

6. INTEGRATION

DEFINITIONS, CONCEPTS AND FORMULAE

1. $\frac{d}{dx}(\sin^2 x) = \sin 2x \Rightarrow \sin^2 x$ is an antiderivative of $\sin 2x$.
2. If $F(x)$ is an antiderivative of $f(x)$ then $F(x) + c$, $c \in \mathbb{R}$ is called indefinite integral of $f(x)$ w.r.t x . It is denoted by $\int f(x) dx$.
3. The process of finding the integral of a function is known as INTEGRATION.
4. $\int 0 dx = c$.
5. $\int 1 dx = x + c$.
6. $\int x^n dx = \frac{x^{n+1}}{n+1} + c$ ($n \neq -1$).
7. $\int \frac{1}{x} dx = \log |x| + c$.
8. $\int \frac{1}{\sqrt{x}} dx = 2\sqrt{x} + c$.
9. $\int |x| dx = \frac{x|x|}{2} + c$.
10. $\int e^x dx = e^x + c$.
11. If $a > 0$, $a \neq 1$ then $\int a^x dx = \frac{a^x}{\log a} + c$
12. $\int \sin x dx = -\cos x + c$
13. $\int \cos x dx = \sin x + c$
14. $\int \sec^2 x dx = \tan x + c$
15. $\int \cosec^2 x dx = -\cot x + c$
16. $\int \sec x \tan x dx = \sec x + c$
17. $\int \cosec x \cot x dx = -\cosec x + c$
18. $\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + c = -\cos^{-1} x + c$
19. $\int \frac{1}{1+x^2} dx = \tan^{-1} x + c = -\cot^{-1} x + c$
20. $\int \frac{1}{x\sqrt{x^2-1}} dx = \sec^{-1} x + c = -\cosec^{-1} x + c$
21. $\int \sinh x dx = \cosh x + c$
22. $\int \cosh x dx = \sinh x + c$
23. $\int \operatorname{Sech}^2 x dx = \tanh x + c$
24. $\int \operatorname{Cosech}^2 x dx = -\coth x + c$
25. $\int \operatorname{Sech} x \tanh x dx = -\operatorname{Sec h} x + c$
26. $\int \operatorname{Cosech} x \operatorname{Cot h} x dx = -\operatorname{cosech} x + c$
27. $\int \frac{1}{\sqrt{1+x^2}} dx = \sinh^{-1} x + c = \log(x + \sqrt{x^2 + 1}) + c$
28. $\int \frac{1}{\sqrt{x^2-1}} dx = \cosh^{-1} x + c = \log(x + \sqrt{x^2 - 1}) + c$
29. $\int \frac{1}{1-x^2} dx = \tanh^{-1} x + c = \frac{1}{2} \log \left| \frac{1+x}{1-x} \right| + c$
30. $\int \frac{f'(x)}{f(x)} dx = \log |f(x)| + c$, $\int \frac{f'(x)}{[f(x)]^2} dx = \frac{-1}{f(x)} + c$
31. $\int \frac{f'(x)}{\sqrt{f(x)}} dx = 2\sqrt{f(x)} + c$
32. $\int [f(x)]^n f'(x) dx = \frac{[f(x)]^{n+1}}{n+1} + c$
33. $\int \tan x dx = \log |\sec x| + c$
34. $\int \cot x dx = \log |\sin x| + c$
35. $\int \sec x dx = \log |\sec x + \tan x| + c$
 $= \log \left| \tan \left(\frac{\pi}{4} + \frac{x}{2} \right) \right| + c$
36. $\int \cosec x dx = \log |\cosec x - \cot x| + c$
 $= \log \left| \tan \frac{x}{2} \right| + c$
37. $\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \left(\frac{x}{a} \right) + c$
38. $\int \frac{1}{\sqrt{a^2 + x^2}} dx = \sinh^{-1} \left(\frac{x}{a} \right) + c$
39. $\int \frac{1}{\sqrt{x^2 - a^2}} dx = \cosh^{-1} \left(\frac{x}{a} \right) + c$
40. $\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a} + c$
41. $\int \frac{1}{a^2 - x^2} dx = \frac{1}{2a} \log \left| \frac{a+x}{a-x} \right| + c$
42. $\int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + c$

43. $\int \sqrt{a^2 - x^2} dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1}\left(\frac{x}{a}\right) + c$

44. $\int \sqrt{a^2 + x^2} dx = \frac{x}{2} \sqrt{a^2 + x^2} + \frac{a^2}{2} \sinh^{-1}\left(\frac{x}{a}\right) + c$

45. $\int \sqrt{x^2 - a^2} dx = \frac{x}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \cosh^{-1}\left(\frac{x}{a}\right) + c$

46. Integration by parts:

$$\int f(x).g(x)dx = f(x) \int g(x)dx - \int [f'(x) \int g(x)dx] dx$$

Note: In integration by parts, the first function will be taken as in the following order: Inverse functions, logarithmic functions, algebraic functions, trigonometric functions and exponential functions. (ILATE rule)

47. $\int e^x \cos bx dx = \frac{e^{ax}}{a^2 + b^2} (a \cos bx + b \sin bx) + c$

48. $\int e^x \sin bx dx = \frac{e^{ax}}{a^2 + b^2} (a \sin bx - b \cos bx) + c$

49. $\int e^x [f(x) + f'(x)] dx = e^x f(x) + c$

50. $\int \log x dx = x \log x - x + c$

51. $\int [x f'(x) + f(x)] dx = x f(x) + c$

52. $I_n = \int \sin^n x dx = \frac{-\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} I_{n-2}$

53. $I_n = \int \cos^n x dx = \frac{+\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} I_{n-2}$

54. $I_n = \int \tan^n x dx = \frac{\tan^{n-1} x}{n-1} - I_{n-2}$

55. $I_n = \int \cot^n x dx = \frac{-\cot^{n-1} x}{n-1} - I_{n-2}$

56. $I_n = \int \sec^n x dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} I_{n-2}$

57. $I_n = \int \operatorname{cosec}^n x dx = \frac{-\operatorname{cosec}^{n-2} x \cot x}{n-1} + \frac{n-2}{n-1} I_{n-2}$

LEVEL - I (VSAQ)

1. Evaluate $\int \left(x + \frac{4}{1+x^2} \right) dx$.

Sol: $\int \left(x + \frac{4}{1+x^2} \right) dx$

$$= \int x dx + 4 \int \frac{1}{1+x^2} dx$$

$$= \frac{x^2}{2} + 4 \tan^{-1} x + c.$$

2. Evaluate $\int \left(\frac{1}{\sqrt{1-x^2}} + \frac{2}{\sqrt{1+x^2}} \right) dx$.

Sol: $\int \left(\frac{1}{\sqrt{1-x^2}} + \frac{2}{\sqrt{1+x^2}} \right) dx$
 $= \int \frac{1}{\sqrt{1-x^2}} dx + 2 \int \frac{1}{\sqrt{1+x^2}} dx$
 $= \sin^{-1} x + 2 \cdot \sinh^{-1} x + c.$

3. Find $\int \sec^2 x \operatorname{cosec}^2 x dx$

Sol: $\int \sec^2 x \operatorname{cosec}^2 x dx$
 $= \int \frac{1}{\cos^2 x \cdot \sin^2 x} dx$
 $= \int \frac{\sin^2 x + \cos^2 x}{\cos^2 x \cdot \sin^2 x} dx$
 $= \int \frac{\sin^2 x}{\cos^2 x \cdot \sin^2 x} dx + \int \frac{\cos^2 x}{\cos^2 x \cdot \sin^2 x} dx$
 $= \int \frac{1}{\cos^2 x} dx + \int \frac{1}{\sin^2 x} dx$
 $= \int \sec^2 x dx + \int \operatorname{cosec}^2 x dx$
 $= \tan x - \cot x + c.$

4. Find $\int \frac{(a^x - b^x)^2}{a^x b^x} dx$.

Sol: $\int \frac{(a^x - b^x)^2}{a^x b^x} dx$
 $= \int \left[\frac{(a^x)^2 - 2a^x b^x + (b^x)^2}{a^x b^x} \right] dx$
 $= \int \left[\frac{(a^x)^2}{a^x b^x} - \frac{2a^x b^x}{a^x b^x} + \frac{(b^x)^2}{a^x b^x} \right] dx$
 $= \int \left[\frac{a^x}{b^x} - 2 + \frac{b^x}{a^x} \right] dx$
 $= \int \left(\frac{a}{b} \right)^x dx - 2 \int 1 dx + \int \left(\frac{b}{a} \right)^x dx$
 $= \frac{\left(\frac{a}{b} \right)^x}{\log \left(\frac{a}{b} \right)} - 2x + \frac{\left(\frac{b}{a} \right)^x}{\log \left(\frac{b}{a} \right)} + c.$

5. Evaluate $\int \frac{1+\cos^2 x}{1-\cos 2x} dx$.

Sol:
$$\begin{aligned} & \int \frac{1+\cos^2 x}{1-\cos 2x} dx \\ &= \int \frac{1+\cos^2 x}{2\sin^2 x} dx \\ &= \frac{1}{2} \int \left(\frac{1}{\sin^2 x} + \frac{\cos^2 x}{\sin^2 x} \right) dx \\ &= \frac{1}{2} \int (\csc^2 x + \cot^2 x) dx \\ &= \int (\csc^2 x + \csc^2 x - 1) dx \\ &= \frac{1}{2} \left[\int 2\csc^2 x - \int 1 dx \right] \\ &= \frac{1}{2} [-2\cot x - x] + c \\ &= -\cot x - \frac{x}{2} + c. \end{aligned}$$

6. Find $\int \sqrt{1-\cos 2x} dx$.

Sol:
$$\begin{aligned} & \int \sqrt{1-\cos 2x} dx \\ &= \int \sqrt{2\sin^2 x} dx \\ &= \sqrt{2} \int \sin x dx \\ &= -\sqrt{2} \cos x + c. \end{aligned}$$

7. Evaluate $\int \frac{1}{1+\cos x} dx$.

Sol:
$$\begin{aligned} & \int \frac{1}{1+\cos x} dx \\ &= \int \frac{1}{1+\cos x} \times \frac{1-\cos x}{1-\cos x} dx \\ &= \int \frac{1-\cos x}{1-\cos^2 x} dx = \int \frac{1-\cos x}{\sin^2 x} dx \\ &= \int \left(\frac{1}{\sin^2 x} - \frac{\cos x}{\sin^2 x} \right) dx \\ &= \int (\csc^2 x - \cot x \csc x) dx \\ &= \int \csc^2 x dx - \int \cot x \csc x dx \\ &= -\cot x + \csc x + c. \end{aligned}$$

8. Evaluate $\int \frac{\sin^4 x}{\cos^6 x} dx$.

Sol:
$$\begin{aligned} & \int \frac{\sin^4 x}{\cos^6 x} dx \\ &= \int \frac{\sin^4 x}{\cos^4 x} \cdot \frac{1}{\cos^2 x} dx \\ &= \int \tan^4 x \sec^2 x dx \\ & \quad \text{Let } \tan x = t \\ & \quad \sec^2 x dx = dt \\ &= \int t^4 dt \\ &= \frac{t^5}{5} + c \\ &= \frac{(\tan x)^5}{5} + c. \end{aligned}$$

9. Evaluate $\int \frac{\log(1+x)}{1+x} dx$.

Sol:
$$\begin{aligned} & \int \frac{\log(1+x)}{1+x} dx \\ & \quad \text{put } \log(1+x) = t \\ & \Rightarrow \frac{1}{1+x} dx = dt \\ &= \int t dt \\ &= \frac{t^2}{2} + c \\ &= \frac{(\log(1+x))^2}{2} + c. \end{aligned}$$

10. Find $\int \left(1 - \frac{1}{x^2}\right) e^{\frac{x+1}{x}} dx$.

Sol:
$$\begin{aligned} & \int \left(1 - \frac{1}{x^2}\right) e^{\frac{x+1}{x}} dx \\ & \quad \text{Put } x + \frac{1}{x} = t \\ & \Rightarrow \left(1 - \frac{1}{x^2}\right) dx = dt \\ &= \int e^t dt \\ &= e^t + c \\ &= e^{\frac{x+1}{x}} + c. \end{aligned}$$

11. Evaluate $\int \frac{e^{\tan^{-1}x}}{1+x^2} dx$.

Sol: $\int \frac{e^{\tan^{-1}x}}{1+x^2} dx$

Put $\tan^{-1} x = t \Rightarrow \frac{1}{1+x^2} dx = dt$

$$= \int e^t dt$$

$$= e^t + c = e^{\tan^{-1}x} + c.$$

12. Evaluate $\int \frac{\sin(\tan^{-1}x)}{1+x^2} dx$.

Sol: $\int \frac{\sin(\tan^{-1}x)}{1+x^2} dx$

Put $\tan^{-1} x = t$

$$\frac{1}{1+x^2} dx = dt$$

$$= \int \sin t dt = -\cos t$$

$$= \cos(\tan^{-1} x) + c.$$

13. Evaluate $\int \frac{2x^3}{1+x^8} dx$.

Sol: $\int \frac{2x^3}{1+x^8} dx$

$$= \int \frac{2x^3}{1+(x^4)^2} dx$$

Let $x^4 = t$
 $4x^3 \cdot dx = dt$

$$2x^3 dx = \frac{dt}{2}$$

$$= \int \frac{1}{1+t^2} \cdot \frac{dt}{2}$$

$$= \frac{1}{2} \tan^{-1}(t) + c$$

$$= \frac{1}{2} \tan^{-1}(x^4) + c.$$

14. Evaluate $\int \frac{x^8}{1+x^{18}} dx$.

Sol: $\int \frac{x^8}{1+x^{18}} dx$

$$= \int \frac{x^8}{1+(x^9)^2} dx$$

Put $x^9 = t \Rightarrow 9x^8 dx = dt$

$$\Rightarrow x^8 dx = \frac{dt}{9}$$

$$= \frac{1}{9} \int \frac{1}{1+t^2} dt$$

$$= \frac{1}{9} \tan^{-1}(t) + c$$

$$= \frac{1}{9} \tan^{-1}(x^9) + c.$$

15. Find $\int e^x \sin x dx$.

Sol: $\int e^x \sin x dx$

$$\therefore \int e^{ax} \sin(bx) dx = \frac{e^{ax}}{a^2+b^2} [a \sin(bx) - b \cos(bx)] + c$$

$$= \frac{e^x}{1^2+1^2} [1 \cdot \sin x - 1 \cdot \cos x] + c$$

$$= \frac{e^x}{2} [\sin x - \cos x] + c.$$

16. Evaluate $\int \frac{1}{(x+3)\sqrt{x+2}} dx$.

Sol: $\int \frac{1}{(x+3)\sqrt{x+2}} dx$

Put $x+2 = t^2$

$x+3 = 1+t^2$

$dx = 2t \cdot dt$

$$= \int \frac{1}{(1+t^2)t} 2t dt$$

$$= 2 \int \frac{1}{1+t^2} dt$$

$$= 2 \tan^{-1} t + c$$

$$= 2 \tan^{-1} (\sqrt{x+2}) + c.$$

17. Find $\int \frac{\cos x + \sin x}{\sqrt{1+\sin 2x}} dx$.

Sol: $\int \frac{\cos x + \sin x}{\sqrt{1+\sin^2 x}} dx$

$$= \int \frac{\cos x + \sin x}{\sqrt{\sin^2 x + \cos^2 x + 2 \sin x \cos x}} dx$$

$$\begin{aligned}
 &= \int \frac{\cos x + \sin x}{\sqrt{(\sin x + \cos x)^2}} dx \\
 &= \int \frac{\cos x + \cancel{\sin x}}{\cos x + \cancel{\sin x}} dx \\
 &= \int 1 dx \\
 &= x + c.
 \end{aligned}$$

