



AIMSTUTORIAL  
(9000 687 600)

Mathematics -IB  
Study Material

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# MATHEMATICS 1B INDEX

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**(A PLACE TO LEARN)**

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1. If Q (h, k) is the foot of perpendicular from P ( $x_1, y_1$ ) on the straight line  $ax + by + c = 0$ .

Then P.T  $\frac{h-x_1}{a} = \frac{k-y_1}{b} = -\frac{(ax_1+by_1+c)}{a^2+b^2}$ . find the foot of perpendicular of (4, 1) w.r.t the line  $3x - 4y + 12 = 0$ .

**Sol:** Q (h, k) is the foot of perpendicular from P ( $x_1, y_1$ ) on the straight line  $ax + by + c = 0 \dots \dots (1)$

slope of the line (1) is  $(m_1) = -\frac{a}{b}$ , slope of PQ is  $(m_2) = \frac{y_2-y_1}{x_2-x_1} = \frac{k-y_1}{h-x_1}$

since PQ is  $\perp$  lar to (1)  $\Rightarrow m_1 \cdot m_2 = -1$

$$\left(-\frac{a}{b}\right) \cdot \left(\frac{k-y_1}{h-x_1}\right) = -1$$

$$\frac{h-x_1}{a} = \frac{k-y_1}{b} \text{ (Applying ratio proportional)}$$

$$\frac{h-x_1}{a} = \frac{k-y_1}{b} = \frac{a(h-x_1)+b(k-y_1)}{a.a+b.b}$$

$$\frac{h-x_1}{a} = \frac{k-y_1}{b} = \frac{(ah-ax_1+bk-by_1)}{a^2+b^2}$$

$$\frac{h-x_1}{a} = \frac{k-y_1}{b} = \frac{(-ax_1-by_1+ah+bk)}{a^2+b^2} \dots \dots \dots (2)$$

since Q(h, k) lies on eq'n (1)  $ah + bk + c = 0 \Rightarrow ah + bk = -c$

sub in eq'n(2)

$$\frac{h-x_1}{a} = \frac{k-y_1}{b} = \frac{(-ax_1-by_1-c)}{a^2+b^2} \quad \therefore \frac{h-x_1}{a} = \frac{k-y_1}{b} = -\frac{(ax_1+by_1+c)}{a^2+b^2}.$$

let (h, k) be the required foot o f perpendicular from (4, 1) w.r.t the line  $3x - 4y + 12 = 0$ .

$$\frac{h-4}{3} = \frac{k-1}{-4} = -\frac{(3(4)-4(1)+12)}{3^2+(-4)^2}$$

$$\Rightarrow \frac{h-4}{3} = \frac{k-1}{-4} = -\frac{(20)}{25}$$

$$\Rightarrow \left\{ \frac{h-4}{3} = -\frac{4}{5} \quad \left| \frac{k-1}{-4} = -\frac{4}{5} \right. \right\}$$

$$\Rightarrow h - 4 = -\frac{12}{5} \quad \& \quad k - 1 = \frac{16}{5}$$

$$h = -\frac{12}{5} + 4 \quad \& \quad k = \frac{16}{5} + 1$$

$$\Rightarrow h = \frac{-12+20}{5} = \frac{8}{5}, k = \frac{16+5}{5} = \frac{21}{5}$$

$$\therefore (h, k) = \left(\frac{8}{5}, \frac{21}{5}\right)$$



2. If  $Q(h, k)$  is image of  $P(x_1, y_1)$  on the straight line  $ax + by + c = 0$ . Then P.T  
 $\frac{h-x_1}{a} = \frac{k-y_1}{b} = -\frac{2(ax_1+by_1+c)}{a^2+b^2}$  find the image of  $(1, 2)$  w.r.t the line  
 $3x + 4y - 1 = 0.$

Sol:  $Q(h, k)$  is image of  $P(x_1, y_1)$  on the straight line

$$ax + by + c = 0 \dots \dots \dots (1)$$

$$\text{slope of the line (1) is } (m_1) = -\frac{a}{b}$$

$$\text{slope of } PQ \text{ is } (m_2) = \frac{y_2-y_1}{x_2-x_1} = \frac{k-y_1}{h-x_1}$$

since  $PQ$  is  $\perp$  lar to (1)  $\Rightarrow m_1 \cdot m_2 = -1$

$$\left(-\frac{a}{b}\right) \cdot \left(\frac{k-y_1}{h-x_1}\right) = -1$$

$$\frac{h-x_1}{a} = \frac{k-y_1}{b} \text{ (Applying ratio propotional)}$$

$$\frac{h-x_1}{a} = \frac{k-y_1}{b} = \frac{a(h-x_1)+b(k-y_1)}{a.a+b.b}$$

$$\frac{h-x_1}{a} = \frac{k-y_1}{b} = \frac{(ah-ax_1+bk-by_1)}{a^2+b^2}$$

$$\frac{h-x_1}{a} = \frac{k-y_1}{b} = \frac{(-ax_1-by_1+ah+bk)}{a^2+b^2} \dots \dots \dots (2)$$

$$M \text{ is the midpoint of } PQ, \quad M \left[ \frac{x_1+x_2}{2}, \frac{y_1+y_2}{2} \right] = \left[ \frac{x_1+h}{2}, \frac{y_1+k}{2} \right]$$

since  $M \left[ \frac{x_1+h}{2}, \frac{y_1+k}{2} \right]$  lies on eq'n (1)  $\Rightarrow a \left( \frac{x_1+h}{2} \right) + b \left( \frac{y_1+k}{2} \right) + c = 0$

$$\Rightarrow ax_1 + ah + by_1 + bk + 2c = 0$$

$$\therefore ah + bk = -ax_1 - by_1 - 2c \quad \text{sub in eq'n (2)}$$

$$\frac{h-x_1}{a} = \frac{k-y_1}{b} = \frac{(-ax_1-by_1-ax_1-by_1-2c)}{a^2+b^2} \quad \therefore \frac{h-x_1}{a} = \frac{k-y_1}{b} = -\frac{2(ax_1+by_1+c)}{a^2+b^2}.$$

let  $(h, k)$  be the image from  $(1, 2)$  w.r.t the line  $3x + 4y - 1 = 0.$

$$\frac{h-1}{3} = \frac{k-2}{-4} = -\frac{2[3(1)+4(2)-1]}{3^2+(-4)^2} \Rightarrow \frac{h-4}{3} = \frac{k-1}{4} = -\frac{2(10)}{25} \Rightarrow \left\{ \begin{array}{l} \frac{h-1}{3} = -\frac{4}{5} \\ \frac{k-2}{4} = -\frac{4}{5} \end{array} \right| \left\{ \begin{array}{l} h-1 = -\frac{12}{5} \\ k-2 = -\frac{16}{5} \end{array} \right.$$

$$\Rightarrow h-1 = -\frac{12}{5} \quad \& \quad k-2 = -\frac{16}{5} \quad \Rightarrow h = -\frac{12}{5} + 1 \quad \& \quad k = -\frac{16}{5} + 2$$

$$h = \frac{-12+5}{5} = \frac{-7}{5}, \quad k = \frac{-16+10}{5} = \frac{-6}{5} \quad \therefore (h, k) = \left( -\frac{7}{5}, -\frac{6}{5} \right).$$



3. Find the circumcentre of the triangle whose vertices are (1, 3), (-3, 5) and (5, -1).

**Sol:** Let the given vertices are A(1, 3), B(-3, 5) and C(5, -1).

Let S(x, y) be the circumcentre of  $\Delta ABC$  then  $SA = SB = SC$

$$(i) \text{ Consider } SA = SB \\ \text{S.O.B} \quad SA^2 = SB^2$$

$$SA^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

$$\begin{aligned} &\Rightarrow (x - 1)^2 + (y - 3)^2 = (x + 3)^2 + (y - 5)^2 \\ &\Rightarrow x^2 + 1 - 2x + y^2 + 9 - 6y = x^2 + 9 + 6x + y^2 + 25 - 10y \\ &\Rightarrow -8 - 8x - 16 + 4y = 0 \\ &-8x + 4y - 24 = 0 \ (\div -4) \Rightarrow 2x - y + 6 = 0 \dots (1) \end{aligned}$$

$$(ii) \text{ Consider } SB = SC \quad \text{S.O.B}$$

$$SA^2 = SB^2$$

$$\begin{aligned} &\Rightarrow (x + 3)^2 + (y - 5)^2 = (x - 5)^2 + (y + 1)^2 \\ &\Rightarrow x^2 + 9 + 6x + y^2 + 25 - 10y = x^2 + 25 - 10x + y^2 + 1 + 2y \\ &\Rightarrow -16 + 16x + 24 - 12y = 0 \\ &16x - 12y + 8 = 0 \ (\div 4) \Rightarrow 4x - 3y + 2 = 0 \dots (2) \end{aligned}$$

Solving (1) and (2)

$$\begin{matrix} 2 & -1 & 6 & 2 \\ 4 & -3 & 2 & 4 \end{matrix}$$

$$(x, y) = \left[ \frac{-2+18}{-6+4}, \frac{24-4}{-6+4} \right] = \left[ \frac{16}{-2}, \frac{20}{-2} \right]$$

Hence coordinate of the circumcentre of  $\Delta ABC$  are (-8, -10).

$[-2, 3], (2, -1), (4, 0) \& (1, 3), (0, -2), (-3, 1)$



4. Find the circumcentre of the triangle whose sides are  $x + y + 2 = 0$ ,

$$5x - y - 2 = 0 \text{ and } x - 2y + 5 = 0.$$

Sol: Let  $\Delta ABC$  whose sides are  $x + y + 2 = 0 \dots \dots (1)$ ,  $5x - y - 2 = 0 \dots \dots (2)$   
 $x - 2y + 5 = 0 \dots \dots (3)$

solving eq'n (1) & (2)

$$\begin{array}{rrrrr} 1 & 1 & 2 & 1 \\ 5 & -1 & -2 & 5 \end{array}$$

$$A(x, y) = \left[ \frac{-2+2}{-1-5}, \frac{10+2}{-1-5} \right] = (0, -2)$$

solving eq'n (2) & (3)

$$\begin{array}{rrrrr} 5 & -1 & -2 & 5 \\ 1 & -2 & 5 & 1 \end{array}$$

$$B(x, y) = \left[ \frac{-5-4}{-10+1}, \frac{-2-25}{-10+1} \right] = (1, 3)$$

solving eq'n (1) & (3)

$$\begin{array}{rrrrr} 1 & 1 & 2 & 1 \\ 1 & -2 & 5 & 1 \end{array}$$

$$C(x, y) = \left[ \frac{5+4}{-2-1}, \frac{2-5}{-2-1} \right] = (-3, 1)$$

A(0, -2), B(1, 3) and C(-3, 1).

Let S(x, y) be the circumcentre of  $\Delta ABC$  then  $SA = SB = SC$

(i) Consider  $SA = SB$     S.O.B.S     $SA^2 = SB^2$

$$\Rightarrow (x - 0)^2 + (y + 2)^2 = (x - 1)^2 + (y - 3)^2$$

$$SA^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

$$\Rightarrow x^2 + 0 + 0 + y^2 + 4 + 4y = x^2 + 1 - 2x + y^2 + 9 - 6y$$

$$\Rightarrow -1 + 2x - 5 + 10y = 0$$

$$2x + 10y - 6 = 0 \quad (\div 2) \quad \Rightarrow 1x + 5y - 3 = 0 \dots \dots (4)$$

(ii) Consider  $SB = SC$     S.O.B.S     $\Rightarrow SA^2 = SC^2$

$$\Rightarrow (x - 1)^2 + (y - 3)^2 = (x + 3)^2 + (y - 1)^2$$

$$\Rightarrow x^2 + 1 - 2x + y^2 + 9 - 6y = x^2 + 9 + 6x + y^2 + 1 - 2y$$

$$\Rightarrow -8 - 8x + 8 - 4y = 0 \quad \Rightarrow -8x - 4y = 0 \quad (\div -4)$$

$$2x + y = 0 \dots \dots (5) \quad \text{Solving (4) and (5)}$$

$$\begin{array}{rrrrr} 1 & 5 & -3 & 1 \\ 2 & 1 & 0 & 2 \end{array}$$

$$(x, y) = \left[ \frac{0+3}{1-10}, \frac{-6-0}{1-10} \right] = \left[ \frac{3}{-9}, \frac{-6}{-9} \right]$$

Hence coordinate of the circumcentre of  $\Delta ABC$  are  $\left( -\frac{1}{3}, \frac{2}{3} \right)$ .



5. Find the circumcentre of the triangle whose sides are  $x + y = 0$ ,

$$2x + y + 5 = 0 \text{ and } x - y = 2$$

Sol: Let  $\Delta ABC$  whose sides are

$$x + y = 0 \dots \text{(1)} \quad \text{Slope (m)} = -\frac{a}{b} = -1$$

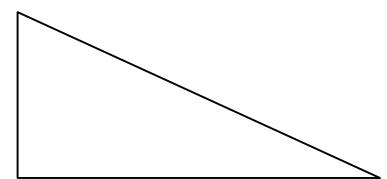
$$2x + y + 5 = 0 \dots \text{(2)} \quad \text{Slope (m)} = -\frac{a}{b} = -\frac{2}{1}$$

$$x - y = 2 \dots \text{(3)} \quad \text{Slope (m)} = -\frac{a}{b} = -\frac{1}{-1} = 1$$

since (1)& (3) are perpendicular lines these lines form a right angled triangle. Circumcentre is the mid point of hypotenuse.

solving eq'n (1) & (2)

$$\begin{array}{cccc} 1 & 1 & 0 & 1 \\ 2 & 1 & 5 & 2 \end{array}$$



$$A(x, y) = \left[ \frac{5-0}{1-2}, \frac{0-5}{1-2} \right] = (-5, 5)$$

solving eq'n(2) & (3)

$$\begin{array}{cccc} 2 & 1 & 5 & 2 \\ 1 & -1 & -2 & 1 \end{array}$$

$$C(x, y) = \left[ \frac{-2+5}{-2-1}, \frac{5+4}{-2-1} \right] = (-1, -3)$$

A (-5, 5) and C (-1, -3)

Circumcentre is the mid point of hypotenuse.

$$= \left[ \frac{x_1+x_2}{2}, \frac{y_1+y_2}{2} \right] = \left[ \frac{-5-1}{2}, \frac{5-3}{2} \right] = (-3, 1)$$



6. Find the Orthocenter of the triangle whose vertices are  $(-5, -7)$ ,  $(13, 2)$  and  $(-5, 6)$ .

Sol: Given vertices are  $(-5, -7)$ ,  $(13, 2)$  and  $(-5, 6)$

Slope of B  $(13, 2)$ , C  $(-5, 6)$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{6 - 2}{-5 - 13} = \frac{4}{-18} = -\frac{2}{9}$$

Since  $AD \perp BC$ , slope of  $AD = \frac{9}{2}$

eq'n of  $AD$  is  $(y - y_1) = m(x - x_1)$

$$A(-5, -7) \quad m = \frac{9}{2}$$

$$\Rightarrow (y + 7) = \frac{9}{2}(x + 5) \Rightarrow 2y + 14 = 9x + 45$$

$$\Rightarrow 9x - 2y + 31 = 0 \dots\dots (1)$$

Slope of A  $(-5, -7)$ , C  $(-5, 6)$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{6 + 7}{-5 + 5} = \frac{13}{0}$$

Since  $BE \perp AC$ , slope of  $BE = 0$

eq'n of  $BE$  is  $(y - y_1) = m(x - x_1)$

$$B(13, 2), \quad m = 0$$

$$\Rightarrow (y - 2) = 0(x - 13)$$

$$\Rightarrow y - 2 = 0$$

$$\Rightarrow y = 2 \dots\dots (2) \quad \text{sub } y = 2 \text{ in (1)}$$

$$\Rightarrow 9x - 2(2) + 31 = 0$$

$$\Rightarrow 9x = -27$$

$$\Rightarrow x = -\frac{27}{9} = -3$$

hence required orthocentre is  $(-3, 2)$



7. If the eqns of the sides of the triangle are  $7x + y - 10 = 0$ ,  
 $x - 2y + 5 = 0$  and  $x + y + 2 = 0$ . find the orthocentre.

**Sol:** Let  $\Delta ABC$  whose sides are

$$7x + y - 10 = 0 \dots\dots (1),$$

$$x - 2y + 5 = 0 \dots\dots (2)$$

$$x + y + 2 = 0 \dots\dots (3)$$

solving eq'n (1) & (2)

$$\begin{array}{rrrrr} 7 & 1 & -10 & 7 \\ 1 & -2 & 5 & 1 \end{array}$$

$$A(x, y) = \left[ \frac{5-20}{-14-1}, \frac{-10-3}{-14-1} \right] = (1, 3)$$

solving eq'n (2) & (3)

$$\begin{array}{rrrrr} 1 & -2 & 5 & 1 \\ 1 & 1 & 2 & 1 \end{array}$$

$$B(x, y) = \left[ \frac{-4-5}{1+2}, \frac{5-2}{1+2} \right] = (-3, 1)$$

solving eq'n (1) & (3)

$$\begin{array}{rrrrr} 7 & 1 & -10 & 7 \\ 1 & 1 & 2 & 1 \end{array}$$

$$C(x, y) = \left[ \frac{2+10}{7-1}, \frac{-10-14}{7-1} \right] =$$

$$(2, -4)$$

A(1, 3), B(-3, 1) and C(2, -4).

$$\text{Now Slope of B } (-3, 1), C (2, -4) \quad m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-4 - 1}{2 + 3} = \frac{-5}{5} = -1$$

Since AD  $\perp$  BC, slope of AD = 1

eq'n of AD is  $(y - y_1) = m(x - x_1)$

$$A(1, 3) \quad m = 1$$

$$\Rightarrow (y - 3) = 1(x - 1) \Rightarrow y - 3 = x - 1 \Rightarrow x - y + 2 = 0 \dots\dots (4)$$

$$\text{Slope of A } (1, 3), C (2, -4) \quad m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-4 - 3}{2 - 1} = \frac{-7}{1}$$

Since BE  $\perp$  AC, slope of BE =  $\frac{1}{7}$

eq'n of BE is  $(y - y_1) = m(x - x_1)$

$$B(-3, 1) \quad m = \frac{1}{7}$$

$$\Rightarrow (y - 1) = \frac{1}{7}(x + 3) \Rightarrow 7y - 7 = x + 3$$

$$\Rightarrow x - 7y + 10 = 0 \dots\dots (5)$$

solving eq'n (4) & (5)

$$\begin{array}{rrrrr} 1 & -1 & 2 & 1 \\ 1 & -7 & 10 & 1 \end{array}$$

$$A(x, y) = \left[ \frac{-10+1}{-7+1}, \frac{2-10}{-7+1} \right] = \left[ \frac{4}{-6}, \frac{-8}{-6} \right]$$

hence required orthocentre is  $(-\frac{2}{3}, \frac{4}{3})$



8. If the eq"ns of the sides of the triangle are  $x + y + 10 = 0$ ,  
 $x - y - 2 = 0$  and  $2x + y - 7 = 0$ . find the orthocentre.

sol: Let  $\Delta ABC$  whose sides are

$$x + y + 10 = 0 \dots \text{ (1)} \quad (\text{m}) = -\frac{a}{b} = -\frac{1}{1} = -1$$

$$x - y - 2 = 0 \dots \text{ (2)} \quad (\text{m}) = -\frac{a}{b} = -\frac{1}{-1} = 1$$

$$2x + y - 7 = 0 \dots \text{ (3)} \quad (\text{m}) = -\frac{a}{b} = -\frac{2}{1} = -2$$

since (1)& (2) are perpendicular lines  
these lines form a right angled triangle.

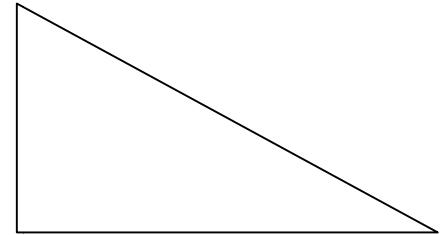
orthocentre is the point of intersection of (1)& (2)

solving eq"n (1) & (2)

$$\begin{array}{cccc} 1 & 1 & 10 & 1 \\ 1 & -1 & -2 & 1 \end{array}$$

$$A(x, y) = \left[ \frac{-2+10}{-1-1}, \frac{10+2}{-1-1} \right]$$

$$= \left[ \frac{8}{-2}, \frac{12}{-2} \right] = [-4, -6]$$



Hence the coordinates of Orthcentre  $\Delta ABC$  are  $(-4, -6)$ .



9. If P and q are the lengths of perpendiculars from the origin to the st lines  
 $x\sec\alpha + y\csc\alpha = a$  and  $x\cos\alpha - y\sin\alpha = a\cos 2\alpha$ ,  $P.T \ 4p^2 + q^2 = a^2$ .

