 \\ -1 & 2 & 1 \\ 3 & 1 & -4 \end{vmatrix} = 2 \begin{vmatrix} 2 & 1 \\ 1 & -4 \end{vmatrix} + 1 \begin{vmatrix} -1 & 1 \\ 3 & -4 \end{vmatrix} + 3 \begin{vmatrix} -1 & 2 \\ 3 & 1 \end{vmatrix} \\ &= 2(-8 - 1) + 1(4 - 3) + 3(-1 - 6) \\ &= 2(-9) + 1(1) + 3(-7) \\ &= -18 + 1 - 21 \\ &= -38 \neq 0. A^{-1} \text{ exists.} \quad \text{Adj}A = \end{aligned}$$

$$\begin{bmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{bmatrix}, \text{cofactor matrix} = \begin{pmatrix} 2 & 1 & -1 & 2 \\ 1 & -4 & 3 & 1 \\ -1 & 3 & 2 & -1 \\ 2 & 1 & -1 & 2 \end{pmatrix}$$

$$\text{cofactor matrix} = \begin{bmatrix} (-8 - 1) & (3 - 4) & (-1 - 6) \\ (3 - 4) & (-8 - 9) & (-3 - 2) \\ (-1 - 6) & (-3 - 2) & (4 - 1) \end{bmatrix}$$

$$= \begin{bmatrix} -9 & -1 & -7 \\ -1 & -17 & -5 \\ -7 & -5 & 3 \end{bmatrix}$$

$$\text{Adj}A = [\text{cofactor}]^T = \begin{bmatrix} -9 & -1 & -7 \\ -1 & -17 & -5 \\ -7 & -5 & 3 \end{bmatrix}$$

$$A^{-1} = \frac{\text{adj}A}{\det A} = \frac{1}{-38} \begin{bmatrix} -9 & -1 & -7 \\ -1 & -17 & -5 \\ -7 & -5 & 3 \end{bmatrix}$$

$$\Rightarrow X = A^{-1} \cdot B$$

$$= \frac{1}{-38} \begin{bmatrix} -9 & -1 & -7 \\ -1 & -17 & -5 \\ -7 & -5 & 3 \end{bmatrix} \begin{bmatrix} 8 \\ 4 \\ 0 \end{bmatrix}$$

$$= \frac{1}{-38} \begin{bmatrix} -72 - 4 + 0 \\ -8 - 68 + 0 \\ -56 - 20 + 0 \end{bmatrix}$$

$$= \frac{1}{-38} \begin{bmatrix} -76 \\ -76 \\ -76 \end{bmatrix} \therefore x = 2, y = 2 \text{ and } z = 2.$$

17. Show that the following system of equations is consistent and solve it completely.

$$x + y + z = 3, 2x + 2y - z = 3, x + y - z = 1;$$

sol: the augmented matrix is

$$[AD] = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 2 & 2 & -1 & 3 \\ 1 & 1 & -1 & 1 \end{bmatrix}$$

on applying $R_2 \rightarrow R_2 - 2R_1, R_3 \rightarrow R_3 - R_1$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 0 & -3 & -3 \\ 0 & 0 & -2 & -2 \end{bmatrix}$$

on applying $R_3 \rightarrow 3R_3 - 2R_2$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 0 & -3 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \dots\dots (1)$$



Comparing with echelon form

Number of non-zero rows in A are 2

$$\therefore \text{rank } (A) = 2$$

Number of non-zero rows in AD are 2

$$\therefore \text{rank } (AD) = 2$$

$$\text{Rank } (A) = \text{rank } (AD) = 2$$

The system is consistent and has infinitely many solutions.

We write equivalent set of equations from ... (1)

$$x + y + z = 3 \dots\dots (2)$$

$$-3z = -3 \dots\dots (3)$$

$$Z = 1$$

$$\text{Hence } z = 1, x + y = 2$$

Let $x = k \Rightarrow y = 2 - k, z = 1, k \in R$ is the solution set.

18. Apply the test of rank to examine whether the following equations are consistent.

$$2x - y + 3z = 8, -x + 2y + z = 4, 3x + y - 4z = 0;$$

sol: the augmented matrix is

$$[AD] = \begin{bmatrix} 2 & -1 & 3 & 8 \\ -1 & 2 & 1 & 4 \\ 3 & 1 & -4 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 & 3 & 8 \\ -1 & 2 & 1 & 4 \\ 3 & 1 & -4 & 0 \end{bmatrix}$$

on interchanging R_1 and R_2

*we transform the above matrix into
an upper triangular matrix.*

on applying $R_2 \rightarrow R_2 + 2R_1$, $R_3 \rightarrow R_3 + 3R_1$

$$\sim \begin{bmatrix} -1 & 2 & 1 & 4 \\ 0 & 3 & 5 & 16 \\ 0 & 7 & -1 & 12 \end{bmatrix}$$

on applying $R_3 \rightarrow 3R_3 - 7R_2$

$$\sim \begin{bmatrix} -1 & 2 & 1 & 4 \\ 0 & 3 & 5 & 16 \\ 0 & 0 & -38 & -76 \end{bmatrix} \dots\dots (1)$$

Comparing with echelon form

Number of non-zero rows in A are 3 \therefore rank (A) = 3

Number of non-zero rows in AD are 3 \therefore rank (AD) = 3

Hence rank (A) = rank [(AD)] = 3.

Thus the system has a unique solution.

We write the system of equations from (1)

$$-x + 2y + z = 4 \dots (2)$$

$$3y + 5z = 16 \dots \dots (3)$$

$$-38z = -76 \dots (4)$$

from (4)

$z = 2$ sub in (3)

$$\Rightarrow 3y = 16 - 10 = 6$$

$$y = 2 \text{ sub } y = 2, z = 2 \text{ in (1)} \Rightarrow x = 2$$

$\therefore x = y = z = 2$ is the solution.

19. Solve the following system of equations

$$x + y + z = 1, 2x + 2y + 3z = 6, x + 4y + 9z = 3;$$

Sol: the augmented matrix is

$$[AD] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 3 & 6 \\ 1 & 4 & 9 & 3 \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 & 0 & x \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & z \end{bmatrix}$$

on applying $R_2 \rightarrow R_2 - 2R_1, R_3 \rightarrow R_3 - R_1$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2-2 & 2-2 & 3-2 & 6-2 \\ 1-1 & 4-1 & 9-1 & 3-1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 4 \\ 0 & 3 & 8 & 2 \end{bmatrix} \quad R_2 \leftrightarrow R_3$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 3 & 8 & 2 \\ 0 & 0 & 1 & 4 \end{bmatrix} \quad R_1 \rightarrow 3R_1 - R_2$$

$$= \begin{bmatrix} 3-0 & 3-3 & 3-8 & 3-2 \\ 0 & 3 & 8 & 2 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 & -5 & 1 \\ 0 & 3 & 8 & 2 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

$$R_1 \rightarrow R_1 + 5R_3, R_2 \rightarrow R_2 - 8R_3$$

$$= \begin{bmatrix} 3+0 & 0+0 & -5+5 & 1+20 \\ 0-0 & 3-0 & 8-8 & 2-32 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 & 0 & 21 \\ 0 & 3 & 0 & -30 \\ 0 & 0 & 1 & 4 \end{bmatrix} \quad R_1 \leftrightarrow \left[\frac{R_1}{3} \right], R_2 \leftrightarrow \left[\frac{R_2}{3} \right]$$

$$= \begin{bmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & -10 \\ 0 & 0 & 1 & 4 \end{bmatrix} \quad \therefore x = 7, y = -10, z = 4$$

$$20x - y + 3z = 5, 4x + 2y - z = 0, -x + 3y + z = 5;$$

Sol: the augmented matrix is

$$[AD] = \begin{bmatrix} 1 & -1 & 3 & 5 \\ 4 & 2 & -1 & 0 \\ -1 & 3 & 1 & 5 \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 & 0 & x \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & z \end{bmatrix}$$

on applying $R_2 \rightarrow R_2 - 4R_1, R_3 \rightarrow R_3 + R_1$

$$= \begin{bmatrix} 1 & -1 & 3 & 5 \\ 4 - 4 & 2 + 4 & -1 - 12 & 0 - 20 \\ -1 + 1 & 3 - 1 & 1 + 3 & 5 + 5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & 3 & 5 \\ 0 & 6 & -13 & -20 \\ 0 & 2 & 4 & 10 \end{bmatrix} R_2 \leftrightarrow \left[\frac{R_3}{2} \right]$$

$$= \begin{bmatrix} 1 & -1 & 3 & 5 \\ 0 & 6 & -13 & -20 \\ 0 & 1 & 2 & 5 \end{bmatrix} R_2 \rightarrow R_2 - 5R_3$$

$$= \begin{bmatrix} 1 & -1 & 3 & 5 \\ 0 - 0 & 6 - 5 & -13 - 10 & -20 - 25 \\ 0 & 1 & 2 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & 3 & 5 \\ 0 & 1 & -23 & -45 \\ 0 & 1 & 2 & 5 \end{bmatrix} R_1 \rightarrow R_1 + R_2, R_3 \rightarrow R_3 - R_2$$

$$= \begin{bmatrix} 1 + 0 & -1 + 1 & 3 - 23 & 5 - 45 \\ 0 & 1 & -23 & -45 \\ 0 - 0 & 1 - 1 & 2 + 23 & 5 + 45 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -20 & -40 \\ 0 & 1 & -23 & -45 \\ 0 & 0 & 25 & 50 \end{bmatrix}$$

$$R_3 \leftrightarrow \left[\frac{R_3}{25} \right]$$

$$= \begin{bmatrix} 1 & 0 & -20 & -40 \\ 0 & 1 & -23 & -45 \\ 0 & 0 & 1 & 2 \end{bmatrix} R_1 \rightarrow R_1 + 20R_3; R_2 \rightarrow R_2 + 23R_3$$

$$= \begin{bmatrix} 1 + 0 & 0 + 0 & -20 + 20 & -40 + 40 \\ 0 & 1 & -23 + 23 & -45 + 46 \\ 0 & 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\therefore x = 0, y = 1, z = 2$$

$$21. 2x - y + 3z = 9, x + y + z = 6, x - y + z = 2;$$

Sol: the augmented matrix is

$$[AD] = \begin{bmatrix} 2 & -1 & 3 & 9 \\ 1 & 1 & 1 & 6 \\ 1 & -1 & 1 & 2 \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 & 0 & x \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & z \end{bmatrix}$$

$$\begin{aligned} R_1 &\leftrightarrow R_1 - R_2 \\ &= \begin{bmatrix} 2-1 & -1-1 & 3-1 & 9-6 \\ 1 & 1 & 1 & 6 \\ 1 & -1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 2 & 3 \\ 1 & 1 & 1 & 6 \\ 1 & -1 & 1 & 2 \end{bmatrix} \end{aligned}$$

on applying $R_2 \rightarrow R_2 - R_1, R_3 \rightarrow R_3 - R_1$

$$= \begin{bmatrix} 1 & -2 & 2 & 3 \\ 1-1 & 1+2 & 1-2 & 6-3 \\ 1-1 & -1+2 & 1-2 & 2-3 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 2 & 3 \\ 0 & 3 & -1 & 3 \\ 0 & 1 & -1 & -1 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - 2R_3$$

$$= \begin{bmatrix} 1 & -2 & 2 & 3 \\ 0-0 & 3-2 & -1+2 & 3+2 \\ 0 & 1 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 2 & 3 \\ 0 & 1 & 1 & 5 \\ 0 & 1 & -1 & -1 \end{bmatrix}$$