18. Evaluate $\int \frac{1}{\sqrt{\sin^{-1}x \cdot \sqrt{1-x^2}}} dx$.

$$\begin{aligned}
 \text{Sol: } &\int \frac{1}{\sqrt{\sin^{-1}x \cdot \sqrt{1-x^2}}} dx \\
 &\text{Let } \sin^{-1} x = t \\
 &\frac{1}{\sqrt{1-x^2}} dx = dt \\
 &= \int \frac{1}{\sqrt{t}} dt \\
 &= 2\sqrt{t} + c \\
 &= 2\sqrt{\sin^{-1} x} + c.
 \end{aligned}$$

19. Find $\int \frac{(1+x)e^x}{\cos^2(xe^x)} dx$.

$$\begin{aligned}
 \text{Sol: } &\int \frac{(1+x)e^x}{\cos^2(xe^x)} dx \\
 &\text{let } x e^x = t \\
 &(1. e^x + x. e^x) dx = dt \\
 &e^x(1+x). dx = dt \\
 &= \int \frac{1}{\cos^2 t} dx \\
 &= \int \sec^2 t dt \\
 &= \tan t + c \\
 &= \tan(xe^x) + c.
 \end{aligned}$$

20. Evaluate $\int \frac{1}{x \log x \log(\log x)} dx$.

$$\begin{aligned}
 \text{Sol: } &\int \frac{1}{x \log x \log(\log x)} dx \\
 &\text{Let } \log(\log x) = t \\
 &\int \frac{1}{\log x} \cdot \frac{1}{x} dx = dt \\
 &= \int \frac{1}{t} dt \\
 &= \log |t| + c \\
 &= \log |\log(\log x)| + c.
 \end{aligned}$$

21. Evaluate $\int \frac{\cot(\log x)}{x} dx$

$$\begin{aligned}
 \text{Sol: } &\int \frac{\cot(\log x)}{x} dx \\
 &\text{Let } \log x = t \\
 &\frac{1}{x} dx = dt
 \end{aligned}$$

$$\begin{aligned}
 &= \int \cot(t) dt \\
 &= \log |\sin t| + c \\
 &= \log |\sin(\log x)| + c.
 \end{aligned}$$

22. Evaluate $\int \log x dx$

$$\begin{aligned}
 \text{Sol: } &\int \log x dx = \int \log x \cdot 1 dx \\
 &u = \log x, v = 1 \\
 &u' = \frac{1}{x}, \int v = \int 1 dx = x
 \end{aligned}$$

By using integration by parts $\int uv = u \int v - \int [u' \int v]$

$$\begin{aligned}
 &= \log x \cdot x - \int \left(\frac{1}{x} \cdot x \right) dx \\
 &= x \log x - \int 1 dx \\
 &= x \log x - x + c.
 \end{aligned}$$

23. Evaluate $\int e^x (\tan x + \sec^2 x) dx$.

$$\begin{aligned}
 \text{Sol: } &\int e^x (\tan x + \sec^2 x) dx \\
 &f(x) = \tan x, f'(x) = \sec^2 x \\
 &= \int e^x [f(x) + f'(x)] dx \\
 &= e^x f(x) + c \\
 &= e^x \tan x + c.
 \end{aligned}$$

24. Evaluate $\int e^x (\sec x + \sec x \tan x) dx$.

$$\begin{aligned}
 \text{Sol: } &\int e^x (\sec x + \sec x \tan x) dx \\
 &f(x) = \sec x, f'(x) = \sec x \tan x \\
 &\int e^x (f(x) + f'(x)) dx = e^x f(x) + c \\
 &= e^x \sec x + c.
 \end{aligned}$$

25. Evaluate $\int e^x (\tan x + \log \sec x) dx$.

Sol: $\int e^x (\tan x + \log \sec x) dx$
 $f(x) = \log(\sec x)$.

$$\begin{aligned} f'(x) &= \frac{1}{\sec x} \sec x \tan x = \tan x \\ &= \int e^x [f(x) + f'(x)] dx \\ &= e^x f(x) + c \\ &= e^x \log(\sec x) + c. \end{aligned}$$

26. Find $\int e^x \left(\frac{1+x \log x}{x} \right) dx$.

Sol: $\int e^x \left(\frac{1+x \log x}{x} \right) dx$
 $= \int e^x \left(\frac{1}{x} + \frac{x \log x}{x} \right) dx$
 $= \int e^x \left(\frac{1}{x} + \log x \right) dx$

$$f(x) = \log x, f'(x) = \frac{1}{x}$$

$$\begin{aligned} &= \int e^x [f(x) + f'(x)] dx \\ &= e^x \cdot f(x) + c \\ &= e^x \cdot \log x + c. \end{aligned}$$

27. Evaluate $\int e^x \left[\frac{1+x}{(2+x)^2} \right] dx$.

Sol: $\int e^x \left[\frac{1+x}{(2+x)^2} \right] dx$
 $= \int e^x \left[\frac{2+x-1}{(2+x)^2} \right] dx$
 $= \int e^x \left[\frac{2+x}{(2+x)^2} - \frac{1}{(2+x)^2} \right] dx$
 $= \int e^x \left[\frac{1}{2+x} + \frac{-1}{(2+x)^2} \right] dx$
 $f(x) = \frac{1}{2+x}, f'(x) = \frac{-1}{(2+x)^2}$

$$\begin{aligned} &= \int e^x [f(x) + f'(x)] dx \\ &= e^x f(x) + c \\ &= e^x \left(\frac{1}{2+x} \right) + c. \end{aligned}$$

28. Evaluate $\int x \tan^{-1} x dx$.

Sol: $\int x \tan^{-1} x dx$

$$\begin{aligned} u &= \tan^{-1} x \Rightarrow u' = \frac{1}{1+x^2} \\ v &= x \Rightarrow \int v dx = \frac{x^2}{2} + c \\ \int uv dx &= u \int v dx - \int (u' \int v dx) dx \\ &= \tan^{-1} x \cdot \frac{x^2}{x} - \int \frac{1}{1+x^2} \cdot \frac{x^2}{2} dx \\ &= \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} \int \frac{x^2}{1+x^2} dx \\ &= \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} \int \left(1 - \frac{1}{1+x^2} \right) dx \\ &= \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} [x - \tan^{-1} x] + c. \end{aligned}$$

29. Evaluate $\int \sqrt{x} \log x dx$

Sol: $\int \sqrt{x} \log x dx$

$$\begin{aligned} u &= \log x \Rightarrow u' = \frac{1}{x} \\ v &= \sqrt{x} \Rightarrow \int v = \frac{\frac{1}{2+1}}{\frac{1}{2}+1} = \frac{2}{3} x^{3/2} \\ \text{by using integration by parts} \\ \int uv &= u \int v - \int (u' \int v) \\ &= \log x \cdot \frac{2}{3} x^{3/2} - \int \frac{1}{x} \cdot \frac{2}{3} x^{3/2} dx \\ &= \frac{2}{3} x^{3/2} \log x - \frac{2}{3} \int x^{\frac{3}{2}-1} dx \\ &= \frac{2}{3} x^{3/2} \log x - \frac{2}{3} \int x^{\frac{1}{2}} dx \\ &= \frac{2}{3} x^{3/2} \log x - \frac{2}{3} \frac{x^{\frac{3}{2}}}{\frac{3}{2}} \\ &= \frac{2}{3} x^{3/2} \log x - \frac{4}{9} x^{3/2} + c \end{aligned}$$

30. Evaluate $\int x \sec^2 x \, dx$.

Sol: $\int x \sec^2 x \, dx$

$$u = x, \quad v = \sec^2 x.$$

$$u' = 1, \quad \int v = \int \sec^2 x \, dx = \tan x$$

By using integration by parts $\int uv = u \int v - \int [u' \int v]$

$$= x \cdot \tan x - \int (1 \cdot \tan x) \, dx$$

$$= x \tan x - \int \tan x \, dx$$

$$= x \tan x - \log|\sec x| + c.$$

31. Evaluate $\int \tan^{-1} x \, dx$.

Sol: $\int \tan^{-1} x \, dx = \int \tan^{-1} x \cdot 1 \, dx$

$$u = \tan^{-1} x, v = 1$$

$$u' = \frac{1}{1+x^2}, \quad \int v = \int 1 \, dx = x$$

By using integration by parts $\int uv = u \int v - \int [u' \int v]$

$$= \tan^{-1} x \cdot x - \int \frac{1}{1+x^2} \cdot x \, dx$$

$$= x \tan^{-1} x - \frac{1}{2} \int \frac{2x}{1+x^2} \, dx$$

$$\therefore \int \frac{f'(x)}{f(x)} \, dx = \log|f(x)| + c$$

$$= x \tan^{-1} x - \frac{1}{2} \log|1+x^2| + c.$$

LEVEL - I (LAQ)

1. Evaluate $\int (6x+5)\sqrt{6-2x^2+x} \, dx$

Sol: Let $6x+5 = k \frac{d}{dx} (6-2x^2+x) + \ell$

$$6x+5 = k[-2(2x)+1] + \ell$$

$$6x+5 = k(-4x) + k + \ell$$

Equating the co-efficients of

$$x \rightarrow 6 = -4k \Rightarrow k = \frac{6}{-4} = \frac{3}{-2}$$

$$c \rightarrow 5 = k + \ell$$

$$5 = \frac{-3}{2} + \ell \Rightarrow \ell = 5 + \frac{3}{2} = \frac{13}{2}$$

$$6x+5 = \frac{-3}{2} \frac{d}{dx} (6-2x^2+x) + \frac{13}{2}$$

$$\therefore \int (6x+5)\sqrt{6-2x^2+x} \, dx$$

$$= \int \left[\frac{-3}{2} \frac{d}{dx} (6-2x^2+x) + \frac{13}{2} \right] \sqrt{6-2x^2+x} \, dx$$

$$= \int \left\{ \frac{-3}{2} \left[\frac{d}{dx} (6-2x^2+x) \right] (6-2x^2+x) + \frac{13}{2} \sqrt{6-2x^2+x} \right\} \, dx$$

$$= \frac{-3}{2} \int (6-2x^2+x)^{1/2} \frac{d}{dx} (6-2x^2+x) \, dx$$

$$+ \frac{13}{2} \int \sqrt{6-2x^2+x} \, dx$$

$$\int f^n(x) \cdot f'(x) \, dx = -2x^2 + x + 6$$

$$= \frac{f^{n+1}(x)}{n+1} = -2 \left(x^2 - \frac{x}{2} - 3 \right)$$

$$= -2 \left[\left(x - \frac{1}{4} \right)^2 - 3 - \left(\frac{1}{4} \right)^2 \right]$$

$$= -2 \left[\left(x - \frac{1}{4} \right)^2 - 3 - \frac{1}{16} \right]$$

$$= -2 \left[\left(x - \frac{1}{4} \right)^2 - \frac{49}{16} \right]$$

$$= 2 \left[\frac{49}{16} - \left(x - \frac{1}{4} \right)^2 \right]$$

$$= \frac{-3}{2} \frac{(6-2x^2+x)^{1/2+1}}{\frac{1}{2}+1} + \frac{13}{2} \int \sqrt{2 \left[\frac{49}{16} - \left(x - \frac{1}{4} \right)^2 \right]} \, dx$$

$$= \frac{-3}{2} \frac{(6-2x^2+x)^{3/2}}{\frac{3}{2}} + \frac{13}{2} \sqrt{2} \int \sqrt{\left(\frac{7}{4} \right)^2 - \left(x - \frac{1}{4} \right)^2} \, dx$$

$$= -(6-2x^2+x)^{3/2} + \frac{13}{\sqrt{2}} \left\{ \frac{\left(x - \frac{1}{4} \right)}{2} \sqrt{\left(\frac{7}{4} \right)^2 - \left(x - \frac{1}{4} \right)^2} \right.$$

$$\left. + \frac{\left(\frac{49}{16} \right)}{2} \sin^{-1} \left(\frac{x - \frac{1}{4}}{\frac{7}{4}} \right) \right\}$$

$$= -(6-2x^2+x)^{\frac{3}{2}} + \frac{13}{\sqrt{2}} \left\{ \frac{4x-1}{8} \sqrt{\left(\frac{7}{4} \right)^2 - \left(x - \frac{1}{4} \right)^2} \right.$$

$$\left. + \frac{49}{32} \sin^{-1} \left(\frac{4x-1}{7} \right) \right\} + c.$$

$$\begin{aligned}
 &= \frac{1}{2} \log |x^2 - x + 1| + \frac{3}{2} \int \frac{1}{\left(x - \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} dx \\
 &= \frac{1}{2} \log |x^2 - x + 1| + \frac{3}{2} \cdot \frac{1}{\frac{\sqrt{3}}{2}} \tan^{-1} \left(\frac{x - \frac{1}{2}}{\frac{\sqrt{3}}{2}} \right) + c \\
 &= \frac{1}{2} \log |x^2 - x + 1| + \sqrt{3} \tan^{-1} \left(\frac{2x - 1}{\sqrt{3}} \right) + c.
 \end{aligned}$$