**Sol:**  $x\sec\alpha + y\csc\alpha = a \Rightarrow \frac{x}{\cos\alpha} + \frac{y}{\sin\alpha} = a$   
 $\Rightarrow x\sin\alpha + y\cos\alpha = a\sin\alpha\cos\alpha$   
 $\Rightarrow x\sin\alpha + y\cos\alpha - a\sin\alpha\cos\alpha = 0$

length of  $\perp$  lar from  $(0,0)$  to above line is  $p = \frac{|c|}{\sqrt{a^2+b^2}}$   
 $\Rightarrow p = \frac{|-a\sin\alpha\cos\alpha|}{\sqrt{\sin^2\alpha+\cos^2\alpha}}$   
 $\Rightarrow p = a\sin\alpha\cos\alpha$   
 $\Rightarrow 2p = a^2\sin\alpha\cos\alpha$   
 $\Rightarrow 2p = a\sin 2\alpha \quad \text{S.O.B.S}$   
 $\Rightarrow 4p^2 = a^2\sin^2 2\alpha \dots\dots (1)$

$q$  is the length of  $\perp$  lar from  $(0,0)$  to line

$$\begin{aligned} x\cos\alpha - y\sin\alpha - a\cos 2\alpha &= 0 \\ q &= \frac{|-a\cos 2\alpha|}{\sqrt{\cos^2\alpha+\sin^2\alpha}} \\ \Rightarrow q &= a\cos 2\alpha \quad \text{S.O.B.S} \\ \Rightarrow q^2 &= a^2\cos^2 2\alpha \dots\dots (2) \end{aligned}$$

now eq''n (1) + (2)

$$4p^2 + q^2 = a^2\sin^2 2\alpha + a^2\cos^2 2\alpha$$

$$\begin{aligned} &= a^2(\sin^2 2\alpha + \cos^2 2\alpha) \\ &= a^2(1) \end{aligned}$$

$$4p^2 + q^2 = a^2$$



10. Find the eq'n of the st lines passing through the point of intersection of the lines

$3x + 2y + 4 = 0$ ,  $2x + 5y = 1$  & whose distance from  $(2, -1)$  is 2.

**Sol:** Given eq'ns

$$3x + 2y + 4 = 0 \dots (1)$$

$$2x + 5y - 1 = 0 \dots (2)$$

solving eq'n (1) & (2)

$$\begin{array}{cccc} 3 & 2 & 4 & 3 \\ 2 & 5 & -1 & 2 \end{array}$$

$$p(x, y) = \left[ \frac{-2-20}{15-4}, \frac{8+3}{15-4} \right] = \left[ \frac{-22}{11}, \frac{11}{11} \right] \\ = (-2, 1)$$

Let 'm' be the slope of the line passing through

P  $(-2, 1)$  is  $(y - y_1) = m(x - x_1)$

$$\Rightarrow (y - 1) = m(x + 2)$$

$$\Rightarrow mx + 2m - y + 1 = 0$$

$$\Rightarrow mx - y + 2m + 1 = 0 \dots (3)$$

Since distance from  $(2, -1)$  to (3) is 2

$$d = \frac{(ax_1 + by_1 + c)}{\sqrt{a^2 + b^2}} \Rightarrow 2 = \frac{(m(2) - 1(-1) + 2m + 1)}{\sqrt{m^2 + (-1)^2}}$$

$$\Rightarrow 2 = \frac{(4m + 2)}{\sqrt{m^2 + 1}} \Rightarrow 2 = \frac{2(2m + 1)}{\sqrt{m^2 + 1}} \quad \text{S.O.B.S}$$

$$\Rightarrow m^2 + 1 = (2m + 1)^2$$

$$\Rightarrow m^2 + 1 = 4m^2 + 4m + 1$$

$$\Rightarrow 4m^2 + 4m + 1 - m^2 - 1 = 0$$

$$\Rightarrow 3m^2 + 4m = 0$$

$$\Rightarrow m(3m + 4) = 0$$

$$m = 0 \text{ or } m = -\frac{4}{3} \quad \text{Required eq'n}$$

case(i) if  $m = 0$

$$(y - 1) = 0(x + 2);$$

$$\Rightarrow y - 1 = 0.$$

case(ii) if  $m = -\frac{4}{3}$

$$\Rightarrow (y - 1) = -\frac{4}{3}(x + 2)$$

$$\Rightarrow 3y - 3 = -4x - 8$$

$$\Rightarrow 4x + 3y - 11 = 0.$$



1. If  $\theta$  is the angle between the pair of lines

$$ax^2 + 2hxy + by^2 = 0, \text{ then P.T } \cos\theta = \frac{|a+b|}{\sqrt{(a-b)^2+4h^2}}.$$

Sol: let  $ax^2 + 2hxy + by^2 = 0$  represent the lines

$$l_1x + m_1y = 0 \dots \dots \dots (1)$$

$$l_2x + m_2y = 0 \dots \dots \dots (2)$$

$$\begin{aligned} \therefore ax^2 + 2hxy + by^2 &\equiv [l_1x + m_1y][l_2x + m_2y] = 0 \\ &\equiv l_1x[l_2x + m_2y] + m_1y[l_2x + m_2y] = 0 \\ &\equiv l_1l_2x^2 + l_1m_2xy + l_2m_1xy + m_1m_2y^2 = 0 \\ &\equiv l_1l_2x^2 + (l_1m_2 + l_2m_1)xy + m_1m_2y^2 = 0 \end{aligned}$$

Comparing both sides  $x^2, y^2$  &  $xy$  coeff, we get

$$l_1l_2 = a, \quad m_1m_2 = b \quad \& \quad l_1m_2 + l_2m_1 = 2h$$

$$\text{now } \cos\theta = \frac{l_1l_2 + m_1m_2}{\sqrt{l_1^2 + m_1^2}\sqrt{l_2^2 + m_2^2}}$$

$$\begin{aligned} \Rightarrow \cos\theta &= \frac{l_1l_2 + m_1m_2}{\sqrt{(l_1l_2)^2 + (l_1m_2)^2 + (l_2m_1)^2 + (m_1m_2)^2}} \\ &= \frac{l_1l_2 + m_1m_2}{\sqrt{(l_1l_2)^2 + (m_1m_2)^2 - 2l_1l_2m_1m_2 + (l_1m_2)^2 + (l_2m_1)^2 + 2l_1l_2m_1m_2}} \\ &= \frac{l_1l_2 + m_1m_2}{\sqrt{(l_1l_2 - m_1m_2)^2 + (l_1m_2 + l_2m_1)^2}} \\ &= \frac{|a+b|}{\sqrt{(a-b)^2 + (2h)^2}} \\ &= \frac{|a+b|}{\sqrt{(a-b)^2 + 4h^2}}. \end{aligned}$$



2. Prove that product of perpendiculars from a point  $(\alpha, \beta)$  to the pair of straight lines

$$ax^2 + 2hxy + by^2 = 0 \text{ is } \left| \frac{a\alpha^2 + 2h\alpha\beta + b\beta^2}{\sqrt{(a-b)^2 + 4h^2}} \right|$$

Sol: let  $ax^2 + 2hxy + by^2 = 0$  represent the lines

$$l_1x + m_1y = 0 \dots \dots \dots (1)$$

$$l_2x + m_2y = 0 \dots \dots \dots (2)$$

$$\therefore ax^2 + 2hxy + by^2 \equiv [l_1x + m_1y][l_2x + m_2y] = 0$$

$$\equiv l_1l_2x^2 + (l_1m_2 + l_2m_1)xy + m_1m_2y^2 = 0$$

Comparing both sides  $x^2, y^2$  &  $xy$  coeff, we get

$$l_1l_2 = a, m_1m_2 = b \text{ & } l_1m_2 + l_2m_1 = 2h$$

w.k.t the length of the  $\perp$  lar from  $(x_1, y_1)$  to the line  $ax + by + c = 0$  is  $\frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$

now product of the  $\perp$  lar from  $(\alpha, \beta)$  to the lines (1) and (2) is

$$\begin{aligned} &= \left( \frac{l_1\alpha + m_1\beta}{\sqrt{l_1^2 + m_1^2}} \right) \left( \frac{l_2\alpha + m_2\beta}{\sqrt{l_2^2 + m_2^2}} \right) \\ &= \frac{l_1\alpha[l_2\alpha + m_2\beta] + m_1\beta[l_2\alpha + m_2\beta]}{\sqrt{(l_1l_2)^2 + (l_1m_2)^2 + (l_2m_1)^2 + (m_1m_2)^2}} \\ &= \frac{l_1l_2\alpha^2 + l_1m_2\alpha\beta + l_2m_1\alpha\beta + m_1m_2\beta^2}{\sqrt{(l_1l_2)^2 + (m_1m_2)^2 - 2l_1l_2m_1m_2 + (l_1m_2)^2 + (l_2m_1)^2 + 2l_1l_2m_1m_2}} \\ &= \frac{l_1l_2\alpha^2 + (l_1m_2 + l_2m_1)\alpha\beta + m_1m_2\beta^2}{\sqrt{(l_1l_2 - m_1m_2)^2 + (l_1m_2 + l_2m_1)^2}} \\ &= \frac{a\alpha^2 + 2h\alpha\beta + b\beta^2}{\sqrt{(a-b)^2 + (2h)^2}} \\ &= \left| \frac{a\alpha^2 + 2h\alpha\beta + b\beta^2}{\sqrt{(a-b)^2 + 4h^2}} \right| \end{aligned}$$



3. If the eq'n  $ax^2 + 2hxy + by^2 = 0$  represent a pair of lines, P.T the combined eq'n of the pair of bisectors bisecting the angle b/w these lines is

$$h(x^2 - y^2) = (a - b)xy.$$

Sol: let  $ax^2 + 2hxy + by^2 = 0$  represent the lines

$$l_1x + m_1y = 0 \dots \dots \dots (1)$$

$$l_2x + m_2y = 0 \dots \dots \dots (2)$$

$$\begin{aligned} \therefore ax^2 + 2hxy + by^2 &\equiv [l_1x + m_1y][l_2x + m_2y] = 0 \\ &\equiv l_1l_2x^2 + l_1m_2xy + l_2m_1xy + m_1m_2y^2 = 0 \\ &\equiv l_1l_2x^2 + (l_1m_2 + l_2m_1)xy + m_1m_2y^2 = 0 \end{aligned}$$

Comparing both sides  $x^2, y^2$  &  $xy$  coeff, we get

$$l_1l_2 = a, \quad m_1m_2 = b \quad \& \quad l_1m_2 + l_2m_1 = 2h$$

Now eq'ns of bisectors of angle b/w (1 & (2) are

$$\left( \frac{l_1x + m_1y}{\sqrt{l_1^2 + m_1^2}} \right) = \pm \left( \frac{l_2x + m_2y}{\sqrt{l_2^2 + m_2^2}} \right)$$

S.O.B.S and cross multiplying, we get

$$\Rightarrow (l_1x + m_1y)^2(l_2^2 + m_2^2) = (l_2x + m_2y)^2(l_1^2 + m_1^2)$$

$$\begin{aligned} \Rightarrow (l_1^2x^2 + m_1^2y^2 + 2l_1m_1xy)(l_2^2 + m_2^2) \\ = (l_2^2x^2 + m_2^2y^2 + 2l_2m_2xy)(l_1^2 + m_1^2) \end{aligned}$$

$$\begin{aligned} \Rightarrow (l_1l_2)^2x^2 + (l_1m_2)^2x^2 + (l_2m_1)^2y^2 + (m_1m_2)^2x^2 + 2l_1m_1l_2^2xy + \\ 2l_1m_1m_2^2xy \end{aligned}$$

$$= (l_1l_2)^2x^2 + (l_2m_1)^2x^2 + (l_1m_2)^2y^2 + (m_1m_2)^2x^2 + 2l_2m_2l_1^2xy + 2l_2m_2m_1^2xy$$

$$\Rightarrow x^2[(l_1m_2)^2 - (l_2m_1)^2] - y^2[(l_1m_2)^2 - (l_2m_1)^2]$$

$$= 2xy[l_1l_2(l_1m_2 - l_2m_1) - m_1m_2((l_1m_2 - l_2m_1))]$$

$$\Rightarrow [(l_1m_2 + l_2m_1)(l_1m_2 - l_2m_1)](x^2 - y^2) = 2xy(l_1m_2 - l_2m_1)[l_1l_2 - m_1m_2]$$

$$\Rightarrow (l_1m_2 + l_2m_1)(x^2 - y^2) = 2xy[l_1l_2 - m_1m_2]$$

$$\Rightarrow 2h(x^2 - y^2) = 2xy(a - b)$$

$$\therefore h(x^2 - y^2) = xy(a - b).$$



4. S.T the area of the triangle formed by the lines  $ax^2 + 2hxy + by^2 = 0$

and  $lx + my + n = 0$  is  $\frac{n^2\sqrt{h^2-ab}}{am^2-2hlm+bl^2}$  Sq. units.

Sol: let  $ax^2 + 2hxy + by^2 = 0$  represent the lines

$$l_1x + m_1y = 0 \dots \dots \dots (1)$$

$$l_2x + m_2y = 0 \dots \dots \dots (2)$$

$$lx + my + n = 0 \dots \dots \dots (3)$$

$$\therefore ax^2 + 2hxy + by^2 \equiv [l_1x + m_1y][l_2x + m_2y] = 0$$

$$\equiv l_1l_2x^2 + (l_1m_2 + l_2m_1)xy + m_1m_2y^2 = 0$$

Comparing both sides  $x^2, y^2$  &  $xy$  coeff, we get

$$l_1l_2 = a, m_1m_2 = b \text{ & } l_1m_2 + l_2m_1 = 2h$$

Solving (1) & (2) we get,  $(0, 0)$

Solving (1) & (3)

$$\begin{array}{cccc} l_1 & m_1 & 0 & l_1 \\ l & m & n & l \end{array}$$

$$A(x_1, y_1) = \left[ \frac{nm_1 - 0}{l_1m - l m_1}, \frac{0 - nl_1}{l_1m - l m_1} \right]$$

Similarly by solving (2) & (3) we get,

$$B(x_2, y_2) = \left[ \frac{nm_2}{l_2m - l m_2}, \frac{nl_2}{l_2m - l m_2} \right]$$

$$\begin{aligned} \text{Now area of } \Delta OAB &= \frac{1}{2} |x_1y_2 - x_2y_1| \\ &= \frac{1}{2} \left| \left( \frac{nm_1}{l_1m - l m_1} \right) \left( \frac{-nl_2}{l_2m - l m_2} \right) - \left( \frac{nm_2}{l_2m - l m_2} \right) \left( \frac{-nl_1}{l_1m - l m_1} \right) \right| \end{aligned}$$

$$= \frac{1}{2} \left| \frac{n^2(l_1m_2 - l_2m_1)}{(l_1m - l m_1)(l_2m - l m_2)} \right|$$

$$= \frac{1}{2} \left| \frac{n^2\sqrt{(l_1m_2 + l_2m_1)^2 - 4l_1l_2 m_1m_2}}{l_1l_2m^2 - l_1m_2lm - l_2m_1lm - m_1m_2l^2} \right|$$

$$= \frac{1}{2} \left| \frac{n^2\sqrt{(2h)^2 - 4ab}}{am^2 - (l_1m_2 + l_2m_1)lm + bl^2} \right|$$

$$= \frac{1}{2} \left| \frac{n^2\sqrt{4h^2 - 4ab}}{am^2 - 2hlm + bl^2} \right|$$

$$= \frac{2}{2} \left| \frac{n^2\sqrt{h^2 - a}}{am^2 - 2hlm + bl^2} \right| = \left| \frac{n^2\sqrt{h^2 - a}}{am^2 - 2hlm + bl^2} \right| \text{ Sq. units}$$



5. If the eq'n  $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ , Represent a pair of lines, P.T  
 (i)  $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$   
 (ii)  $h^2 \geq ab, g^2 \geq ac, f^2 \geq bc$ .

Sol: Let  $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

represent the lines  $l_1x + m_1y + n_1 = 0 \dots (1)$ ,  $l_2x + m_2y + n_2 = 0 \dots (2)$

$$\therefore ax^2 + 2hxy + by^2 + 2gx + 2fy + c \equiv [l_1x + m_1y + n_1][l_2x + m_2y + n_2] = 0$$

$$\equiv l_1l_2x^2 + l_1m_2xy + l_1n_2x + l_2m_1xy + m_1m_2y^2 + m_1n_2y + l_2n_1x + m_2n_1y + n_1n_2 = 0$$

$$\equiv l_1l_2x^2 + (l_1m_2 + l_2m_1)xy + m_1m_2y^2 + (l_1n_2 + l_2n_1)x + (m_1n_2 + m_2n_1)y + n_1n_2 = 0$$

Comparing both sides  $x^2, xy, y^2, x, y - \text{coeff & constant we get},$

$$l_1l_2 = a, m_1m_2 = b, l_1m_2 + l_2m_1 = 2h$$

$$l_1n_2 + l_2n_1 = 2g, m_1n_2 + m_2n_1 = 2f \text{ And } n_1n_2 = c$$

$$\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ 0 & 0 & 0 \end{vmatrix} \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ 0 & 0 & 0 \end{vmatrix} = 0 (R_1 \leftrightarrow R_2)$$

$$\Rightarrow \begin{vmatrix} l_1 & l_2 & 0 \\ m_1 & m_2 & 0 \\ n_1 & n_2 & 0 \end{vmatrix} (-) \begin{vmatrix} l_2 & m_2 & n_2 \\ l_1 & m_1 & n_1 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} l_1l_2 + l_1l_2 & l_1m_2 + l_2m_1 & l_1n_2 + l_2n_1 \\ m_1l_2 + l_2m_1 & m_1m_2 + m_2m_1 & m_1n_2 + m_2n_1 \\ n_1l_2 + n_2n_1 & n_1m_2 + n_2m_1 & n_1n_2 + n_2n_1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 2a & 2h & 2g \\ 2h & 2b & 2f \\ 2g & 2f & 2c \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

$$\Rightarrow a(bc - f^2) - h(hc - gf) + g(hf - bg) = 0$$

$$\therefore abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

$$\text{now } h^2 - ab = \left[ \frac{l_1m_2 + l_2m_1}{2} \right]^2 - l_1l_2m_1m_2$$

$$= \frac{(l_1m_2 + l_2m_1)^2 - 4l_1l_2m_1m_2}{4}$$

$$= \frac{(l_1m_2 - l_2m_1)^2}{2} \geq 0 \therefore h^2 - ab \geq 0 \Rightarrow h^2 \geq ab$$

Similarly, we can show that  $g^2 \geq ac, f^2 \geq bc$ .



6. If the eq'n  $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$   
Represent a pair of lines, P.T

(i)  $h^2 = ab$ , (ii)  $af^2 = bg^2$  (iii) the distance b/w parallel lines is  $2\sqrt{\frac{g^2-ac}{a(a+b)}}$  or

$$2\sqrt{\frac{f^2-bc}{b(a+b)}}$$

Sol: Let  $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$   
represent the lines

$$lx + my + n_1 = 0 \dots \dots \dots (1)$$

$$lx + my + n_2 = 0 \dots \dots \dots (2)$$

$$\therefore ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

$$\equiv [lx + my + n_1][lx + my + n_2] = 0$$

$$\equiv l^2x^2 + lmx + ln_2x + lmxy + m^2y^2 + mn_2y + ln_1x + mn_1y + n_1n_2 = 0$$

$$\equiv l^2x^2 + 2lmxy + m^2y^2 + l(n_1 + n_2)x + m(n_1 + n_2)y + n_1n_2 = 0$$

Comparing both sides  $x^2, xy, y^2, x, y$  - coeff and constant

we get,  $l^2 = a, m^2 = b, 2lm = 2h$

$$l(n_1 + n_2) = 2g, m(n_1 + n_2) = 2f \text{ & } n_1n_2 = c$$

$$\text{now } h^2 = (lm)^2 = l^2m^2 = ab \Rightarrow h^2 = ab$$

$$af^2 = l^2 \left[ \frac{m(n_1+n_2)}{2} \right]^2 = m^2 \left[ \frac{l(n_1+n_2)}{2} \right]^2 = bg^2$$

$$\Rightarrow af^2 = bg^2$$

distance b/w parellal lines (1) & (2)

$$\frac{|C_1 - C_2|}{\sqrt{a^2 + b^2}} = \frac{|n_1 - n_2|}{\sqrt{l^2 + m^2}}$$

$$= \frac{\sqrt{(n_1 + n_2)^2 - 4n_1n_2}}{\sqrt{a+b}} \dots \dots \dots (\text{A})$$

$$= \sqrt{\frac{\left(\frac{2g}{l}\right)^2 - 4c}{a+b}} = \sqrt{\frac{4g^2 - 4l^2c}{l^2(a+b)}} = 2\sqrt{\frac{g^2 - ac}{a(a+b)}}$$

$$\text{or} = \sqrt{\frac{\left(\frac{2f}{m}\right)^2 - 4c}{a+b}} = \sqrt{\frac{4f^2 - 4m^2c}{m^2(a+b)}} = 2\sqrt{\frac{f^2 - bc}{b(a+b)}}$$



1. Find the angle b/w line joining the origin to the point of intersection of the curve  $x^2 + 2xy + y^2 + 2x + 2y - 5 = 0$  And the line  $3x - y + 1 = 0$ .