Aims tutorial

$$R_1 \rightarrow R_1 + 2R_2, R_3 \rightarrow R_3 - R_2$$

$$= \begin{bmatrix} 1+0 & -2+2 & 2+2 & 3+10 \\ 0 & 1 & 1 & 5 \\ 0-0 & 1-1 & -1-1 & -1-5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 4 & 13 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & -2 & -6 \end{bmatrix}$$

$$R_3 \leftrightarrow \left[\begin{smallmatrix} R_3 \\ 2 \end{smallmatrix} \right]$$

$$= \begin{bmatrix} 1 & 0 & 4 & 13 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$R_1 \rightarrow R_1 - 4R_3; R_2 \rightarrow R_2 - R_3$$

$$= \begin{bmatrix} 1 & 0 & 4-4 & 13-12 \\ 0 & 1 & 1-1 & 5-3 \\ 0 & 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix} \therefore x = 1, y = 2, z = 3$$

1. If $A+B+C = 180^\circ$, then show that
 $\sin 2A - \sin 2B + \sin 2C = 4\cos A \sin B \cos C.$

A: Given: $A+B+C = 180^\circ$
 $\sin 2A - \sin 2B + \sin 2C$
 $= 2\cos\left(\frac{2A+2B}{2}\right)\sin\left(\frac{2A-2B}{2}\right) + 2\sin C \cos C$
 $= 2\cos(A+B)\sin(A-B) + 2\sin C \cos C$
 $= 2\cos(180^\circ - C)\sin(A-B) + 2\sin C \cos C$
 $= -2\cos C \sin(A-B) + 2\sin C \cos C$
 $= 2\cos C [-\sin(A-B) + \sin C]$
 $= 2\cos C [\sin\{180^\circ - (A+B)\} - \sin(A-B)]$
 $= 2\cos C [\sin(A+B) - \sin(A-B)]$

$\therefore \sin C - \sin D$
 $= 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$

$= 2\cos C \cdot 2\cos A \sin B$
 $= 4\cos A \sin B \cos C$

2. If $A+B+C = 90^\circ$, then prove that
 $\cos 2A + \cos 2B + \cos 2C = 1 + 4 \sin A \sin B \sin C.$

A: Given $A+B+C = 90^\circ$
Now $\cos 2A + \cos 2B + \cos 2C$
 $= 2\cos(A+B)\cos(A-B) + 1 - 2\sin^2 C$
 $= 2\cos(90^\circ - C)\cos(A-B) + 1 - 2\sin^2 C$
 $= 2\sin C \cos(A-B) + 1 - 2\sin^2 C$
 $= 1 + 2\sin C [\cos(A-B) - \sin C]$
 $= 1 + 2\sin C [\cos(A-B) - \sin\{90^\circ - (A+B)\}]$
 $= 1 + 2\sin C [\cos(A-B) - \cos(A+B)]$
 $= 1 + 2\sin C \cdot 2\sin A \sin B$
 $= 1 + 4 \sin A \sin B \sin C.$

3. If $A+B+C = 270^\circ$, show that
 $\cos 2A + \cos 2B + \cos 2C = 1 - 4 \sin A \sin B \sin C$

Given: $A+B+C = 270^\circ$
Now $\cos 2A + \cos 2B + \cos 2C$
 $= 2\cos(A+B)\cos(A-B) + 1 - 2\sin^2 C$
 $= 2\cos(270^\circ - C)\cos(A-B) + 1 - 2\sin^2 C$
 $= -2\sin C \cos(A-B) + 1 - 2\sin^2 C$
 $= 1 - 2\sin C [\cos(A-B) + \sin\{270^\circ - (A+B)\}]$
 $= 1 - 2\sin C [\cos(A-B) - \cos(A+B)]$
 $\therefore \sin(270^\circ - \theta) = -\cos \theta$
 $= 1 - 2\sin C \cdot 2\sin A \sin B$
 $= 1 - 4 \sin A \sin B \sin C$

4. $A + B + C = 180^\circ$ Then show that
 $\cos A + \cos B + \cos C = 1 + 4 \sin A/2 \sin B/2 \sin C/2.$

A: $\cos A + \cos B + \cos C$
 $= 2 \cos \frac{A+B}{2} \cos \frac{A-B}{2} + \cos C$
 $\because A + B + C = 180^\circ$
 $\cos \frac{A+B}{2} = \sin \frac{C}{2}$
 $= 2 \sin \frac{C}{2} \cos \frac{A-B}{2} + 1 - 2\sin^2 \frac{C}{2}$
 $= 1 + 2\sin \frac{C}{2} \left(\cos \frac{A-B}{2} - \sin \frac{C}{2} \right)$
 $= 1 + 2\sin \frac{C}{2} \left(\cos \frac{A-B}{2} - \cos \frac{A+B}{2} \right)$
 $= 1 + 2\sin \frac{C}{2} \cdot 2\sin \frac{A}{2} \sin \frac{B}{2}$
 $= 1 + 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$

5. If $A+B+C = 0^\circ$, prove that

$$\sin A + \sin B - \sin C = -4\cos \frac{A}{2} \cos \frac{B}{2} \sin \frac{C}{2}$$

A: Given $A+B+C = 0^\circ$

$$\therefore \sin A + \sin B - \sin C$$

$$\begin{aligned} &= 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) - 2\sin\frac{C}{2}\cos\frac{C}{2} \\ &= 2\sin\left(\frac{-C}{2}\right)\cos\left(\frac{A-B}{2}\right) - 2\sin\frac{C}{2}\cos\frac{C}{2} \\ &= -2\sin\frac{C}{2}\cos\left(\frac{A-B}{2}\right) - 2\sin\frac{C}{2}\cos\frac{C}{2} \\ &= -2\sin\frac{C}{2}\left[\cos\left(\frac{A-B}{2}\right) + \cos\frac{C}{2}\right] \\ &= -2\sin\frac{C}{2}\left[\cos\left(\frac{A-B}{2}\right) + \cos\left\{-\left(\frac{A+B}{2}\right)\right\}\right] \\ &= -2\sin\frac{C}{2}\left[\cos\left(\frac{A-B}{2}\right) + \cos\left(\frac{A+B}{2}\right)\right] \\ &= -2\sin\frac{C}{2}\cdot 2\cos\frac{A}{2}\cos\frac{B}{2} \\ &= -4\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}. \end{aligned}$$

6. If $A+B+C = 180^\circ$, prove that

$$\begin{aligned} \cos^2\left(\frac{A}{2}\right) + \cos^2\left(\frac{B}{2}\right) + \cos^2\left(\frac{C}{2}\right) &= 2\left(1 + \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right) \\ \text{A: Given } A+B+C &= 180^\circ \\ \therefore \cos^2\frac{A}{2} + \cos^2\frac{B}{2} + \cos^2\frac{C}{2} & \\ &= \cos^2\frac{A}{2} + 1 - \sin^2\frac{B}{2} + \cos^2\frac{C}{2} \\ &= 1 + \left(\cos^2\frac{A}{2} - \sin^2\frac{B}{2}\right) + \cos^2\frac{C}{2} \\ &= 1 + \cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) + 1 - \sin^2\frac{C}{2} \\ &= 2 + \sin\frac{C}{2}\cos\left(\frac{A-B}{2}\right) - \sin^2\frac{C}{2} \\ &= 2 + \sin\frac{C}{2}\left[\cos\left(\frac{A-B}{2}\right) - \sin\frac{C}{2}\right] \quad \boxed{\because \cos\left(\frac{A+B}{2}\right) = \sin\frac{C}{2}} \\ &= 2 + \sin\frac{C}{2}\left[\cos\left(\frac{A-B}{2}\right) - \cos\left(\frac{A+B}{2}\right)\right] \\ &= 2 + \sin\frac{C}{2}\cdot 2\sin\frac{A}{2}\sin\frac{B}{2} = 2[1 + \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}] \end{aligned}$$

7. If $A+B+C = \pi$, prove that

$$\sin^2 A + \sin^2 B + \sin^2 C = 2 + 2\cos A \cos B \cos C.$$

A: Given: $A+B+C = \pi$

$$\begin{aligned} \text{Now } \sin^2 A + \sin^2 B + \sin^2 C & \\ &= 1 - \cos^2 A + \sin^2 B + \sin^2 C \\ &= 1 - (\cos^2 A - \sin^2 B) + \sin^2 C \\ &= 1 - \cos(A+B) \cos(A-B) + 1 - \cos^2 C \\ &= 2 - \cos(\pi - C) \cos(A-B) - \cos^2 C \\ &= 2 + \cos C \cos(A-B) - \cos^2 C \\ &= 2 + \cos C [\cos(A-B) - \cos C] \\ &= 2 + \cos C [\cos(A-B) - \cos\{\pi - (A+B)\}] \\ &= 2 + \cos C [\cos(A-B) + \cos(A+B)] \\ &= 2 + \cos C 2\cos A \cos B \\ &= 2 + 2\cos A \cos B \cos C. \end{aligned}$$

8. If $A+B+C = \pi$, then show that

$$\sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2} = 1 + 4\sin\left(\frac{\pi-A}{4}\right)\sin\left(\frac{\pi-B}{4}\right)\sin\left(\frac{\pi-C}{4}\right)$$