4. Evaluate $\int \frac{x+1}{x^2+3x+12} dx$.

Sol: Let $x+1 = A \frac{d}{dx}(x^2+3x+12) + B$

$$\Rightarrow x+1 = A(2x+3) + B$$

Comparing both sides,

$$1 = 2A \Rightarrow A = \frac{1}{2}$$

$$1 = 3A + B \Rightarrow B = -\frac{1}{2}$$

Now $\int \frac{x+1}{x^2+3x+12} dx$

$$= \frac{1}{2} \int \frac{2x+3}{x^2+3x+12} dx$$

$$\frac{1}{2} \int \frac{dx}{x^2+3x+12}$$

$$\begin{aligned}
 &= \frac{1}{2} \log |x^2 + 3x + 12| - \frac{1}{2} \int \frac{dx}{\left(x + \frac{3}{2}\right)^2 + \left(\frac{\sqrt{39}}{2}\right)^2} \\
 &= \frac{1}{2} \log |x^2 + 3x + 12| - \frac{1}{\left(\frac{\sqrt{39}}{2}\right)} \tan^{-1} \left(\frac{x + \frac{3}{2}}{\left(\frac{\sqrt{39}}{2}\right)} \right) + c
 \end{aligned}$$

$$= \frac{1}{2} \log |x^2 + 3x + 12| - \frac{1}{\sqrt{39}} \tan^{-1} \left(\frac{2x+3}{\sqrt{39}} \right) + c.$$

5. Evaluate $\int \frac{2x+5}{\sqrt{x^2-2x+10}} dx$.

Sol: Given $\int \frac{2x+5}{\sqrt{x^2-2x+10}} dx$

Let $2x+5 = A \frac{d}{dx}(x^2-2x+10) + B$

$$2x+5 = A(2x-2) + B \dots\dots\dots (1)$$

Coefficient of x put $x = 0$

$$2 = 2A \quad 5 = -2A + B$$

$$A = 1 \quad 5 = -2 + B$$

$$B = 5 + 2 = 7$$

A, B in (1)

$$2x+5 = (2x-2) + 7$$

$$\int \frac{2x+5}{\sqrt{x^2-2x+10}} dx = \int \frac{(2x-2)+7}{\sqrt{x^2-2x+10}} dx$$

$$= \int \frac{2x-2}{\sqrt{x^2-2x+10}} dx + \int \frac{7}{\sqrt{x^2-2x+10}} dx$$

$$= 2\sqrt{x^2-2x+10} + 7 \int \frac{dx}{\sqrt{(x-1)^2+3^2}} + c$$

$$= 2\sqrt{x^2-2x+10} + 7 \sinh^{-1} \left(\frac{x-1}{3} \right) + c.$$

6. Evaluate $\int \frac{1}{(1+x)\sqrt{3+2x-x^2}} dx$

Sol: Let $1+x = \frac{1}{t} \Rightarrow x = \frac{1}{t} - 1 = \frac{1-t}{t}$

$$dx = -\frac{1}{t^2} dt \quad 3+2x-x^2$$

$$= 3 + 2\left(\frac{1-t}{t}\right) - \left(\frac{1-t}{t}\right)^2$$

$$= 3 + \frac{2(1-t)}{t} - \frac{1+t^2-2t}{t^2}$$

$$= \frac{3t^2 + 2t(1-t) - (1+t^2-2t)}{t^2}$$

$$= \frac{3t^2 + 2t - 2t^2 - 1 - t^2 + 2t}{t^2}$$

$$= \frac{4t-1}{t^2}$$

$$\int \frac{1}{(1+x)\sqrt{3+2x-x^2}} dx$$

$$\begin{aligned}
 &= \int \frac{1}{t} \sqrt{\frac{4t-1}{t^2}} \cdot \frac{-1}{t^2} dt \\
 &= -\int \frac{1}{\sqrt{4t-1}} dt \\
 &= -\frac{2\sqrt{4t-1}}{4} + c \quad \because \int \frac{1}{\sqrt{ax+b}} = \frac{2\sqrt{ax+b}}{a} \\
 &= -\frac{1}{2} \sqrt{4 \cdot \frac{1}{1+x} - 1} + c \\
 &= -\frac{1}{2} \sqrt{\frac{4-1-x}{1+x}} + c \\
 &= -\frac{1}{2} \sqrt{\frac{3-x}{1+x}} + c
 \end{aligned}$$

7. Evaluate $\int \frac{1}{4\cos x + 3\sin x} dx$.

Sol: $\int \frac{1}{4\cos x + 3\sin x} dx$

multiply and divide with $\sqrt{4^2 + 3^2} = 5$ in Dr

$$\begin{aligned}
 &= \int \frac{1}{5\left(\frac{4}{5}\cos x + \frac{3}{5}\sin x\right)} dx \\
 &\text{Let } \cos \alpha = \frac{4}{5}, \sin \alpha = \frac{3}{5} \\
 &= \frac{1}{5} \int \frac{1}{\cos x \cos \alpha + \sin x \sin \alpha} dx \\
 &= \frac{1}{5} \int \frac{1}{\cos(x-\alpha)} dx \\
 &= \frac{1}{5} \int \sec(x-\alpha) dx \\
 &= \frac{1}{5} \log |\sec(x-\alpha) + \tan(x-\alpha)| + c
 \end{aligned}$$

where $\tan \alpha = \frac{3}{4}$, $\alpha = \tan^{-1}\left(\frac{3}{4}\right)$.

8. Evaluate $\int \frac{1}{\sin x + \sqrt{3}\cos x} dx$.

Sol: $\int \frac{1}{\sin x + \sqrt{3}\cos x} dx$

$$\begin{aligned}
 &= \int \frac{1}{2\left(\frac{1}{2}\sin x + \frac{\sqrt{3}}{2}\cos x\right)} dx \\
 &= \frac{1}{2} \int \frac{1}{\left(\sin x \cos \frac{\pi}{3} + \cos x \sin \frac{\pi}{3}\right)} dx \\
 &= \frac{1}{2} \int \frac{1}{\sin\left(x + \frac{\pi}{3}\right)} dx = \frac{1}{2} \int \csc\left(x + \frac{\pi}{3}\right) dx \\
 &= \frac{1}{2} \log \left| \tan\left(\frac{\pi}{2} + \frac{x}{3}\right) \right| + c.
 \end{aligned}$$

9. Evaluate $\int \frac{9\cos x - \sin x}{4\sin x + 5\cos x} dx$.

Sol: Given $\int \frac{9\cos x - \sin x}{4\sin x + 5\cos x} dx$

Let $Nr = A \frac{d}{dx}(dr) + B(dr)$

$$\begin{aligned}
 9\cos x - \sin x &= A \frac{d}{dx}(4\sin x + 5\cos x) + B(4\sin x + 5\cos x) \\
 9\cos x - \sin x &= A(4\cos x - 5\sin x) + B(4\sin x + 5\cos x)
 \end{aligned}$$

Coefficient of $\cos x = 4A + 5B = 9$
 Coefficient of $\sin x = -5A + 4B = -1$.
 Solving $4A + 5B - 9 = 0$, $-5A + 4B + 1 = 0$

$$\begin{array}{rrrr}
 5 & -9 & 4 & 5 \\
 4 & 1 & -5 & 4
 \end{array}$$

$$\frac{A}{5+36} = \frac{B}{45-4} = \frac{C}{16+25}$$

$$\frac{A}{41} = \frac{B}{41} = \frac{C}{41} \Rightarrow A = \frac{41}{41} = 1, B = \frac{41}{41} = 1$$

$$9(\cos x - \sin x) = (4\sin x + 5\cos x) + (4\cos x - 5\sin x)$$

$$\int \frac{9\cos x - \sin x}{4\sin x + 5\cos x} dx$$

$$= \int \frac{(4\sin x + 5\cos x) + (4\cos x - 5\sin x)}{4\sin x + 5\cos x} dx$$

$$= \int \frac{4\sin x + 5\cos x}{4\sin x + 5\cos x} dx + \int \frac{4\cos x - 5\sin x}{4\sin x + 5\cos x} dx$$

$$= \int 1 dx + \int \frac{4\cos x - 5\sin x}{4\sin x + 5\cos x} dx$$

$$= x + \log |4\sin x + 5\cos x| + c.$$

10. Evaluate $\int \frac{\cos x + 3 \sin x + 7}{\cos x + \sin x + 1} dx$.

Sol: $Nr = A(Dr) + B \frac{d}{dx}(Dr) + C$

$$\cos x + 3 \sin x + 7 = A(\cos x + \sin x + 1) + B \frac{d}{dx}(\cos x + \sin x + 1) + C$$

$$\cos x + 3 \sin x + 7 = A(\cos x + \sin x + 1) + B(-\sin x + \cos x) + C$$

Equating the co.coefficients of

$$\cos x \rightarrow 1 = A + B \quad \Rightarrow A + B - 1 = 0 \quad \dots(1)$$

$$\sin x \rightarrow 3 = A - B \quad \Rightarrow A - B - 3 = 0.$$

$$C \rightarrow 7 = A + C \quad \text{adding}$$

$$7 = 2 + C \quad 2A - 4 = 0$$

$$C = 5 \quad 2A = 4 \Rightarrow A = 2$$

sub in (1)

$$2 + B - 1 = 0$$

$$B + 1 = 0$$

$$B = -1$$

$$\therefore \cos x + 3 \sin x + 7 = 2(\cos x + \sin x + 1) - 1 \frac{d}{dx}(\cos x + \sin x + 1) + 5$$

$$\int \frac{\cos x + 3 \sin x + 7}{\cos x + \sin x + 1} dx = \int \frac{2(\cos x + \sin x + 1) - 1 \frac{d}{dx}(\cos x + \sin x + 1) + 5}{\cos x + \sin x + 1} dx$$

$$= 2 \int \frac{\cos x + \sin x + 1}{\cos x + \sin x + 1} dx - 1 \int \frac{\frac{d}{dx}(\cos x + \sin x + 1)}{\cos x + \sin x + 1} dx + 5 \int \frac{1}{\cos x + \sin x + 1} dx$$

$$= 2 \int 1 dx - \log |\cos x + \sin x + 1| + 5 \int \frac{1}{\cos x + \sin x + 1} dx \quad \dots(2)$$

$$\int \frac{1}{\cos x + \sin x + 1} dx$$

$$t = \tan x/2$$

$$\cos x = \frac{1-t^2}{1+t^2}$$

$$dx = \frac{2 dt}{1+t^2}$$

$$\sin x = \frac{2t}{1+t^2}$$

$$= \int \frac{1}{\frac{1-t^2}{1+t^2} + \frac{2t}{1+t^2} + 1} \cdot \frac{2dt}{1+t^2}$$

$$= \int \frac{1}{\frac{1-t^2+2t+1+t^2}{1+t^2}} \cdot \frac{2dt}{1+t^2}$$

$$= \int \frac{1}{2+2t} \cdot 2dt$$

$$= \int \frac{1}{2(1+t)} \cdot 2dt$$

$$= \int \frac{1}{1+t} dt$$

$$= \log |1+t| + c$$

$$= \log |1+\tan(x/2)| + c$$

Sub. in (2)

$$\int \frac{\cos x + 3 \sin x + 7}{\cos x + \sin x + 1} dx = 2x - \log |\cos x + \sin x + 1| + 5 \cdot \log \left| 1 + \tan \frac{x}{2} \right| + c.$$

11. Evaluate $\int \frac{2\sin x + 3\cos x + 4}{3\sin x + 4\cos x + 5} dx$.

Sol: $Nr = A(Dr) + B \frac{d}{dx} (Dr) + C$

$$2\sin x + 3\cos x + 4 = A(3\sin x + 4\cos x + 5) + B \frac{d}{dx} (3\sin x + 4\cos x + 5) + C$$

$$2\sin x + 3\cos x + 4 = A(3\sin x + 4\cos x + 5) + B(3\cos x - 4\sin x) + C$$

Equating the co-efficients of

$$\cos x \rightarrow 3 = 4A + 3B \Rightarrow 4A = 3B = -3 = 0$$

$$\sin x \rightarrow 2 = 3A - 4B \Rightarrow 3A - 4B - 2 = 0$$

$$C \rightarrow 4 = 5A + C \quad \text{solve}$$

$$\begin{array}{r} 3 - 3 & 4 & 3 \\ -4 & -2 & 3 - 4 \end{array}$$

$$\frac{A}{-6 - 12} = \frac{B}{-9 + 8} = \frac{1}{-16 - 9}$$

$$\frac{A}{-18} = \frac{B}{-1} = \frac{1}{-25}$$

$$\frac{A}{-18} = \frac{1}{-25} \quad \frac{B}{-1} = \frac{1}{-25}$$

$$A = \frac{18}{25} \quad B = \frac{1}{25}$$

$$4 = 5\left(\frac{18}{25}\right) + C$$

$$C = 4 - \frac{18}{5} = \frac{20 - 18}{5} = \frac{2}{5}$$

$$2\sin x + 3\cos x + 4 = \frac{18}{25}(3\sin x + 4\cos x + 5) + \frac{1}{25} \frac{d}{dx}(3\sin x + 4\cos x + 5) + \frac{2}{5}$$

$$\int \frac{2\sin x + 3\cos x + 4}{3\sin x + 4\cos x + 5} dx = \int \frac{\frac{18}{25}(3\sin x + 4\cos x + 5) + \frac{1}{25} \frac{d}{dx}(3\sin x + 4\cos x + 5) + \frac{2}{5}}{3\sin x + 4\cos x + 5} dx$$

$$= \frac{18}{25} \int \frac{3\sin x + 4\cos x + 5}{3\sin x + 4\cos x + 5} dx + \frac{1}{25} \int \frac{\frac{d}{dx}(3\sin x + 4\cos x + 5)}{3\sin x + 4\cos x + 5} dx + \frac{2}{5} \int \frac{1}{3\sin x + 4\cos x + 5} dx$$

$$= \frac{18}{25} [1 \cdot dx + \frac{1}{25} \log |3\sin x + 4\cos x + 5|] + \frac{2}{5} \int \frac{1}{3\sin x + 4\cos x + 5} dx$$

$$= \frac{18}{25} x + \frac{1}{25} \log |3\sin x + 4\cos x + 5| + \frac{2}{5} \int \frac{1}{3\sin x + 4\cos x + 5} dx \quad (1)$$

$$\int \frac{1}{3\sin x + 4\cos x + 5} dx$$

$$t = \tan x/2$$

$$\sin x = \frac{2t}{1+t^2}$$

$$dx = \frac{2dt}{1+t^2}$$

$$\cos x = \frac{1+t^2}{1+t^2}$$

$$= \int \frac{1}{3\left(\frac{2t}{1+t^2}\right) + 4\left(\frac{1-t^2}{1+t^2}\right) + 5} \cdot \frac{2dt}{1+t^2}$$