*Sol:* Given curve  $S \equiv x^2 + 2xy + y^2 + 2x + 2y - 5 = 0 \dots (1)$

Line  $3x - y + 1 = 0 \dots (2)$

$\Rightarrow 1 = -3x + y \dots (3)$

homogenising (1) using (3) the combined eq'n is

$$(x^2 + 2xy + y^2) + (2x)(1) + (2y)(1) - 5(1)^2 = 0$$

$$\Rightarrow (x^2 + 2xy + y^2) + 2x(-3x + y) + 2y(-3x + y) - 5(-3x + y)^2 = 0$$

$$\Rightarrow (x^2 + 2xy + y^2) - 6x^2 + 2xy - 6xy + 2y^2 - 5(9x^2 + y^2 - 6xy) = 0$$

$$\Rightarrow x^2 + 2xy + y^2 - 6x^2 - 4xy + 4y^2 - 45x^2 - 5y^2 + 30xy = 0$$

$$\Rightarrow -50x^2 + 28xy - 2y^2 = 0 \quad (\div by -2)$$

$$\Rightarrow 25x^2 - 14xy + y^2 = 0$$

$$a = 25, 2h = -14, b = 1 \left[ \begin{array}{l} \text{compare with} \\ ax^2 + 2hxy + by^2 = 0 \end{array} \right]$$

$$\cos\theta = \frac{|a+b|}{\sqrt{(a-b)^2+4h^2}} = \frac{25+1}{\sqrt{(25-1)^2+(-14)^2}}$$

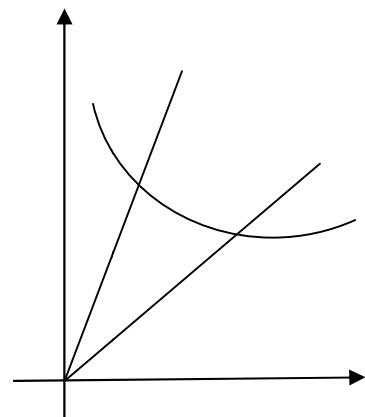
$$= \frac{26}{\sqrt{24^2+14^2}}$$

$$= \frac{26}{\sqrt{576+196}}$$

$$= \frac{26}{\sqrt{772}}$$

$$= \frac{26}{\sqrt{4.193}} = \frac{26}{2\sqrt{193}}$$

$$\theta = \cos^{-1} \left( \frac{13}{\sqrt{193}} \right)$$



Q.No: 20



Homogenizing





2. Find the angle b/w line joining the origin to the point of intersection of the curve  $7x^2 - 4xy + 8y^2 + 2x - 4y - 8 = 0$  And the line  $3x - y = 2$ .

*Sol:* Given curve  $S \equiv 7x^2 - 4xy + 8y^2 + 2x - 4y - 8 = 0 \dots (1)$

Line  $3x - y = 2 \dots (2)$

$$\Rightarrow 1 = \frac{3x-y}{2} \dots (3)$$

homogenising (1) using (3) the combined eq'n is

$$(7x^2 - 4xy + 8y^2) + (2x)(1) - (4y)(1) - 8(1)^2 = 0$$

$$\Rightarrow (7x^2 - 4xy + 8y^2) + \frac{2x(3x-y)}{2} - \frac{4y(3x-y)}{2} - 8\left(\frac{3x-y}{2}\right)^2 = 0$$

$$\Rightarrow (7x^2 - 4xy + 8y^2) + x(3x-y) - 2y(3x-y) - 8\frac{(3x-y)^2}{4} = 0$$

$$\Rightarrow (7x^2 - 4xy + 8y^2) + 3x^2 - xy - 6xy + 2y^2 - 2(9x^2 + y^2 - 6xy) = 0$$

$$\Rightarrow 7x^2 - 4xy + 8y^2 + 3x^2 - 7xy + 2y^2 - 18x^2 - 2y^2 + 12xy = 0$$

$$\Rightarrow -8x^2 + xy + 8y^2 = 0$$

$$a = -8, \quad 2h = 1, \quad b = 8 \left[ \begin{array}{l} \text{compare with} \\ ax^2 + 2hxy + by^2 = 0 \end{array} \right]$$

coefficient of  $x^2$  + coefficient of  $y^2 = -8 + 8 = 0$

$(a + b = 0)$

(1) & (2) mutually perpendicular





3. Find the value of 'k' if line joining the origin to the point of intersection of the curve  $2x^2 - 2xy + 3y^2 + 2x - y - 1 = 0$  And the line  $x + 2y = k$  mutually perpendicular.

**SOL:** Given curve  $2x^2 - 2xy + 3y^2 + 2x - y - 1 = 0 \dots (1)$

$$\text{Line } x + 2y = k \dots (2)$$

$$\Rightarrow 1 = \frac{x+2y}{k} \dots (3)$$

homogenising (1) using (3) the combined eq'n is

$$(2x^2 - 2xy + 3y^2) + (2x)(1) - (y)(1) - 1(1)^2 = 0$$

$$\Rightarrow (2x^2 - 2xy + 3y^2) + \left(2x\left(\frac{x+2y}{k}\right) - (y)\left(\frac{x+2y}{k}\right) - 1\left(\frac{x+2y}{k}\right)^2\right) = 0$$

$$\Rightarrow (2x^2 - 2xy + 3y^2) + \frac{[2x^2 + 4xy - xy - 2y^2]}{k} - 1 \frac{[x^2 - 4xy + 4y^2]}{k^2} = 0$$

$$\Rightarrow x^2 \left[2 + \frac{2}{k} - \frac{1}{k^2}\right] + y \left[3 - \frac{2}{k} - \frac{4}{k^2}\right] + 2xy \left[-2 + \frac{3}{k} + \frac{4}{k^2}\right] = 0$$

$$a = 2 + \frac{2}{k} - \frac{1}{k^2}, \quad b = 3 - \frac{2}{k} - \frac{4}{k^2} \quad \left[ \begin{matrix} \text{compare with} \\ ax^2 + 2hxy + by^2 = 0 \end{matrix} \right]$$

since (1) & (2) mutually perpendicular

coefficient of  $x^2$  + coefficient of  $y^2 = 0$

$$(a + b = 0)$$

$$\Rightarrow 2 + \frac{2}{k} - \frac{1}{k^2} + 3 - \frac{2}{k} - \frac{4}{k^2} = 0$$

$$\Rightarrow 5 - \frac{5}{k^2} = 0$$

$$\Rightarrow 5 = \frac{5}{k^2}$$

$$\Rightarrow k^2 = \frac{5}{5} = 1$$

$$k = \pm 1$$



Q.No: 20



Homogenizing





4. Show that the lines joining the origin to the point of intersection of the curve

$x^2 - xy + y^2 + 3x + 3y - 2 = 0$  And the line

$x - y - \sqrt{2} = 0$ . are mutually perpendicular.

Sol: Given curve  $x^2 - xy + y^2 + 3x + 3y - 2 = 0 \dots (1)$

Line  $x - y - \sqrt{2} = 0 \dots (2)$

$$\Rightarrow x - y = \sqrt{2} \text{ or } \frac{x-y}{\sqrt{2}} = 1 \dots (3)$$

homogenising (1) using (3) the combined eq'n is

$$(x^2 - xy + y^2) + 3(x + y)(1) - 2(1)^2 = 0$$

$$\Rightarrow (x^2 - xy + y^2) + 3(x + y)\left(\frac{x-y}{\sqrt{2}}\right) - 5\left(\frac{x-y}{\sqrt{2}}\right)^2 = 0$$

$$\Rightarrow (x^2 - xy + y^2) + 3\left[\frac{x^2 - y^2}{\sqrt{2}}\right] - \frac{2(x^2 + y^2 - 2xy)}{2} = 0$$

$$\Rightarrow x^2 - xy + y^2 + \frac{3}{\sqrt{2}}x^2 - \frac{3}{\sqrt{2}}y^2 - x^2 - y^2 + 2xy = 0$$

$$\Rightarrow \frac{3}{\sqrt{2}}x^2 + xy - \frac{3}{\sqrt{2}}y^2 = 0$$

$$a = \frac{3}{\sqrt{2}}, 2h = 1, b = \frac{3}{\sqrt{2}} \quad \left[ \begin{array}{l} \text{compare with} \\ ax^2 + 2hxy + by^2 = 0 \end{array} \right]$$

$\therefore$  coefficient of  $x^2$  + coefficient of  $y^2$

$$\frac{3}{\sqrt{2}} - \frac{3}{\sqrt{2}} = 0$$

$\therefore$  OA, OB are perpendicular.



Q.No: 20



Homogenizing



5. Write down the equation of the pair of st lines joining the origin to the point of intersection of the curve

$$3x^2 + 4xy - 4y^2 - 11x + 2y + 6 = 0 \text{ And the line}$$

$$6x - y + 8 = 0.$$

*Sol:* Given curve  $3x^2 + 4xy - 4y^2 - 11x + 2y + 6 = 0 \dots (1)$

$$\text{Line } 6x - y + 8 = 0 \dots (2)$$

$$\Rightarrow 8 = -6x + y \quad \text{or } 1 = \left[ \frac{-6x+y}{8} \right] \dots (3)$$

homogenising (1) using (3) the combined eq'n is

$$(3x^2 + 4xy - 4y^2) - (11x - 2y)(1) + 6(1)^2 = 0$$

$$\Rightarrow (3x^2 + 4xy - 4y^2) - (11x - 2y) \left[ \frac{-6x+y}{8} \right] + 6 \left[ \frac{-6x+y}{8} \right]^2 = 0 \text{ M B sides by 64}$$

$$\Rightarrow 64(3x^2 + 4xy - 4y^2) - 8[-66x^2 + 11xy + 12xy - 36y^2] + 6(36x^2 + y^2 - 12xy) = 0$$

$$\Rightarrow 936x^2 - 256xy + 256xy - 234y^2 = 0$$

$$\Rightarrow -50x^2 + 28xy - 2y^2 = 0 \quad (\div by -2)$$

$$\Rightarrow 468x^2 - 117y^2 = 0$$

$$\Rightarrow 4x^2 - y^2 = 0 \dots \dots (3)$$

$$a = 4, 2h = 0, b = -1 \left[ \begin{array}{l} \text{compare with} \\ ax^2 + 2hxy + by^2 = 0 \end{array} \right]$$

The equation of pair of angular bisectors of (3)

$$h(x^2 - y^2) = (a - b)xy$$

$$\Rightarrow 0(x^2 - y^2) = (4 + 1)xy$$

$$\Rightarrow 0 = 5xy$$

$$\Rightarrow xy = 0 \Rightarrow x = 0 \text{ or } y = 0$$

Which equations are of coordinates axes.

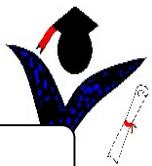
$\therefore$  the pair of lines are equally inclined to the coordinate axes.



Q.No: 20



Homogenizing





6. Find the condition for the chord  $lx + my = 1$  of the circle  $x^2 + y^2 = a^2$  to (i) subtend a right angle at the origin. (ii) to coincide.

Sol: Given circle  $x^2 + y^2 = a^2 \dots \dots (1)$

chord  $lx + my = 1 \dots \dots (2)$

homogenising (1) using (2) the combined eq'n is

$$(x^2 + y^2) = a^2(1)^2$$

$$\Rightarrow (x^2 + y^2) = a^2(lx + my)^2$$

$$\Rightarrow (x^2 + y^2) = a^2[l^2x^2 + m^2y^2 + 2lmxy]^2$$

$$\Rightarrow (x^2 + y^2) = a^2l^2x^2 + a^2m^2y^2 + a^22lmxy$$

$$\Rightarrow x^2 - a^2l^2x^2 + y^2 - a^2m^2y^2 - a^22lmxy = 0$$

$$\Rightarrow (1 - a^2l^2)x^2 + (1 - a^2m^2)y^2 - a^22lmxy = 0 \left[ \begin{array}{l} \text{compare with} \\ ax^2 + 2hxy + by^2 = 0 \end{array} \right]$$

(i) Condition for subtends a right angled coefficient of  $x^2$  + coefficient of  $y^2 = 0$

$$\Rightarrow (1 - a^2l^2) + (1 - a^2m^2) = 0$$

$$\Rightarrow 2 - a^2l^2 - a^2m^2 = 0$$

$$\Rightarrow 2 = a^2l^2 + a^2m^2$$

$$\therefore a^2(l^2 + m^2) = 2$$

(ii) Condition for coincide  $h^2 = ab$

$$\Rightarrow (a^2lm)^2 = (1 - a^2l^2)(1 - a^2m^2)$$

$$\Rightarrow a^4l^2m^2 = 1 - a^2l^2 - a^2m^2 + a^4l^2m^2$$

$$\Rightarrow 1 = a^2l^2 + a^2m^2$$

$$a^2(l^2 + m^2) = 1$$



Q.No: 20



Homogenizing





1. S.T that the pair of st lines  $6x^2 - 5xy - 6y^2 = 0$ ,  
 $6x^2 - 5xy - 6y^2 + x + 5y - 1 = 0$  form a square.

Sol: Given pair of st lines

$$6x^2 - 5xy - 6y^2 = 0$$

$$6x^2 - 5xy - 6y^2 + x + 5y - 1 = 0$$

$$\begin{aligned} \text{Now } 6x^2 - 5xy - 6y^2 &= 6x^2 - 9xy + 4xy - 6y^2 \\ &= 3x(2x - 3y) + 2y(2x - 3y) \\ &= (2x - 3y)(3x + 2y) \end{aligned}$$

$$\Rightarrow 2x - 3y = 0 \dots (1) \Rightarrow m_1 = \frac{2}{3} \quad [m = -a/b]$$

$$3x + 2y = 0 \dots \dots (2) \Rightarrow m_2 = -\frac{3}{2} \quad [m_1 m_2 = -1]$$

$$\begin{aligned} \text{Let } 6x^2 - 6xy - 6y^2 + x + 5y - 1 &\equiv (2x - 3y + l)(3x + 2y + m) \\ &\equiv [6x^2 + 4xy + 2mx - 9xy - 6y^2 - 3my] \\ &\quad + 3lx + 2ly + lm \end{aligned}$$

$$\equiv [6x^2 - 5xy - 6y^2(3l + 2m)x + (2l - 3m)y + lm]$$

*equating the coefficients of x and y,*

$$3l + 2m = 1 \dots (a), 2l - 3m = 5 \dots (b)$$

Solving (a) & (b)

$$\begin{array}{cccc} 3 & 2 & -1 & 3 \\ 2 & -3 & -5 & 2 \end{array} \quad (l, m) = \left[ \frac{-10-3}{-9-4}, \frac{-2+15}{-9-4} \right] = (1, -1)$$

So the lines represented by

$$6x^2 - 6xy - 6y^2 + x + 5y - 1 = 0 \text{ are}$$

$$2x - 3y + 1 = 0 \dots (3) \Rightarrow m_3 = \frac{2}{3}$$

$$3x + 2y - 1 = 0 \dots (4) \Rightarrow m_4 = -\frac{3}{2}$$

$$[m_3 m_4 = -1]$$

Eq'n (1), (3) and (2), (4) are parallel lines

Eq'n (1), (2) and (3), (4) are perpendicular lines  
so the figure form a rectangle.

Now the distance b/w Parallel lines (1), (3) is

$$d_1 = \frac{|C_1 - C_2|}{\sqrt{a^2 + b^2}} = \frac{|0+1|}{\sqrt{2^2 + 3^2}} = \frac{1}{\sqrt{13}}$$

Also the distance b/w Parallel lines (2), (4) is

$$d_2 = \frac{|C_1 - C_2|}{\sqrt{a^2 + b^2}} = \frac{|0-1|}{\sqrt{3^2 + 2^2}} = \frac{1}{\sqrt{13}}$$

$\Rightarrow d_1 = d_2$  thus the figure formed is a square.



Q.No: 20



Homogenizing





1. If a ray makes angles  $\alpha, \beta, \gamma$  and  $\delta$  with four diagonals of a cube, then find

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = \frac{4}{3}.$$

Sol: let 'OABCDEFG' be the given cube with edge '1' and its vertices are

$$O(0, 0, 0)$$

$$A(1, 0, 0)$$

$$B(0, 1, 0)$$

$$C(0, 0, 1)$$

$$D(1, 1, 0)$$

$$E(1, 0, 1)$$

$$F(0, 1, 1)$$

$$G(1, 1, 1)$$

Direction ratios  $[x_2 - x_1, y_2 - y_1, z_2 - z_1]$

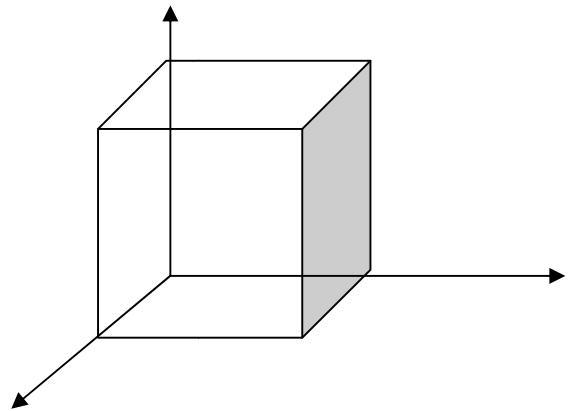
D.r's of diagonals Dc's of diagonals

$$\text{Dr's of OG } (1, 1, 1) \quad \text{Dc's of OG } \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

$$\text{Dr's of AF } (-1, 1, 1) \quad \text{Dc's of AF } \left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

$$\text{Dr's of BE } (1, -1, 1) \quad \text{Dc's of BE } \left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

$$\text{Dr's of CD } (1, 1, -1) \quad \text{Dc's of CD } \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$$



Let  $(l, m, n)$  be the DC's of the ray which make angles  $\alpha, \beta, \gamma$  and  $\delta$  with four diagonals of a cube OG, AF, BE and CD respectively.

$$\text{Now } \cos\alpha = |l_1 l_2 + m_1 m_2 + n_1 n_2|$$

$$\cos\alpha = \left| l \left(\frac{1}{\sqrt{3}}\right) + m \left(\frac{1}{\sqrt{3}}\right) + n \left(\frac{1}{\sqrt{3}}\right) \right| = \left| \frac{l+m+n}{\sqrt{3}} \right|$$

$$\text{Similarly we get, } \cos\beta = \left| \frac{-l+m+n}{\sqrt{3}} \right|,$$

$$\cos\gamma = \left| \frac{l-m+n}{\sqrt{3}} \right|, \cos\delta = \left| \frac{l+m-n}{\sqrt{3}} \right|$$

$$\text{Now } \cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta$$

$$= \left( \frac{l+m+n}{\sqrt{3}} \right)^2 + \left( \frac{-l+m+n}{\sqrt{3}} \right)^2 + \left( \frac{l-m+n}{\sqrt{3}} \right)^2 + \left( \frac{l+m-n}{\sqrt{3}} \right)^2$$

$$= \frac{4}{3}(l^2 + m^2 + n^2)$$

$$(l^2 + m^2 + n^2) = 1$$

$$= \frac{4}{3}(1) = \frac{4}{3}$$

Q.No:21

DC'S AND DR'S



2. Find the angle b/w the diagonals of a cube.

Sol: let 'OABCDEFG' be the given cube with edge '1' and its vertices are

$$O(0, 0, 0)$$

$$A(1, 0, 0)$$

$$B(0, 1, 0)$$

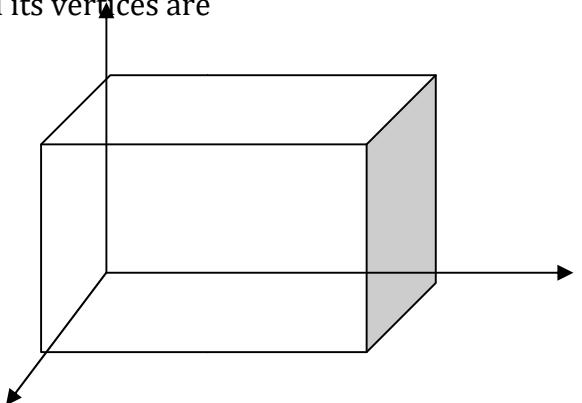
$$C(0, 0, 1)$$

$$D(1, 1, 0)$$

$$E(1, 0, 1)$$

$$F(0, 1, 1)$$

$$G(1, 1, 1)$$



Direction ratios  $[x_2 - x_1, y_2 - y_1, z_2 - z_1]$

D.r's of diagonals Dc's of diagonals

$$\text{Dr's of OG } (1, 1, 1) \quad \text{Dc's of OG } \left( \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right)$$

$$\text{Dr's of AF } (-1, 1, 1) \quad \text{Dc's of AF } \left( -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right)$$

$$\text{Dr's of BE } (1, -1, 1) \quad \text{Dc's of BE } \left( \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right)$$

$$\text{Dr's of CD } (1, 1, -1) \quad \text{Dc's of CD } \left( \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \right)$$