A: Given $A+B+C = \pi$

$$\begin{aligned} &= 1 + 4\sin\left(\frac{\pi-A}{4}\right)\sin\left(\frac{\pi-B}{4}\right)\sin\left(\frac{\pi-C}{4}\right) \\ &= 1 + 2\left[\sin\left(\frac{\pi-A}{4}\right)\sin\left(\frac{\pi-B}{4}\right)\right]\sin\left(\frac{\pi-C}{4}\right) \\ &= 1 + 2\left[\cos\left(\frac{\pi-A-\pi+B}{4}\right) - \cos\left(\frac{\pi-A+\pi-B}{4}\right)\right]\sin\left(\frac{\pi+C}{4}\right) \\ &\quad \because 2\sin A 2\sin B = \cos(A-B) - \cos(A+B) \\ &= 1 + 2\left[\cos\left\{-\left(\frac{A-B}{4}\right)\right\} - \cos\left\{\frac{\pi}{2} - \left(\frac{A+B}{4}\right)\right\}\right]\sin\left(\frac{A+B}{4}\right) \\ &= 1 + 2\left[\cos\left(\frac{A-B}{4}\right) - \sin\left(\frac{A+B}{4}\right)\right]\sin\left(\frac{A+B}{4}\right) \\ &= 2\sin\left(\frac{A+B}{4}\right)\cos\left(\frac{A-B}{4}\right) + 1 - 2\sin^2\left(\frac{A+B}{4}\right) \end{aligned}$$

$$\begin{aligned} &\quad \because 2\sin A \cos B = \sin(A+B) + \sin(A-B) \\ &\quad 1 - 2\sin^2 A = \cos 2A. \end{aligned}$$

$$= \sin\left(\frac{A}{4} + \frac{B}{4} + \frac{A}{4} - \frac{B}{4}\right) + \sin\left(\frac{A}{4} + \frac{B}{4} - \frac{A}{4} + \frac{B}{4}\right) + \cos 2\left(\frac{A+B}{4}\right)$$

$$= \sin \frac{A}{2} + \sin \frac{B}{2} + \cos \left(\frac{A+B}{2} \right)$$

$$\therefore \cos \left(\frac{A+B}{2} \right) = \cos \left(\frac{\pi - C}{2} \right) = \sin \frac{C}{2}$$

$$= \sin \frac{A}{2} + \sin \frac{B}{2} + \sin \frac{C}{2}.$$

9. If $A+B+C = \pi$, prove that

$$\cos \frac{A}{2} + \cos \frac{B}{2} + \cos \frac{C}{2} = 4 \cos \left(\frac{\pi - A}{4} \right) \cos \left(\frac{\pi - B}{4} \right) \cos \left(\frac{\pi - C}{4} \right)$$

A: Given: $A+B+C = \pi$

$$\begin{aligned} & 4 \cos \left(\frac{\pi - A}{4} \right) \cos \left(\frac{\pi - B}{4} \right) \cos \left(\frac{\pi - C}{4} \right) \\ &= 2 \left[2 \cos \left(\frac{\pi - A}{4} \right) \cos \left(\frac{\pi - B}{4} \right) \right] \cos \left(\frac{\pi - C}{4} \right) \\ &= 2 \left[\cos \left(\frac{\pi - A + \pi - B}{4} \right) + \cos \left(\frac{\pi - A - \pi + B}{4} \right) \right] \cos \left(\frac{\pi - C}{4} \right) \end{aligned}$$

$$\therefore 2 \cos A \cos B = \cos(A+B) + \cos(A-B)$$

$$\begin{aligned} &= 2 \left[\cos \left\{ \frac{\pi}{2} - \left(\frac{A+B}{4} \right) \right\} + \cos \left\{ - \left(\frac{A-B}{4} \right) \right\} \right] \cos \left(\frac{A+B}{4} \right) \\ &= 2 \left[\sin \left(\frac{A+B}{4} \right) + \cos \left(\frac{A-B}{4} \right) \right] \cos \left(\frac{A+B}{4} \right) \end{aligned}$$

$$= 2 \cos \left(\frac{A+B}{4} \right) \cos \left(\frac{A-B}{4} \right) + 2 \sin \left(\frac{A+B}{4} \right) \cos \left(\frac{A+B}{4} \right)$$

$$\therefore 2 \sin A \cos A = \sin 2A$$

$$= \cos \left(\frac{A}{4} + \frac{B}{4} + \frac{A}{4} - \frac{B}{4} \right) + \cos \left(\frac{A}{4} + \frac{B}{4} - \frac{A}{4} + \frac{B}{4} \right) + \sin 2 \left(\frac{A+B}{4} \right)$$

$$= \cos \frac{A}{2} + \cos \frac{B}{2} + \sin \left(\frac{A+B}{2} \right)$$

$$= \cos \frac{A}{2} + \cos \frac{B}{2} + \cos \frac{C}{2}.$$

$$10. \text{In triangle ABC, prove that } \cos \frac{A}{2} + \cos \frac{B}{2} - \cos \frac{C}{2} = 4 \cos \left(\frac{\pi + A}{4} \right) \cos \left(\frac{\pi + B}{4} \right) \cos \left(\frac{\pi - C}{4} \right).$$

A : Given that $A + B + C = \pi$

$$\begin{aligned} & 4 \cos \left(\frac{\pi + A}{4} \right) \cos \left(\frac{\pi + B}{4} \right) \cos \left(\frac{\pi - C}{4} \right) \\ &= 2 \left[2 \cos \left(\frac{\pi + A}{4} \right) \cos \left(\frac{\pi + B}{4} \right) \right] \cos \left(\frac{\pi - C}{4} \right) \end{aligned}$$

$$= 2 \left[\cos \left(\frac{\pi}{4} + \frac{A}{4} + \frac{\pi}{4} + \frac{B}{4} \right) + \cos \left(\frac{\pi}{4} + \frac{A}{4} - \frac{\pi}{4} - \frac{B}{4} \right) \right] \cos \left(\frac{A+B}{4} \right)$$

$$= 2 \left[\cos \left(\frac{\pi}{2} + \frac{A+B}{4} \right) + \cos \frac{A-B}{4} \right] \cos \left(\frac{A+B}{4} \right)$$

$$= 2 \left[-\sin \left(\frac{A+B}{4} \right) + \cos \left(\frac{A-B}{4} \right) \right] \cos \left(\frac{A+B}{4} \right)$$

$$= 2 \cos \left(\frac{A+B}{4} \right) \cos \left(\frac{A-B}{4} \right) - 2 \sin \left(\frac{A+B}{4} \right) \cos \left(\frac{A+B}{4} \right)$$

$$= \cos \left(\frac{A}{4} + \frac{B}{4} + \frac{A}{4} - \frac{B}{4} \right) + \cos \left(\frac{A}{4} + \frac{B}{4} - \frac{A}{4} + \frac{B}{4} \right) - \sin 2 \left(\frac{A+B}{4} \right)$$

$$= \cos \frac{A}{2} + \cos \frac{B}{2} - \sin \left(\frac{A+B}{2} \right)$$

$$= \cos \frac{A}{2} + \cos \frac{B}{2} - \cos \frac{C}{2}.$$

11. If $A + B + C = 2\pi$, then prove that
 $\sin A - \sin B + \sin C - \sin D$

$$= -4 \cos \frac{A+B}{2} \sin \frac{A+C}{2} \cos \frac{A+D}{2}$$

A : Given that $A + B + C + D = 2\pi$

L.H.S. : $\sin A - \sin B + \sin C - \sin D$

$$\begin{aligned} &= 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right) + 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right) \\ &= 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right) + 2\cos\left(\pi - \left(\frac{A+B}{2}\right)\right)\sin\left(\frac{C-D}{2}\right) \\ &= 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right) - 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{C-D}{2}\right) \\ &= 2\cos\left(\frac{A+B}{2}\right)\left[\sin\left(\frac{A-B}{2}\right) - \sin\left(\frac{C-D}{2}\right)\right] \\ &= 2\cos\left(\frac{A+B}{2}\right)\left[2\cos\left(\frac{\frac{A-B}{2} + \frac{C-D}{2}}{2}\right)\sin\left(\frac{\frac{A-B}{2} - \frac{C-D}{2}}{2}\right)\right] \\ &= 4\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A+C-(B+D)}{4}\right)\sin\left(\frac{A-B-C-D}{4}\right) \\ &= 4\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{2\pi-(B+D)-(B+D)}{4}\right)\sin\left(\frac{2\pi-(B+C)-(B+C)}{4}\right) \end{aligned}$$

($\because A + B + C + D = 2\pi$)

$$\begin{aligned} &= 4\cos\left(\frac{A+B}{2}\right)\cos\left[\frac{\pi}{2} - \left(\frac{B+D}{2}\right)\right]\sin\left[\frac{\pi}{2} - \left(\frac{B+C}{2}\right)\right] \\ &= 4\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{B+D}{2}\right)\cos\left(\frac{B+C}{2}\right) \\ &= 4\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{2\pi-(A+C)}{2}\right)\cos\left(\frac{2\pi-(A+D)}{2}\right) \\ &= 4\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A+C}{2}\right)\left(-\cos\left(\frac{A+D}{2}\right)\right) \\ &= -4\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A+C}{2}\right)\cos\left(\frac{A+D}{2}\right) \end{aligned}$$

R.H.S. Hence proved.

12. If $A + B + C + D = 2\pi$,
prove that $\cos 2A + \cos 2B + \cos 2C + \cos 2D$
 $= 4 \cos(A+B) \cos(A+C) \cos(A+D)$

A: Given that $A + B + C + D = 2\pi$

L.H.S. $\cos 2A + \cos 2B + \cos 2C + \cos 2D$

$$\begin{aligned} &\Rightarrow 2\cos(A+B)\cos(A-B) + 2\cos(C+D)\cos(C-D) \\ &\Rightarrow 2\cos(A+B)\cos(A-B) + 2\cos(2\pi-(A+B))\cos(C-D) \\ &\Rightarrow 2\cos(A+B)\cos(A-B) + 2\cos(A+B)\cos(C-D) \\ &\Rightarrow 2\cos(A+B)[\cos(A-B) + \cos(C-D)] \\ &= 2\cos(A+B)\left[2\cos\left\{\frac{(A+C)-(B+D)}{2}\right\}\right. \\ &\quad \left.\cos\left\{\frac{(A+D)-(B+C)}{2}\right\}\right] \\ &= 2\cos(A+B)\left[2\cos\left\{\frac{A+C-\{360^0-(A+C)\}}{2}\right\}\right. \\ &\quad \left.\cos\left\{\frac{A+D-\{360^0-(A+D)\}}{2}\right\}\right] \\ &= 2\cos(A+B)[2\cos(A+C-180^0)\cos(A+D-180^0)] \\ &\quad \because \cos(-\theta) = \cos \theta, \cos(180^0 - \theta) = -\cos \theta \\ &= 2\cos(A+B)[2\cos\{180^0-(A+C)\}\cos\{180^0-(A+D)\}] \\ &= 2\cos(A+B)[2\{-\cos(A+C)\}\{-\cos(A+D)\}] \\ &= 4\cos(A+B)\cos(A+C)\cos(A+D) \\ &= \text{RHS} \\ &\text{Hence proved.} \end{aligned}$$

13.If A, B, C are angles of a triangle, prove that $\cos 2A + \cos 2B + \cos 2C = -4\cos A \cos B \cos C - 1$.

A: Given $A + B + C = 180^\circ$.

$$\begin{aligned} & \cos 2A + \cos 2B + \cos 2C \\ &= 2\cos(A+B)\cos(A-B) + \cos 2C \\ &= 2\cos(180^\circ - C)\cos(A-B) + \cos 2C \\ &= -2\cos C \cos(A+B) + 2\cos^2 C - 1 \\ &= -1 - 2\cos C [\cos(A-B) - \cos C] \\ &= -1 - 2\cos C [\cos(A-B) - \cos \{180^\circ - (A+B)\}] \\ &= -1 - 2\cos C [\cos(A-B) + \cos(A+B)] \\ &= -1 - 2\cos C \cdot 2\cos A \cos B \\ &= -1 - 4\cos A \cos B \cos C. \end{aligned}$$

14.If A, B, C are angles of a triangle, prove that $\sin A + \sin B - \sin C = 4\sin \frac{A}{2} \sin \frac{B}{2} \cos \frac{C}{2}$.