$$= \int \frac{1}{\frac{6t+4-4t^2+5+5t^2}{1+t^2}} \cdot \frac{2dt}{1+t^2}$$

$$= \int \frac{1}{t^2+6t+9} 2dt$$

$$= 2 \int \frac{1}{(t+3)^2} dt$$

$$= 2\left(\frac{-1}{t+3}\right) + c$$

$$= \frac{-2}{\tan\left(\frac{x}{2}\right) + 3} + c$$

sub. in (1)

$$\therefore \int \frac{2\sin x + 3\cos x + 4}{3\sin x + 4\cos x + 5} dx = \frac{18}{25}x$$

$$+ \frac{1}{25} \log |\sin x + 4\cos x + 5| + \frac{2}{5} \left[\frac{-2}{\tan\left(\frac{x}{2}\right) + 3} \right] + c$$

$$12. \text{ Evaluate } \int \frac{2x+3}{(x+3)(x^2+4)} dx.$$

$$\text{Sol: Let } \frac{2x+3}{(x+3)(x^2+4)} = \frac{A}{x+3} + \frac{Bx+C}{x^2+4}$$

$$\Rightarrow 2x+3 = A(x^2+4) + (Bx+c)(x+3)$$

$$\text{Put } x = -3 \quad \text{Coefficient } x^2$$

$$-6+3 = A(9+4) \quad A+B=0$$

$$-3 = A(13) \quad \frac{-3}{13} + B = 0$$

$$A = \frac{-3}{13}$$

$$B = \frac{3}{13}$$

$$\text{Put } x = 0$$

$$\Rightarrow 4A + 3C = 3.$$

$$3c = 3 + \frac{12}{13} = \frac{51}{13} \Rightarrow 3c = \frac{51}{13} \Rightarrow C = \frac{17}{13}$$

$$\int \frac{2x+3}{(x+3)(x^2+4)} dx$$

$$= \int \left(\frac{-\frac{3}{13}}{(x+3)} + \frac{\frac{3}{13}x + \frac{17}{13}}{x^2+4} \right) dx$$

$$\begin{aligned} &= \frac{-3}{13} \int \frac{1}{(x+3)} dx + \frac{3}{13} \int \frac{1}{x^2+4} dx + \frac{17}{13} \int \frac{1}{x^2+4} dx \\ &= \frac{-3}{13} \log |x+3| + \frac{3}{13} \cdot \frac{1}{2} \int \frac{2x}{x^2+4} dx + \frac{17}{13} \cdot \frac{1}{2} \cdot \tan^{-1}\left(\frac{x}{2}\right) \\ &= \frac{-3}{13} \log |x+3| + \frac{3}{26} \log |x^2+4| + \frac{17}{26} \tan^{-1}\frac{x}{2} + c. \end{aligned}$$

13. If $I_n = \int \sin^n x dx$, then show that

$$I_n = \frac{-\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} I_{n-2} \text{ and hence find } I_4$$

$$\text{Sol: } I_n = \int \sin^n x dx$$

$$= \int \sin^{n-1} x \cdot \sin x dx$$

$$u = \sin^{n-1} x, v = \sin x$$

$$\int v = \int \sin x dx$$

$$= -\cos x$$

By using integration by parts, $\int uv = u \int v - \int [u' v]$

$$I_n = \sin^{n-1} x (-\cos x) - \int (n-1) \sin^{n-2} x \cdot \cos x (-\cos x) dx$$

$$I_n = -\sin^{n-1} x \cdot \cos x + (n-1) \int \sin^{n-2} x \cdot \cos^2 x dx$$

$$I_n = -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x (1-\sin^2 x) dx$$

$$I_n = -\sin^{n-1} x \cos x + (n-1) \int (\sin^{n-2} x - \sin^n x) dx$$

$$I_n = -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x dx - (n-1) \int \sin^n x dx$$

$$I_n = -\sin^{n-1} x \cos x + (n-1) I_{n-2} - (n-1) I_n$$

$$I_n + (n-1) I_n = -\sin^{n-1} x \cos x + (n-1) I_{n-2}$$

$$I_n [1 + n - 1] = -\sin^{n-1} x \cos x + (n-1) I_{n-2}$$

$$I_n (n) = -\sin^{n-1} x \cos x + (n-1) I_{n-2}$$

$$I_n = \frac{-\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} I_{n-2}$$

$$I_4 = \int \sin^4 x dx$$

$$= \frac{-\sin^3 x \cos x}{4} + \frac{3}{4} I_2$$

$$= \frac{-\sin^3 x \cos x}{4} + \frac{3}{4} \left[\frac{-\sin x \cos x}{2} + \frac{1}{2} I_0 \right]$$

$$= \frac{-\sin^3 x \cos x}{4} - \frac{3}{8} \sin x \cos x + \frac{3}{8} I_0$$

$$[I_0 = \int \sin^0 x dx = \int 1 dx = x + c]$$

$$= \frac{-\sin^3 x \cos x}{4} - \frac{3}{8} \sin x \cos x + \frac{3}{8} x + c.$$

14. If $I_n = \int \cos^n x dx$, then show that

$$I_n = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} I_{n-2} \text{ and hence find } I_4$$

Sol: $I_n = \int \cos^n x dx$

$$= \int \cos^{n-1} x \cdot \cos x dx$$

$$u = \cos^{n-1} x, v = \cos x$$

$$\int v = \int \cos x dx$$

$$= \sin x$$

By using integration by parts, $\int uv = u \int v - \int [u' \int v]$

$$I_n = \cos^{n-1} x (\sin x) - \int (n-1) \cos^{n-2} x \cdot -\sin x (\sin x) dx$$

$$I_n = \cos^{n-1} x \cdot \sin x + (n-1) \int \cos^{n-2} x \cdot \sin^2 x dx$$

$$I_n = \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x (1-\cos^2 x) dx$$

$$I_n = \cos^{n-1} x \sin x + (n-1) \int (\cos^{n-2} x - \cos^n x) dx$$

$$I_n = \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x dx - (n-1) \int \cos^n x dx$$

$$I_n = \cos^{n-1} x \sin x + (n-1) I_{n-2} - (n-1) I_n$$

$$I_n + (n-1) I_n = \cos^{n-1} x \sin x + (n-1) I_{n-2}$$

$$I_n [1 + n - 1] = \cos^{n-1} x \sin x + (n-1) I_{n-2}$$

$$I_n (n) = -\cos^{n-1} x \sin x + (n-1) I_{n-2}$$

$$I_n = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} I_{n-2}$$

$$I_4 = \int \cos^4 x dx$$

$$= \frac{\cos^3 x \sin x}{4} + \frac{3}{4} I_2$$

$$= \frac{\cos^3 x \sin x}{4} + \frac{3}{4} \left[\frac{\cos x \sin x}{2} + \frac{1}{2} I_0 \right]$$

$$= \frac{\cos^3 x \sin x}{4} - \frac{3}{8} \cos x \sin x + \frac{3}{8} I_0$$

$$[I_0 = \int \cos^0 x dx = \int 1 dx = x + c]$$

$$= \frac{\cos^3 x \sin x}{4} - \frac{3}{8} \cos x \sin x + \frac{3}{8} x + c$$

15. Find the reduction formula for $\int \tan^n x dx$ and hence find $\int \tan^6 x dx$.

Sol: $I_n = \int \tan^n x dx$

$$= \int \tan^{n-2} x \cdot \tan^2 x dx$$

$$= \int \tan^{n-2} x (\sec^2 x - 1) dx$$

$$= \int \tan^{n-2} x \cdot \sec^2 x - \tan^{n-2} x dx$$

$$= \int \tan^{n-2} x \cdot \sec^2 x dx - \int \tan^{n-2} x dx$$

$$\tan x = t$$

$$\text{differentiating}$$

$$\sec^2 x dx = dt$$

$$= \int t^{n-2} dt - I_{n-2}$$

$$= \frac{t^{n-1}}{n-1} - I_{n-2}$$

$$= \frac{\tan^{n-1} x}{n-1} - I_{n-2}$$

case (1) If n is even, I_n ends with I_0 .

$$I_0 = \int \tan^0 x dx = \int 1 dx = x + c$$

case (2) If n is odd, I_n ends with I_1

$$I_1 = \int \tan^1 x dx = \log |\sec x| + c$$

$$I_6 = \int \tan^6 x dx$$

$$= \frac{\tan^5 x}{5} - I_4$$

$$= \frac{\tan^5 x}{5} - \left[\frac{\tan^3 x}{3} - I_2 \right]$$

$$= \frac{\tan^5 x}{5} - \frac{\tan^3 x}{3} + I_2$$

$$= \frac{\tan^5 x}{5} - \frac{\tan^3 x}{3} + \frac{\tan x}{1} - I_0$$

$$= \frac{\tan^5 x}{5} - \frac{\tan^3 x}{3} + \frac{\tan x}{1} - x + c$$

16. Obtain reduction formula for $I_n = \int \cot^n x dx$, deduce the value of $\int \cot^4 x dx$.

Sol: $I_n = \int \cot^n x dx$

$$= \int \cot^{n-2} x \cdot \cot^2 x dx$$

$$= \int \cot^{n-2} x (\cosec^2 x - 1) dx$$

$$= \int \cot^{n-2} x \cosec^2 x - \int \cot^{n-2} x dx$$

$$\text{Put } \cot x = t$$

$$-\cosec^2 x dx = dt$$

$$\cosec^2 x dx = -dt$$

$$= - \int t^{n-2} dt \Rightarrow - \frac{t^{n-1}}{n-1}$$

$$I_n = \frac{\cot^{n-1} x}{n-1} - I_{n-2}$$

$$\text{now } I_4 = \int \cot^4 x dx$$

$$= -\frac{\cot^{4-1} x}{4-1} x - I_2$$

$$= -\frac{\cot^3 x}{3} + \cot x + \int \cot^0 x dx$$

$$= -\frac{\cot^3 x}{3} + \cot x + \int 1 dx$$

$$= -\frac{\cot^3 x}{3} + \cot x + x + C.$$

17. If $I_n = \int \sec^n x dx$, then prove that

$$I_n = \frac{\sec^{n-2} x \cdot \tan x}{n-1} + \frac{n-2}{n-1} I_{n-2}$$

$$\text{Sol: } I_n = \int \sec^n x dx$$

$$= \int \sec^{n-2} x \cdot \sec^2 x dx$$

$$u = \sec^{n-2} x, v = \sec^2 x.$$

$$\int v = \int \sec^2 x dx = \tan x$$

By using integration by parts, $\int uv = u \int v - \int [u' \int v]$

$$I_n = \sec^{n-2} x \cdot \tan x - \int (n-2) \sec^{n-3} x \sec x \tan x \cdot \tan x \cdot dx$$

$$I_n = \sec^{n-2} x \tan x - (n-2) \int \sec^{n-2} x \cdot \tan^2 x \cdot dx$$

$$I_n = \sec^{n-2} x \tan x - (n-2) \int \sec^{n-2} x \cdot (\sec^2 x - 1) dx$$

$$I_n = \sec^{n-2} x \tan x - (n-2) \int (\sec^n x - \sec^{n-2} x) dx$$

$$I_n = \sec^{n-2} x \tan x - (n-2) \int \sec^n x \cdot dx + (n-2) \int \sec^{n-2} x \cdot dx$$

$$I_n = \sec^{n-2} x \tan x - (n-2) I_n + (n-2) I_{n-2}$$

$$I_n + (n-2) I_n = \sec^{n-2} x \tan x + (n-2) I_{n-2}$$

$$(1+n-2) I_n = \sec^{n-2} x \tan x + (n-2) I_{n-2}$$

$$I_n [n-1] = \sec^{n-2} x \tan x + (n-2) I_{n-2}$$

$$I_n = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} I_{n-2}$$

18. Obtain the reduction formulae for

$$I_n = \int \cosec^n x dx, n \text{ being a positive integer}$$

$n \geq 2$ and deduce the value of $\int \cosec^5 x dx$.

$$\text{Sol: } I_n = \int \cosec^n x dx$$

$$I_n = \int \cosec^{n-2} x \cdot \cosec^2 x dx$$

$$u = \cosec^{n-2} x \Rightarrow u' = (n-2) \cosec^{n-3} x (-\cosec x \cot x)$$

$$v = \cosec^2 x \Rightarrow \int v = -\cot x$$

By using integration by parts

$$\int uv \cdot dx = u \int v - \int (u' \int v)$$

$$I_n = \cosec^{n-2} x (-\cot x) - \int (n-2) \cosec^{n-3} x (-\cosec x \cot x) (-\cot x) dx$$

$$I_n = -\cosec^{n-2} x (\cot x) - (n-2) \int \cosec^{n-2} x \cdot \cot^2 x dx$$

$$I_n = -\cosec^{n-2} x \cdot \cot x - (n-2) \int \cosec^{n-2} x \cdot (\cosec^2 x - 1) dx$$

$$I_n = -\cosec^{n-2} x \cdot \cot x - (n-2) \int (\cosec^n x - \cosec^{n-2} x) dx$$

$$I_n = -\cosec^{n-2} x \cdot \cot x - (n-2) [I_n - I_{n-2}]$$

$$I_n = -\cosec^{n-2} x \cdot \cot x - (n-2) I_n + (n-2) I_{n-2}$$

$$I_n + (n-2) I_n = -\cosec^{n-2} x \cdot \cot x + (n-2) I_{n-2}$$

$$I_n (1+n-2) = -\cosec^{n-2} x \cdot \cot x + (n-2) I_{n-2}$$

$$I_n (n-1) = -\cosec^{n-2} x \cdot \cot x + (n-2) I_{n-2}$$

$$I_n = \frac{-\cosec^{n-2} x \cot x}{n-1} + \frac{n-2}{n-1} I_{n-2}$$

$$\text{Now, } I_5 = \int \cosec^5 x dx$$

$$= -\frac{1}{4} \cosec^3 x \cdot \cot x + \frac{3}{4} I_3$$

$$= -\frac{1}{4} \cosec^3 x \cot x + \frac{3}{4} \left(\frac{-1}{2} \cosec x \cot x + \frac{1}{2} I_1 \right)$$

$$= -\frac{1}{4} \cosec^3 x \cot x - \frac{3}{8} \cosec x \cdot \cot x + \frac{3}{8} \int \cosec x dx$$

$$= -\frac{1}{4} \cosec^3 x \cot x - \frac{3}{8} \cosec x \cdot \cot x + \frac{3}{8} \log \left| \tan \frac{x}{2} \right| + C$$

LEVEL - II (VSAQ)

1. Evaluate $\int \frac{x^2 + 3x - 1}{2x} dx$.

Sol: $\int \frac{x^2 + 3x - 1}{2x} dx$

$$= \int \left(\frac{x^2}{2x} + \frac{3x}{2x} - \frac{1}{2x} \right) dx$$

$$= \frac{1}{2} \int x dx + \frac{3}{2} \int 1 dx - \frac{1}{2} \int \frac{1}{x} dx$$

$$= \frac{1}{2} \left(\frac{x^2}{2} \right) + \frac{3}{2} x - \frac{1}{2} \log|x| + C.$$