If ' $\theta$ ' is the angle between the diagonals OG and AF then

$$\cos \theta = |l_1 l_2 + m_1 m_2 + n_1 n_2|$$

$$= \left| -\frac{1}{\sqrt{3}} \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}} \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}} \frac{1}{\sqrt{3}} \right|$$

$$= \left| -\frac{1}{3} + \frac{1}{3} + \frac{1}{3} \right| \quad \therefore \theta = \cos^{-1} \left( \frac{1}{3} \right)$$



3. Find the angle b/w the lines whose direction cosines satisfy the eq''ns

$$3l + m + 5n = 0, \quad 6mn - 2nl + 5lm = 0.$$

**Sol:** Given eq''ns  $3l + m + 5n = 0 \dots (1)$

$$6mn - 2nl + 5lm = 0. \dots (2)$$

From (1)  $\Rightarrow m = -(3l + 5n)$  Sub 'm' value in (2)

$$\Rightarrow -6[3l + 5n]n - 2nl - 5l[3l + 5n] = 0$$

$$\Rightarrow -18ln - 30n^2 - 2ln - 15l^2 - 25ln = 0$$

$$\Rightarrow -15l^2 - 45ln - 30n^2 = 0 \quad \div by -15$$

$$\Rightarrow l^2 + 3ln + 2n^2 = 0$$

$$\Rightarrow l^2 + 2ln + ln + 2n^2 = 0$$

$$\Rightarrow l(l + 2n) + n(l + 2n) = 0$$

$$\Rightarrow (l + 2n)(l + n) = 0$$

$$(l + 2n) = 0 \dots \dots (3) \quad \text{and} \quad (l + n) = 0 \dots \dots (4)$$

Solving (1) & (3)

$$\begin{array}{ccccccc} 3 & 1 & 5 & 3 & 1 & 5 \\ 1 & 0 & 2 & 1 & 0 & 2 \end{array}$$

$$\frac{l}{2-0} = \frac{m}{5-6} = \frac{n}{0-1}$$

$$\Rightarrow \frac{l}{2} = \frac{m}{-1} = \frac{n}{-1}$$

$$\begin{aligned} Dr's \text{ of } 1^{st} \text{ line } (a_1, b_1, c_1) \\ = (2, -1, -1) \end{aligned}$$

Solving (1) & (4)

$$\begin{array}{ccccccc} 3 & 1 & 5 & 3 & 1 & 5 \\ 1 & 0 & 1 & 1 & 0 & 1 \end{array}$$

$$\frac{l}{1-0} = \frac{m}{5-3} = \frac{n}{0-1}$$

$$\Rightarrow \frac{l}{1} = \frac{m}{2} = \frac{n}{-1}$$

$$\begin{aligned} Dr's \text{ of } 2^{nd} \text{ line } (a_2, b_2, c_2) \\ = (1, 2, -1) \end{aligned}$$

$$\cos \theta = \frac{|a_1a_2 + b_1b_2 + c_1c_2|}{\sqrt{a_1^2 + b_1^2 + c_1^2} \sqrt{a_2^2 + b_2^2 + c_2^2}} \Rightarrow \cos \theta = \frac{(2)(1) + (-1)(2) + (-1)(-1)}{\sqrt{2^2 + 1^2 + 1^2} \sqrt{1^2 + 2^2 + 1^2}}$$

$$\Rightarrow \cos \theta = \frac{2-2+1}{\sqrt{6}\sqrt{6}} = \cos \theta = \frac{1}{6} \quad \therefore \theta = \cos^{-1} \left( \frac{1}{6} \right)$$

Q.No:21

DC'S AND DR'S



4. Find the angle b/w the lines whose direction cosines satisfy the eq"ns

$$l + m + n = 0, \quad l^2 + m^2 - n^2 = 0.$$

Sol: Given eq"ns  $l + m + n = 0 \dots (1)$ ,  $l^2 + m^2 - n^2 = 0 \dots (2)$

From (1)  $\Rightarrow l = -(m + n)$  Sub 'l' value in (2)

$$\Rightarrow (m + n)^2 + m^2 - n^2 = 0$$

$$\Rightarrow m^2 + n^2 + 2mn + m^2 - n^2 = 0$$

$$\Rightarrow 2m^2 + 2mn = 0$$

$$\Rightarrow 2m(m + n) = 0$$

$$\Rightarrow 2m = 0 \text{ and } m + n = 0$$

$$m = 0 \dots \dots (3) \text{ and } m + n = 0 \dots \dots (4).$$

Solving (1) & (3)

$$\begin{matrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{matrix}$$

$$\frac{l}{0-1} = \frac{m}{0-0} = \frac{n}{1-0}$$

$$\Rightarrow \frac{l}{-1} = \frac{m}{0} = \frac{n}{1}$$

$$\begin{aligned} Dr's \text{ of } 1^{st} \text{ line } (a_1, b_1, c_1) \\ = (-1, 0, 1) \end{aligned}$$

Solving (1) & (4)

$$\begin{matrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 \end{matrix}$$

$$\frac{l}{0-0} = \frac{m}{0-1} = \frac{n}{1-0}$$

$$\Rightarrow \frac{l}{0} = \frac{m}{-1} = \frac{n}{1}$$

$$\begin{aligned} Dr's \text{ of } 2^{nd} \text{ line } (a_2, b_2, c_2) \\ = (0, -1, 1) \end{aligned}$$

$$\cos \theta = \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2} \sqrt{a_2^2 + b_2^2 + c_2^2}}$$

$$\Rightarrow \cos \theta = \frac{(-1)(0) + (0)(-1) + (1)(1)}{\sqrt{-1^2 + 0^2 + 1^2} \sqrt{0^2 + -1^2 + 1^2}}$$

$$\Rightarrow \cos \theta = \frac{0+0+1}{\sqrt{2}\sqrt{2}}$$

$$\cos \theta = \frac{1}{2}$$

$$\therefore \theta = 60 \text{ or } \frac{\pi}{3}$$

Q.No:21

DC'S AND DR'S



5. Find the direction cosines satisfy the eq"ns  $l + m + n = 0$ ,  
 $mn - 2nl - 2lm = 0$ .

Sol: Given eq"ns  $l + m + n = 0 \dots (1)$        $mn - 2nl - 2lm = 0 \dots (2)$

From (1)  $\Rightarrow l = -(m + n)$       Sub 'l' value in (2)

$$\Rightarrow mn + 2n(m + n) + 2(m + n)m = 0$$

$$\Rightarrow mn + 2mn + 2n^2 + 2m^2 + 2mn = 0$$

$$\Rightarrow 2m^2 + 5mn + 2n^2 = 0$$

$$\Rightarrow 2m^2 + 4mn + mn + 2n^2 = 0$$

$$\Rightarrow 2m(m + 2n) + n(m + 2n) = 0$$

$$\Rightarrow (m + 2n)(2m + n) = 0$$

$$m + 2n = 0 \dots \dots (3) \quad \text{and} \quad 2m + n = 0 \dots \dots (4)$$

<p>Solving (1) &amp; (3)</p> $\begin{array}{cccccc} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 0 & 1 & 2 \end{array}$ $\frac{l}{2-1} = \frac{m}{0-2} = \frac{n}{1-0}$ $\Rightarrow \frac{l}{1} = \frac{m}{-2} = \frac{n}{1}$ <p><i>Dr's of 1<sup>st</sup> line</i> <math>(a_1, b_1, c_1)</math>  <math>= (1, -2, 1)</math></p> <p><i>Dc's of 1<sup>st</sup> line</i> <math>(l_1, m_1, n_1)</math>  <math>= (\frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}})</math></p>	<p>Solving (1) &amp; (4)</p> $\begin{array}{cccccc} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 1 & 0 & 2 & 1 \end{array}$ $\frac{l}{1-2} = \frac{m}{0-1} = \frac{n}{2-0}$ $\Rightarrow \frac{l}{-1} = \frac{m}{-1} = \frac{n}{2}$ <p><i>Dr's of 2<sup>nd</sup> line</i> <math>(a_2, b_2, c_2)</math>  <math>= (-1, -1, 2)</math></p> <p><i>Dc's of 2<sup>nd</sup> line</i> <math>(l_2, m_2, n_2)</math>  <math>= (\frac{-1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}})</math></p>
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6. Show that the lines whose d.c's are given by

$l + m + n = 0, 2mn + 3nl - 5lm = 0$ . are mutually perpendicular.

Sol: given eq'ns

$$l + m + n = 0 \dots \dots (1)$$

$$2mn + 3nl - 5lm = 0 \dots \dots (2)$$

From (1)  $\Rightarrow l = -(m + n)$

Sub 'l' value in (2)

$$\Rightarrow 2mn - 3n(m + n) + 5(m + n)m = 0$$

$$\Rightarrow 2mn - 3mn - 3n^2 + 5m^2 + 5mn = 0$$

$$\Rightarrow 5m^2 + 4mn - 3n^2 = 0 \quad \div \text{by } n^2$$

$$\Rightarrow 5\left[\frac{m}{n}\right]^2 + 4\left[\frac{m}{n}\right] - 3 = 0$$

Let  $\frac{m_1}{n_1}, \frac{m_2}{n_2}$  be the roots

Product of the roots  $= \frac{c}{a}$

$$\Rightarrow \left[\frac{m_1}{n_1}\right] \left[\frac{m_2}{n_2}\right] = -\frac{3}{5} \Rightarrow \frac{m_1 m_2}{-3} = \frac{n_1 n_2}{5} \dots \dots (A)$$

From (1)  $\Rightarrow m = -(l + n)$

Sub 'm' value in (2)

$$\Rightarrow -2(l + n)n + 3nl + 5l(l + n) = 0$$

$$\Rightarrow -2ln - 2n^2 + 3nl + 5l^2 + 5ln = 0$$

$$\Rightarrow 5l^2 + 6ln - 2n^2 = 0 \quad \div \text{by } n^2$$

$$\Rightarrow 5\left[\frac{l}{n}\right]^2 + 6\left[\frac{l}{n}\right] - 2 = 0$$

Let  $\frac{l_1}{n_1}, \frac{l_2}{n_2}$  be the roots

Product of the roots  $= \frac{c}{a}$

$$\Rightarrow \left[\frac{l_1}{n_1}\right] \left[\frac{l_2}{n_2}\right] = \frac{2}{5} \Rightarrow \frac{l_1 l_2}{-2} = \frac{n_1 n_2}{5} \dots \dots (B)$$

From (A) & (B)

$$\Rightarrow \frac{l_1 l_2}{-2} = \frac{m_1 m_2}{-3} = \frac{n_1 n_2}{5} = k(\text{say})$$

$$\Rightarrow l_1 l_2 = -2k, m_1 m_2 = -3k, n_1 n_2 = 5k$$

$$\cos\theta = |l_1 l_2 + m_1 m_2 + n_1 n_2| = -2k - 3k + 5k = 0$$

$\theta = 90^\circ \therefore \text{the two lines are perpendicular to each other.}$



7. S.T the points  $(4, 7, 8)$ ,  $(2, 3, 4)$ ,  $(-1, -2, 1)$ ,  $(1, 2, 5)$  are the vertices of a parallelogram.

**Sol:** let the Given points  $A(4, 7, 8)$ ,  $B(2, 3, 4)$ ,  $C(-1, -2, 1)$ ,  $D(1, 2, 5)$

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

$$A(4, 7, 8) \quad B(2, 3, 4)$$

$$AB = \sqrt{(2 - 4)^2 + (3 - 7)^2 + (4 - 8)^2}$$

$$= \sqrt{(-2)^2 + (-4)^2 + (-4)^2}$$

$$= \sqrt{4 + 16 + 16} = \sqrt{36} = 6$$

$$B(2, 3, 4) \quad C(-1, -2, 1)$$

$$BC = \sqrt{(-1 - 2)^2 + (-2 - 3)^2 + (1 - 4)^2}$$

$$= \sqrt{(-3)^2 + (-5)^2 + (-3)^2}$$

$$BC = \sqrt{9 + 25 + 9} = \sqrt{43}$$

$$C(-1, -2, 1) \quad D(1, 2, 5)$$

$$CD = \sqrt{(1 + 1)^2 + (2 + 2)^2 + (5 - 1)^2}$$

$$CD = \sqrt{(2)^2 + (4)^2 + (4)^2}$$

$$CD = \sqrt{4 + 16 + 16} = \sqrt{36} = 6$$

$$D(1, 2, 5) \quad A(4, 7, 8)$$

$$DA = \sqrt{(4 - 1)^2 + (7 - 2)^2 + (8 - 5)^2}$$

$$DA = \sqrt{(3)^2 + (5)^2 + (5)^2}$$

$$DA = \sqrt{9 + 25 + 9} = \sqrt{43}$$

$$AB = CD, BC = DA$$

$\therefore A, B, C, D$  are the vertices of the parallelogram.



8. If  $(6, 10, 10), (1, 0, -5), (6, -10, 0)$  Are vertices of a triangle find the D.r's of its sides. Determine whether it is right angled or isosceles.

Sol: let  $A(6, 10, 10), B(1, 0, -5), C(6, -10, 0)$  Are vertices of a triangle

Direction ratios  $[x_2 - x_1, y_2 - y_1, z_2 - z_1]$

$$A(6, 6, 10) \quad B(1, 0, -5)$$

$$\text{D.r's of } AB \Rightarrow [1 - 6, 0 - 10, -5 - 10]$$

$$= [-5, -10, -15] = [1, 2, 3]$$

$$B(1, 0, -5), C(6, -10, 0)$$

$$\text{D.r's of } BC \Rightarrow [6 - 1, -10 - 0, 0 + 5]$$

$$= [5, -10, 5] = [1, -2, 1]$$

$$A(6, 10, 10) \quad C(6, -10, 0)$$

$$\text{D.r's of } AC \Rightarrow [6 - 6, -10 - 10, 0 - 10]$$

$$= [0, -20, -10] = [0, 2, 1]$$

$$\cos \angle ABC = \frac{|a_1a_2 + b_1b_2 + c_1c_2|}{\sqrt{a_1^2 + b_1^2 + c_1^2} \sqrt{a_2^2 + b_2^2 + c_2^2}}$$

$$\cos \angle ABC = \frac{|1(1+2(-2)+3(1)|}{\sqrt{1^2+2^2+3^2} \sqrt{1^2+(-2)^2+1^2}}$$

$$\cos \angle ABC = \frac{|1-4+3|}{\sqrt{1+4+9} \sqrt{1+4+1}}$$

$$\cos \angle ABC = \frac{0}{\sqrt{14} \sqrt{6}} = 0$$

$$\angle B = \frac{\pi}{2}$$

$\therefore$  The given triangle is a right angled.

Q.No:21

DC'S AND DR'S





1. If  $\sqrt{1-x^2} + \sqrt{1-y^2} = a(x-y)$ , then P.T  $\frac{dy}{dx} = \frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$ .

Sol: let

$$x = \sin A \Rightarrow A = \sin^{-1} x, \quad y = \sin B \Rightarrow B = \sin^{-1} y$$

$$\Rightarrow \sqrt{1-x^2} + \sqrt{1-y^2} = a(x-y)$$

$$\Rightarrow \sqrt{1-\sin^2 A} + \sqrt{1-\sin^2 B} = a(\sin A - \sin B)$$

$$\Rightarrow \sqrt{\cos^2 A} + \sqrt{\cos^2 B} = a(\sin A - \sin B)$$

$$\Rightarrow \cos A + \cos B = a(\sin A - \sin B)$$

$$\Rightarrow 2\cos\left[\frac{A+B}{2}\right]\cos\left[\frac{A-B}{2}\right] = a2\cos\left[\frac{A+B}{2}\right]\sin\left[\frac{A-B}{2}\right]$$

$$\Rightarrow \cos\left[\frac{A-B}{2}\right] = a\sin\left[\frac{A+B}{2}\right]$$

$$\Rightarrow \frac{\cos\left[\frac{A-B}{2}\right]}{\sin\left[\frac{A+B}{2}\right]} = a \Rightarrow \cot\left[\frac{A-B}{2}\right] = a$$

$$\Rightarrow \frac{A-B}{2} = \cot^{-1}(a)$$

$$\Rightarrow A - B = 2\cot^{-1}(a)$$

$$\Rightarrow \sin^{-1} x - \sin^{-1} y = 2\cot^{-1}(a) \quad \text{Diff w.r.t } x'$$

$$\Rightarrow \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}} \frac{dy}{dx} = 0$$

$$\Rightarrow -\frac{1}{\sqrt{1-x^2}} \frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$$





$$2. \text{ If } y = \tan^{-1} \left[ \frac{\sqrt{1+x^2} + \sqrt{1-x^2}}{\sqrt{1+x^2} - \sqrt{1-x^2}} \right], \text{ find } \frac{dy}{dx}.$$

Sol: let  $x^2 = \cos 2\theta \Rightarrow \theta = \frac{1}{2} \cos^{-1}(x^2)$

$$y = \tan^{-1} \left[ \frac{\sqrt{1+x^2} + \sqrt{1-x^2}}{\sqrt{1+x^2} - \sqrt{1-x^2}} \right]$$

$$y = \tan^{-1} \left[ \frac{\sqrt{1+\cos 2\theta} + \sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta} - \sqrt{1-\cos 2\theta}} \right]$$

$$y = \tan^{-1} \left[ \frac{\sqrt{2\cos^2 \theta} + \sqrt{2\sin^2 \theta}}{\sqrt{2\cos^2 \theta} - \sqrt{2\sin^2 \theta}} \right]$$

$$y = \tan^{-1} \left[ \frac{\sqrt{2}(\cos \theta + \sin \theta)}{\sqrt{2}(\cos \theta - \sin \theta)} \right] \quad (\div \cos \theta)$$

$$y = \tan^{-1} \left[ \frac{\frac{\cos \theta}{\cos \theta} + \frac{\sin \theta}{\cos \theta}}{\frac{\cos \theta}{\cos \theta} - \frac{\sin \theta}{\cos \theta}} \right] = \tan^{-1} \left[ \frac{1 + \tan \theta}{1 - \tan \theta} \right]$$

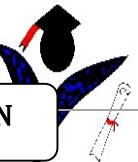
$$y = \tan^{-1} \left[ \tan^{-1} \left( \frac{\pi}{4} + \theta \right) \right] = \frac{\pi}{4} + \theta$$

$$y = \frac{\pi}{4} + \frac{1}{2} \cos^{-1}(x^2) \quad \text{diff w.r.t } x'$$

$$\frac{dy}{dx} = 0 + \frac{1}{2} \left[ -\frac{1}{\sqrt{1-(x^2)^2}} \right] \frac{d}{dx}(x^2)$$

$$\frac{dy}{dx} = 0 + \frac{1}{2} \left[ -\frac{1}{\sqrt{1-(x^2)^2}} \right] (2x)$$

$$\frac{dy}{dx} = \left[ \frac{-x}{\sqrt{1-x^4}} \right]$$





3. If  $y = x^{\tan x} + (\sin x)^{\cos x}$ , find  $\frac{dy}{dx}$ .

$$\text{Sol: let } y = P + Q \quad \text{diff w.r.t } x' \quad \frac{dy}{dx} = \frac{dP}{dx} + \frac{dQ}{dx} \dots (1)$$

where  $P = x^{\tan x}$  Applying log on B.S

$$\Rightarrow \log P = \log x^{\tan x}$$

$$\Rightarrow \log P = \tan x \cdot \log x \quad [(uv)' = uv' + vu'] \quad \text{diff w.r.t } x'$$

$$\Rightarrow \frac{1}{P} \frac{dP}{dx} = \tan x \frac{d}{dx}(\log x) + \log x \frac{d}{dx}(\tan x)$$

$$\Rightarrow \frac{1}{P} \frac{dP}{dx} = \tan x \cdot \frac{1}{x} + \log x \cdot \sec^2 x$$

$$\Rightarrow \frac{dP}{dx} = P \left[ \frac{\tan x}{x} + \log x \cdot \sec^2 x \right]$$

$$\Rightarrow \frac{dP}{dx} = x^{\tan x} \left[ \frac{\tan x}{x} + \log x \cdot \sec^2 x \right] \dots (2)$$

where  $Q = \sin x^{\cos x}$  Applying log on B.S

$$\Rightarrow \log Q = \log \sin x^{\cos x}$$

$$\Rightarrow \log Q = \cos x \cdot \log(\sin x) \quad [(uv)' = uv' + vu'] \quad \text{diff w.r.t } x'$$

$$\Rightarrow \frac{1}{Q} \frac{dQ}{dx} = \cos x \frac{d}{dx}[\log(\sin x)] + \log(\sin x) \frac{d}{dx}(\cos x)$$

$$\Rightarrow \frac{1}{Q} \frac{dQ}{dx} = \cos x \cdot \frac{1}{\sin x} \cdot \cos x + \log \sin x \cdot (-\sin x)$$

$$\Rightarrow \frac{dQ}{dx} = Q \left[ \frac{\cos^2 x}{\sin x} - \log(\sin x) \cdot \sin x \right]$$

$$\Rightarrow \frac{dQ}{dx} = \sin x^{\cos x} \left[ \frac{\cos^2 x}{\sin x} - \log(\sin x) \cdot \sin x \right] \dots (3)$$

$$\text{sub eq''n(2)&(3)in (1)} \quad \frac{dy}{dx} = \frac{dP}{dx} + \frac{dQ}{dx}$$

$$\frac{dy}{dx} = x^{\tan x} \left[ \frac{\tan x}{x} + \log x \cdot \sec^2 x \right] + \sin x^{\cos x} \left[ \frac{\cos^2 x}{\sin x} - \log(\sin x) \cdot \sin x \right]$$