A: Given $A + B + C = 180^\circ$.

$$\begin{aligned} & \sin A + \sin B - \sin C \\ &= 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) - 2\sin\frac{C}{2}\cos\frac{C}{2} \\ &= 2\cos\frac{C}{2}\cos\left(\frac{A-B}{2}\right) - 2\sin\frac{C}{2}\cos\frac{C}{2} \\ &\quad \because \sin\left(\frac{A+B}{2}\right) = \sin\left(\frac{180^\circ - C}{2}\right) = \cos\frac{C}{2} \\ &= 2\cos\frac{C}{2}\left[\cos\left(\frac{A-B}{2}\right) - \sin\frac{C}{2}\right] \\ &= 2\cos\frac{C}{2}\left[\cos\left(\frac{A-B}{2}\right) - \cos\left(\frac{A+B}{2}\right)\right] \\ &= 2\cos\frac{C}{2} \cdot 2\sin\frac{A}{2}\sin\frac{B}{2} \\ &= 4\sin\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}. \end{aligned}$$

15.If A, B, C are angles of a triangle, then prove that $\sin^2 A + \sin^2 B - \sin^2 C = 2\sin A \sin B \cos C$.

A: Given $A + B + C = 180^\circ$.

$$\begin{aligned} & \sin^2 A + \sin^2 B - \sin^2 C \\ &= (1 - \cos^2 A) + \sin^2 B - \sin^2 C \\ &= 1 - (\cos^2 A - \sin^2 B) - \sin^2 C \\ &= 1 - \cos(A+B)\cos(A-B) - \sin^2 C \\ &\quad \therefore \cos^2 A - \sin^2 B = \cos(A+B)\cos(A-B) \\ &= -\cos(180^\circ - C)\cos(A-B) + (1 - \cos^2 C) \\ &= 1 + \cos C \cos(A-B) + \cos^2 C \\ &\quad \therefore \cos C (180^\circ - \theta) = -\cos \theta. \\ &= \cos C [\cos(A-B) + \cos C] \\ &= \cos C [\cos(A-B) + \cos \{180^\circ - (A+B)\}] \\ &= \cos C [\cos(A-B) - \cos(A+B)] \\ &= \cos C \cdot 2\sin A \sin B. \\ &\quad \therefore \cos(A-B) - \cos(A+B) = 2\sin A \sin B. \\ &= 4\sin A \sin B \cos C. \end{aligned}$$

16.If A, B, C are the angles of a triangle, then prove that $\sin^2 \frac{A}{2} + \sin^2 \frac{B}{2} - \sin^2 \frac{C}{2} = 1 - 2\cos \frac{A}{2} \cos \frac{B}{2} \sin \frac{C}{2}$

$$\begin{aligned} & \sin^2 \frac{A}{2} + \sin^2 \frac{B}{2} - \sin^2 \frac{C}{2} \\ &= 1 - \cos^2 \frac{A}{2} + \sin^2 \frac{B}{2} - \sin^2 \frac{C}{2} \\ &= 1 - \left(\cos^2 \frac{A}{2} - \sin^2 \frac{B}{2}\right) - \sin^2 \frac{C}{2} \\ &= 1 - \cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) + \sin^2 \frac{C}{2} \\ &= 1 - \sin \frac{C}{2} \left[\cos\left(\frac{A-B}{2}\right) + \sin \frac{C}{2}\right] \\ &= 1 - \sin \frac{C}{2} \left[\cos\left(\frac{A-B}{2}\right) + \cos\left(\frac{A+B}{2}\right)\right] \\ &= 1 - 2\sin \frac{C}{2} \cos \frac{A}{2} \cos \frac{B}{2} \\ &= 1 - 4\cos \frac{A}{2} \cos \frac{B}{2} \sin \frac{C}{2}. \end{aligned}$$

17. If $A + B + C = 0^\circ$, then prove that

$$\cos^2 A + \cos^2 B + \cos^2 C = 1 + 2\cos A \cos B \cos C.$$

A: Given $A + B + C = 0^\circ$.

$$\cos^2 A + \cos^2 B + \cos^2 C.$$

$$= \cos^2 A + (1 - \sin^2 B) + \cos^2 C.$$

$$= 1 + (\cos^2 A - \sin^2 B) + \cos^2 C$$

$$= 1 + \cos(A + B) \cos(A - B) + \cos^2 C$$

$$\therefore \cos^2 A - \sin^2 B = \cos(A + B) \cos(A - B).$$

$$= 1 + \cos(-C) \cos(A - B) + \cos^2 C.$$

$$= 1 + \cos C \cos(A - B) + \cos^2 C.$$

$$\therefore \cos(-\theta) = \cos\theta.$$

$$= 1 + \cos C [\cos(A - B) + \cos C].$$

$$= 1 + \cos C \{ \cos(A - B) + \cos(-(A + B)) \}.$$

$$= 1 + \cos C [\cos(A - B) + \cos(A + B)]$$

$$= 1 + \cos C [2\cos A \cos B].$$

$$= 1 + 2 \cos A \cos B \cos C.$$

18. If $A + B + C = \frac{3\pi}{2}$, then prove that

$$\sin 2A + \sin 2B - \sin 2C = -4 \sin A \sin B \cos C.$$

A: Given $A + B + C = \frac{3\pi}{2}$.

$$\sin 2A + \sin 2B - \sin 2C$$

$$= 2 \sin(A + B) \cos(A - B) - 2 \sin C \cos C.$$

$$= 2 \sin(270^\circ - C) \cos(A - B) - 2 \sin C \cos C.$$

$$= -2 \cos C \cos(A - B) - 2 \sin C \cos C.$$

$$\therefore \sin(270^\circ - \theta) = -\cos\theta.$$

$$= -2 \cos C [\cos(A - B) + \sin C].$$

$$= -2 \cos C \{ \cos(A - B) + \sin(270^\circ - (A + B)) \}.$$

$$= -2 \cos C [\cos(A - B) - \cos(A + B)].$$

$$= -2 \cos C \cdot 2 \sin A \sin B.$$

$$= -4 \sin A \sin B \cos C.$$

19. If $A+B+C = 2S$, then prove that

$$\sin(S - A) + \sin(S - B) + \sin C = 4 \cos\left(\frac{S-A}{2}\right) \cos\left(\frac{S-B}{2}\right) \sin\frac{C}{2}.$$

A: Given: $A+B+C = 2S$

$$\text{Now } \sin(S - A) + \sin(S - B) + \sin C$$

$$= 2 \sin\left(\frac{S-A+S-B}{2}\right) \cos\left(\frac{S-A-S+B}{2}\right) + \sin C$$

$$= 2 \sin\frac{C}{2} \cos\left(\frac{-A+B}{2}\right) + 2 \sin\frac{C}{2} \cos\frac{C}{2}$$

$$= 2 \sin\frac{C}{2} \left[\cos\left(\frac{-A+B}{2}\right) + \cos\frac{C}{2} \right]$$

$$= 2 \sin\frac{C}{2} \left[\cos\left(\frac{2S-A-B}{2}\right) + \cos\left(\frac{-A+B}{2}\right) \right]$$

$$= 2 \sin\frac{C}{2} \cdot 2 \cos\left(\frac{2S-A-B-A+B}{4}\right) \cos\left(\frac{2S-A-B+A-B}{4}\right)$$

$$= 4 \sin\frac{C}{2} \cos\left(\frac{S-A}{2}\right) \cos\left(\frac{S-B}{2}\right)$$

$$= 4 \cos\left(\frac{S-A}{2}\right) \cos\left(\frac{S-B}{2}\right) \sin\frac{C}{2}.$$

- 1. Find the shortest distance between the skew lines $\bar{r} = 6\bar{i} + 2\bar{j} + 2\bar{k} + t(\bar{i} - 2\bar{j} + 2\bar{k})$ and $\bar{r} = -4\bar{i} - \bar{k} + s(3\bar{i} - 2\bar{j} - 2\bar{k})$.**

Given two skew lines are

$$\bar{r} = 6\bar{i} + 2\bar{j} + 2\bar{k} + t(\bar{i} - 2\bar{j} + 2\bar{k}),$$

$$\bar{r} = -4\bar{i} - \bar{k} + s(3\bar{i} - 2\bar{j} - 2\bar{k})$$

We know that the shortest distance between the skew lines $\bar{r} = \bar{a} + t\bar{b}$ and $\bar{r} = \bar{c} + s\bar{d}$ is

$$\frac{|[\bar{a} - \bar{c} \quad \bar{b} \quad \bar{d}]|}{|\bar{b} \times \bar{d}|}$$

$$\text{Here } \bar{a} = 6\bar{i} + 2\bar{j} + 2\bar{k}, \quad \bar{b} = \bar{i} - 2\bar{j} + 2\bar{k},$$

$$\bar{c} = -4\bar{i} - \bar{k} \quad \bar{d} = 3\bar{i} - 2\bar{j} - 2\bar{k}.$$

$$\bar{a} - \bar{c} = 10\bar{i} + 2\bar{j} + 3\bar{k}$$

$$[\bar{a} - \bar{c} \quad \bar{b} \quad \bar{d}] = \begin{vmatrix} 10 & 2 & 3 \\ 1 & -2 & 2 \\ 3 & -2 & -2 \end{vmatrix} = 10(4+4)-2(-2-6)+3(-2+6) = 80 + 16 + 12 = 108$$

$$\bar{b} \times \bar{d} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 1 & -2 & 2 \\ 3 & -2 & -2 \end{vmatrix} = \bar{i}(4+4) - \bar{j}(-2-6) + \bar{k}(-2+6) = 8\bar{i} + 8\bar{j} + 4\bar{k} = 4(2\bar{i} + 2\bar{j} + \bar{k})$$

$$\bar{b} \times \bar{d} = 4\sqrt{4+4+1} = 4(3) = 12$$

∴ Shortest distance between the skew lines

$$= \frac{|108|}{12} = 9 \text{ units.}$$

- 2. If $A = (1, -2, -1)$, $B = (4, 0, -3)$, $C = (1, 2, -1)$ and $D = (2, -4, -5)$, find the shortest distance between the lines AB and CD.**

A: Given points are A (1, -2, -1) B (4, 0, -3) C(1, 2, -1) D (2, -4, -5).