2. Evaluate $\int \frac{1}{\cosh x + \sinh x} dx$.

Sol: $\int \frac{1}{\cosh x + \sinh x} dx$

$$= \int \frac{\cosh^2 x - \sinh^2 x}{\cosh x + \sinh x} dx$$

$$= \int \frac{(\cosh x + \sinh x)(\cosh x - \sinh x)}{(\cosh x + \sinh x)} dx$$

$$= \int \cosh x dx - \int \sinh x dx$$

$$= \sinh x - \cosh x + C.$$

3. Evaluate $\int \frac{x^6 - 1}{1+x^2} dx$.

Sol: $\int \frac{x^6 - 1}{1+x^2} dx$

$$\begin{array}{r} x^2 + 1 \end{array} \begin{array}{r} x^6 - 1 \\ x^4 - x^2 + 1 \\ \hline -x^4 - 1 \\ -x^4 - x^2 \\ \hline x^2 - 1 \\ x^2 + 1 \\ \hline -2 \end{array}$$

$$\begin{aligned} &= \int \left(x^4 - x^2 + 1 - \frac{2}{1+x^2} \right) dx \\ &= \int x^4 dx - \int x^2 dx + \int 1 dx - 2 \int \frac{1}{1+x^2} dx \\ &= \frac{x^5}{5} - \frac{x^3}{3} + x - 2 \tan^{-1} x + C. \end{aligned}$$

4. Evaluate $\int \frac{1}{1-\sin x} dx$.

Sol: $\int \frac{1}{1-\sin x} dx$

$$= \int \frac{1+\sin x}{(1-\sin x)(1+\sin x)} dx$$

$$= \int \frac{1+\sin x}{1-\sin^2 x} dx$$

$$= \int \frac{1+\sin x}{\cos^2 x} dx$$

$$= \int \frac{1}{\cos^2 x} dx + \frac{\sin x}{\cos^2 x} dx$$

$$= \int \frac{1}{\cos^2 x} dx + \int \frac{\sin x}{\cos^2 x} dx$$

$$= \int \sec^2 x dx + \int \sec x \cdot \tan x dx$$

$$= \tan x + \sec x + C.$$

5. Find $\int \frac{1}{\sqrt{1+5x}} dx$.

Sol: $\int \frac{1}{\sqrt{1+5x}} dx$

$$\therefore \int \frac{1}{\sqrt{ax+b}} dx = \frac{2\sqrt{ax+b}}{a} + C$$

$$= \frac{2\sqrt{1+5x}}{5} + C.$$

6. Evaluate $\int \frac{a \cdot x^{n-1}}{bx^n + c} dx$.

Sol: $\int \frac{a \cdot x^{n-1}}{bx^n + c} dx$

let $bx^n + c = t$
 $b \cdot n \cdot x^{n-1} \cdot dx = dt$
 $x^{n-1} \cdot dx = \frac{dt}{bn}$

$$\begin{aligned} &= \int \frac{a}{t} \frac{dt}{bn} \\ &= \frac{a}{bn} \int \frac{1}{t} dt \\ &= \frac{a}{bn} \log|t| + k \\ &= \frac{a}{bn} \log|bx^n + c| + k. \end{aligned}$$

7. Find $\int x^3 \sin(x^4) dx$.

Sol: $\int x^3 \sin(x^4) dx$

$$\begin{aligned} & \text{let } x^4 = t \\ & 4x^3 dx = dt \\ & x^3 dx = \frac{dt}{4} \\ & = \int \sin t \cdot \frac{dt}{4} \\ & = \frac{1}{4} (-\cos t) + c \\ & = \frac{-1}{4} \cos(x^4) + c. \end{aligned}$$

8. Evaluate $\int \frac{1}{\sqrt{x}} \cos(\sqrt{x}) dx$.

Sol: $\int \frac{1}{\sqrt{x}} \cos(\sqrt{x}) dx$

$$\begin{aligned} & \text{Let } \sqrt{x} = t \\ & \frac{1}{2\sqrt{x}} dx = dt \\ & \frac{1}{\sqrt{x}} dx = 2dt \\ & = \int \cos t \cdot 2dt \\ & = 2 \int \cos t \cdot dt \\ & = 2 \sin t + c \\ & = 2 \sin \sqrt{x} + c. \end{aligned}$$

9. Evaluate $\int \frac{1}{1+e^x} dx$.

Sol: $\int \frac{1}{1+e^x} dx$

$$\begin{aligned} & \text{Let } 1+e^x = t \\ & e^x dx = dt \\ & dx = \frac{dt}{e^x} = \frac{dt}{t-1} \\ & = \int \frac{1}{t-1} \cdot \frac{dt}{t-1} \\ & = \int \left(\frac{1}{t-1} - \frac{1}{t} \right) dt \\ & = \log|t-1| - \log|t| + c \\ & = \log|e^x| - \log|1+e^x| + c \\ & = x \log e - \log|1+e^x| + c \\ & = x - \log|1+e^x| + c. \end{aligned}$$

10. Evaluate $\int \frac{1}{1+(2x+1)^2} dx$.

Sol: $\int \frac{1}{1+(2x+1)^2} dx$

$$\begin{aligned} & \text{Let } 2x+1 = t \\ & 2 dx = dt \\ & dx = dt/2 \\ & = \int \frac{1}{1+t^2} \cdot \frac{dt}{2} \\ & = \frac{1}{2} \int \frac{1}{1+t^2} dt \\ & = \frac{1}{2} \tan^{-1}(t) + c = \frac{1}{2} \tan^{-1}(2x+1) + c. \end{aligned}$$

11. Find $\int \frac{(1+\log x)^n}{x} dx$.

Sol: $\int \frac{(1+\log x)^n}{x} dx$

$$\begin{aligned} & \text{Let } 1+\log x = t \\ & \frac{1}{x} dx = dt \\ & = \int t^n \cdot dt \\ & = \frac{t^{n+1}}{n+1} + c \\ & = \frac{(1+\log x)^{n+1}}{n+1} + c. \end{aligned}$$

12. Evaluate $\int \sqrt{\cos x} \cdot \sin x \cdot dx$.

Sol: $\int \sqrt{\cos x} \cdot \sin x \cdot dx$

$$\begin{aligned} & \text{Let } \cos x = t \\ & -\sin x dx = dt \\ & \sin x dx = -dt \\ & = \int \sqrt{t} (-dt) \\ & = - \int t^{1/2} dt \\ & = - \frac{t^{1/2+1}}{(1/2)+1} + c \\ & = - \frac{2}{3} t^{3/2} + c \\ & = - \frac{2}{3} (\cos x)^{3/2} + c. \end{aligned}$$

13. Find $\int \frac{\sec^2 x}{(1+\tan x)^3} dx$.

Sol: $\int \frac{\sec^2 x}{(1+\tan x)^3} dx$

$$= \int \frac{1}{t^3} dt$$

$$= \int t^{-3} dt$$

$$= \frac{t^{-3+1}}{-3+1} + C$$

$$= \frac{-1}{2t^2} + C$$

$$= \frac{-1}{2(1+\tan x)^2} + C.$$

Let $1 + \tan x = t$
 $\sec^2 x \cdot dx = dt$

14. Evaluate $\int \frac{\operatorname{cosec}^2 x}{(a+b \cot x)^5} dx$.

Sol: $\int \frac{\operatorname{cosec}^2 x}{(a+b \cot x)^5} dx$

Let $a+b \cot x = t$
 $b(-\operatorname{cosec}^2 x) dx = dt$

$$\operatorname{cosec}^2 x \cdot dx = \frac{-1}{b} dt$$

$$= \int \frac{1}{t^5} \cdot \frac{dt}{-b}$$

$$= \frac{-1}{b} \int t^{-5} dt$$

$$= \frac{-1}{b} \left(\frac{t^{-5+1}}{-5+1} \right) + C$$

$$= \frac{-1}{b} \left(\frac{t^{-4}}{-4} \right) + C$$

$$= \frac{1}{4b} \frac{1}{t^4} + C$$

$$= \frac{1}{4b} \frac{1}{(a+b \cot x)^4} + C.$$

15. Evaluate $\int \frac{x^2+1}{x^4+1} dx$.

Sol: $\int \frac{x^2+1}{x^4+1} dx$

$$= \int \frac{x^2 \left(1 + \frac{1}{x^2} \right)}{x^4 \left(x^2 + \frac{1}{x^2} \right)} dx$$

$$= \int \frac{1 + \frac{1}{x^2}}{x^2 + \frac{1}{x^2} - 2 + 2} dx$$

$$= \int \frac{1 + \frac{1}{x^2}}{(x - \frac{1}{x})^2 + 2} dx$$

$$\text{Put } x - \frac{1}{x} = t \Rightarrow \left(1 + \frac{1}{x^2} \right) dx = dt$$

$$\left(\int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) \right)$$

$$= \int \frac{1}{t^2 + (\sqrt{2})^2} dt = \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{t}{\sqrt{2}} \right) + C$$

$$= \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{x - \frac{1}{x}}{\sqrt{2}} \right) + C$$

$$= \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{x^2 - 1}{\sqrt{2}x} \right) + C.$$

16. Evaluate $\int \cos^3 x \cdot dx$.

Sol: $\int \cos^3 x \cdot dx$

$$= \int \frac{\cos 3x + 3 \cos x}{4} dx$$

$$(\cos 3x = 4\cos^3 x - 3\cos x \\ 4\cos^3 x = \cos 3x + 3 \cos x)$$

$$= \frac{1}{4} \int \cos 3x \cdot dx + \frac{3}{4} \int \cos x \cdot dx$$

$$= \frac{1}{4} \left(\frac{\sin 3x}{3} \right) + \frac{3}{4} (\sin x) + C.$$

17. Find $\int \cos x \cos 3x . dx$.

Sol: $\int \cos x \cos 3x . dx$

$$\begin{aligned} &= \frac{1}{2} \int 2 \cos 3x \cos x . dx \\ &= \frac{1}{2} \int [\cos(3x + x) + \cos(3x - x)]. dx \\ &= \frac{1}{2} \int (\cos 4x + \cos 2x) dx \\ &= \frac{1}{2} \int \cos 4x . dx + \frac{1}{2} \int \cos 2x . dx \\ &= \frac{1}{2} \left(\frac{\sin 4x}{4} \right) + \frac{1}{2} \left(\frac{\sin 2x}{2} \right) + c. \end{aligned}$$

18. Evaluate $\int \frac{1}{\sqrt{4-9x^2}} dx$.

Sol: $\int \frac{1}{\sqrt{4-9x^2}} dx$

$$\begin{aligned} &= \int \frac{1}{\sqrt{9\left(\frac{4}{9}-x^2\right)^2}} dx \\ &= \int \frac{1}{3\sqrt{\left(\frac{2}{3}\right)^2-x^2}} dx \\ &= \frac{1}{3} \int \frac{1}{\sqrt{\left(\frac{2}{3}\right)^2-x^2}} dx & \therefore \int \frac{1}{\sqrt{a^2-x^2}} dx \\ &= \frac{1}{3} \sin^{-1} \left(\frac{x}{\left(\frac{2}{3}\right)} \right) + c & = \sin^{-1} x + c \\ &= \frac{1}{3} \sin^{-1} \left(\frac{x}{\left(\frac{2}{3}\right)} \right) + c \\ &= \frac{1}{3} \sin^{-1} \left(\frac{3x}{2} \right) + c. \end{aligned}$$

19. Find $\int \frac{1}{\sqrt{25+9x^2}} dx$.

Sol: $\int \frac{1}{\sqrt{25+9x^2}} dx$

$$\begin{aligned} &= \int \frac{1}{\sqrt{9\left[\frac{25}{9}+x^2\right]}} dx \\ &= \frac{1}{3} \int \frac{1}{\sqrt{\left(\frac{5}{3}\right)^2+x^2}} dx & \therefore \int \frac{1}{\sqrt{a^2+x^2}} dx \\ &= \sinh^{-1} x + c \end{aligned}$$

$$\begin{aligned} &= \frac{1}{3} \sinh^{-1} \left(\frac{x}{\left(\frac{5}{3}\right)} \right) + c \\ &= \frac{1}{3} \sinh^{-1} \left(\frac{3x}{5} \right) + c. \end{aligned}$$

20. Evaluate $\int \frac{1}{8+2x^2} dx$.

Sol: $\int \frac{1}{8+2x^2} dx$

$$\begin{aligned} &= \int \frac{1}{2(4+x^2)} dx \\ &= \frac{1}{2} \int \frac{1}{2^2+x^2} dx & \therefore \int \frac{1}{a^2+x^2} dx \\ &= \frac{1}{a} \tan^{-1} \frac{x}{a} + c \\ &= \frac{1}{2} \left[\frac{1}{2} \tan^{-1} \left(\frac{x}{2} \right) \right] + c \\ &= \frac{1}{4} \tan^{-1} \left(\frac{x}{2} \right) + c. \end{aligned}$$