4. If  $y = (\sin x)^{\log x} + x^{\sin x}$ , find  $\frac{dy}{dx}$ .

Sol: let  $y = P + Q \quad \text{diff w.r.t } x'$

$$\frac{dy}{dx} = \frac{dP}{dx} + \frac{dQ}{dx} \dots (1)$$

where  $P = \sin x^{\log x}$  Applying log on B.S  $\Rightarrow \log P = \log \sin x^{\log x}$

$\Rightarrow \log P = \log x \cdot \log \sin x \quad [(uv)' = uv' + vu'] \quad \text{diff w.r.t } x'$

$$\Rightarrow \frac{1}{P} \frac{dP}{dx} = \log x \frac{d}{dx} (\log \sin x) + \log \sin x \frac{d}{dx} (\log x)$$

$$\Rightarrow \frac{1}{P} \frac{dP}{dx} = \log x \cdot \frac{1}{\sin x} \cdot \cos x + \log \sin x \cdot \frac{1}{x}$$

$$\Rightarrow \frac{dP}{dx} = P \left[ \log x \cdot \cot x + (\log \sin x) \frac{1}{x} \right]$$

$$\Rightarrow \frac{dP}{dx} = \sin x^{\log x} \left[ \log x \cdot \cot x + \frac{\log \sin x}{x} \right] \dots (2)$$

where  $Q = x^{\sin x}$  Applying log on B.S  $\Rightarrow \log Q = \log x^{\sin x}$

$\Rightarrow \log Q = \sin x \cdot \log x \quad [(uv)' = uv' + vu'] \quad \text{diff w.r.t } x'$

$$\Rightarrow \frac{1}{Q} \frac{dQ}{dx} = \sin x \frac{d}{dx} [\log x] + \log x \frac{d}{dx} (\sin x)$$

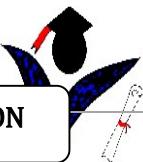
$$\Rightarrow \frac{1}{Q} \frac{dQ}{dx} = \sin x \cdot \frac{1}{x} + \log x \cdot (\cos x)$$

$$\Rightarrow \frac{dQ}{dx} = Q \left[ \frac{\sin x}{x} + \log x \cdot \cos x \right]$$

$$\Rightarrow \frac{dQ}{dx} = x^{\sin x} \left[ \frac{\sin x}{x} + \log x \cdot \cos x \right] \dots (3)$$

$$\text{sub eq''n(2)&(3)in (1)} \quad \frac{dy}{dx} = \frac{dP}{dx} + \frac{dQ}{dx}$$

$$\frac{dy}{dx} = \sin x^{\log x} \left[ \log x \cdot \cot x + \frac{\log \sin x}{x} \right] + x^{\sin x} \left[ \frac{\sin x}{x} + \log x \cdot \cos x \right]$$





$$5. \text{ If } x^y + y^x = a^b, \text{ S.T } \frac{dy}{dx} = - \left[ \frac{yx^{y-1} + y^x \log y}{x^y \log x + xy^{x-1}} \right]$$

Sol: let  $P = x^y, Q = y^x$

$$\Rightarrow P + Q = a^b \text{ diff w.r.t } x' \quad \frac{dP}{dx} + \frac{dQ}{dx} = 0 \dots (1)$$

where  $P = x^y$  Applying log on B.S  $\Rightarrow \log P = \log x^y$

$\Rightarrow \log P = y \cdot \log x$   $[(uv)' = uv' + vu'] \text{ diff w.r.t } x'$

$$\Rightarrow \frac{1}{P} \frac{dP}{dx} = y \cdot \frac{d}{dx} (\log x) + \log x \frac{d}{dx} (y)$$

$$\Rightarrow \frac{1}{P} \frac{dP}{dx} = y \cdot \frac{1}{x} + \log x \cdot \frac{dy}{dx}$$

$$\Rightarrow \frac{dP}{dx} = P \left[ y \cdot x^{-1} + \log x \cdot \frac{dy}{dx} \right]$$

$$\Rightarrow \frac{dP}{dx} = x^y \left[ y \cdot x^{-1} + \log x \cdot \frac{dy}{dx} \right]$$

$$\Rightarrow \frac{dP}{dx} = \left[ y \cdot x^{y-1} + x^y \log x \cdot \frac{dy}{dx} \right] \dots (2)$$

where  $Q = y^x$  Applying log on B.S  $\Rightarrow \log Q = \log y^x$

$\Rightarrow \log Q = x \cdot \log y$   $[(uv)' = uv' + vu'] \text{ diff w.r.t } x'$

$$\Rightarrow \frac{1}{Q} \frac{dQ}{dx} = x \cdot \frac{d}{dx} (\log y) + \log y \frac{d}{dx} (x)$$

$$\Rightarrow \frac{1}{Q} \frac{dQ}{dx} = x \cdot \frac{1}{y} \frac{dy}{dx} + \log y \cdot 1$$

$$\Rightarrow \frac{dQ}{dx} = Q \left[ x \cdot y^{-1} \frac{dy}{dx} + \log y \right] = y^x \left[ x \cdot y^{-1} \frac{dy}{dx} + \log y \right]$$

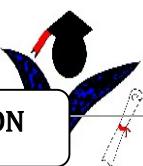
$$\Rightarrow \frac{dQ}{dx} = \left[ x \cdot y^{x-1} \frac{dy}{dx} + y^x \log y \right] \dots (3)$$

$$\text{sub eq''n(2)&(3)in (1)} \quad \frac{dP}{dx} + \frac{dQ}{dx} = 0$$

$$y \cdot x^{y-1} + x^y \log x \cdot \frac{dy}{dx} + x \cdot y^{x-1} \frac{dy}{dx} + y^x \log y = 0$$

$$\frac{dy}{dx} (x^y \log x + xy^{x-1}) + (yx^{y-1} + y^x \log y) = 0$$

$$\frac{dy}{dx} = - \left[ \frac{yx^{y-1} + y^x \log y}{x^y \log x + xy^{x-1}} \right]$$





6. If  $y = x\sqrt{a^2 + x^2} + a^2 \log[x + \sqrt{a^2 + x^2}]$ , find  $\frac{dy}{dx}$ .

$$sol: y = x\sqrt{a^2 + x^2} + a^2 \log[x + \sqrt{a^2 + x^2}]$$

diff w.r.t 'x'

$$\Rightarrow \frac{dy}{dx} = x \cdot \frac{d}{dx}(\sqrt{a^2 + x^2}) + \sqrt{a^2 + x^2} \cdot \frac{dx}{dx} \\ + a^2 \cdot \frac{d}{dx} \left\{ \log[x + \sqrt{a^2 + x^2}] \right\}$$

$$\Rightarrow \frac{dy}{dx} = x \cdot \frac{1}{2\sqrt{a^2+x^2}} \frac{d}{dx}(a^2 + x^2) + \sqrt{a^2 + x^2} \cdot 1 \\ + a^2 \cdot \left\{ \frac{1}{[x + \sqrt{a^2 + x^2}]} \right\} \frac{d}{dx}[x + \sqrt{a^2 + x^2}]$$

$$\Rightarrow \frac{dy}{dx} = x \cdot \frac{1}{2\sqrt{a^2+x^2}} (0 + 2x) + \sqrt{a^2 + x^2} \\ + a^2 \cdot \left\{ \frac{1}{[x + \sqrt{a^2 + x^2}]} \right\} \left[ 1 + \frac{1}{2\sqrt{a^2 + x^2}} (0 + 2x) \right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{x^2}{\sqrt{a^2+x^2}} + \sqrt{a^2 + x^2} \\ + a^2 \cdot \left\{ \frac{1}{[x + \sqrt{a^2 + x^2}]} \right\} \left[ \frac{\sqrt{a^2 + x^2} + x}{\sqrt{a^2 + x^2}} \right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{x^2}{\sqrt{a^2+x^2}} + \sqrt{a^2 + x^2} + \frac{a^2}{\sqrt{a^2+x^2}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{a^2+x^2}{\sqrt{a^2+x^2}} + \sqrt{a^2 + x^2}$$

$$\Rightarrow \frac{dy}{dx} = \sqrt{a^2 + x^2} + \sqrt{a^2 + x^2}$$

$$\therefore \frac{dy}{dx} = 2\sqrt{a^2 + x^2}$$





7. If  $f(x) = \sin^{-1} \sqrt{\frac{x-\beta}{\alpha-\beta}}$  &  $g(x) = \tan^{-1} \left[ \sqrt{\frac{x-\beta}{\alpha-x}} \right]$ , then P.T  
 $f'(x) = g'(x)$

**Sol:** let  $\theta = \sin^{-1} \sqrt{\frac{x-\beta}{\alpha-\beta}}$  ... (1)

$$\Rightarrow \sin \theta = \sqrt{\frac{x-\beta}{\alpha-\beta}}$$

and we know that

$$\cos \theta = \sqrt{1 - \sin^2 \theta}$$

$$\Rightarrow \cos \theta = \sqrt{1 - \left[ \sqrt{\frac{x-\beta}{\alpha-\beta}} \right]^2}$$

$$\Rightarrow \cos \theta = \sqrt{1 - \frac{x-\beta}{\alpha-\beta}}$$

$$\Rightarrow \cos \theta = \sqrt{\frac{\alpha-\beta-x+\beta}{\alpha-\beta}} = \sqrt{\frac{\alpha-x}{\alpha-\beta}}$$

Now  $\tan \theta = \frac{\sin \theta}{\cos \theta}$

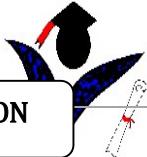
$$\Rightarrow \tan \theta = \frac{\sqrt{\frac{x-\beta}{\alpha-\beta}}}{\sqrt{\frac{\alpha-x}{\alpha-\beta}}} = \sqrt{\frac{x-\beta}{\alpha-x}}$$

$$\Rightarrow \theta = \tan^{-1} \sqrt{\frac{x-\beta}{\alpha-x}} = g(x) \dots (2)$$

from eq'n(1)&(2)

$$f(x) = g(x)$$

$$\text{diff w.r.t } x' + \Rightarrow f'(x) = g'(x)$$





8. If  $y = \frac{(1-2x)^{\frac{2}{3}}(1+3x)^{\frac{3}{4}}}{(1-6x)^{\frac{5}{6}}(1+7x)^{-\frac{6}{7}}}$ , find  $\frac{dy}{dx}$ .

Sol:  $y = \frac{(1-2x)^{\frac{2}{3}}(1+3x)^{\frac{3}{4}}}{(1-6x)^{\frac{5}{6}}(1+7x)^{-\frac{6}{7}}}$  applying log on B.S

$$\Rightarrow \log y = \log \left[ \frac{(1-2x)^{\frac{2}{3}}(1+3x)^{\frac{3}{4}}}{(1-6x)^{\frac{5}{6}}(1+7x)^{-\frac{6}{7}}} \right]$$

$$\Rightarrow \log y = \log(1-2x)^{\frac{2}{3}} + \log(1+3x)^{\frac{3}{4}} \\ - \log(1-6x)^{\frac{5}{6}} - \log(1+7x)^{-\frac{6}{7}}$$

$$\Rightarrow \log y = \frac{2}{3} \log(1-2x) + \frac{3}{4} \log(1+3x) \\ - \frac{5}{6} \log(1-6x) + \frac{6}{7} \log(1+7x) \quad \text{diff w.r.t } 'x'$$

$$\Rightarrow \frac{1}{y} \cdot \frac{dy}{dx} = \frac{2}{3} \cdot \frac{1}{1-2x} (-2) + \frac{3}{4} \cdot \frac{1}{1+3x} (3) \\ - \frac{5}{6} \cdot \frac{1}{1-6x} (-6) + \frac{6}{7} \cdot \frac{1}{1+7x} (7)$$

$$\frac{dy}{dx} = y \left[ -\frac{4}{3} \frac{1}{(1-2x)} + \frac{9}{4} \frac{1}{(1+3x)} + \frac{5}{(1-6x)} + \frac{1}{(1+7x)} \right]$$





1. If the tangent at any point on the curve  $x^{2/3} + y^{2/3} = a^{2/3}$  intersects the coordinate axes in A, B. Show that the length AB is constant.

**Sol:** Given equation of the curve is  $x^{2/3} + y^{2/3} = a^{2/3}$

Parametric equations of this curve are

$$x = a \cos^3 \theta, y = a \sin^3 \theta.$$

Let P( $a \cos^3 \theta, a \sin^3 \theta$ ) be any point on it.

By parametric differentiation,

$$\begin{aligned}\frac{dy}{dx} &= \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{a \cdot 3\sin^2 \theta (\cos \theta)}{a \cdot 3\cos^2 \theta (-\sin \theta)} \\ &= \frac{-\sin \theta}{\cos \theta}\end{aligned}$$

Equation of tangent at P is

$$y - a \sin^3 \theta = \frac{-\sin \theta}{\cos \theta} (x - a \cos^3 \theta)$$

$$\frac{y}{\sin \theta} - a \sin^2 \theta = \frac{-x}{\cos \theta} + a \cos^2 \theta$$

$$\frac{x}{\cos \theta} + \frac{y}{\sin \theta} = a(\cos^2 \theta + \sin^2 \theta)$$

$$\frac{x}{a \cos \theta} + \frac{y}{a \sin \theta} = 1$$

Let this tangent meets the coordinate axes at A, B

then A = ( $a \cos \theta, 0$ ), B = (0,  $a \sin \theta$ )

$$\text{Now } AB = \sqrt{(-a \cos \theta)^2 + (a \sin \theta)^2}$$

$$= \sqrt{a^2 (\cos^2 \theta + \sin^2 \theta)}$$

AB = a which is a constant.

2. If the tangent at any point P on the curve  $x^m y^n = a^{m+n}$  ( $m, n \neq 0$ ) meets the coordinate axes in A, B. Show that AP : PB is constant.

**Sol:** We use the formula

$$\begin{aligned}AP : PB &= a - x_1 : x_1 - 0 \\ &= a - x_1 : x_1\end{aligned}$$

Given equation of the curve is

$$x^m y^n = a^{m+n}$$

Taking logarithms on both sides,

$$m \log x + n \log y = (m+n) \log a$$

differentiating w.r.t. x,

$$m \cdot \frac{1}{x} + n \cdot \frac{1}{y} \frac{dy}{dx} = 0.$$

$$\frac{dy}{dx} = \frac{-my}{nx}$$

$$\text{slope of tangent at P} = \left( \frac{dy}{dx} \right)_{(x_1, y_1)} = \frac{-my_1}{nx_1}$$

Equation of tangent at P is

$$y - y_1 = \frac{-my_1}{nx_1} (x - x_1)$$

$$\Rightarrow (y - y_1) \frac{n}{y_1} = \frac{-m}{x_1} (x - x_1)$$

$$\Rightarrow \frac{ny}{y_1} - n = \frac{-mx}{x_1} + m$$

$$\Rightarrow \frac{mx}{x_1} + \frac{ny}{y_1} = m + n$$

$$\Rightarrow \frac{x}{(m+n)x_1} + \frac{y}{(m+n)y_1} = 1$$

Let this tangent meets x and y axes at A, B.

$$\text{then } A\left(\frac{(m+n)x_1}{m}, 0\right), B\left(0, \frac{(m+n)y_1}{n}\right)$$

$$\text{Now } AP : PB = \frac{(m+n)x_1}{m} - x_1 : x_1 - 0$$

$$= \frac{mx_1 + nx_1 - mx_1}{m} : x_1$$

$$= \frac{n}{m} : 1$$

= n : m which is a constant.



3. Show that the curves  $y^2 = 4(x + 1)$  and  $y^2 = 36(9 - x)$  intersect orthogonally.

**Sol:** Solving  $y^2 = 4(x + 1)$ ,  $y^2 = 36(9 - x)$

$$4(x + 1) = 36(9 - x)$$

$$\Rightarrow x + 1 = 9(9 - x)$$

$$\Rightarrow x + 1 = 81 - 9x$$

$$\Rightarrow x + 9x = 80$$

$$\Rightarrow x = \frac{80}{10} = 8$$

Put  $x = 8$  in  $y^2 = 4(x + 1)$

$$y^2 = 4(8 + 1) = 4(9) = 36$$

$$\therefore y = \pm 6$$

The points of intersection are P(8, 6), Q(8, -6)

At the point P(8, 6):

$$y^2 = 4(x + 1)$$

differentiating w.r.t. x

$$2y \frac{dy}{dx} = 4 \Rightarrow \frac{dy}{dx} = \frac{4}{2y} = \frac{2}{y}$$

slope of the tangent at P to the first curve is

$$m_1 = \frac{2}{6} = \frac{1}{3}$$

$$y^2 = 36(9 - x)$$

$$2y \frac{dy}{dx} = -36 \Rightarrow \frac{dy}{dx} = \frac{-18}{y}$$

slope of the tangent at P to the second curve is

$$m_2 = \frac{-18}{6} = -3$$

$$\text{Here } m_1 m_2 = \frac{1}{3} (-3) = -1$$

$\therefore$  Angle between the tangents at P is  $90^\circ$ .

so the given two curves intersect orthogonally at

P. Similarly, we can show that, the given two curves intersect orthogonally at the other point Q.

4. Find the angle between the curves  $y^2 = 8x$ ,  $4x^2 + y^2 = 32$

**Sol:**  $y^2 = 8x \quad \dots (1)$ ,  $4x^2 + y^2 = 32 \quad \dots (2)$

Substituting  $y^2 = 8x$  in Equation (2)

$$4x^2 + 8x = 32$$

$$4x^2 + 8x - 32 = 0 \quad \div 4$$

$$x^2 + 2x - 8 = 0$$

$$x^2 + 4x - 2x - 8 = 0$$

$$x(x+4) - 2(x+4) = 0$$

$$(x-2)(x+4) = 0$$

$$x = 2 \text{ (or) } x = -4$$

$$x = 2 \Rightarrow y^2 = 16 \Rightarrow y = \pm 4$$

$$x = -4 \Rightarrow y^2 = -32, y \text{ is not real}$$

$\therefore$  The point of intersection of the two curves is P(2, 4) and Q(2, -4)

Equation of the first curve is  $y^2 = 8x$

Differentiating w.r.t x

$$2y \frac{dy}{dx} = 8$$

$$\frac{dy}{dx} = \frac{8}{2y} = \frac{4}{y}$$

Equation of the second curve is  $4x^2 + y^2 = 32$

$$8x + 2y \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-8x}{2y} = -\frac{4x}{y}$$

**Case (i) :** At P(2, 4)

$$m_1 = \frac{4}{4} = 1$$

$$m_2 = \frac{-8}{4} = -2$$

If  $\theta$  is the angle between the two curves then,

$$\tan \theta = \left| \frac{m_2 - m_1}{1 + m_1 m_2} \right| = \left| \frac{1 + 2}{1 - 2} \right| = 3$$

$$\theta = \tan^{-1}(3)$$

**Case (ii) :** At Q(2, -4)

$$m_1 = \frac{4}{-4} = -1$$

$$m_2 = \frac{-8}{-4} = 2$$

$$\tan \theta = \left| \frac{-1 - 2}{1 - 2} \right| = 3$$

$$\theta = \tan^{-1}(3).$$



5. Find the angle between the curves  $2y^2 - 9x = 0$ ,  $3x^2 + 4y = 0$  (in the 4<sup>th</sup> quadrant).

**Sol:** Given curves are  $2y^2 - 9x = 0$ ,  $3x^2 + 4y = 0$ .

$$4y = -3x^2 \Rightarrow y = -\frac{3}{4}x^2$$

$2y^2 - 9x = 0$  becomes

$$2\left(\frac{9}{16}x^4\right) - 9x = 0$$

$$\frac{x^4}{8} - x = 0$$

$$x^4 - 8x = 0$$

$$x(x^3 - 8) = 0$$

$$x = 0, x = 2$$

If  $x = 0$ ,  $y = 0$

$$\text{If } x = 2, y = -\frac{3}{4}(2^2) = -3.$$

Points of intersection are P(0, 0), Q(2, -3)

Q(2, -3) lies in 4<sup>th</sup> quadrant.