Equation of the line passing through the points A (1, -2, -1) and B (4, 0, -3) is

$$\begin{aligned} \bar{r} &= \bar{i} - 2\bar{j} - \bar{k} + t[4\bar{i} + 0\bar{j} - 3\bar{k} - (\bar{i} - 2\bar{j} - \bar{k})] \\ &= \bar{i} - 2\bar{j} - \bar{k} + t(3\bar{i} + 2\bar{j} - 2\bar{k}) \end{aligned}$$

comparing this with $\bar{r} = \bar{a} + t\bar{b}$

$$\bar{a} = \bar{i} - 2\bar{j} - \bar{k}, \quad \bar{b} = 3\bar{i} + 2\bar{j} - 2\bar{k}$$

Equation of the line passing through the points C(1, 2, -1) and D (2, -4, -5) is

$$\begin{aligned} \bar{r} &= \bar{i} + 2\bar{j} - \bar{k} + s[(2\bar{i} - 4\bar{j} - 5\bar{k}) - (\bar{i} + 2\bar{j} - \bar{k})] \\ &= \bar{i} + 2\bar{j} - \bar{k} + s(\bar{i} - 6\bar{j} - 4\bar{k}) \end{aligned}$$

Comparing with $\bar{r} = \bar{c} + s\bar{d}$

$$\bar{c} = \bar{i} + 2\bar{j} - \bar{k} \quad \bar{d} = \bar{i} - 6\bar{j} - 4\bar{k}$$

Shortest distance between the given skew lines is

$$\frac{|[\bar{a} - \bar{c} \quad \bar{b} \quad \bar{d}]|}{|\bar{b} \times \bar{d}|}$$

$$\bar{a} - \bar{c} = -4\bar{j}$$

$$[\bar{a} - \bar{c} \quad \bar{b} \quad \bar{d}] = \begin{vmatrix} 0 & -4 & 0 \\ 3 & 2 & -2 \\ 1 & -6 & -4 \end{vmatrix} = 0 + 4(-12 + 2) + 0 = 40$$

$$\Rightarrow |[\bar{a} - \bar{c} \quad \bar{b} \quad \bar{d}]| = |40| = 40$$

$$\begin{aligned} \bar{b} \times \bar{d} &= \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 3 & 2 & -2 \\ 1 & -6 & -4 \end{vmatrix} \\ &= \bar{i}(-8 - 12) - \bar{j}(-12 + 2) + \bar{k}(-18 - 2) \\ &= -20\bar{i} + 10\bar{j} - 20\bar{k} \\ &= -10(2\bar{i} - \bar{j} + 2\bar{k}) \end{aligned}$$

$$|\bar{b} \times \bar{d}| = 10\sqrt{2^2 + (-1)^2 + 2^2} = 10 \times 3 = 30$$

Shortest distance between the skew

$$\text{lines} = \frac{40}{30} = \frac{4}{3} \text{ units.}$$

3. For any vectors \bar{a} , \bar{b} , \bar{c} prove that

$$\text{i) } (\bar{a} \times \bar{b}) \times \bar{c} = (\bar{a} \cdot \bar{c}) \bar{b} - (\bar{b} \cdot \bar{c}) \bar{a}$$

$$\text{ii) } \bar{a} \times (\bar{b} \times \bar{c}) = (\bar{a} \cdot \bar{c}) \bar{b} - (\bar{a} \cdot \bar{b}) \bar{c}.$$

A: **Part1:** To show that $(\bar{a} \times \bar{b}) \times \bar{c} = (\bar{a} \cdot \bar{c}) \bar{b} - (\bar{b} \cdot \bar{c}) \bar{a}$

Suppose that

i) \bar{a} , \bar{b} , \bar{c} are non-zero vectors.

ii) \bar{a} is not parallel to \bar{b} .

iii) \bar{c} is not perpendicular to the plane containing \bar{a} , \bar{b} .

Taking \bar{a} , \bar{b} , \bar{c} satisfying the above, as follows:

$$\bar{a} = a_1 \bar{i}$$

$$\bar{b} = b_1 \bar{i} + b_2 \bar{j}$$

$$\bar{c} = c_1 \bar{i} + c_2 \bar{j} + c_3 \bar{k}$$

$$\bar{a} \times \bar{b} = a_1 \bar{i} \times (b_1 \bar{i} + b_2 \bar{j})$$

$$= a_1 b_1 \bar{i} \times \bar{i} + a_1 b_2 \bar{i} \times \bar{j}$$

$$= \bar{0} + a_1 b_2 \bar{k}$$

$$= a_1 b_2 \bar{k}.$$

$$\text{Now } (\bar{a} \times \bar{b}) \times \bar{c} = a_1 b_2 \bar{k} \times (c_1 \bar{i} + c_2 \bar{j} + c_3 \bar{k})$$

=

$$a_1 b_2 c_1 \bar{k} \times \bar{i} + a_1 b_2 c_2 \bar{k} \times \bar{j} + a_1 b_2 c_3 \bar{k} \times \bar{k}$$

$$= a_1 b_2 c_1 \bar{j} - a_1 b_2 c_2 \bar{i} \quad \dots \dots \dots (1)$$

$$\therefore \bar{k} \times \bar{k} = \bar{0}$$

$$\bar{a} \cdot \bar{c} = a_1 \bar{i} \cdot (c_1 \bar{i} + c_2 \bar{j} + c_3 \bar{k})$$

$$= a_1 c_1$$

$$\bar{b} \cdot \bar{c} = (b_1 \bar{i} + b_2 \bar{j}) \cdot (c_1 \bar{i} + c_2 \bar{j} + c_3 \bar{k})$$

$$= b_1 c_1 + b_2 c_2$$

$$\text{Now } (\bar{a} \cdot \bar{c}) \bar{b} - (\bar{b} \cdot \bar{c}) \bar{a}$$

$$= a_1 c_1 (b_1 \bar{i} + b_2 \bar{j}) - (b_1 c_1 + b_2 c_2) a_1 \bar{i}$$

$$= a_1 b_1 c_1 \bar{i} + a_1 b_2 c_1 \bar{j} - a_1 b_1 c_1 \bar{i} - a_1 b_2 c_1 \bar{i}$$

$$= a_1 b_2 c_1 \bar{j} - a_1 b_2 c_2 \bar{i} \quad \dots \dots \dots (2)$$

From (1) and (2)

$$(\bar{a} \times \bar{b}) \times \bar{c} = (\bar{a} \cdot \bar{c}) \bar{b} - (\bar{b} \cdot \bar{c}) \bar{a}.$$

Part 2: To prove that $\bar{a} \times (\bar{b} \times \bar{c}) = (\bar{a} \cdot \bar{c}) \bar{b} - (\bar{a} \cdot \bar{b}) \bar{c}$

$$\text{Now } \bar{a} \times (\bar{b} \times \bar{c}) = -(\bar{b} \times \bar{c}) \times \bar{a}$$

$$= -\{(\bar{b} \cdot \bar{a}) \bar{c} - (\bar{c} \cdot \bar{a}) \bar{b}\} \text{ from part 1.}$$

$$= (\bar{c} \cdot \bar{a}) \bar{b} - (\bar{a} \cdot \bar{b}) \bar{c}$$

4. If $\bar{a} = \bar{i} - 2\bar{j} + \bar{k}$, $\bar{b} = 2\bar{i} + \bar{j} + \bar{k}$, $\bar{c} = \bar{i} + 2\bar{j} - \bar{k}$,

find $\bar{a} \times (\bar{b} \times \bar{c})$ and $|(\bar{a} \times \bar{b}) \times \bar{c}|$.

A: Given

$$\bar{a} = \bar{i} - 2\bar{j} + \bar{k}, \bar{b} = 2\bar{i} + \bar{j} + \bar{k}, \bar{c} = \bar{i} + 2\bar{j} - \bar{k}$$

$$\bar{a} \times (\bar{b} \times \bar{c}) = (\bar{a} \cdot \bar{c}) \bar{b} - (\bar{a} \cdot \bar{b}) \bar{c}$$

$$= \{(\bar{i} - 2\bar{j} + \bar{k}) \cdot (\bar{i} + 2\bar{j} - \bar{k})\} \bar{b} -$$

$$\{(\bar{i} - 2\bar{j} + \bar{k}) \cdot (2\bar{i} + \bar{j} + \bar{k})\} \bar{c}$$

$$= (1 - 4 - 1) (2\bar{i} + \bar{j} + \bar{k}) - (2 - 2 + 1) (\bar{i} + 2\bar{j} - \bar{k})$$

$$= (-4) (2\bar{i} + \bar{j} + \bar{k}) - 1 (\bar{i} + 2\bar{j} - \bar{k})$$

$$= -8\bar{i} - 4\bar{j} - 4\bar{k} - \bar{i} - 2\bar{j} + \bar{k}$$

$$= -9\bar{i} - 6\bar{j} - 3\bar{k}$$

$$(\bar{a} \times \bar{b}) \times \bar{c} = (\bar{a} \cdot \bar{c}) \bar{b} - (\bar{b} \cdot \bar{c}) \bar{a}$$

$$= (1 - 4 - 1) (2\bar{i} + \bar{j} + \bar{k}) - (2 + 2 -$$

$$1) (\bar{i} - 2\bar{j} + \bar{k})$$

$$= -8\bar{i} - 4\bar{j} - 4\bar{k} - 3\bar{i} + 6\bar{j} - 3\bar{k}$$

$$= -11\bar{i} + 2\bar{j} - 7\bar{k}$$

$$\therefore |(\bar{a} \times \bar{b}) \times \bar{c}| = \sqrt{121 + 4 + 49}$$

$$= \sqrt{174}$$

5. If $\bar{a} = 2\bar{i} + \bar{j} - 3\bar{k}$, $\bar{b} = \bar{i} - 2\bar{j} + \bar{k}$,
 $\bar{c} = -\bar{i} + \bar{j} - 4\bar{k}$ and $\bar{d} = \bar{i} + \bar{j} + \bar{k}$ then
compute $[(\bar{a} \times \bar{b}) \times (\bar{c} \times \bar{d})]$.

A: We know that $(\bar{a} \times \bar{b}) \times (\bar{c} \times \bar{d})$

$$= [\bar{a} \bar{c} \bar{d}] \bar{b} - [\bar{b} \bar{c} \bar{d}] \bar{a}$$

$$[\bar{a} \bar{c} \bar{d}] = \begin{vmatrix} 2 & 1 & -3 \\ -1 & 1 & -4 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 2(1+4) - 1(-1+4) - 3(-1-1) \\ = 10 - 3 + 6 = 13$$

$$[\bar{b} \bar{c} \bar{d}] = \begin{vmatrix} 1 & -2 & 1 \\ -1 & 1 & -4 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 1(1+4) + 2(-1+4) + 1(-1-1) \\ = 5 + 6 - 2 \\ = 9$$

$$\therefore (\bar{a} \times \bar{b}) \times (\bar{c} \times \bar{d})$$

$$= [\bar{a} \bar{c} \bar{d}] \bar{b} - [\bar{b} \bar{c} \bar{d}] \bar{a} \\ = 13(\bar{i} - 2\bar{j} + \bar{k}) - 9(2\bar{i} + \bar{j} - 3\bar{k}) \\ = 13\bar{i} - 26\bar{j} + 13\bar{k} - 18\bar{i} - 9\bar{j} + 27\bar{k} \\ = -5\bar{i} - 35\bar{j} + 40\bar{k} \\ = 5(-\bar{i} - 7\bar{j} + 8\bar{k})$$

$$|(\bar{a} \times \bar{b}) \times (\bar{c} \times \bar{d})| = 5\sqrt{1+49+64} \\ = 5\sqrt{117}$$

6. If $\bar{a} = 2\bar{i} + 3\bar{j} + 4\bar{k}$, $\bar{b} = \bar{i} + \bar{j} - \bar{k}$ and
 $\bar{c} = \bar{i} - \bar{j} + \bar{k}$ then compute $\bar{a} \times (\bar{b} \times \bar{c})$ and
verify that it is perpendicular to \bar{a} .