21. Find $\int \frac{x^2}{\sqrt{1-x^6}} dx$.

Sol: $\int \frac{x^2}{\sqrt{1-x^6}} dx$

$$= \int \frac{x^2}{\sqrt{1-(x^3)^2}} dx$$

Let $x^3 = t$
 $3x^2 .dx = dt$

$$x^2 dx = \frac{dt}{3}$$

$$= \int \frac{1}{\sqrt{1-t^2}} \frac{dt}{3}$$

$$= \frac{1}{3} \sin^{-1}(t) + c$$

$$= \frac{1}{3} \sin^{-1}(x^3) + c.$$

22. Find $\int \frac{x^2}{\sqrt{x+5}} dx$.

Sol: $\int \frac{x^2}{\sqrt{x+5}} dx$

Let $x+5 = t^2$
 $x = t^2 - 5$
 $dx = 2t.dt$

$$= \int \frac{(t^2-5)^2}{\sqrt{t^2}} 2t.dt$$

$$= 2 \int (t^4 - 10t^2 + 25) dt$$

$$= 2 \left(\frac{t^5}{5} - \frac{10t^3}{3} + 25t \right) + c$$

$$= \frac{2}{5} (x+5)^{5/2} - \frac{20}{3} (x+5)^{3/2} + 50(x+5)^{1/2} + c.$$

23. Find $\int \frac{1}{\sqrt{2x-3x^2+1}} dx$.

Sol: $\int \frac{1}{\sqrt{2x-3x^2+1}} dx$

Now $-3x^2 + 2x + 1$

$$= -3 \left[x^2 - \frac{2}{3}x - \frac{1}{3} \right]$$

$$= -3 \left[\left(x - \frac{1}{3} \right)^2 - \frac{1}{3} - \left(\frac{1}{3} \right)^2 \right]$$

$$= -3 \left[\left(x - \frac{1}{3} \right)^2 - \frac{1}{3} - \frac{1}{9} \right]$$

$$= -3 \left[\left(x - \frac{1}{3} \right)^2 - \frac{4}{9} \right]$$

$$= 3 \left[\frac{4}{9} - \left(x - \frac{1}{3} \right)^2 \right]$$

$$= \int \frac{1}{\sqrt{3 \left[\frac{4}{9} - \left(x - \frac{1}{3} \right)^2 \right]}} dx$$

$$= \frac{1}{\sqrt{3}} \int \frac{1}{\sqrt{\left(\frac{2}{3} \right)^2 - \left(x - \frac{1}{3} \right)^2}} dx$$

$$= \frac{1}{\sqrt{3}} \sin^{-1} \left(\frac{x - \frac{1}{3}}{\frac{2}{3}} \right) + c$$

$$= \frac{1}{\sqrt{3}} \sin^{-1} \left(\frac{3x-1}{2} \right) + c.$$

24. Evaluate $\int \frac{1}{\sqrt{x^2 + 2x + 10}} dx$.

Sol: $\int \frac{1}{\sqrt{x^2 + 2x + 10}} dx$

$$\begin{aligned} x^2 + 2x + 10 \\ &= (x+1)^2 + 10 - 1 \\ &= (x+1)^2 + 9 \\ &= (x+1)^2 + 3^2 \end{aligned}$$

$$= \int \frac{1}{\sqrt{(x+1)^2 + 3^2}} dx$$

$$= \sinh^{-1} \left(\frac{x+1}{3} \right) + c.$$

25. Find $\int \sqrt{3+8x-3x^2} .dx$.

Sol: $\int \sqrt{3+8x-3x^2} .dx$

Now $-3x^2 + 8x + 3$

$$= -3 \left(x^2 - \frac{8}{3}x - 1 \right)$$

$$= -3 \left[\left(x - \frac{4}{3} \right)^2 - 1 - \left(\frac{4}{3} \right)^2 \right]$$

$$\begin{aligned}
&= -3 \left[\left(x - \frac{4}{3} \right)^2 - 1 - \frac{16}{9} \right] \\
&= -3 \left[\left(x - \frac{4}{3} \right)^2 - \frac{25}{9} \right] \\
&= 3 \left[\frac{25}{9} - \left(x - \frac{4}{3} \right)^2 \right] \\
&= \int \sqrt{3 \left(\frac{25}{9} - \left(x - \frac{4}{3} \right)^2 \right)} dx \\
&= \sqrt{3} \int \sqrt{\left(\frac{5}{3} \right)^2 - \left(x - \frac{4}{3} \right)^2} dx \\
&= \sqrt{3} \left\{ \left(x - \frac{4}{3} \right) \sqrt{\left(\frac{5}{3} \right)^2 - \left(x - \frac{4}{3} \right)^2} + \frac{25}{9} \sin^{-1} \left[\frac{x - \frac{4}{3}}{\frac{5}{3}} \right] \right\} + C \\
&= \sqrt{3} \left\{ \left(\frac{3x-4}{6} \right) \sqrt{\left(\frac{5}{3} \right)^2 - \left(x - \frac{4}{3} \right)^2} + \frac{25}{18} \sin^{-1} \left(\frac{3x-4}{5} \right) \right\} + C.
\end{aligned}$$

26. Find $\int x.e^x dx$.

Sol: $\int x.e^x dx$

$$u = x, v = e^x$$

$$\int v = \int e^x dx = e^x$$

By using integration by parts $\int uv = u \int v - \int [u'v]$

$$\begin{aligned}
&= x.e^x - \int (1.e^x) dx \\
&= x.e^x - \int e^x dx \\
&= x.e^x - e^x + C.
\end{aligned}$$

27. Find $\int x.\sin^2 x dx$.

Sol: $\int x.\sin^2 x dx$

$$\begin{aligned}
&= \int x \left(\frac{1-\cos 2x}{2} \right) dx \\
&= \int \left(\frac{x}{2} - \frac{1}{2} x \cos 2x \right) dx \\
&= \frac{1}{2} \int x dx - \frac{1}{2} \int x \cos 2x dx \\
&\quad u = x; v = \cos 2x, \\
&\quad \int v = \int \cos 2x dx = \frac{\sin 2x}{2}
\end{aligned}$$

$$\begin{aligned}
&\text{By using integration by parts } \int uv = u \int v - \int [u'v] \\
&= \frac{1}{2} \left(\frac{x^2}{2} \right) - \frac{1}{2} \left\{ x \cdot \frac{\sin 2x}{2} - \int 1 \cdot \frac{\sin 2x}{2} dx \right\} \\
&= \frac{x^2}{4} - \frac{x \sin 2x}{4} + \frac{1}{4} \int \sin 2x dx \\
&= \frac{x^2}{4} - \frac{x \sin 2x}{4} + \frac{1}{4} \left(\frac{-\cos 2x}{2} \right) + C \\
&= \frac{x^2}{4} - \frac{x \sin 2x}{4} - \left(\frac{\cos 2x}{8} \right) + C.
\end{aligned}$$

28. Find $\int \sin^{-1} x dx$.

Sol: $\int \sin^{-1} x dx = \int \sin^{-1} x \cdot 1 dx.$

$$u = \sin^{-1} x, v = 1$$

$$\int v = \int 1 dx = x$$

By using integration by parts $\int uv = u \int v - \int [u'v]$

$$\begin{aligned}
&= \sin^{-1} x \cdot x - \int \frac{1}{\sqrt{1-x^2}} \cdot x dx \\
&= x \sin^{-1} x - \frac{1}{2} \int \frac{-2x}{\sqrt{1-x^2}} dx \\
&= x \sin^{-1} x + \frac{1}{2} \int \frac{2\sqrt{1-x^2}}{x} dx \\
&= x \sin^{-1} x + \sqrt{1-x^2} + C.
\end{aligned}$$

29. Find $\int e^x (\sin x + \cos x) dx$.

Sol: $\int e^x (\sin x + \cos x) dx$

$$f(x) = \sin x, f'(x) = \cos x$$

$$= \int e^x [f(x) + f'(x)] dx$$

$$= e^x f(x) + C$$

$$= e^x \sin x + C.$$

30. Evaluate $\int \cos \sqrt{x} dx$.

Sol: $\int \cos \sqrt{x} dx$

$$\text{Let } \sqrt{x} = t$$

$$\begin{aligned}
\frac{1}{2\sqrt{x}} dx &= dt \\
dx &= 2\sqrt{x} dt \\
&= 2t dt
\end{aligned}$$

$$\begin{aligned}
&= \int \cos t \cdot 2t dt \\
&\quad u = 2t, v = \cos t \\
&\quad \int v = \int \cos t dt = \sin t
\end{aligned}$$

By using integration by parts $\int uv = u \int v - \int [u' \int v]$

$$\begin{aligned} &= 2t \cdot \sin t - \int 2 \cdot \sin t \cdot dt \\ &= 2t \cdot \sin t - 2(-\cos t) + c \\ &= 2\sqrt{x} \sin \sqrt{x} + 2\cos \sqrt{x} + c. \end{aligned}$$

31. Find $\int x^n \log x \, dx$.

Sol: $\int x^n \log x \, dx$ $u = \log x, v = x^n$

$$\int v = \int x^n \, dx = \frac{x^{n+1}}{n+1}$$

By using integration by parts $\int uv = u \int v - \int [u' \int v]$

$$\begin{aligned} &= \log x \cdot \frac{x^{n+1}}{n+1} - \int \frac{1}{x} \cdot \frac{x^{n+1}}{n+1} \cdot dx \\ &= \log x \cdot \frac{x^{n+1}}{n+1} - \frac{1}{n+1} \int x^n \, dx \\ &= \log x \frac{x^{n+1}}{n+1} - \frac{1}{n+1} \left(\frac{x^{n+1}}{n+1} \right) + c. \end{aligned}$$

32. Find $\int \frac{1+\sin^2 x}{1+\cos 2x} \, dx$.

Sol: $\int \frac{1+\sin^2 x}{1+\cos 2x} \, dx$

$$\begin{aligned} &= \int \frac{1+\sin^2 x}{2\cos^2 x} \, dx \\ &= \int \left(\frac{1}{2\cos^2 x} + \frac{\sin^2 x}{2\cos^2 x} \right) dx \\ &= \frac{1}{2} \int \sec^2 x \, dx + \frac{1}{2} \int \tan^2 x \, dx \\ &= \frac{1}{2} \tan x + \frac{1}{2} \int (\sec^2 x - 1) dx \\ &= \frac{1}{2} \tan x + \frac{1}{2} (\tan x - x) + c \\ &= \frac{1}{2} \tan x + \frac{1}{2} \tan x - \frac{x}{2} + c \\ &= \tan x - \frac{x}{2} + c. \end{aligned}$$

33. Find $\int \frac{e^{\sin^{-1} x}}{\sqrt{1-x^2}} \, dx$.

Sol: $\int \frac{e^{\sin^{-1} x}}{\sqrt{1-x^2}} \, dx$

Let $\sin^{-1} x = t$

$$\frac{1}{\sqrt{1-x^2}} \cdot dx = dt$$

$$\begin{aligned} &= \int e^t \cdot dt \\ &= e^t + c \\ &= e^{\sin^{-1} x} + c. \end{aligned}$$

34. Evaluate $\int \frac{1-\tan x}{1+\tan x} \, dx$.

Sol: $\int \frac{1-\tan x}{1+\tan x} \, dx$

$$\begin{aligned} &= \int \frac{1-\frac{\sin x}{\cos x}}{1+\frac{\sin x}{\cos x}} \cdot dx \\ &= \int \frac{\cos x - \sin x}{\cos x + \sin x} \, dx \end{aligned}$$

Let $\cos x + \sin x = t$
 $(-\sin x + \cos x) dx = dt$

$$\begin{aligned} &= \int \frac{1}{t} \cdot dt \\ &= \log |t| + c \\ &= \log |\cos x + \sin x| + c. \end{aligned}$$

35. Find $\int \frac{1}{e^x + e^{-x}} \, dx$.

Sol: $\int \frac{1}{e^x + e^{-x}} \, dx$

$$\begin{aligned} &= \int \frac{1}{e^x + \frac{1}{e^x}} \, dx \\ &= \int \frac{1}{\frac{(e^x)^2 + 1}{e^x}} \cdot \frac{e^x}{e^x} \, dx \\ &= \int \frac{e^x}{(e^x)^2 + 1} \, dx \\ &= \int \frac{1}{t^2 + 1} dt \end{aligned}$$

Let $e^x = t$
 $e^x dx = dt$

$$\begin{aligned} &= \tan^{-1}(t) + c \\ &= \tan^{-1}(e^x) + c. \end{aligned}$$

36. Evaluate $\int \frac{x-1}{(x-2)(x-3)} \, dx$.

Sol: $\int \frac{x-1}{(x-2)(x-3)} \, dx$

$$\begin{aligned} &= \int \left(\frac{-1}{x-2} + \frac{2}{x-3} \right) dx \\ &= -\int \frac{1}{x-2} dx + 2 \int \frac{1}{x-3} dx \\ &= -\log|x-2| + 2 \log|x-3| + c. \end{aligned}$$

37. Evaluate $\int \frac{dx}{(x-1)(x-2)}$.

Sol:
$$\begin{aligned} & \int \frac{dx}{(x-1)(x-2)} dx \\ &= \int \left(\frac{1}{x-2} - \frac{1}{x-1} \right) dx \\ &= \int \frac{1}{x-2} dx - \int \frac{1}{x-1} dx \\ &= \log|x-2| - \log|x-1| + c \\ &= \log \left| \frac{x-2}{x-1} \right| + c. \end{aligned}$$

38. Evaluate $\int \frac{\sin 2x}{a \cos^2 x + b \sin^2 x} dx$.

Sol:
$$\begin{aligned} & \int \frac{\sin 2x}{a \cos^2 x + b \sin^2 x} dx \\ & \text{Let } a \cos^2 x + b \sin^2 x = t \\ & [a 2 \cos x (-\sin x) + b 2 \sin x (\cos x)] dx = 1.dt \\ & 2 \sin x \cos x (b-a) . dx = dt \\ & (b-a) \sin 2x dx = dt \\ & = \int \frac{1}{t} dt = \log|t| + c \\ & = \log|a \cos^2 x + b \sin^2 x| + c. \end{aligned}$$

39. Evaluate $\int e^x \left(\frac{1-\sin x}{1-\cos x} \right) dx$.

Sol:
$$\begin{aligned} & \int e^x \left(\frac{1-\sin x}{1-\cos x} \right) dx \\ &= \int e^x \left(\frac{1-\sin x}{2 \sin^2(x/2)} \right) dx \\ &= \int e^x \left[\frac{1}{2 \sin^2(x/2)} - \frac{\sin x}{2 \sin^2(x/2)} \right] dx \\ &= \int e^x \left[\frac{1}{2 \sin^2(x/2)} - \frac{2 \sin(x/2) \cos(x/2)}{2 \sin^2(x/2)} \right] dx \\ &= \int e^x \left[\frac{1}{2} \operatorname{cosec}^2(x/2) - \cot(x/2) \right] dx \\ & \text{Let } f(x) = -\cot(x/2) \end{aligned}$$

$$f'(x) = -[-\operatorname{cosec}^2(x/2)] \cdot \frac{1}{2} dx$$

$$= \operatorname{cosec}^2 \frac{x}{2} \cdot \frac{1}{2} dx$$

$$= \int e^x [f(x) + f'(x)].dx$$

$$= e^x f(x) + c$$

$$= e^x [-\cot x/2] + c.$$

40. Evaluate $\int \frac{x e^x}{(1+x)^2} dx$.

Sol:
$$\begin{aligned} & \int \frac{x e^x}{(1+x)^2} dx \\ &= \int \frac{e^x(x)}{(1+x)^2} dx \\ &= \int e^x \left[\frac{1+x-1}{(1+x)^2} \right] dx \\ &= \int e^x \left[\frac{1+x}{(1+x)^2} - \frac{1}{(1+x)^2} \right] dx \\ &= \int e^x \left[\frac{1}{1+x} + \frac{-1}{(1+x)^2} \right] dx \\ & f(x) = \frac{1}{1+x}, f'(x) = \frac{-1}{(1+x)^2} \\ &= \int e^x [f(x) + f'(x)] dx \\ &= e^x \cdot f(x) + c = e^x \frac{1}{1+x} + c. \end{aligned}$$

41. Evaluate $\int \frac{\sin \theta}{\sqrt{2 - \cos^2 \theta}} d\theta$.

Sol:
$$\int \frac{\sin \theta}{\sqrt{2 - \cos^2 \theta}} d\theta$$

Let $\cos \theta = t$
 $-\sin \theta . d\theta = dt$
 $\sin \theta . d\theta = -dt$

$$\begin{aligned} &= \int \frac{1}{\sqrt{2-t^2}} (-dt) = - \int \frac{1}{\sqrt{(\sqrt{2})^2 - t^2}} dt \\ &= -\sin^{-1} \left(\frac{t}{\sqrt{2}} \right) + c \\ &= -\sin^{-1} \left(\frac{\cos \theta}{\sqrt{2}} \right) + c. \end{aligned}$$

LEVEL - II (LAQ)

1. $\int \sqrt{\frac{5-x}{x-2}} dx$ on (2, 5).