Differentiating  $2y^2 - 9x = 0$  with respect to x,

$$2(2y) \frac{dy}{dx} - 9 = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{9}{4y}$$

∴ Slope of the tangent to the curve  $2y^2 - 9x = 0$  at Q is

$$m_1 = \frac{9}{4(-3)} = \frac{-3}{4}$$

Differentiating  $3x^2 + 4y = 0$  with respect to x,

$$3(2x) + 4 \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-6x}{4} = \frac{-3x}{2}$$

Slope of the tangent to the curve  $3x^2 + 4y = 0$  at

$$Q \text{ is } m_2 = \frac{-3}{2}(2) = -3$$

Let  $\theta$  be the angle between the tangent to the given two curves at Q.

$$\tan \theta = \frac{m_1 - m_2}{1 + m_1 m_2}$$

$$= \frac{\frac{-3}{4} + 3}{1 + \frac{9}{4}}$$

$$= \frac{9}{13}$$

$$\therefore \theta = \tan^{-1} \left( \frac{9}{13} \right).$$

6. At any point 't' on the curve  $x = a(t + \sin t)$ ,  $y = a(1 - \cos t)$ , find the length of tangent, normal, subtangent and subnormal.

**Sol:** Given curve is  $x = a(t + \sin t)$ ,  $y = a(1 - \cos t)$ . Let P(a(t + sin t), a(1 - cos t)) be any point on it. By parametric differentiation,

$$\frac{dy}{dx} = \frac{\left( \frac{dy}{dt} \right)}{\left( \frac{dx}{dt} \right)} = \frac{a(0 + \sin t)}{a(1 + \cos t)} = \frac{2 \sin \frac{t}{2} \cos \frac{t}{2}}{2 \cos^2 \frac{t}{2}} = \tan \frac{t}{2}$$

$$\text{Slope of the tangent at P} = \tan \frac{t}{2}$$

Length of the tangent at P

$$= \left| \frac{y_1 \sqrt{1+m^2}}{m} \right| = \left| \frac{a(1-\cos t) \sqrt{1+\tan^2 \frac{t}{2}}}{\tan \frac{t}{2}} \right|$$

$$= \left| \frac{a \cdot 2 \sin^2 \frac{t}{2} \times \cos \frac{t}{2}}{\cos \frac{t}{2} \sin \frac{t}{2}} \right|$$

$$= \left| 2a \sin \frac{t}{2} \right| \text{ units.}$$

Length of the normal at P

$$= |y_1 \sqrt{1+m^2}|$$

$$= \left| a(1-\cos t) \sqrt{1+\tan^2 \frac{t}{2}} \right|$$

$$= \left| a \cdot 2 \sin^2 \frac{t}{2} \cdot \sec \frac{t}{2} \right| \text{ units.}$$

Length of the subtangent at P

$$= \left| \frac{y_1}{m} \right| = \left| \frac{a(1-\cos t)}{\tan \frac{t}{2}} \right|$$

$$= \left| \frac{a \cdot 2 \sin^2 \frac{t}{2}}{\sin \frac{t}{2}} \cos \frac{t}{2} \right|$$



=  $|a \sin t|$  units.

Length of subnormal at P

$$= |y_1 \cdot m|$$

$$= \left| a(1 - \cos t) \cdot \tan \frac{t}{2} \right|$$

$$= \left| a \cdot 2 \sin^2 \frac{t}{2} \cdot \tan \frac{t}{2} \right| \text{ units.}$$

- 7. Show that the square of the length of sub tangent at any point on the curve  $by^2 = (x + a)^3$ ,  $b \neq 0$ , varies with the length of the subnormal at that point.**

**Sol:** Given curve is  $by^2 = (x + a)^3$  ----- (1)

Let P( $x_1, y_1$ ) be any point on it,

$$by_1^2 = (x_1 + a)^3 \text{ ----- (2)}$$

differentiating  $by^2 = (x + a)^3$  w.r.t. x

$$b \cdot 2y \frac{dy}{dx} = 3(x + a)^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{3(x + a)^2}{2by}$$

$$\text{Slope of the tangent at P} = \frac{3(x_1 + a)^2}{2by_1}$$

$$\text{Length of subtangent at P} = \left| \frac{y_1}{m} \right| = \left| \frac{\frac{y_1}{3(x_1 + a)^2}}{\frac{2by_1}{3(x_1 + a)^2}} \right|$$

$$= \left| \frac{2by_1^2}{3(x_1 + a)^2} \right|$$

Length of the subnormal at P =  $|y_1 \cdot m|$

$$= \left| y_1 \cdot \frac{3(x_1 + a)^2}{2by_1} \right| =$$

$$\left| \frac{3(x_1 + a)^2}{2b} \right|$$

$$\text{Consider } \frac{(L.S.T)^2}{L.S.N} = \frac{\left\{ \frac{2by_1^2}{3(x_1 + a)^2} \right\}^2}{\left\{ \frac{3(x_1 + a)^2}{2b} \right\}} = \frac{8b^3 y_1^4}{27(x_1 + a)^6}$$

$$= \frac{8b^3 y_1^4}{27(by_1^2)^2}$$

$$= \frac{8b}{27} \text{ which is a constant, say k}$$

$$\Rightarrow (L.S.T.)^2 = k \cdot (L.S.N)$$

$\Rightarrow (L.S.T.)^2$  varies as L.S.N.

- 8. Show that the condition for the orthogonality of the curves  $ax^2 + by^2 = 1$  and  $a_1x^2 + b_1y^2 = 1$**

$$\text{is } \frac{1}{a} \cdot \frac{1}{b} = \frac{1}{a_1} \cdot \frac{1}{b_1}.$$

**Sol:** Given equations of the curves are

$$ax^2 + by^2 = 1 \text{ ----- (1)}$$

$$a_1x^2 + b_1y^2 = 1 \text{ ----- (2)}$$

Let P( $x_1, y_1$ ) be the point of intersection of (1) & (2)

$$ax_1^2 + by_1^2 = 1$$

$$a_1x_1^2 + b_1y_1^2 = 1$$

$$(a - a_1)x_1^2 + (b - b_1)y_1^2 = 0$$

$$\Rightarrow \frac{b - b_1}{a - a_1} = \frac{-x_1^2}{y_1^2} \text{ ----- (3)}$$

differentiating  $ax^2 + by^2 = 1$

$$\Rightarrow a \cdot 2x + b \cdot 2y \frac{dy}{dx} = 0$$

$$\Rightarrow by \frac{dy}{dx} = -ax$$

$$\Rightarrow \frac{dy}{dx} = \frac{-ax}{by}$$

$$\text{slope of the tangent at P to (1) is } m_1 = \frac{-ax_1}{by_1}$$

$$\text{similarly, slope of tangent at P to (2) is } m_2 = \frac{-a_1x_1}{b_1y_1}$$

since (1), (2) cut orthogonally at P,

$$m_1 m_2 = -1$$

$$\Rightarrow \left( \frac{-ax_1}{by_1} \right) \left( \frac{-a_1x_1}{b_1y_1} \right) = -1.$$

$$\Rightarrow \frac{aa_1}{bb_1} \frac{x_1^2}{y_1^2} = -1$$

$$\Rightarrow \frac{aa_1}{bb_1} \left\{ \frac{-(b - b_1)}{a - a_1} \right\} = -1 \text{ from (3)}$$

$$\Rightarrow \frac{b - b_1}{a - a_1} = \frac{a - a_1}{aa_1}$$

$$\Rightarrow \frac{1}{b_1} - \frac{1}{b} = \frac{1}{a_1} - \frac{1}{a}$$

$$\Rightarrow \frac{1}{a} - \frac{1}{b} = \frac{1}{a_1} - \frac{1}{b_1}.$$



9. Show that the equation of tangent to the

$$\text{curve } \left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 2 \text{ at } (a, b) \text{ is } \frac{x}{a} + \frac{y}{b} = 2.$$

**Sol:** Given curve is  $\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 2$

Differentiating with respect to x,

$$\frac{1}{a^n} \cdot nx^{n-1} + \frac{1}{b^n} ny^{n-1} \frac{dy}{dx} = 0$$

$$\frac{1}{b^n} ny^{n-1} \frac{dy}{dx} = \frac{-nx^{n-1}}{a^n}$$

$$\frac{dy}{dx} = \frac{-nx^{n-1}}{a^n} \cdot \frac{b^n}{ny^{n-1}}$$

∴ slope of the tangent to the curve at P(a, b) is

$$m = \frac{-na^{n-1}}{a^n} \cdot \frac{b^n}{nb^{n-1}} = \frac{-b}{a}$$

Now equation of tangent at P is

$$y - b = \frac{-b}{a}(x - a)$$

$$\frac{y}{b} - 1 = \frac{-x}{a} + 1$$

$$\frac{x}{a} + \frac{y}{b} = 2.$$

10. Find the angle between the curves given below.  $x + y + 2 = 0$ ,  $x^2 + y^2 - 10y = 0$

**Sol:** Given curves

$$x + y + 2 = 0 \quad \text{--- (1)}$$

$$x^2 + y^2 - 10y = 0 \quad \text{--- (2)}$$

From equation (1)

$$x = -(y + 2), \text{ Substituting in (2)}$$

$$(y + 2)^2 + y^2 - 10y = 0$$

$$y^2 + 4y + 4 + y^2 - 10y = 0$$

$$2y^2 - 6y + 4 = 0 \quad \div 2$$

$$y^2 - 3y + 2 = 0$$

$$y^2 - 2y - y + 2 = 0$$

$$y(y - 2) - 1(y - 2) = 0$$

$$(y - 1)(y - 2) = 0$$

$$y = 1 \text{ (or)} y = 2$$

$$y = 1 \Rightarrow x = -(1 + 2) = -3$$

$$y = 2 \Rightarrow x = -(2 + 2) = -4$$

∴ The points of intersection of the curves are P (-3, 1) and Q (-4, 2)

Equation of the curve is  $x^2 + y^2 - 10y = 0$

Differentiating w.r.t x

$$2x + 2y \frac{dy}{dx} - 10 \frac{dy}{dx} = 0$$

$$2(y - 5) \frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = \frac{-x}{y-5}$$

Equation of the line is  $x + y + 2 = 0$

$$1 + \frac{dy}{dx} = 0, \quad \frac{dy}{dx} = -1$$

**Case (i) :** At P(-3, 1)

$$\left( \frac{dy}{dx} \right)_{P(-3,1)} = \frac{-x}{y-5}$$

$$m_1 = \frac{3}{1-5} = \frac{-3}{4}$$

$$\left( \frac{dy}{dx} \right)_{P(-3,1)} = -1$$

$$m_2 = -1$$

If  $\theta$  is the angle between the curves then

$$\tan \theta = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right|$$

$$= \left| \frac{\frac{-3}{4} - 1}{1 + \frac{3}{4}} \right| = \left| \frac{-3+4}{4+3} \right| = \frac{1}{7}$$

$$\theta = \tan^{-1} \left( \frac{1}{7} \right)$$

**Case (ii) :** At Q(-4, 2)

$$m_1 = \frac{4}{2-5} = \frac{-4}{3}, \quad m_2 = -1$$

$$\tan \theta = \left| \frac{\frac{-4}{3} + 1}{1 + \frac{4}{3}} \right| = \left| \frac{-4+3}{3+4} \right| = \frac{1}{7}$$

$$\theta = \tan^{-1} \left( \frac{1}{7} \right).$$



11. Find the angle between the curves  $xy = 2$  and  $x^2 + 4y = 0$

Sol: Given curves  $xy = 2 \Rightarrow x = \frac{2}{y}$  ——— (1)  
 $x^2 + 4y = 0$  ——— (2)

The points of intersection of (1) and (2) is

$$\Rightarrow \left(\frac{2}{y}\right)^2 + 4y = 0$$

$$\Rightarrow \frac{4}{y^2} + 4y = 0$$

$$\Rightarrow 4 + 4y^3 = 0 \quad \div 4$$

$$\Rightarrow 1 + y^3 = 0$$

$$\Rightarrow y^3 = -1$$

$$\Rightarrow y = -1$$

$$\Rightarrow x = -2$$

$$P(-2, -1)$$

$$xy = 2$$

Differentiating w.r.t x

$$x \frac{dy}{dx} + y \cdot 1 = 0$$

$$\frac{dy}{dx} = \frac{-y}{x}$$

$$\left. \frac{dy}{dx} \right|_{P(-2,-1)} = \frac{-1}{2}$$

$$x^2 + 4y = 0$$

Differentiating w.r.t x

$$2x + 4 \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-2x}{4}$$

$$\left. \frac{dy}{dx} \right|_{(-2,-1)} = 1$$

Let  $\theta$  is the angle between the curves then,

$$\tan \theta = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right|$$

$$\tan \theta = \left| \frac{\frac{-1}{2} - 1}{1 - \frac{1}{2}} \right|$$

$$= \left| \frac{-1 - 2}{2 - 1} \right|$$

$$= 3$$

$$\theta = \tan^{-1}(3)$$

12. Find the angle between the curves given below.

$$y^2 = 4x, x^2 + y^2 = 5$$

Sol:  $y^2 = 4x$  ——— (1),  $x^2 + y^2 = 5$  ——— (2)

Substituting  $y^2 = 4x$  in Equation (2)

$$x^2 + 4x = 5$$

$$x^2 + 4x - 5 = 0$$

$$(x - 1)(x + 5) = 0$$

$$x = 1 \text{ (or) } x = -5$$

$$x = 1 \quad y = \sqrt{4} = \pm 2$$

$$x = -5 \Rightarrow y = \sqrt{-20} \text{ is not real}$$

∴ The point of intersection of the two curves is P(1, 2) and Q(1, -2)

Equation of the second curve is  $y^2 = 4x$

Differentiating w.r.t x

$$2y \frac{dy}{dx} = 4$$

$$\frac{dy}{dx} = \frac{4}{2y} = \frac{2}{y}$$

Equation of the second curve is  $x^2 + y^2 = 5$

Differentiating w.r.t x

$$2x + 2y \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-2x}{2y} = \frac{-x}{y}$$

**Case (i) :** At P(1, 2)

$$m_1 = \frac{2}{2} = 1$$

$$m_2 = \frac{-1}{2}$$

If  $\theta$  is the angle between two curves then

$$\tan \theta = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right|$$

$$\tan \theta = \left| \frac{1 + \frac{1}{2}}{1 - \frac{1}{2}} \right| = \left| \frac{\frac{3}{2}}{\frac{1}{2}} \right| = 3$$

$$\theta = \tan^{-1}(3)$$



Case (ii): At Q (1, -2)

$$m_1 = \frac{2}{-2} = -1, \quad m_2 = \frac{-1}{-2} = \frac{1}{2}$$

$$\tan \theta = \left| \frac{-1 - \frac{1}{2}}{1 - \frac{1}{2}} \right| = \left| \frac{-2 - 1}{2 - 1} \right| = 3$$

$$\theta = \tan^{-1}(3).$$

13. Find the length of subtangent, subnormal at a point 't' on the curve  $x = a(\cos t + t \sin t)$ ,  $y = a(\sin t - t \cos t)$ .

Sol: Given curve is  $x = a(\cos t + t \sin t)$ ,  $y = a(\sin t - t \cos t)$   
Let P( $a(\cos t + t \sin t)$ ,  $a(\sin t - t \cos t)$ ) be a point on it.

By parametric differentiation,

$$\begin{aligned}\frac{dy}{dx} &= \frac{\left( \frac{dy}{dt} \right)}{\left( \frac{dx}{dt} \right)} = \frac{a[\cos t - (1 \cdot \cos t + t \sin t)]}{a[-\sin t + 1 \cdot \sin t + t \cdot \cos t]} \\ &= \frac{t \cdot \sin t}{t \cdot \cos t} \\ &= \tan t\end{aligned}$$

$$\begin{aligned}\text{Length of subtangent at } P &= \left| \frac{y_1}{m} \right| \\ &= \left| \frac{a(\sin t - t \cos t)}{\tan t} \right| \\ &= |a(\sin t - t \cos t) \cot t|.\end{aligned}$$

$$\begin{aligned}\text{Length of subnormal at } P &= |y_1 \cdot m| \\ &= |a(\sin t - t \cos t) \tan t|.\end{aligned}$$

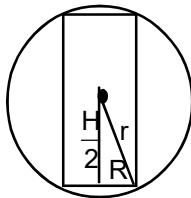


1. Show that when the curved surface of a right circular cylinder inscribed in a sphere of radius  $r$  is maximum, then the height of the cylinder is  $\sqrt{2} r$ .

**Sol:** Let  $R$  be the base radius,  $H$  be the height of the right circular cylinder inscribed in a sphere of radius  $r$ .

$$R^2 + \left(\frac{H}{2}\right)^2 = r^2$$

$$R^2 = r^2 - \frac{H^2}{4}$$



Curved surface area of the cylinder

$$C = 2\pi RH$$

$$C^2 = 4\pi^2 R^2 H^2$$

$$f(H) = 4\pi^2 \left(r^2 - \frac{H^2}{4}\right) H^2$$

$$= 4\pi^2 \left[r^2 H^2 - \frac{H^4}{4}\right]$$

differentiating w.r.t.  $H$ ,

$$\text{i)} f'(H) = 4\pi^2 \left[r^2 \cdot 2H - \frac{4H^3}{4}\right]$$

$$= 4\pi^2 [2r^2 H - H^3]$$

$$\text{ii) for minimum or maximum, } f'(H) = 0.$$

$$2Hr^2 - H^3 = 0$$

$$H = 0, \quad H^2 = 2r^2$$

is not possible  $H = \sqrt{2} r$ .

$$\text{iii) } f''(H) = 4\pi^2 [2r^2 - 3H^2]$$

iv) verification:

$$\text{At } H = \sqrt{2} r, \quad f''(H) = 4\pi^2 [2r^2 - 3(2r^2)] < 0$$

$f(H)$  is maximum at  $H = \sqrt{2} r$ .

Hence, for maximum curved surface area, then the height of the cylinder is  $\sqrt{2} r$

2. Prove that the radius of the right circular cylinder of greatest curved surface area which can be inscribed in a given cone is half of that of the cone.

**Sol:** Let  $O$  be the centre of the circular base of the cone and its height be  $H$ . Let  $r$  be the radius of the circular base of the cone.

Then  $AO = H$ ,  $OC = R$ .

Let a cylinder with radius  $r$  (OE) be inscribed in the given cone. Let its height be  $h$ .

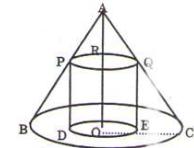
$$\text{i.e., } RO = QE = PD = h$$

Now the triangle  $AOC$  and  $QEC$  are similar. Therefore,

$$\frac{QE}{OA} = \frac{EC}{OC}$$

$$\text{i.e., } \frac{h}{H} = \frac{R-r}{R}$$

$$\therefore h = \frac{H(R-r)}{R} = H \left(1 - \frac{r}{R}\right) \dots\dots\dots (1)$$



Let  $S$  denote the curved surface area of the chosen cylinder. Then

$$S = 2\pi rh = 2\pi rH \left(1 - \frac{r}{R}\right)$$

$$= 2\pi H \left(r - \frac{r^2}{R}\right)$$

As the cone is fixed one, the value of  $R$  and  $H$  are constants. Thus  $S$  is function of  $r$  only.

$$\text{i) } \frac{dS}{dr} = 2\pi H \left(1 - \frac{2r}{R}\right)$$

$$\text{ii) for minimum or maximum } \frac{dS}{dr} = 0$$

$$2\pi H \left(1 - \frac{2r}{R}\right) = 0$$

$$\text{Since } H \neq 0, \quad 1 - \frac{2r}{R} = 0$$

$$1 = \frac{2r}{R}$$

$$r = \frac{R}{2}$$

$$\text{iii) } \frac{d^2S}{dr^2} = \frac{-4\pi H}{R}.$$

iv) verification:

$$\text{At } r = \frac{R}{2}, \quad \frac{d^2S}{dr^2} = \frac{-4\pi H}{R} < 0 \text{ for all } r$$

$$\therefore S \text{ is maximum at } r = \frac{R}{2}.$$



3. A window is in the shape of a rectangle surmounted by a semicircle. If the perimeter of the window be 20ft find the maximum area.