A: Given vectors are $\bar{a} = 2\bar{i} + 3\bar{j} + 4\bar{k}$

$$\bar{b} = \bar{i} + \bar{j} - \bar{k}$$

$$\bar{c} = \bar{i} - \bar{j} + \bar{k}$$

$$\text{Consider } \bar{b} \times \bar{c} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{vmatrix} \\ = \bar{i}(1-1) - \bar{j}(1-1) + \bar{k}(-1-1)$$

$$= -2\bar{j} - 2\bar{k}$$

$$\text{Now } \bar{a} \times (\bar{b} \times \bar{c}) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 2 & 3 & 4 \\ 0 & -2 & -2 \end{vmatrix}$$

$$= \bar{i}(-6+8) - \bar{j}(-4-0) + \bar{k}(-4-0)$$

$$= 2\bar{i} + 4\bar{j} - 4\bar{k}$$

Now consider

$$[\bar{a} \times (\bar{b} \times \bar{c})] \cdot \bar{a} = (2\bar{i} + 3\bar{j} + 4\bar{k}) \cdot (2\bar{i} + 3\bar{j} + 4\bar{k}) \\ = 4 + 12 - 16 \\ = 16 - 16 = 0$$

$$\Rightarrow [\bar{a} \times (\bar{b} \times \bar{c})] \cdot \bar{a} = 0$$

$\Rightarrow \bar{a} \times (\bar{b} \times \bar{c})$ is perpendicular to \bar{a} .

Hence proved.

7. If $\bar{a} = \bar{i} - 2\bar{j} - 3\bar{k}$, $\bar{b} = 2\bar{i} + \bar{j} - \bar{k}$ and $\bar{c} = \bar{i} + 3\bar{j} - 2\bar{k}$, verify that $\bar{a} \times (\bar{b} \times \bar{c}) \neq (\bar{a} \times \bar{b}) \times \bar{c}$.

A: Given :

$$\begin{aligned}\bar{a} &= \bar{i} - 2\bar{j} - 3\bar{k}, \bar{b} = 2\bar{i} + \bar{j} - \bar{k} \text{ and } \bar{c} = \bar{i} + 3\bar{j} - 2\bar{k} \\ \bar{a} \times (\bar{b} \times \bar{c}) &= (\bar{a} \cdot \bar{c}) \bar{b} - (\bar{a} \cdot \bar{b}) \bar{c} \\ &= \{(\bar{i} - 2\bar{j} - 3\bar{k}) \cdot (\bar{i} + 3\bar{j} - 2\bar{k})\} (2\bar{i} + \bar{j} - \bar{k}) \\ &\quad - \{(\bar{i} - 2\bar{j} - 3\bar{k}) \cdot (2\bar{i} + \bar{j} - \bar{k})\} (\bar{i} + 3\bar{j} - 2\bar{k}) \\ &= \{(1(1)-2(3)-3(-2)) (2\bar{i} + \bar{j} - \bar{k}) \\ &\quad - \{1(2)-2(1)-3(1)\} (\bar{i} + 3\bar{j} - 2\bar{k}) \\ &= (1-6+6)(2\bar{i} + \bar{j} - \bar{k}) - (2-2+3)(\bar{i} + 3\bar{j} - 2\bar{k}) \\ &= 2\bar{i} + \bar{j} - \bar{k} - 3\bar{i} - 9\bar{j} + 6\bar{k} \\ &= -\bar{i} - 8\bar{j} + 5\bar{k} \dots\dots\dots(1) \\ (\bar{a} \times \bar{b}) \times \bar{c} &= (\bar{a} \cdot \bar{c}) \bar{b} - (\bar{b} \cdot \bar{c}) \bar{a} \\ &= \{(1(1)-2(3)-3(-2)) (2\bar{i} + \bar{j} - \bar{k}) \\ &\quad - \{2(1) + 1(3) - 1(-2)\} (\bar{i} - 2\bar{j} - 3\bar{k}) \\ &= (1-6+6)(2\bar{i} + \bar{j} - \bar{k}) - (2+3+2)(\bar{i} - 2\bar{j} - 3\bar{k}) \\ &= 2\bar{i} + \bar{j} - \bar{k} - 7\bar{i} + 14\bar{j} + 21\bar{k} \\ &= 5\bar{i} + 15\bar{j} + 20\bar{k} \dots\dots\dots(2)\end{aligned}$$

From (1) and (2)

$$\bar{a} \times (\bar{b} \times \bar{c}) \neq (\bar{a} \times \bar{b}) \times \bar{c}.$$

8. Find the cartesian equation of the plane passing through the points A(2, 3, -1), B(4, 5, 2) and C(3, 6, 5).

A: Let O be any origin.

Let $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ be the position vector of any point in the plane of ΔABC .

$$\begin{aligned}\bar{AP} &= \bar{OP} - \bar{OA} \\ &= (x-2)\bar{i} (y-3)\bar{j} + (z+1)\bar{k} \\ \bar{AB} &= \bar{OB} - \bar{OA} \\ &= 2\bar{i} + 2\bar{j} + 3\bar{k} \\ \bar{AC} &= \bar{OC} - \bar{OA} \\ &= \bar{i} + 3\bar{j} + 6\bar{k}\end{aligned}$$

The vectors \bar{AP} , \bar{AB} , \bar{AC} are coplanar.

$$\Rightarrow [\bar{AP} \bar{AB} \bar{AC}] = 0$$

$$\Rightarrow \begin{vmatrix} x-2 & y-3 & z+1 \\ 2 & 2 & 3 \\ 1 & 3 & 6 \end{vmatrix} = 0$$

$$\begin{aligned}\Rightarrow (x-2)(12-9) - (y-3)(12-3) + (z+1)(6-2) &= 0 \\ \Rightarrow 3x - 6 - 9y + 27 + 4z + 4 &= 0 \\ \Rightarrow 3x - 9y + 4z + 25 &= 0.\end{aligned}$$

9. Find the equation of the plane passing through the point A = (3, -2, -1) and parallel to the vector $\bar{b} = \bar{i} - 2\bar{j} + 4\bar{k}$ and $\bar{c} = 3\bar{i} + 2\bar{j} - 5\bar{k}$.

A: Let P be any point on the plane with position vector $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$.

Vector equation of the plane passing through the point A = (3, -2, -1) and parallel to the vectors

$$\bar{b} = \bar{i} - 2\bar{j} + 4\bar{k} \text{ and } \bar{c} = 3\bar{i} + 2\bar{j} - 5\bar{k} \text{ is}$$

$$[\bar{AP} \bar{b} \bar{c}] = 0$$

Its cartesian equation is

$$\begin{vmatrix} x-3 & y+2 & z+1 \\ 1 & -2 & 4 \\ 3 & 2 & -5 \end{vmatrix} = 0$$

$$\begin{aligned}\Rightarrow (x-3)(10-8) - (y+2)(-5-12) + (z+1)(2+6) &= 0 \\ \Rightarrow 2(x-3) + 17(y+2) + 8(z+1) &= 0 \\ \Rightarrow 2x - 6 + 17y + 34 + 8z + 8 &= 0 \\ \Rightarrow 2x + 17y + 8z + 36 &= 0.\end{aligned}$$

10. Show that the points (5,-1,1) (7, -4, 7), (1, - 6, 10) and (- 1, - 3, 4) are the vertices of a rhombus by vectors.

A: Let O be the origin and ABCD be the given figure.

$$\therefore \overline{OA} = 5\vec{i} - \vec{j} + \vec{k}, \overline{OB} = 7\vec{i} - 4\vec{j} + 7\vec{k}$$

$$\overline{OC} = \vec{i} - 6\vec{j} + 10\vec{k}, \overline{OD} = -\vec{i} - 3\vec{j} + 4\vec{k}$$

$$\text{Now } \overline{AB} = \overline{OB} - \overline{OA} = 2\vec{i} - 3\vec{j} + 6\vec{k}$$

$$|\overline{AB}| = \sqrt{4 + 9 + 36} = 7$$

$$\overline{BC} = \overline{OC} - \overline{OB} = -6\vec{i} - 2\vec{j} + 3\vec{k}$$

$$|\overline{BC}| = \sqrt{36 + 4 + 9} = 7$$

$$\overline{CD} = \overline{OD} - \overline{OC} = -2\vec{i} + 3\vec{j} - 6\vec{k}$$

$$|\overline{CD}| = \sqrt{4 + 9 + 36} = 7$$

$$\overline{DA} = \overline{OA} - \overline{OD} = 6\vec{i} + 2\vec{j} - 3\vec{k}$$

$$|\overline{DA}| = \sqrt{36 + 4 + 9} = 7$$

$$\therefore AB = BC = CD = DA$$

$$\overline{AC} = \overline{OC} - \overline{OA} = -4\vec{i} - 5\vec{j} + 9\vec{k}$$

$$|\overline{AC}| = \sqrt{16 + 25 + 81} = \sqrt{122}$$

$$\overline{BD} = \overline{OD} - \overline{OB} = -8\vec{i} + \vec{j} - 3\vec{k}$$

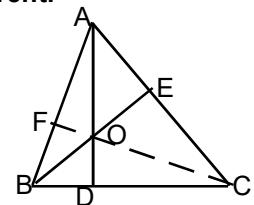
$$|\overline{BD}| = \sqrt{64 + 1 + 9} = \sqrt{74}$$

$$\therefore AC \neq BD$$

Hence the given points are the vertices of a rhombus.

11. Using scalar product, prove that the altitudes of a triangle are concurrent.

A:



Let ABC be the triangle

Let the altitudes AD and BE intersect at O.

Join C to O and extend it to CF.

To prove that altitudes are concurrent, it is enough to prove that $CF \perp AB$.

Let O be the origin and $\vec{a}, \vec{b}, \vec{c}$ be the position vectors of A, B, C respectively.