Sol: Given $\int \sqrt{\frac{5-x}{x-2}} dx = \int \sqrt{\frac{5-x}{x-2} \cdot \frac{5-x}{5-x}} dx$

$$= \int \frac{5-x}{\sqrt{5x-10-x^2+2x}} dx$$

$$= \int \frac{5-x}{\sqrt{7x-10-x^2}} dx$$

Let $5-x = A \frac{d}{dx}(7x-10-x^2) + B$

$$\Rightarrow 5-x = A(7-2x) + B$$

Coefficient of $x \Rightarrow -2A = -1$

$$\Rightarrow A = \frac{1}{2}$$

Put $x = 0 \Rightarrow 7A + B = 5$

$$\Rightarrow 7\left(\frac{1}{2}\right) + B = 5$$

$$\Rightarrow B = 5 - \frac{7}{2} = \frac{3}{2}$$

$$\therefore 5-x = \frac{1}{2}(7-2x) + \frac{3}{2}$$

then $\int \frac{5-x}{\sqrt{7x-10-x^2}} dx = \int \frac{\frac{1}{2}(7-2x) + \frac{3}{2}}{\sqrt{7x-10-x^2}} dx$

$$= \int \frac{1}{2} \frac{(7-2x)}{\sqrt{7x-10-x^2}} dx + \frac{3}{2} \int \frac{1}{\sqrt{7x-10-x^2}} dx$$

$$\therefore -x^2 + 7x - 10 = -[x^2 - 7x + 10]$$

$$= - \left[\left(x - \frac{7}{2} \right)^2 + 10 - \frac{49}{4} \right]$$

$$= - \left[\left(x - \frac{7}{2} \right)^2 - \frac{9}{4} \right]$$

$$= \frac{9}{4} - \left[x - \frac{7}{2} \right]^2$$

$$= \frac{1}{2} \int \frac{(7-2x)}{\sqrt{7x-10-x^2}} dx + \frac{3}{2} \int \frac{1}{\sqrt{\left(\frac{3}{2}\right)^2 - \left(x - \frac{7}{2}\right)^2}} dx$$

$$\left\{ \int \frac{f'(x)}{\sqrt{f(x)}} dx = 2\sqrt{f(x)} + C, \int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}(x/a) + C \right\}$$

$$= \frac{1}{2} \cdot 2\sqrt{7x-10-x^2} + \frac{3}{2} \sin^{-1}\left[\frac{x-\frac{7}{2}}{\frac{3}{2}}\right] + C$$

$$= \sqrt{7x-10-x^2} + \frac{3}{2} \sin^{-1}\left[\frac{2x-7}{3}\right] + C.$$

13. Evaluate $\int \frac{x+1}{x^2+3x+12} dx$.

Sol: Let $x+1 = A \frac{d}{dx}(x^2+3x+12) + B$

$$\Rightarrow x+1 = A(2x+3) + B$$

Comparing both sides,

$$1 = 2A \Rightarrow A = \frac{1}{2}$$

$$1 = 3A + B \Rightarrow B = -\frac{1}{2}$$

Now $\int \frac{x+1}{x^2+3x+12} dx$

$$= \frac{1}{2} \int \frac{2x+3}{x^2+3x+12} dx - \frac{1}{2} \int \frac{dx}{x^2+3x+12}$$

$$= \frac{1}{2} \log|x^2+3x+12| - \frac{1}{2} \int \frac{dx}{\left(x+\frac{3}{2}\right)^2 + \left(\frac{\sqrt{39}}{2}\right)^2}$$

$$= \frac{1}{2} \log|x^2+3x+12| - \frac{1}{\left(\frac{\sqrt{39}}{2}\right)} \tan^{-1}\left(\frac{x+\frac{3}{2}}{\left(\frac{\sqrt{39}}{2}\right)}\right) + C$$

$$= \frac{1}{2} \log|x^2+3x+12| - \frac{1}{\sqrt{39}} \tan^{-1}\left(\frac{2x+3}{\sqrt{39}}\right) + C.$$

3. Evaluate $\int \frac{1}{1+\sin x + \cos x} dx$.

Sol: $\int \frac{1}{1+\sin x + \cos x} dx$

Let $\tan \frac{x}{2} = t$. $\sin x = \frac{2t}{1+t^2}$

$$\Rightarrow dx = \frac{2dt}{1+t^2} \quad \cos x = \frac{1-t^2}{1+t^2}$$

$$\int \frac{1}{1+\sin x + \cos x} dx$$

$$= \int \frac{1}{1+\frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2}} \cdot \frac{2dt}{1+t^2}$$

$$= \int \frac{1+t^2}{1+t^2+2t+1-t^2} \cdot \frac{2dt}{1+t^2}$$

$$= 2 \int \frac{1}{2+2t} dt$$

$$= \frac{1}{2} \int \frac{1}{1+t} dt = \log |1+t| + c$$

$$= \log \left| 1 + \tan \frac{x}{2} \right| + c.$$

4. Find $\int \frac{1}{3\cos x + 4\sin x + 6} dx$

Sol: Let $t = \tan \left(\frac{x}{2} \right)$ $\sin x = \frac{2t}{1+t^2}$

$$dx = \frac{2dt}{1+t^2} \quad \cos x = \frac{1-t^2}{1+t^2}$$

$$\int \frac{1}{3\cos x + 4\sin x + 6} dx$$

$$= \int \frac{1}{3\left(\frac{1-t^2}{1+t^2}\right) + 4\left(\frac{2t}{1+t^2}\right) + 6} \cdot \frac{2dt}{1+t^2}$$

$$= \int \frac{1}{3-3t^2+8t+6+6t^2} \cdot \frac{2dt}{1+t^2}$$

$$= \int \frac{1}{3t^2+8t+9} \cdot 2dt$$

$$3t^2+8t+9$$

$$= 3 \left[t^2 + \frac{8}{3}t + 3 \right]$$

$$= 3 \left[\left(t + \frac{4}{3} \right)^2 + 3 - \left(\frac{4}{3} \right)^2 \right]$$

$$= 3 \left[\left(t + \frac{4}{3} \right)^2 + 3 - \frac{16}{9} \right]$$

$$= 3 \left[\left(t + \frac{4}{3} \right)^2 + \frac{11}{9} \right]$$

$$= \int \frac{1}{3 \left[\left(t + \frac{4}{3} \right)^2 + \frac{11}{9} \right]} \cdot 2dt$$

$$= \frac{2}{3} \int \frac{1}{\left(t + \frac{4}{3} \right)^2 + \left(\frac{\sqrt{11}}{3} \right)^2} dt$$

$$= \frac{2}{3} \cdot \frac{1}{\sqrt{11}} \tan^{-1} \left[\frac{t + \frac{4}{3}}{\frac{\sqrt{11}}{3}} \right] + c$$

$$= \frac{2}{\sqrt{11}} \tan^{-1} \left(\frac{3t+4}{\sqrt{11}} \right) + c$$

$$= \frac{2}{\sqrt{11}} \tan^{-1} \left(\frac{3 \tan \left(\frac{x}{2} \right) + 4}{\sqrt{11}} \right) + c.$$

5. Evaluate $\int \frac{2\cos x + 3\sin x}{4\cos x + 5\sin x} dx$.

Sol: Nr = A(Dr) + B $\frac{d}{dx}(Dr)$

$$2\cos x + 3\sin x = A(4\cos x + 5\sin x)$$

$$+ B \frac{d}{dx}(4\cos x + 5\sin x)$$

$$2\cos x + 3\sin x = A(4\cos x + 5\sin x) + B(-4\sin x + 5\cos x)$$

equating the co-efficients of

$$\cos x \rightarrow 2 = 4A + 5B, \Rightarrow 4A + 5B - 2 = 0$$

$$\sin x \rightarrow 3 = 5A - 4B, \Rightarrow 5A - 4B - 3 = 0$$

solve

$$\begin{array}{rrrr} 5 & -2 & 4 & 5 \\ -4 & -3 & 5 & -4 \end{array}$$

$$\frac{A}{-15 - 8} = \frac{B}{-10 + 12} = \frac{1}{-16 - 25}$$

$$\frac{A}{-23} = \frac{B}{+2} = \frac{1}{-14}$$

$$A = \frac{-23}{-41} = \frac{23}{41}, \quad B = \frac{2}{-14}$$

$$\therefore 2\cos x + 3\sin x = \frac{23}{41}(4\cos x + 5\sin x) - \frac{2}{41} \frac{d}{dx}(4\cos x + 5\sin x)$$

$$\therefore \int \frac{2\cos x + 3\sin x}{4\cos x + 5\sin x} dx$$

$$= \int \frac{\frac{23}{41}(4\cos x + 5\sin x) - \frac{2}{41} \frac{d}{dx}(4\cos x + 5\sin x)}{4\cos x + 5\sin x} dx$$

$$= \frac{23}{41} \int \frac{4\cos x + 5\sin x}{4\cos x + 5\sin x} dx - \frac{2}{41} \int \frac{\frac{d}{dx}(4\cos x + 5\sin x)}{4\cos x + 5\sin x} dx$$

$$= \frac{23}{41} \int 1 dx - \frac{2}{41} \log |4\cos x + 5\sin x| + C$$

$$= \frac{23}{41} x - \frac{2}{41} \log |4\cos x + 5\sin x| + C$$

6. Evaluate $\int \frac{\sin x \cos x}{\cos^2 x + 3\cos x + 2} dx$.

Sol: Given $\int \frac{\sin x \cos x}{\cos^2 x + 3\cos x + 2} dx$

$$\begin{aligned} \text{Put } \cos x &= t \\ -\sin x dx &= dt \\ \Rightarrow \sin x dx &= -dt \end{aligned}$$

$$= \int \frac{t}{t^2 + 3t + 2} (-dt)$$

$$= - \int \frac{t}{(t+1)(t+2)} dt$$

$$= \text{Let } \frac{-t}{(t+1)(t+2)} = \frac{A}{t+1} + \frac{B}{t+2}$$

$$-t = A(t+2) + B(t+1)$$

$$\text{Put } t = -1, \quad \text{Put } t = -2,$$

$$1 = A(-1+2) \quad 2 = B(-2+1)$$

$$\Rightarrow A = 1 \quad 2 = -B \Rightarrow B = -2$$

$$= \int \frac{t}{(t+1)(t+2)} dt = \int \left(\frac{1}{t+1} - \frac{2}{t+2} \right) dt$$

$$\int \frac{1}{t+1} dt - 2 \int \frac{1}{t+2} dt$$

$$= \log |t+1| - 2\log |t+2| + C$$

$$= \log |\cos x + 1| - 2\log |\cos x + 2| + C.$$

7. Show that $\int \sqrt{a^2 - x^2} dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \left(\frac{x}{a} \right) + C$

Sol: $I = \int \sqrt{a^2 - x^2} dx$

$$= \int \sqrt{a^2 - x^2} \cdot 1 dx$$

$$u = \sqrt{a^2 - x^2}, \quad v = 1$$

$$\int v = \int 1 dx = x$$

By using integration by parts, $\int uv = u \int v - \int [u' \int v]$

$$I = \sqrt{a^2 - x^2} \cdot x - \int \frac{1}{2\sqrt{a^2 - x^2}} (-2x) \cdot x dx$$

$$I = \sqrt{a^2 - x^2} \cdot x - \int \frac{-x^2}{\sqrt{a^2 - x^2}} dx$$

$$I = x\sqrt{a^2 - x^2} - \int \frac{a^2 - x^2 - a^2}{\sqrt{a^2 - x^2}} dx$$

$$I = x\sqrt{a^2 - x^2} - \int \left(\frac{a^2 - x^2}{\sqrt{a^2 - x^2}} - \frac{a^2}{\sqrt{a^2 - x^2}} \right) dx$$

$$I = x\sqrt{a^2 - x^2} - \int \left(\sqrt{a^2 - x^2} - \frac{a^2}{\sqrt{a^2 - x^2}} \right) dx$$

$$I = x\sqrt{a^2 - x^2} - \int \sqrt{a^2 - x^2} dx + a^2 \int \frac{1}{\sqrt{a^2 - x^2}} dx$$

$$I = x\sqrt{a^2 - x^2} - I + a^2 \cdot \sin^{-1} \left(\frac{x}{a} \right)$$

$$I + I = x\sqrt{a^2 - x^2} + a^2 \cdot \sin^{-1} \left(\frac{x}{a} \right)$$

$$2I = x\sqrt{a^2 - x^2} + a^2 \sin^{-1} \left(\frac{x}{a} \right)$$

$$I = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \left(\frac{x}{a} \right) + C$$

8. Evaluate $\int \frac{1}{4\cos x + 3\sin x} dx$.

Sol: $\int \frac{1}{4\cos x + 3\sin x} dx$

Multiply and divide with $\sqrt{4^2 + 3^2} = 5$ in Dr

$$= \int \frac{1}{5\left(\frac{4}{5}\cos x + \frac{3}{5}\sin x\right)} dx$$

$$\text{Let } \cos \alpha = \frac{4}{5}, \sin \alpha = \frac{3}{5}$$

$$= \frac{1}{5} \int \frac{1}{\cos x \cos \alpha + \sin x \sin \alpha} dx$$

$$= \frac{1}{5} \int \frac{1}{\cos(x - \alpha)} dx$$

$$= \frac{1}{5} \int \sec(x - \alpha) dx$$

$$= \frac{1}{5} \log |\sec(x - \alpha) + \tan(x - \alpha)| + C$$

$$\text{where } \tan \alpha = \frac{3}{4}, \alpha = \tan^{-1}\left(\frac{3}{4}\right).$$