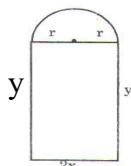
**Sol:** Let  $2x$  be the length,  $y$  be the breath of a rectangle surmounted by a semicircle of radius ' $r$ '

Given that the Perimeter of the semicircle is 20ft

$$\Rightarrow 2x + 2y + \pi r = 20$$

$$\Rightarrow 2y = 20 - 2x - \pi r$$

$$\Rightarrow y = \frac{20 - 2x - \pi r}{2}$$



Total area = area of a rectangle + area of a semicircle.

$$\text{Total area } (A) = 2xy + \frac{1}{2} \pi r^2$$

$$= 2xy + \frac{1}{2} \pi x^2 \quad (2r = 2x) \\ (r = x)$$

$$= 2x \frac{(20 - 2x - \pi x)}{2} + \frac{1}{2} \pi x^2$$

$$A = x(20 - 2x - \pi x) + \frac{\pi}{2} x^2$$

$$= 20x - 2x^2 - \pi x^2 + \frac{\pi x^2}{2}$$

Differentiating w.r.t 'x'

$$\text{i) } \frac{dA}{dx} = 20 - 4x - 2\pi x + \frac{2\pi x}{2} \\ = 20 - 4x - 2\pi x + \pi x \\ = 20 - 4x - \pi x.$$

$$\text{ii) for a maximum or minimum } \frac{dA}{dx} = 0$$

$$\Rightarrow 20 - 4x - \pi x = 0 \\ \Rightarrow 4x + \pi x = 20 \\ \Rightarrow x(4 + \pi) = 20$$

$$\Rightarrow x = \frac{20}{4 + \pi}$$

$$\text{iii) } \frac{d^2 A}{dx^2} = -4 - \pi$$

iv) verification:

$$\text{At } x = \frac{20}{4 + \pi}, \frac{d^2 A}{dx^2} = -4 - \pi < 0$$

A is maximum at  $x = \frac{20}{4 + \pi}$

$$\Rightarrow y = \frac{20 - \frac{40}{\pi + 4} - \frac{20\pi}{\pi + 4}}{2}$$

$$= \frac{20\pi + 80 - 40 - 20\pi}{2(\pi + 4)} = \frac{40}{2(\pi + 4)}$$

$$= \frac{20}{\pi + 4}$$

∴ Maximum area

$$= 2 \left( \frac{20}{\pi + 4} \right) \cdot \left( \frac{20}{\pi + 4} \right) + \frac{\pi}{2} \left( \frac{20}{\pi + 4} \right)^2$$

$$= \left( \frac{20}{\pi + 4} \right)^2 \left( 2 + \frac{\pi}{2} \right)$$

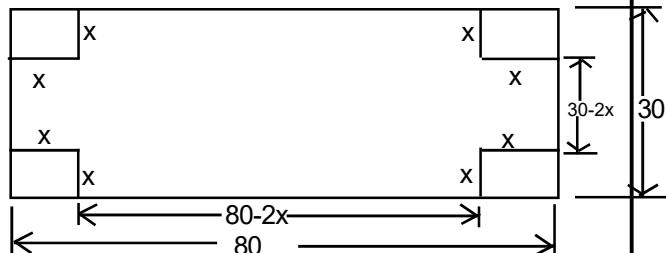
$$= \frac{400}{(\pi + 4)^2} \left( \frac{4 + \pi}{2} \right)$$

$$= \frac{200}{\pi + 4} \text{ sq. units}$$



4. From a rectangular sheet of dimensions  $30 \text{ cm} \times 80 \text{ cm}$  four equal squares of side  $x \text{ cm}$  are removed at the corners, and the sides are then turned up so as to form an open rectangular box. Find  $x$ , so that volume of the box is the greatest.

**Sol:**



Given dimensions of the rectangular sheet are 80, 30.

Given that  $x$  be the side of the square cut out. Now the dimensions of the open rectangular box are

$$\ell = 80 - 2x, b = 30 - 2x, h = x$$

volume of the box,

$$\begin{aligned} v &= \ell bx \\ &= (80 - 2x)(30 - 2x)x \\ &= (2400 - 220x + 4x^2)x \end{aligned}$$

let  $f(x) = 4x^3 - 220x^2 + 2400x$   
differentiating w.r.t.  $x$ ,

i)  $f'(x) = 12x^2 - 440x + 2400$

ii) For max or min,  $f'(x) = 0$

$$\Rightarrow 12x^2 - 440x + 2400 = 0 \quad \div 4$$

$$\Rightarrow 3x^2 - 110x + 600 = 0$$

$$\Rightarrow 3x^2 - 90x - 20x + 600 = 0$$

$$\Rightarrow 3x(x - 30) - 20(x - 30) = 0$$

$$x = 30 \text{ is impossible or } x = \frac{20}{3}$$

iii)  $f''(x) = 24x - 440$

iv) Verification:

$$\text{If } x = \frac{20}{3}, f''(x) = 24\left(\frac{20}{3}\right) - 440 = 160 - 440 = -280 < 0$$

Hence, the volume of the box is maximum

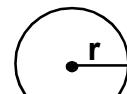
when  $x = \frac{20}{3} \text{ cm.}$

5. A wire of length  $\ell$  is cut into two parts which are bent respectively in the form a square and a circle. What are the lengths of the pieces of the wire so that the sum of the areas is the least.

**Sol:** A wire of length  $\ell$  is cut into two parts:  $x, \ell - x$ . Let  $a$  be the side of the square and  $r$  be the radius of the circle.

$$4a = x$$

$$2\pi r = \ell - x$$



$$a = \frac{x}{4}$$

$$r = \frac{\ell - x}{2\pi}$$

Sum of the areas

$$S = a^2 + \pi r^2$$

$$= \left(\frac{x}{4}\right)^2 + \pi\left(\frac{\ell - x}{2\pi}\right)^2$$

$$= \frac{x^2}{16} + \frac{1}{4\pi}(\ell - x)^2$$

differentiating w.r.t.  $x$ ,

i)  $\frac{dS}{dx} = \frac{2x}{16} + \frac{1}{4\pi} 2(\ell - x)(-1)$

ii) For max or min. of  $S$ ,  $\frac{dS}{dx} = 0$

$$\Rightarrow \frac{x}{8} - \frac{(\ell - x)}{2\pi} = 0$$

$$\Rightarrow \pi x - 4\ell + 4x = 0$$

$$\Rightarrow x(\pi + 4) = 4\ell$$

$$\Rightarrow x = \frac{4\ell}{\pi + 4}$$

iii)  $\frac{d^2S}{dx^2} = \frac{2}{16} + \frac{2}{4\pi}$

iv) Verification:

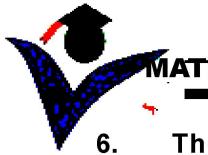
$$\text{at } x = \frac{4\ell}{\pi + 4}, \frac{d^2S}{dx^2} > 0$$

$$\therefore S \text{ is min. at } x = \frac{4\ell}{\pi + 4}$$

$$\text{Length of other part} = \ell - x = \ell - \frac{4\ell}{\pi + 4} = \frac{\pi\ell}{\pi + 4}$$

Hence  $S$  is least when the lengths of the pieces

are  $\frac{4\ell}{\pi + 4}$  and  $\frac{\pi\ell}{\pi + 4}$ .



6. The profit function  $P(x)$  of a company selling  $x$  items per day is given by  $P(x) = (150 - x)x - 1000$ . Find the number of items that the company should manufacture to get maximum profit. Also find the maximum profit.

**Sol:** Given profit function is

$$P(x) = (150 - x)x - 1000$$

$$\text{i)} P'(x) = (150 - x)1 + x(-1)$$

$$P'(x) = 150 - 2x$$

$$\text{ii)} \text{For min. or max. } P'(x) = 0$$

$$\Rightarrow 150 - 2x = 0$$

$$\Rightarrow x = 75$$

$$\text{iii)} P''(x) = -2$$

iv) Verification:

$$\text{At } x = 75, P''(x) = -2 < 0.$$

$\therefore P(x)$  is max. at  $x = 75$ .

$\therefore$  The number of items is  $x = 75$

And max. profit =  $(150 - 75)75 - 1000$

$$= 75(75) - 1000 = 4625.$$

7. Find the absolute maximum and absolute minimum of  $f(x) = 2x^3 - 3x^2 - 36x + 2$  on the interval  $[0, 5]$ .

**Sol:** Given  $f(x) = 2x^3 - 3x^2 - 36x + 2$  on  $[0, 5]$

$$f'(x) = 6x^2 - 6x - 36$$

for minimum (or) maximum,

$$f'(x) = 0$$

$$\Rightarrow 6x^2 - 6x - 36 = 0$$

$$\Rightarrow x^2 - x - 6 = 0$$

$$\Rightarrow (x - 3)(x + 2) = 0$$

$$\Rightarrow x - 3 = 0, x + 2 = 0$$

$$\Rightarrow x = 3, x = -2 \notin [0, 5]$$

The values,

$$f(0) = 2$$

$$\begin{aligned} f(5) &= 2(5)^3 - 3(5)^2 - 36(5) + 2 \\ &= 250 - 75 - 180 + 2 \\ &= -3 \end{aligned}$$

$$\begin{aligned} f(3) &= 2(3)^3 - 3(3)^2 - 36(3) + 2 \\ &= 54 - 27 - 108 + 2 \\ &= -79. \end{aligned}$$

Absolute maximum = max. of  $\{f(0), f(5), f(3)\}$

$$= \max. \{2, -3, -79\}$$

Absolute minimum = min. of  $\{f(0), f(5), f(3)\}$

$$= \min. \{2, -3, -79\}$$

$$= -79.$$

8. Find the positive integers  $x$  and  $y$  such that  $x + y = 60$  and  $xy^3$  is maximum.

**Sol:** Given positive integers  $x, y$

$$\text{Given } x + y = 60$$

$$y = 60 - x$$

$$\text{Given } f(x, y) = xy^3$$

$$f(x) = x(60 - x)^3$$

$$\begin{aligned} \text{i)} f'(x) &= 1.(60 - x)^3 + x 3(60 - x)^2 (-1) \\ &= (60 - x)^2 [60 - x - 3x] \\ &= (60 - x)^2 [60 - 4x]. \end{aligned}$$

$$\text{ii) for minimum (or) maximum, } f'(x) = 0$$

$$\Rightarrow (60 - x)^2 (60 - 4x) = 0$$

$$\Rightarrow 60 - x = 0, \quad 60 - 4x = 0$$

$$\Rightarrow x = 60, \quad 4x = 60$$

$$x = 15.$$

$$\begin{aligned} \text{iii)} f''(x) &= 2(60 - x) (-1) (60 - 4x) + (60 - x)^2 (-4) \\ &= 4(45)^2 < 0 \end{aligned}$$

$\therefore f(x)$  is maximum at  $x = 15$ .

$$\begin{aligned} y &= 60 - x \\ &= 60 - 15 \\ &= 45 \end{aligned}$$

$$\therefore x = 15, y = 45.$$

**QUESTION NO.: 11**

1. If the distance from P to the points (2, 3) and (2, -3) are in the ratio 2 : 3, then find the equation of locus of P.

**Sol:** Let P(x<sub>1</sub>, y<sub>1</sub>) be any point on the locus.

Given two points are A(2, 3), B(2, -3).

Given geometric property: PA : PB = 2 : 3.

$$\frac{PA}{PB} = \frac{2}{3}$$

$$\Rightarrow 3PA = 2PB$$

$$\text{Squaring on both sides } \Rightarrow 9(PA)^2 = 4(PB)^2 \Rightarrow \infty$$

$$\Rightarrow 9[x_1^2 - 4x_1 + 4 + y_1^2 - 6y_1 + 9] \\ = 4[x_1^2 - 4x_1 + 4 + y_1^2 + 6y_1 + 9]$$

$$5x_1^2 + 5y_1^2 - 20x_1 - 78y_1 + 65 = 0$$

Hence, the required equation of the locus is  
 $5x^2 + 5y^2 - 20x - 78y + 65 = 0$ .

2. Find the equation of the locus of P, if the ratio of the distances from P to A(5, -4) and B(7, 6) is 2 : 3.

**Sol:** Let P(x<sub>1</sub>, y<sub>1</sub>) be any point on the locus.

Given geometric property PA : PB = 2 : 3

$$\frac{PA}{PB} = \frac{2}{3}$$

$$\Rightarrow 3AP = 2PB$$

$$\text{Squaring on both sides } \Rightarrow 9(PA)^2 = 4(PB)^2$$

$$\Rightarrow 9[(x_1 - 5)^2 + (y_1 + 4)^2] = 4[(x_1 - 7)^2 + (y_1 - 6)^2]$$

$$\Rightarrow 9[x_1^2 + 25 - 10x_1 + y_1^2 + 16 + 8y_1] \\ = 4(x_1^2 + 49 - 14x_1 + y_1^2 + 36 - 12y_1)$$

$$\Rightarrow 9x_1^2 + 225 - 90x_1 + 9y_1^2 + 144 + 72y_1 \\ = 4x_1^2 + 196 - 56x_1 + 4y^2 + 144 - 48y_1.$$

$$5x_1^2 + 5y_1^2 + 34x_1 + 120y_1 + 29 = 0$$

Hence the required equation of locus is

$$5x^2 + 5y^2 - 34x + 120y + 29 = 0.$$

3. A(5,3) and B(3, -2) are two fixed points. Find the equation of locus of P, so that the area of triangle DPAB is 9.

**Sol:** Let P(x<sub>1</sub>, y<sub>1</sub>) be any point on the locus.

Given two points are A(5, 3), B(3, -2).

G.G.P : Area of  $\Delta PAB = 9$

$$\frac{1}{2} \begin{vmatrix} x_1 & 5 & 3 & x_1 \\ y_1 & 3 & -2 & y_1 \end{vmatrix} = 9.$$

$$\Rightarrow \frac{1}{2} |3x_1 - 5y_1 - 10 - 9 + 3y_1 + 2x_1| = 9.$$

$$\Rightarrow |5x_1 - 2y_1 - 19| = 18$$

Squaring on both sides,

$$\Rightarrow (5x_1 - 2y_1 - 19)^2 = (18)^2$$

$$\Rightarrow 25x_1^2 - 20x_1y_1 + 4y_1^2 + 361 - 190x_1 + 76y_1 = 324.$$

Hence, the required equation of the locus is

$$25x^2 - 20xy + 4y^2 - 190x + 76y + 37 = 0.$$

4. A(2, 3) and B(-3, 4) are two given points. Find the equation of locus of P so that the area of the triangle PAB is 8.5.

**Sol:** Let P(x<sub>1</sub>, y<sub>1</sub>) be a point on the locus.

Given two points A(2, 3), B(-3, 4)

Given geometric condition to be satisfied by P is that area of  $\Delta PAB = 8.5$ .

$$\Rightarrow \frac{1}{2} |x_1(3-4) + 2(4-y_1) - 3(y_1-3)| = 8.5$$

$$\Rightarrow \frac{1}{2} |-x_1 + 8 - 2y_1 - 3y_1 + 9| = \frac{85}{10}$$

$$\Rightarrow |-x_1 - 5y_1 + 17| = 17$$

$$\Rightarrow x_1 - 5y_1 + 17 = 17 \text{ or } -x_1 - 5y_1 + 17 = -17.$$

$$\text{i.e., } x_1 + 5y_1 = 0 \text{ or } x_1 + 5y_1 = 34$$

Hence the required equation of locus is  
 $(x + 5y)(x + 5y - 34) = 0$

$$\Rightarrow x^2 + 5xy - 34x + 5xy + 25y^2 - 170y = 0$$

$$\Rightarrow x^2 + 10xy + 25y^2 - 34x - 170y = 0.$$

- 5. Find the equation of locus of P, if the line segment joining (2,3), (-1,5) subtends a right angle at P.**

**Sol:** Let P( $x_1, y_1$ ) be any point on the locus.

Given two points are A(2, 3), B(-1, 5).

$$\Rightarrow \text{Slope of AP} \times \text{Slope of BP} = -1.$$

$$\Rightarrow \left( \frac{y_2 - y_1}{x_2 - x_1} \right) \left( \frac{y_1 - 5}{x_1 + 1} \right) = -1$$

$$\Rightarrow (y_1 - 3)(y_1 - 5) = -(x_1 - 2)(x_1 + 1).$$

$$\Rightarrow (x_1 + 1)(x_1 - 2) + (y_1 - 3)(y_1 - 5) = 0.$$

$$\Rightarrow x_1^2 + x_1 - 2x_1 - 2 + y_1^2 - 3y_1 - 5y_1 + 15 = 0$$

$$\Rightarrow x_1^2 + y_1^2 - x_1 - 8y_1 + 13 = 0.$$

Hence, the required equation of the locus is

$$x^2 + y^2 - x - 8y + 13 = 0.$$

- 6. The ends of the hypotenuse of a right angled triangle are (0, 6) and (6, 0). Find the equation of locus of its third vertex.**

**Sol:** Let P( $x_1, y_1$ ) be any point on the Locus.

Given ends of the hypotenuse are A(0, 6), B(6, 0).

Given condition :  $\angle APB = 90^\circ$ .

$$\text{Slope of AP} \times \text{Slope of BP} = -1.$$

$$\Rightarrow \left( \frac{y_1 - 6}{x_1 - 0} \right) \left( \frac{y_1 - 0}{x_1 - 6} \right) = -1.$$

$$\Rightarrow y_1(y_1 - 6) = -x_1(x_1 - 6).$$

$$\Rightarrow x_1(x_1 - 6) + y_1(y_1 - 6) = 0.$$

$$\Rightarrow x_1^2 + y_1^2 - 6x_1 - 6y_1 = 0.$$

Hence, the required equation of the locus is

$$x^2 + y^2 - 6x - 6y = 0.$$

- 7.A(1, 2), B(2, -3) and C(-2, 3) are three points. A point P moves such that  $PA^2 + PB^2 = 2PC^2$ . Show that the equation to the locus of P is  $7x - 7y + 4 = 0$ .**

**Sol:** Let P( $x_1, y_1$ ) be any point on the locus.

Given three points are A(1, 2), B(2, -3), C(-2, 3).

Given condition  $PA^2 + PB^2 = 2PC^2$ .

$$\Rightarrow (x_1 - 1)^2 + (y_1 - 2)^2 + (x_1 - 2)^2 + (y_1 + 3)^2 \\ = 2[(x_1 + 2)^2 + (y_1 - 3)^2].$$

$$x_1^2 - 2x_1 + 1 + y_1^2 - 4y_1 + 4 + x_1^2 - 4x_1 + 4 + y_1^2 + 6y_1 + 9 \\ = 2[x_1^2 + 4x_1 + 4 + y_1^2 - 6y_1 + 9]$$

$$\Rightarrow 2x_1^2 + 2y_1^2 - 6x_1 - 2y_1 + 18$$

$$= 2x_1^2 + 8x_1 + 8 + 2y_1^2 - 12y_1 + 18$$

$$\Rightarrow -6x_1 + 8 + 2y_1 + 18 = 8x_1 - 12y_1 + 26$$

$$\Rightarrow 14x_1 - 14y_1 + 8 = 0 \quad (\div 2)$$

$$\Rightarrow 7x_1 - 7y_1 + 4 = 0$$

Hence the required equation of the locus is

$$7x - 7y + 4 = 0.$$

- 8. Find the equation of the locus of point 'P' such that the distance of P from the origin is twice the distance of P from A (1, 2).**

**Sol:** Let P( $x_1, y_1$ ) be any point on the locus.

Given points are O(0, 0), A(1, 2).

Given condition :  $OP = 2PA$

$$\Rightarrow \sqrt{(x_1 - 0)^2 + (y_1 - 0)^2} = 2\sqrt{(x_1 - 1)^2 + (y_1 - 2)^2}$$

Squaring on both sides,

$$\Rightarrow x_1^2 + y_1^2 = 4[(x_1 - 1)^2 + (y_1 - 2)^2]$$

$$\Rightarrow x_1^2 + y_1^2 = 4[x_1^2 - 2x_1 + 1 + y_1^2 - 4y_1 + 4]$$

$$\Rightarrow 3x_1^2 + 3y_1^2 - 8x_1 - 16y_1 + 20 = 0.$$

Hence the required equation of the locus is

$$3x^2 + 3y^2 - 8x - 16y + 20 = 0.$$

9. Find the equation of locus of the point, the sum of whose distances from (0, 2) and (0, -2) is 6 units.

Sol: Let P(x<sub>1</sub>, y<sub>1</sub>) be any point on the locus.

Given two points are A(0, 2), B(0, -2).

Given Condition : PA + PB = 6.  $\Rightarrow$  PA = 6 - PB

$$\Rightarrow \sqrt{x_1^2 + (y_1 - 2)^2} = 6 - \sqrt{x_1^2 + (y_1 + 2)^2}$$

Squaring on both sides,

$$\begin{aligned} \Rightarrow x_1^2 + y_1^2 - 4y_1 + 4 &= 36 + x_1^2 + y_1^2 + 4y_1 + 4 \\ &\quad - 12\sqrt{x_1^2 + y_1^2 + 4y_1 + 4} \end{aligned}$$

$$\Rightarrow -8y_1 - 36 = -12\sqrt{x_1^2 + y_1^2 + 4y_1 + 4} \quad (\div -4)$$

$$2y_1 + 9 = 3\sqrt{x_1^2 + y_1^2 + 4y_1 + 4} \quad \text{S.O.B}$$

$$\Rightarrow 4y_1^2 + 36y_1 + 81 = 9x_1^2 + 9y_1^2 + 36y_1 + 36$$

$$\Rightarrow 9x_1^2 + 5y_1^2 = 45 \quad (\div 45)$$

$$\Rightarrow \frac{x_1^2}{5} + \frac{y_1^2}{9} = 1$$

Hence, the required equation of the locus is

$$\frac{x^2}{5} + \frac{y^2}{9} = 1.$$

10. Find the equation of locus of P, if A (2, 3) B (2, -3) and PA + PB = 8

Sol: Let P(x<sub>1</sub>, y<sub>1</sub>) be any point on the locus.