Now $AD \perp BC$ and A, O, D are collinear.

$$\Rightarrow \overline{OA} \perp \overline{BC}$$

$$\Rightarrow \overline{OA} \cdot \overline{BC} = 0$$

$$\Rightarrow \overline{OA} \cdot (\overline{OC} - \overline{OB}) = 0$$

$$\Rightarrow \vec{a} \cdot (\vec{c} - \vec{b}) = 0$$

$$\Rightarrow \vec{a} \cdot \vec{c} - \vec{a} \cdot \vec{b} = 0 \dots\dots\dots(1)$$

Also $BE \perp CA$ and B, O, E are collinear.

$$\Rightarrow \overline{OB} \perp \overline{CA}$$

$$\Rightarrow \overline{OB} \cdot \overline{CA} = 0$$

$$\Rightarrow \overline{OB} \cdot (\overline{OA} - \overline{OC}) = 0$$

$$\Rightarrow \vec{b} \cdot (\vec{a} - \vec{c}) = 0$$

$$\Rightarrow \vec{b} \cdot \vec{a} - \vec{b} \cdot \vec{c} = 0 \dots\dots\dots(2)$$

$$(1) \text{ and } (2) \quad \vec{a} \cdot \vec{c} - \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{b} - \vec{b} \cdot \vec{c} = 0$$

$$\Rightarrow (\vec{a} - \vec{b}) \cdot \vec{c} = 0$$

$$\Rightarrow (\overline{OA} - \overline{OB}) \cdot \overline{OC} = 0$$

$$\Rightarrow \overline{BA} \cdot \overline{OC} = 0$$

$$BA \perp OC$$

$$\Rightarrow AB \perp CF$$

Hence the altitudes of a triangle are concurrent.

1. If $a = 13, b = 14, c = 15$, S.T $R = \frac{65}{8}, r = 4, r_1 = \frac{21}{2}, r_2 = 12, r_3 = 14$

sol: Given $a = 13, b = 14, c = 15$

$$S = \frac{a+b+c}{2} = \frac{13+14+15}{2} = \frac{42}{2} = 21$$

$$S = \frac{a+b+c}{2}$$

$$\begin{aligned}\Delta &= \sqrt{(s)(s-a)(s-b)(s-c)} \\ &= \sqrt{(21)(21-13)(21-14)(21-15)} \\ &= \sqrt{(21)(8)(7)(6)} \\ &= \sqrt{7.3.4.2.7.3.2} \\ &= \sqrt{(7.7).(3.3).(4.4)} \\ &= \sqrt{(7.3.4)^2} = 84\end{aligned}$$

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)}$$

$$R = \frac{abc}{4\Delta} = \frac{13.14.15}{4.84} = \frac{65}{8}$$

$$R = \frac{abc}{4\Delta}$$

$$r = \frac{\Delta}{s} = \frac{84}{21} = 4$$

$$r_1 = \frac{\Delta}{(s-a)} = \frac{84}{21-13} = \frac{84}{8} = \frac{21}{2}$$

$$r_2 = \frac{\Delta}{(s-b)} = \frac{84}{21-14} = \frac{84}{7} = 12$$

$$r_3 = \frac{\Delta}{(s-c)} = \frac{84}{21-15} = \frac{84}{6} = 14$$



2. If $r=1, r_1 = 2, r_2 = 3, r_3 = 6$ find a, b, c .

Sol: given $r=1, r_1 = 2, r_2 = 3, r_3 = 6$

We know that

$$\Delta = \sqrt{r \cdot r_1 \cdot r_2 \cdot r_3}$$

$$= \sqrt{1 \cdot 2 \cdot 3 \cdot 6} = \sqrt{6 \cdot 6} = \sqrt{6^2} = 6$$

$$S = \frac{\Delta}{r}$$

$$S = \frac{6}{1} = 6$$

$$= \frac{6}{1} = 6$$

$$a = s - \frac{\Delta}{r_1}$$

$$a = 6 - \frac{6}{2} = 6 - 3 = 3$$

$$b = s - \frac{\Delta}{r_2}$$

$$b = 6 - \frac{6}{3} = 6 - 2 = 4.$$

$$c = s - \frac{\Delta}{r_3}$$

$$c = 6 - \frac{6}{6} = 6 - 1 = 5.$$



3. If $r_1 = 8, r_2 = 12, r_3 = 24$ find a, b, c .

Sol: given $r_1 = 8, r_2 = 12, r_3 = 24$

We know that $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}$

$$\frac{1}{r} = \frac{1}{8} + \frac{1}{12} + \frac{1}{24}$$

$$\frac{1}{r} = \frac{3+2+1}{24} = \frac{6}{24} = \frac{1}{4} \Rightarrow r = 4$$

$$\Delta = \sqrt{r \cdot r_1 \cdot r_2 \cdot r_3}$$

$$= \sqrt{4 \cdot 8 \cdot 12 \cdot 24} = \sqrt{4 \cdot 4 \cdot 2 \cdot 12 \cdot 12 \cdot 2}$$

$$= \sqrt{4^2 \cdot 2^2 \cdot 12^2} = 4 \cdot 2 \cdot 12 = 8 \cdot 12 = 96$$

$$\Rightarrow S = \frac{\Delta}{r}$$

$$= \frac{96}{4} = 24$$

$$a = s - \frac{\Delta}{r_1}$$

$$a = 24 - \frac{96}{8} = 24 - 12 = 12$$

$$b = s - \frac{\Delta}{r_2}$$

$$b = 24 - \frac{96}{12} = 24 - 8 = 16.$$

$$c = s - \frac{\Delta}{r_3}$$

$$c = 24 - \frac{96}{24} = 24 - 4 = 20$$



$$4. \text{ S.T } \frac{r_1}{bc} + \frac{r_1}{ca} + \frac{r_1}{ab} = \frac{1}{r} - \frac{1}{2R}$$

$$\text{Sol: } \frac{r_1}{bc} + \frac{r_1}{ca} + \frac{r_1}{ab} = \sum \frac{r_1}{bc}$$

$$a=2R \sin A$$

$$= \sum \frac{ar_1}{abc} = \sum \frac{2R \sin A \cdot s \cdot \tan \frac{A}{2}}{abc}$$

$$r_1 = s \cdot \tan \frac{A}{2}$$

$$= \frac{2RS}{abc} \sum (2 \sin \frac{A}{2} \cos \frac{A}{2}) \frac{\sin A/2}{\cos A/2}$$

$$\sin A = 2 \sin \frac{A}{2} \cos \frac{A}{2}$$

$$\tan \frac{A}{2} = \frac{\sin A/2}{\cos A/2}$$

$$= \frac{4RS}{abc} \sum (\sin^2 \frac{A}{2}) = \frac{S}{\Delta} \sum \left(\frac{1-\cos A}{2} \right)$$

$$\frac{4R}{abc} = \frac{S}{\Delta}$$

$$\frac{S}{\Delta} = \frac{1}{r}$$

$$= \frac{1}{2r} [1 - \cos A + 1 - \cos B + 1 - \cos C]$$

$$\sin^2 \frac{A}{2} = \frac{1 - \cos A}{2}$$

$$= \frac{1}{2r} [3 - (\cos A + \cos B + \cos C)]$$

$$(\cos A + \cos B + \cos C) = 1 + 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$$

$$= \frac{1}{2r} [3 - (1 + 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2})]$$

$$= \frac{1}{2r} \left[2 - \frac{4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}}{R} \right]$$

$$r = 4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$$

$$= \frac{1}{2r} \left[2 - \frac{r}{R} \right] = \frac{1}{2r} \cdot 2 - \frac{1}{2r} \cdot \frac{r}{R}$$

$$= \frac{1}{r} - \frac{1}{2R}$$



5. S.T $r + r_3 + r_1 - r_2 = 4R \cos B$

Sol: W.K.T $r = 4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$,

$$r_1 = 4R \sin \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}$$

$$r_2 = 4R \cos \frac{A}{2} \sin \frac{B}{2} \cos \frac{C}{2}$$

$$r_3 = 4R \cos \frac{A}{2} \cos \frac{B}{2} \sin \frac{C}{2}$$

$$r + r_3 = 4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} + 4R \cos \frac{A}{2} \cos \frac{B}{2} \sin \frac{C}{2}$$

$$= 4R \sin \frac{C}{2} \left\{ \sin \frac{A}{2} \sin \frac{B}{2} + \cos \frac{A}{2} \cos \frac{B}{2} \right\}$$

$$\cos \frac{A-B}{2} = \cos \frac{A}{2} \cos \frac{B}{2} + \sin \frac{A}{2} \sin \frac{B}{2}$$

$$= 4R \sin \frac{C}{2} \left\{ \cos \frac{A-B}{2} \right\} \dots\dots (1)$$

$$r_1 - r_2 = 4R \sin \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2} - 4R \cos \frac{A}{2} \sin \frac{B}{2} \cos \frac{C}{2}$$

$$= 4R \cos \frac{C}{2} \left\{ \sin \frac{A}{2} \cos \frac{B}{2} - \cos \frac{A}{2} \sin \frac{B}{2} \right\}$$

$$\sin \frac{A-B}{2} = \sin \frac{A}{2} \cos \frac{B}{2} - \cos \frac{A}{2} \sin \frac{B}{2}$$

$$= 4R \cos \frac{C}{2} \left\{ \sin \frac{A-B}{2} \right\} \dots\dots (2)$$

L.H.S $\Rightarrow r + r_3 + r_1 - r_2$

$$= 4R \sin \frac{C}{2} \left\{ \cos \frac{A+B}{2} \right\} + 4R \cos \frac{C}{2} \left\{ \sin \frac{A-B}{2} \right\}$$

$$= 4R \left[\sin \frac{C}{2} \cos \frac{A-B}{2} + \cos \frac{C}{2} \sin \frac{A-B}{2} \right]$$

$$\sin \frac{A-B}{2} = \sin \frac{A}{2} \cos \frac{B}{2} - \cos \frac{A}{2} \sin \frac{B}{2}$$

$$= 4R \left[\sin \frac{A-B+C}{2} \right]$$

$$A + B + C = \pi$$

$$= 4R \left[\sin \left(\frac{\pi}{2} - B \right) \right]$$

$$= 4R \cos B \quad \text{R.H.S}$$



6. In ΔABC , $\frac{\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2}}{\cot A + \cot B + \cot C} = \frac{(a+b+c)^2}{a^2+b^2+c^2}$

Sol:

$$\text{Consider Nr} \Rightarrow \cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2}$$

$$= \frac{s(s-a)}{\Delta} + \frac{s(s-b)}{\Delta} + \frac{s(s-c)}{\Delta} \quad \cot\left(\frac{A}{2}\right) = \frac{s(s-a)}{\Delta}$$

$$= \frac{s}{\Delta}(s-a+s-b+s-c)$$

$$= \frac{s}{\Delta}(3s-(a+b+c)) \quad a+b+c = 2s$$

$$= \frac{s}{\Delta}(3s-2s)$$

$$= \frac{s}{\Delta}(s) = \frac{s^2}{\Delta} = \frac{(a+b+c)^2}{4\Delta}$$

$$\text{Consider Dr} \Rightarrow \cot A + \cot B + \cot C$$

$$= \frac{\cos A}{\sin A} + \frac{\cos B}{\sin B} + \frac{\cos C}{\sin C}$$

$$\cos A = \frac{-a^2 + b^2 + c^2}{2bc}$$

$$= \frac{-a^2 + b^2 + c^2}{2bc \sin A} + \frac{a^2 - b^2 + c^2}{2c \sin B} + \frac{a^2 + b^2 - c^2}{2a \sin C}$$

$$= \frac{-a^2 + b^2 + c^2}{4(\frac{1}{2}bc \sin A)} + \frac{a^2 - b^2 + c^2}{4(\frac{1}{2}ac \sin B)} + \frac{a^2 + b^2 - c^2}{4(\frac{1}{2}ab \sin C)}$$

$$\Delta = \frac{1}{2}(bc \sin A)$$

$$= \frac{-a^2 + b^2 + c^2 + a^2 - b^2 + c^2 + a^2 + b^2 - c^2}{4\Delta}$$

$$= \frac{a^2 + b^2 + c^2}{4\Delta}$$

$$\text{Now Nr/Dr} \quad \frac{\frac{(a+b+c)^2}{4\Delta}}{\frac{a^2 + b^2 + c^2}{4\Delta}} = \frac{(a+b+c)^2}{a^2 + b^2 + c^2}$$

$$\therefore \frac{\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2}}{\cot A + \cot B + \cot C} = \frac{(a+b+c)^2}{a^2 + b^2 + c^2}$$