9. Evaluate $\int \frac{1}{5 + 4\cos x} dx$.

Sol: $\int \frac{1}{5 + 4\cos x} dx$

$$\text{Let } t = \tan x/2$$

$$dx = \frac{2dt}{1+t^2}, \cos x = \frac{1-t^2}{1+t^2}$$

$$= \int \frac{1}{5 + 4\left(\frac{1-t^2}{1+t^2}\right)} \cdot \frac{2dt}{1+t^2}$$

$$= \int \frac{1}{5 + 5t^2 + 4 - 4t^2} \cdot \frac{2dt}{1+t^2}$$

$$= 2 \int \frac{1}{9+t^2} dt$$

$$= 2 \cdot \frac{1}{3} \tan^{-1}\left(\frac{t}{3}\right) + C$$

$$= \frac{2}{3} \tan^{-1}\left[\frac{\tan\left(\frac{x}{2}\right)}{3}\right] + C.$$

10. Evaluate $\int \frac{1}{2 - 3\cos 2x} dx$.

Sol: $\int \frac{1}{2 - 3\cos 2x} dx$

$$\text{Let } t = \tan x$$

$$dx = \frac{dt}{1+t^2}, \cos 2x = \frac{1-t^2}{1+t^2}$$

$$= \int \frac{1}{2 - 3\left(\frac{1-t^2}{1+t^2}\right)} \cdot \frac{dt}{1+t^2}$$

$$= \int \frac{1}{\frac{2+2t^2 - 3+3t^2}{1+t^2}} \cdot \frac{dt}{1+t^2}$$

$$= \int \frac{1}{5t^2 - 1} dt$$

$$= \int \frac{1}{5\left(t^2 - \frac{1}{5}\right)} dt$$

$$= \frac{1}{5} \int \frac{1}{t^2 - \left(\frac{1}{\sqrt{5}}\right)^2} dt$$

$$= \frac{1}{5} \cdot \frac{1}{2 \cdot \frac{1}{\sqrt{5}}} \log \left| \frac{t - \frac{1}{\sqrt{5}}}{t + \frac{1}{\sqrt{5}}} \right| + C$$

$$= \frac{1}{2\sqrt{5}} \log \left| \frac{\sqrt{5}t - 1}{\sqrt{5}t + 1} \right| + C$$

$$= \frac{1}{2\sqrt{5}} \log \left| \frac{\sqrt{5} \tan x - 1}{\sqrt{5} \tan x + 1} \right| + C.$$

11. Evaluate $\int \frac{\sin x \cos x}{\cos^2 x + 3 \cos x + 2} dx$.

Sol: $\int \frac{\sin x \cos x}{\cos^2 x + 3 \cos x + 2} dx$

$$\begin{aligned} &\text{Let } \cos x = t \\ &-\sin x \cdot dx = dt \\ &\sin x \cdot dx = -dt \end{aligned}$$

$$= \int \frac{t}{t^2 + 3t + 2} \cdot -dt$$

$$= \int \frac{-t}{(t+1)(t+2)} dt$$

$$\begin{aligned}
 &= \int \left[\frac{1}{t+1} + \frac{-2}{t+2} \right] dt \\
 &= \int \frac{1}{t+1} dt - 2 \int \frac{1}{t+2} dt \\
 &= \log |t+1| - 2 \log |t+2| + c \\
 &= \log |\cos x + 1| - 2 \log |\cos x + 2| + c.
 \end{aligned}$$

12. Evaluate $\int x \cos^{-1} x . dx$.

Sol: $\int x \cos^{-1} x . dx$

$$\begin{aligned}
 u &= \cos^{-1} x, v = x \\
 \int v &= \int x . dx = \frac{x^2}{2}
 \end{aligned}$$

By using integration by parts, $\int uv = u \int v - \int [u' \int v]$

$$\begin{aligned}
 &= \cos^{-1} x \cdot \frac{x^2}{2} - \int \frac{-1}{\sqrt{1-x^2}} \cdot \frac{x^2}{2} . dx \\
 &= \cos^{-1} x \cdot \frac{x^2}{2} + \frac{1}{2} \int \frac{x^2}{\sqrt{1-x^2}} dx \\
 &\quad x = \sin \theta \\
 &\quad dx = \cos \theta . d\theta \\
 &= \cos^{-1} x \cdot \frac{x^2}{2} + \frac{1}{2} \int \frac{\sin^2 \theta}{\sqrt{1-\sin^2 \theta}} . \cos \theta . d\theta \\
 &= \cos^{-1} x \cdot \frac{x^2}{2} + \frac{1}{2} \int \sin^2 \theta . d\theta \\
 &= \cos^{-1} x \cdot \frac{x^2}{2} + \frac{1}{2} \int \frac{1-\cos 2\theta}{2} d\theta \\
 &= \cos^{-1} x \cdot \frac{x^2}{2} + \frac{1}{4} \int (1-\cos 2\theta) . d\theta \\
 &= \cos^{-1} x \cdot \frac{x^2}{2} + \frac{1}{4} \left(\theta - \frac{\sin 2\theta}{2} \right) + c \\
 &= \cos^{-1} x \cdot \frac{x^2}{2} + \frac{1}{4} \left(\theta - \frac{2 \sin \theta \cos \theta}{2} \right) + c \\
 &= \cos^{-1} x \cdot \frac{x^2}{2} + \frac{1}{4} (\sin^{-1} x - x \sqrt{1-x^2}) + c.
 \end{aligned}$$

13. Using integration by parts, evaluate $\int e^x \cos x dx$.

Sol: $I = \int e^x \cos x dx$

$$\begin{aligned}
 u &= \cos x, v = e^x \\
 \int v &= \int e^x dx = e^x
 \end{aligned}$$

By using integration by parts, $\int uv = u \int v - \int [u' \int v]$

$$I = \cos x . e^x - \int (-\sin x) e^x . dx$$

$$\begin{aligned}
 I &= e^x \cos x + \int \sin x . e^x . dx \\
 u &= \sin x, v = e^x
 \end{aligned}$$

$$\int v = \int e^x . dx = e^x$$

By using integration by parts, $\int uv = u \int v - \int [u' \int v]$

$$I = e^x \cos x + \sin x e^x - \int \cos x . e^x . dx$$

$$I = e^x \cos x + e^x \sin x - I$$

$$I + I = e^x [\cos x + \sin x]$$

$$2I = e^x [\cos x + \sin x]$$

$$I = \frac{e^x}{2} [\cos x + \sin x] + c$$

14. Evaluate $\int \frac{1}{(1-x)(4+x^2)} dx$.

Sol: $\int \frac{1}{(1-x)(4+x^2)} dx$

$$\frac{1}{(1-x)(4+x^2)} = \frac{A}{1-x} + \frac{Bx+C}{x^2+4}$$

$$1 = A(x^2+4) + (Bx+C)(1-x) \quad \dots (1)$$

$$Put \ 1-x=0, x=1 \text{ in (1)}$$

$$1 = A(1+4) + 0$$

$$1 = 5A \Rightarrow A = 1/5$$

$$Put x=0 \text{ in (1)}$$

$$1 = A(0+4) + (B.0+C)(1-0)$$

$$1 = 4A + C$$

$$C = 1 - 4A = 1 - 4 \left(\frac{1}{5} \right)$$

$$\Rightarrow C = \frac{1}{5}$$

Equating the co-efficients of x^2 in (1)

$$0 = A - B$$

$$A = B$$

$$\therefore B = \frac{1}{5}$$

$$\therefore \frac{1}{(1-x)(4+x^2)} = \frac{\frac{1}{5}}{1-x} + \frac{\frac{1}{5}x+\frac{1}{5}}{x^2+4}$$

$$\int \frac{1}{(1-x)(4+x^2)} dx$$

$$= \frac{1}{5} \int \frac{1}{1-x} dx + \frac{1}{5} \int \frac{x+1}{x^2+4} dx$$

$$= \frac{1}{5} \left[\log|1-x| \right]_{-1} + \frac{1}{5} \int \frac{x}{x^2+4} + \frac{1}{x^2+4} dx$$

$$\begin{aligned}
 &= \frac{-1}{5} \log |1-x| + \frac{1}{5} \int \frac{x}{x^2+4} dx + \frac{1}{5} \int \frac{1}{x^2+2^2} dx \\
 &= \frac{-1}{5} \log |1-x| + \frac{1}{5} \left(\frac{1}{2} \right) \int \frac{2x}{x^2+4} dx + \frac{1}{5} \int \frac{1}{x^2+2^2} dx \\
 &= \frac{-1}{5} \log |1-x| + \frac{1}{10} \log |x^2+4| + \frac{1}{5} \cdot \frac{1}{2} \tan^{-1} \left(\frac{x}{2} \right) + C \\
 &= \frac{-1}{5} \log |1-x| + \frac{1}{10} \log |x^2+4| + \frac{1}{10} \tan^{-1} \left(\frac{x}{2} \right) + C.
 \end{aligned}$$

15. Evaluate $\int \frac{1}{a \sin x + b \cos x} dx$.

Sol: $\int \frac{1}{a \sin x + b \cos x} dx$

multiply and divide with $\sqrt{a^2+b^2}$ in Denominator

$$= \int \frac{1}{\sqrt{a^2+b^2} \left[\frac{a}{\sqrt{a^2+b^2}} \sin x + \frac{b}{\sqrt{a^2+b^2}} \cos x \right]} dx$$

Let $\frac{a}{\sqrt{a^2+b^2}} = \sin \alpha, \frac{b}{\sqrt{a^2+b^2}} = \cos \alpha$

$$\begin{aligned}
 &= \frac{1}{\sqrt{a^2+b^2}} \int \frac{1}{\cos x \cos \alpha + \sin x \sin \alpha} dx \\
 &= \frac{1}{\sqrt{a^2+b^2}} \int \frac{1}{\cos(x-\alpha)} dx \\
 &= \frac{1}{\sqrt{a^2+b^2}} \int \sec(x-\alpha) dx \\
 &= \frac{1}{\sqrt{a^2+b^2}} \log |\sec(x-\alpha) + \tan(x-\alpha)| + C
 \end{aligned}$$

where $\alpha = \tan^{-1} \left(\frac{a}{b} \right)$.

16. Evaluate $\int \frac{1}{4 \sin^2 x + 9 \cos^2 x} dx$.

Sol: $\int \frac{1}{4 \sin^2 x + 9 \cos^2 x} dx$

Divide Nr & Dr with $\cos^2 x$

$$\begin{aligned}
 &= \int \frac{\frac{1}{\cos^2 x}}{4 \frac{\sin^2 x}{\cos^2 x} + 9 \cdot \frac{\cos^2 x}{\cos^2 x}} dx \\
 &= \int \frac{\sec^2 x}{4 \tan^2 x + 9} dx
 \end{aligned}$$

Let $\tan x = t$
 $\sec^2 x \cdot dx = dt$

$$\begin{aligned}
 &= \int \frac{1}{4t^2 + 9} dt \\
 &= \int \frac{1}{4(t^2 + \frac{9}{4})} dt \\
 &= \frac{1}{4} \int \frac{1}{t^2 + (\frac{3}{2})^2} dt \\
 &= \frac{1}{4} \cdot \frac{1}{(\frac{3}{2})} \tan^{-1} \left[\frac{t}{(\frac{3}{2})} \right] + C \\
 &= \frac{1}{6} \tan^{-1} \left(\frac{2t}{3} \right) + C \\
 &= \frac{1}{6} \tan^{-1} \left(\frac{2 \tan x}{3} \right) + C.
 \end{aligned}$$

17. If $I_{m,n} = \int \sin^m x \cos^n x dx$ **then prove that**

$$I_{m,n} = \frac{\cos^{n-1} x \sin^{m+1} x}{m+n} + \frac{n-1}{m+n} \cdot I_{m,n-2}$$

Sol: $I_{m,n} = \int \sin^m x \cos^n x dx$

$$= \int \sin^m x \cos^{n-1} x \cos x dx$$

$$u = \cos^{n-1} x, v = \sin^m x \cos x$$

$$\int v = \int \sin^m x \cos x dx$$

$$f(x) = \sin x, f'(x) = \cos x$$

$$= \frac{\sin^{m+1} x}{m+1}$$

By using integratin by parts, $\int uv = u \int v - \int [u' \int v]$

$$I_{m,n} = \cos^{n-1}x \cdot \frac{\sin^{m+1}x}{m+1} - \int (n-1) \cos^{n-2}x (-\sin x) \cdot \frac{\sin^{m+1}x}{m+1} dx$$

$$I_{m,n} = \frac{\cos^{n-1}x \cdot \sin^{m+1}x}{m+1} + \frac{n-1}{m+1} \int \cos^{n-2}x \cdot \sin^m x \cdot \sin^2 x dx$$

$$I_{m,n} = \frac{\cos^{n-1}x \cdot \sin^{m+1}x}{m+1} + \frac{n-1}{m+1} \int \cos^{n-2}x \cdot \sin^m x (1 - \cos^2 x) dx$$

$$I_{m,n} = \frac{\cos^{n-1}x \cdot \sin^{m+1}x}{m+1} + \frac{n-1}{m+1} \int (\cos^{n-2}x \cdot \sin^m x - \cos^n x \sin^m x) dx$$

$$I_{m,n} = \frac{\cos^{n-1}x \cdot \sin^{m+1}x}{m+1} + \frac{n-1}{m+1} \int \sin^m x \cos^{n-2} x dx - \frac{n-1}{m+1} \int \sin^m x \cos^n x dx$$

$$I_{m,n} = \frac{\cos^{n-1}x \cdot \sin^{m+1}x}{m+1} + \frac{n-1}{m+1} I_{m,n-2} - \frac{n-1}{m+1} I_{m,n}$$

$$I_{m,n} + \frac{n-1}{m+1} I_{m,n} = \frac{\cos^{n-1}x \cdot \sin^{m+1}x}{m+1} + \frac{n-1}{m+1} I_{m,n-2}$$

$$I_{m,n} \left[1 + \frac{n-1}{m+1} \right] = \frac{\cos^{n-1}x \cdot \sin^{m+1}x}{m+1} + \frac{n-1}{m+1} I_{m,n-2}$$

$$I_{m,n} \left[\frac{m+1+n-1}{m+1} \right] = \frac{\cos^{n-1}x \cdot \sin^{m+1}x}{m+1} + \frac{n-1}{m+1} I_{m,n-2}$$

$$I_{m,n} (m+n) = \cos^{n-1}x \cdot \sin^{m+1}x + (n-1) I_{m,n-2}$$

$$I_{m,n} = \frac{\cos^{n-1}x \cdot \sin^{m+1}x}{m+n} + \frac{n-1}{m+n} I_{m,n-2}$$

* * * * *