7. If P_1, P_2, P_3 are the altitudes drawn from vertices A, B, C to the opposite sides of a triangle then S.T (i) $\frac{1}{P_1} + \frac{1}{P_2} + \frac{1}{P_3} = \frac{1}{r}$
(ii) $\frac{1}{P_1} + \frac{1}{P_2} - \frac{1}{P_3} = \frac{1}{r_3}$ (iii) $P_1 \cdot P_2 \cdot P_3 = \frac{8\Delta^3}{abc} = \frac{(abc)^2}{8R^3}$

$$\text{Sol: } \Delta = \frac{1}{2} aP_1, \Delta = \frac{1}{2} bP_2, \Delta = \frac{1}{2} cP_3$$

$$\Rightarrow P_1 = \frac{2\Delta}{a}, P_2 = \frac{2\Delta}{b}, P_3 = \frac{2\Delta}{c}$$

$$(i) \frac{1}{P_1} + \frac{1}{P_2} + \frac{1}{P_3} = \frac{a}{2\Delta} + \frac{b}{2\Delta} + \frac{c}{2\Delta}$$

$$= \frac{a+b+c}{2\Delta} = \frac{2s}{2\Delta} = \frac{1}{r}$$

$$\boxed{\frac{s}{\Delta} = \frac{1}{r}}$$

$$(ii) \frac{1}{P_1} + \frac{1}{P_2} - \frac{1}{P_3} = \frac{a}{2\Delta} + \frac{b}{2\Delta} - \frac{c}{2\Delta}$$

$$= \frac{a+b-c}{2\Delta} = \frac{2s-2c}{2\Delta} = \frac{s-c}{\Delta} = \frac{1}{r_3}$$

$$(iii) P_1 \cdot P_2 \cdot P_3 = \frac{2\Delta}{a} \cdot \frac{2\Delta}{b} \cdot \frac{2\Delta}{c} = \frac{8\Delta^3}{abc}$$

$$\boxed{\Delta = \frac{abc}{4R}}$$

$$= \frac{8 \left(\frac{abc}{4R} \right)^3}{abc} = \frac{8(abc)^2}{64R^3} = \frac{(abc)^2}{8R^3}$$



$$8. S.T \frac{ab-r_1r_2}{r_3} = \frac{bc-r_2r_3}{r_1} = \frac{ab-r_3r_1}{r_2}$$

Sol: $ab - r_1r_2$

$$= (2R \sin A)(2R \sin B)$$

$$- \left(4R \sin \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2} \right) \left(4R \cos \frac{A}{2} \sin \frac{B}{2} \cos \frac{C}{2} \right)$$

$$= 4R^2 \sin A \sin B - \left[4R^2 \cos^2 \frac{C}{2} \right] \left(2 \sin \frac{A}{2} \cos \frac{B}{2} \right) \left(2 \sin \frac{B}{2} \cos \frac{C}{2} \right)$$

$$= 4R^2 \sin A \sin B - \left[4R^2 \cos^2 \frac{C}{2} \right] (\sin A)(\sin B)$$

$$= 4R^2 \sin A \sin B \left[1 - \cos^2 \frac{C}{2} \right]$$

$$= 4R^2 \sin A \sin B \left[\sin^2 \frac{C}{2} \right]$$

$$\text{Now } \frac{ab-r_1r_2}{r_3} = \frac{4R^2 \sin A \sin B \left[\sin^2 \frac{C}{2} \right]}{4R \cos \frac{A}{2} \cos \frac{B}{2} \sin \frac{C}{2}}$$

$$= \frac{4R^2 \left(2 \sin \frac{A}{2} \cos \frac{B}{2} \right) \left(2 \sin \frac{B}{2} \cos \frac{C}{2} \right) \left[\sin^2 \frac{C}{2} \right]}{4R \cos \frac{A}{2} \cos \frac{B}{2} \sin \frac{C}{2}}$$

$$= 4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} = r$$

$$\text{simmilarly } \frac{bc-r_2r_3}{r_1} = \frac{ab-r_3r_1}{r_2} = r$$



9. $a \cos^2 \frac{A}{2} + b \cos^2 \frac{B}{2} + c \cos^2 \frac{C}{2} = S + \frac{\Delta}{R}$

Sol:

$$\text{L.H.S} \Rightarrow a \cos \frac{A}{2} + b \cos \frac{B}{2} + c \cot \frac{C}{2}$$

$$\Rightarrow \sum a \cos \frac{A}{2} = \sum a \left(\frac{1+\cos A}{2} \right)$$

$$\Rightarrow \frac{1}{2} \sum (a + a \cos A)$$

$$\Rightarrow \frac{1}{2} \sum (a) + \frac{1}{2} \sum (\cos A)$$

$$\Rightarrow \frac{a+b+c}{2} + \frac{1}{2} \sum (2R \sin A \cos A)$$

$$\Rightarrow s + \frac{R}{2} \sum (\sin 2A)$$

$$\Rightarrow s + \frac{R}{2} (\sin 2A + \sin 2B + \sin 2C)$$

$$\Rightarrow s + \frac{R}{2} (4 \sin A \sin B \sin C)$$

$$\Rightarrow s + \frac{2R^2 \sin A \sin B \sin C}{R}$$

$$\Rightarrow s + \frac{\Delta}{R} \text{ R.H.S}$$



$$10. \sin^2 \frac{A}{2} + \sin^2 \frac{B}{2} + \sin^2 \frac{C}{2} = 1 - \frac{r}{2R}.$$

Sol:

$$\text{L.H.S} \Rightarrow \sin^2 \frac{A}{2} + \sin^2 \frac{B}{2} + \sin^2 \frac{C}{2}$$

$$\Rightarrow \frac{1-\cos A}{2} + \frac{1-\cos B}{2} + \frac{1-\cos C}{2}$$

$$\Rightarrow \frac{1}{2} [3 - (\cos A + \cos B + \cos C)]$$

$$\Rightarrow \frac{1}{2} [3 - (2\cos \left(\frac{A+B}{2}\right) + \cos \left(\frac{A-B}{2}\right) + \cos C)]$$

$$\Rightarrow \frac{1}{2} [3 - (2\sin \frac{C}{2} + \cos \left(\frac{A-B}{2}\right) + 1 - 2\sin^2 \frac{A}{2})]$$

$$\Rightarrow \frac{1}{2} [3 - 1 - 2\sin \frac{C}{2} \{ \cos \left(\frac{A-B}{2}\right) - \sin \frac{A}{2} \}]$$

$$\Rightarrow \frac{1}{2} [2 - 2\sin \frac{C}{2} \{ \cos \left(\frac{A-B}{2}\right) - \cos \left(\frac{A+B}{2}\right) \}]$$

$$\Rightarrow \frac{1}{2} [3 - 1 - 2\sin \frac{C}{2} \{ 2\sin \frac{A}{2} \sin \frac{B}{2} \}]$$

$$\Rightarrow \frac{1}{2} [2 - 4\sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}]$$

$$\Rightarrow \frac{1}{2} \left[2 - \frac{4R\sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}}{R} \right]$$

$$\Rightarrow \frac{1}{2} \left[2 - \frac{r}{R} \right]$$

$$= \frac{2}{2} - \frac{r}{2R}$$

$$= 1 - \frac{r}{2R}$$

R.H.S



$$\begin{aligned}
 \text{Or } & \Rightarrow \sin^2 \frac{A}{2} + \sin^2 \frac{B}{2} + \sin^2 \frac{C}{2} \\
 & \Rightarrow 1 - \cos^2 \frac{A}{2} + \sin^2 \frac{B}{2} + \sin^2 \frac{C}{2} \\
 & \Rightarrow 1 - \left\{ \cos^2 \frac{A}{2} - \sin^2 \frac{B}{2} \right\} + \sin^2 \frac{C}{2} \\
 & \Rightarrow \left[1 - \cos \left(\frac{A+B}{2} \right) \cos \left(\frac{A-B}{2} \right) + \sin^2 \frac{C}{2} \right] \\
 & \Rightarrow \left[1 - \sin \frac{C}{2} \cos \left(\frac{A-B}{2} \right) + \sin^2 \frac{C}{2} \right] \\
 & \Rightarrow \left[1 - \sin \frac{C}{2} \left\{ \cos \left(\frac{A-B}{2} \right) - \sin \frac{C}{2} \right\} \right] \\
 & \Rightarrow \left[1 - \sin \frac{C}{2} \left\{ \cos \left(\frac{A-B}{2} \right) - \cos \left(\frac{A+B}{2} \right) \right\} \right] \\
 & \Rightarrow \left[1 - \sin \frac{C}{2} \left\{ 2 \sin \frac{A}{2} \sin \frac{B}{2} \right\} \right] \\
 & \Rightarrow 1 - \frac{4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}}{2R} = 1 - \frac{r}{2R}
 \end{aligned}$$



$$11. a^3 \cos(B - C) + b^3 \cos(C - A) + c^3 \cos(A - B) = 3abc$$

$$\text{Sol: } \sum a^3 \cos(B - C) = \sum a \cdot a^2 \cos(B - C)$$

$$\begin{aligned} &= a^2 \sum 2R \sin A \cdot \cos(B - C) \\ &= a^2 \sum 2R \sin(B + C) \cdot \cos(B - C) \end{aligned}$$

$$\begin{aligned} &= Ra^2 \sum [\sin(B + C + B - C) + \sin(B + C - B + C)] \\ &= Ra^2 \sum [\sin(2B) + \sin(2C)] \end{aligned}$$

$$\begin{aligned} &= Ra^2 \sum [2\sin B \cos B + 2\sin C \cos C] \\ &= a^2 \sum [(2R \sin B) \cos B + (2R \sin C) \cos C] \end{aligned}$$

$$= a^2 \sum [(b) \cos B + (c) \cos C]$$

$$= a^2 [bc \cos B + cc \cos C] + b^2 [cc \cos C + ac \cos A] + c^2 [ac \cos A + bc \cos B]$$

$$= ab[ac \cos B + bc \cos A] + bc[bc \cos C + cc \cos B] + ca[ac \cos C + cc \cos A]$$

$$= ab[c] + bc[a] + ca[b]$$

$$= abc$$