Given points A(2, 3) B (2, -3)

Given PA + PB = 8  $\Rightarrow$  PA = 8 - PB

$$\sqrt{(x_1 - 2)^2 + (y_1 - 3)^2} = 8 - \sqrt{(x_1 - 2)^2 + (y_1 + 3)^2}$$

Squaring on both sides.

$$(x_1 - 2)^2 + (y_1 - 3)^2 = 64 + (x_1 - 2)^2 + (y_1 + 3)^2 - 16\sqrt{(x_1 - 2)^2 + (y_1 + 3)^2}$$

$$y_1^2 - 6y_1 + 9 = 64 + y_1^2 + 6y_1 + 9 - 16\sqrt{(x_1 - 2)^2 + (y_1 + 3)^2}$$

$$-12y_1 - 64 = -16\sqrt{(x_1 - 2)^2 + (y_1 + 3)^2} \quad \div (-4)$$

$$3y_1 + 16 = 4\sqrt{(x_1 - 2)^2 + (y_1 + 3)^2}$$

Squaring on both sides

$$\Rightarrow (3y_1 + 16)^2 = 16[(x_1 - 2)^2 + (y_1 + 3)^2]$$

$$\Rightarrow 9y_1^2 + 96y_1 + 256 = 16[x^2 - 4x_1 + 4 + y_1^2 + 6y_1 + 9]$$

$$\Rightarrow 9y_1^2 + 96y_1 + 256 =$$

$$16x_1^2 - 64x_1 - 64x_1 + 64 + 16y_1^2 + 96y_1 + 144$$

$$\Rightarrow 16x_1^2 + 7y_1^2 - 64x_1 - 48 = 0.$$

The required equation of locus of P is

$$16x^2 + 7y^2 - 64x - 48 = 0.$$

11. Find the equation of locus of a point, the difference of whose distances from (-5, 0) and (5, 0) is 8 units.

Sol: Given condition : |PA - PB| = 8.

$$PA - PB = \pm 8 \quad \Rightarrow PA = PB \pm 8$$

$$\Rightarrow \sqrt{(x_1 + 5)^2 + y_1^2} = \sqrt{(x_1 - 5)^2 + y_1^2} \pm 8$$

Squaring on both sides,

$$\Rightarrow x_1^2 + 10x_1 + 25 + y_1^2 = 64 + x_1^2 - 10x_1 + 25 + y_1^2$$

$$\pm 16\sqrt{x_1^2 - 10x_1 + 25 + y_1^2}$$

$$\Rightarrow 20x_1 - 64 = \pm 16\sqrt{x_1^2 + y_1^2 - 10x_1 + 25} \quad (\div 4)$$

$$\Rightarrow 5x_1 - 16 = \pm 4\sqrt{x_1^2 + y_1^2 - 10x_1 + 25}$$

Again squaring on both sides,

$$\Rightarrow 25x_1^2 - 160x_1 + 256 = 16x_1^2 + 16y_1^2 - 160x_1 + 400$$

$$\Rightarrow 9x_1^2 - 16y_1^2 = 144 \quad (\div 144)$$

$$\Rightarrow \frac{x_1^2}{16} - \frac{y_1^2}{9} = 1$$

Hence, the required equation of the locus is

$$\frac{x^2}{16} - \frac{y^2}{9} = 1.$$

12. Find the equation of locus of P, if A(4, 0), B(-4, 0) and |PA - PB| = 4.

Sol: Let P(x<sub>1</sub>, y<sub>1</sub>) be any point on the locus.

Given two points are A(4, 0), B(-4, 0).

Given Condition : |PA - PB| = 4.

$$PA - PB = \pm 4. \Rightarrow PA = PB \pm 4.$$

$$\Rightarrow \sqrt{(x_1 - 4)^2 + y_1^2} = \sqrt{(x_1 + 4)^2 + y_1^2} \pm 4.$$

Squaring on both sides,

$$\Rightarrow x_1^2 - 8x_1 + 16 + y_1^2 = x_1^2 + 8x_1 + 16 + y_1^2 \pm 8\sqrt{x_1^2 + 8x_1 + 16 + y_1^2}$$

$$\Rightarrow -16x_1 - 16 = \pm 8\sqrt{x_1^2 + y_1^2 + 8x_1 + 16} \quad \div (-8)$$

$$\Rightarrow 2x_1 + 2 = \mp \sqrt{x_1^2 + y_1^2 + 8x_1 + 16}$$

Again squaring on both sides,

$$\Rightarrow 4x_1^2 + 8x_1 + 4 = x_1^2 + y_1^2 + 8x_1 + 16$$

$$\Rightarrow 3x_1^2 - y_1^2 = 12 \quad \div (12)$$

$$\Rightarrow \frac{x_1^2}{4} - \frac{y_1^2}{12} = 1$$

Hence the required equation of the locus is

$$\frac{x^2}{4} - \frac{y^2}{12} = 1.$$

- 1. Find the transformed equation of  $2x^2 + 4xy + 5y^2 = 0$ , when the origin is shifted to (3, 4) by the translation of axes.**

**Sol:** Let (X, Y) be the new coordinates of (x, y) when axes are translated to the point (h, k) = (3, 4).

∴ Transformation equations are

$$\begin{aligned} x &= X + h, & y &= Y + k \\ \Rightarrow x &= X + 3, & y &= Y + 4 \end{aligned}$$

Given curve equation is  $2x^2 + 4xy + 5y^2 = 0$ .

∴ Transformed equation is

$$2(X + 3)^2 + 4(X + 3)(Y + 4) + 5(Y + 4)^2 = 0.$$

$$\Rightarrow 2(X^2 + 6X + 9) + 4(XY + 4X + 3Y + 12) + 5(Y^2 + 8Y + 16) = 0.$$

$$\Rightarrow 2X^2 + 4XY + 5Y^2 + 28X + 52Y + 146 = 0.$$

- 2. If the transformed equation of a curve is  $X^2 + 3XY - 2Y^2 + 17X - 7Y - 11 = 0$ , when the origin is shifted to (2, 3). Find the original equation of the curve.**

**Sol:** Let (X, Y) be the new coordinates of the point (x, y) when the axes are translated to the point (h, k).

Here (h, k) = (2, 3).

∴ Transformation equations are

$$\begin{aligned} x &= X + h, & y &= Y + k \\ \Rightarrow x &= X + 2, & y &= Y + 3 \\ \Rightarrow x &= X - 2, & Y &= Y - 3 \end{aligned}$$

Given transformed equation of the curve is

$$X^2 + 3XY - 2Y^2 + 17X - 7Y - 11 = 0.$$

∴ The original equation of the curve is

$$\Rightarrow (x-2)^2 + 3(x-2)(y-3) - 2(y-3)^2 + 17(x-2) - 7(y-3) - 11 = 0$$

$$\Rightarrow x^2 - 4x + 4 + 3(xy - 3x - 2y + 6) - 2(y^2 - 6y + 9) + 17x - 34 - 7y + 21 - 11 = 0.$$

$$\Rightarrow x^2 + 3xy - 2y^2 + 4x - y - 20 = 0.$$

- 3. Find the transformed equation of  $3x^2 + 10xy + 3y^2 = 9$  when the axes are rotated through an angle  $\frac{\pi}{4}$ .**

**Sol:** Let (X, Y) be the new coordinates of the point (x, y) when the axes are rotated through an angle  $\frac{\pi}{4}$ .

$$\text{Here } \theta = \frac{\pi}{4}$$

$$\Rightarrow \cos \frac{\pi}{4} = \frac{1}{\sqrt{2}}, \sin \frac{\pi}{4} = \frac{1}{\sqrt{2}}.$$

Transformation equations are

$$x = X \cos \theta - Y \sin \theta, y = X \sin \theta + Y \cos \theta$$

$$\Rightarrow x = X \cdot \frac{1}{\sqrt{2}} - Y \cdot \frac{1}{\sqrt{2}}, y = X \cdot \frac{1}{\sqrt{2}} + Y \cdot \frac{1}{\sqrt{2}}$$

$$\Rightarrow x = \frac{X - Y}{\sqrt{2}}, y = \frac{X + Y}{\sqrt{2}}$$

Given equation of the curve is  $3x^2 + 10xy + 3y^2 = 9$ .

∴ Transformed equation of the curve

$$\text{is } 3\left(\frac{X - Y}{\sqrt{2}}\right)^2 + 10\left(\frac{X - Y}{\sqrt{2}}\right)\left(\frac{X + Y}{\sqrt{2}}\right) + 3\left(\frac{X + Y}{\sqrt{2}}\right)^2 = 9$$

$$\Rightarrow 3(X^2 - 2XY + Y^2) + 10(X^2 - Y^2) + 3(X^2 + 2XY + Y^2) = 2(9)$$

$$\Rightarrow 16X^2 - 4Y^2 = 18 \quad \div 2 \\ 8X^2 - 2Y^2 = 9$$

Hence the required transformed equation is

$$8X^2 - 2Y^2 = 9.$$

- 4. Find the transformed equation of  $x^2 + 2\sqrt{3}xy - y^2 = 2a^2$  when the axes are rotated through an angle  $\frac{\pi}{6}$ .**

**Sol:** Let  $(X, Y)$  be the new coordinates of the point  $(x, y)$

when the axes are rotated through an angle  $\frac{\pi}{6}$ .

Given curve equation is  $x^2 + 2\sqrt{3}xy - y^2 = 0$

$$\theta = \frac{\pi}{6} \Rightarrow \cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}, \sin \frac{\pi}{6} = \frac{1}{2}$$

∴ Transformation equations are

$$x = X \cos \theta - Y \sin \theta, \quad y = X \sin \theta + Y \cos \theta$$

$$\Rightarrow x = X \frac{\sqrt{3}}{2} - Y \frac{1}{2}, \quad y = X \frac{1}{2} + Y \frac{\sqrt{3}}{2}$$

$$x = \frac{\sqrt{3}X - Y}{2}, \quad y = \frac{X + \sqrt{3}Y}{2}$$

The required transformed equation is

$$\left(\frac{\sqrt{3}X - Y}{2}\right)^2 + 2\sqrt{3}\left(\frac{\sqrt{3}X - Y}{2}\right)\left(\frac{X + \sqrt{3}Y}{2}\right) - \left(\frac{X + \sqrt{3}Y}{2}\right)^2 = 2a^2$$

$$\Rightarrow 3X^2 - 2\sqrt{3}XY + Y^2 + 2\sqrt{3}(\sqrt{3}X^2 + 2XY - \sqrt{3}Y^2) - (X^2 + 2\sqrt{3}XY + 3Y^2) = 4(2a^2)$$

$$\Rightarrow 8X^2 - 8Y^2 = 8a^2 \quad \div 8$$

$$\Rightarrow X^2 - Y^2 = a^2$$

- 6. Find the transformed equation of  $x \cos \alpha + y \sin \alpha = p$ , when the axes are rotated through an angle  $\alpha$ .**

**Sol:** Let  $(X, Y)$  be the new coordinates of the point  $(x, y)$  when the axes are rotated through an angle  $\alpha$ .

Here  $\theta = \alpha$ .

∴ Transformation equations are

$$x = X \cos \theta - Y \sin \theta, \quad y = X \sin \theta + Y \cos \theta$$

$$x = X \cos \alpha - Y \sin \alpha, \quad y = X \sin \alpha + Y \cos \alpha$$

Given equation is  $x \cos \alpha + y \sin \alpha = p$

∴ Required transformed equation is

$$(X \cos \alpha - Y \sin \alpha) \cos \alpha + (X \sin \alpha + Y \cos \alpha) \sin \alpha = p.$$

$$\Rightarrow X \cos^2 \alpha - Y \sin \alpha \cos \alpha + X \sin^2 \alpha + Y \cos \alpha \sin \alpha = p.$$

$$\Rightarrow X(\cos^2 \alpha + \sin^2 \alpha) = p. \quad \because \cos^2 \alpha + \sin^2 \alpha = 1.$$

$$\Rightarrow X = p$$

- 6. When the axes are rotated through an angle  $45^\circ$ , the transformed equation of a curve is  $17X^2 - 16XY + 17Y^2 = 225$ . Find the original equation of the curve.**

**Sol:** Let  $(X, Y)$  be the new coordinates of the point  $(x, y)$  when the axes are rotated through an angle  $\theta$ . Here  $\theta = 45^\circ$

$$\Rightarrow \cos 45^\circ = \frac{1}{\sqrt{2}}, \sin 45^\circ = \frac{1}{\sqrt{2}}$$

∴ Transformation equations are

$$X = x \cos \theta + y \sin \theta, \quad Y = -x \sin \theta + y \cos \theta$$

$$\Rightarrow X = x \cdot \frac{1}{\sqrt{2}} + y \cdot \frac{1}{\sqrt{2}}, \quad Y = -x \cdot \frac{1}{\sqrt{2}} + y \cdot \frac{1}{\sqrt{2}}$$

$$\Rightarrow X = \frac{x + y}{\sqrt{2}}, \quad Y = \frac{-x + y}{\sqrt{2}}$$

Given transformed equation is

$$17X^2 - 16XY + 17Y^2 = 225.$$

∴ The required original equation is

$$17\left(\frac{x+y}{\sqrt{2}}\right)^2 - 16\left(\frac{x+y}{\sqrt{2}}\right)\left(\frac{-x+y}{\sqrt{2}}\right) + 17\left(\frac{-x+y}{\sqrt{2}}\right)^2 = 225$$

$$\Rightarrow 17(x^2 + 2xy + y^2) - 16(y^2 - x^2) + 17(x^2 - 2xy + y^2) = 2(225)$$

$$\Rightarrow 50x^2 + 18y^2 = 450$$

÷ 2

$$\Rightarrow 25x^2 + 9y^2 = 225.$$

**7. Find the point to which the origin is to be shifted by translation of axes so as to remove the first degree terms from the equation**

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0, \text{ where } h^2 \neq ab.$$

**Sol:** Let  $(X, Y)$  be the new coordinates of the point  $(x, y)$  when the axes are translated to the point  $(\alpha, \beta)$ . So the transformation equations are

$$x = X + \alpha, y = Y + \beta.$$

Given curve equation is

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0.$$

$\therefore$  Transformed equation of the curve is

$$a(X + \alpha)^2 + 2h(X + \alpha)(Y + \beta) + b(Y + \beta)^2 + 2g(X + \alpha) + 2f(Y + \beta) + c = 0.$$

$$\Rightarrow aX^2 + 2hXY + bY^2 + 2X(a\alpha + h\beta + g) + 2Y(h\alpha + b\beta + f) + (a\alpha^2 + 2ha\beta + b\beta^2 + 2g\alpha + 2f\beta + c) = 0.$$

since the first degree terms are to be eliminated, equating coefficients of  $X$  and  $Y$  to zero.

$$a\alpha + h\beta + g = 0$$

$$h\alpha + b\beta + f = 0$$

By the cross multiplication rule, we get

$$\frac{\alpha}{hf - bg} = \frac{\beta}{gh - af} = \frac{1}{ab - h^2}$$

$$\begin{matrix} h & g & a & h \\ b & f & h & b \end{matrix}$$

$$\Rightarrow (\alpha, \beta) = \left( \frac{hf - bg}{ab - h^2}, \frac{gh - af}{ab - h^2} \right)$$

Hence the required point of translation is

$$\left( \frac{hf - bg}{ab - h^2}, \frac{gh - af}{ab - h^2} \right).$$

**8. Show that the axes are to be rotated through**

**an angle of  $\frac{1}{2} \tan^{-1} \left( \frac{2h}{a-b} \right)$  so as to remove  $xy$  term from the equation  $ax^2 + 2hxy + by^2 = 0$  if  $a \neq b$  and through an angle  $\frac{\pi}{4}$  if  $a = b$ .**

**Sol:** Let  $(X, Y)$  be the new coordinates of the point  $(x, y)$  when the axes are rotated through an angle  $\theta$  in the anticlockwise direction.

$\therefore$  Transformation equations are

$$x = X \cos \theta - Y \sin \theta, \quad y = X \sin \theta + Y \cos \theta$$

Given equation of the curve is  $ax^2 + 2hxy + by^2 = 0$ .

$\therefore$  The transformed equation is

$$a(X \cos \theta - Y \sin \theta)^2 + 2h(X \cos \theta - Y \sin \theta)(X \sin \theta + Y \cos \theta) + b(X \sin \theta + Y \cos \theta)^2 = 0.$$

$$\Rightarrow a(X^2 \cos^2 \theta - 2XY \cos \theta \sin \theta + Y^2 \sin^2 \theta) + 2h(X^2 \cos^2 \theta - XY \cos^2 \theta - XY \sin^2 \theta - Y^2 \sin^2 \theta) + b(X^2 \sin^2 \theta + 2XY \sin \theta \cos \theta + Y^2 \cos^2 \theta) = 0.$$

since  $xy$  term is to be eliminated, coefficient of  $XY = 0$ .

$$\Rightarrow -2a \cos \theta \sin \theta + 2h \cos^2 \theta - 2h \sin^2 \theta + 2b \sin \theta \cos \theta = 0.$$

$$\Rightarrow 2h(\cos^2 \theta - \sin^2 \theta) = (a - b) 2 \sin \theta \cos \theta$$

$$\Rightarrow 2h \cos 2\theta = (a - b) \sin 2\theta$$

If  $a \neq b$

$$\frac{2h}{a-b} = \frac{\sin 2\theta}{\cos 2\theta}$$

$$\Rightarrow \tan 2\theta = \frac{2h}{a-b}$$

$$\Rightarrow 2\theta = \tan^{-1} \left( \frac{2h}{a-b} \right)$$

$$\Rightarrow \theta = \frac{1}{2} \tan^{-1} \left( \frac{2h}{a-b} \right)$$

If  $a = b$

$$2h \cos 2\theta = 0$$

$$\Rightarrow \cos 2\theta = 0$$

$$\Rightarrow 2\theta = \frac{\pi}{2}$$

$$\Rightarrow \theta = \frac{\pi}{4}$$

**Mathematics - IB**

1. Find the points on the line  $3x - 4y - 1 = 0$  which are at a distance of 5 units from the point  $(3, 2)$ .

A: Slope of the line  $3x - 4y - 1 = 0$  is

$$m = \tan \theta = \left( -\frac{a}{b} \right) = \left( -\frac{3}{-4} \right) = \frac{3}{4}$$

$$\text{Here } \theta \in Q_1 \Rightarrow \cos \theta = \frac{4}{5}, \sin \theta = \frac{3}{5}$$

Given point  $(x_1, y_1) = (3, 2)$ ,  $r = \pm 5$

Required points on the given line which are at a distance of 5 units from  $(3, 2)$  are

$$\begin{aligned} &= (x_1 + r \cos \theta, y_1 + r \sin \theta) \\ &= \left( 3 + (\pm 5) \left( \frac{4}{5} \right), 2 + (\pm 5) \left( \frac{3}{5} \right) \right) \\ &= (3 \pm 4, 2 \pm 3) \\ &= (3+4, 2+3), (3-4, 2-3) = (7, 5), (-1, -1). \end{aligned}$$

2. A straight line through  $Q(\sqrt{3}, 2)$  makes an angle of  $\frac{\pi}{6}$  with x-axis in positive direction. If this straight line intersects  $\sqrt{3}x - 4y + 8 = 0$  at P, find the distance of PQ.

Sol: Given  $Q(\sqrt{3}, 2) = (x_1, y_1)$ ,  $\theta = \frac{\pi}{6}$

Let  $PQ = r$

The coordinates of any point on the line are

$$\begin{aligned} &= (x_1 + r \cos \theta, y_1 + r \sin \theta) \\ &= \left( \sqrt{3} + r \cos \frac{\pi}{6}, 2 + r \sin \frac{\pi}{6} \right) \end{aligned}$$

$$P = \left( \sqrt{3} + \frac{r\sqrt{3}}{2}, 2 + \frac{r}{2} \right)$$

P lies on the line  $\sqrt{3}x - 4y + 8 = 0$

$$\Rightarrow \sqrt{3} \left( \sqrt{3} + \frac{r\sqrt{3}}{2} \right) - 4 \left( 2 + \frac{r}{2} \right) + 8 = 0$$

$$\Rightarrow 3 + \frac{3r}{2} - 8 - \frac{4r}{2} + 8 = 0$$

$$\Rightarrow 3 - \frac{r}{2} = 0$$

$$\Rightarrow \frac{r}{2} = 3$$

$$\Rightarrow r = 6$$

$$\Rightarrow PQ = 6.$$

**AIMSTUTORIAL**

3. A straight line with slope 1 passes through  $Q(-3, 5)$  and meets the straight line  $x + y - 6 = 0$  at P. Find the distance PQ.

Sol: Slope  $m = 1$

$$\tan q = 1 \quad P q = 45^\circ$$

$$Q(-3, 5) = (x_1, y_1)$$

$$\text{Let } PQ = r$$

$$\begin{aligned} \text{Coordinates of } P &= (x_1 + r \cos \theta, y_1 + r \sin \theta) \\ &= (-3 + r \cos 45^\circ, 5 + r \sin 45^\circ) \end{aligned}$$

$$= \left( -3 + \frac{r}{\sqrt{2}}, 5 + \frac{r}{\sqrt{2}} \right)$$

P lies on the line  $x + y - 6 = 0$

$$\Rightarrow -3 + \frac{r}{\sqrt{2}} + 5 + \frac{r}{\sqrt{2}} - 6 = 0$$

$$\Rightarrow \frac{2r}{\sqrt{2}} = 4$$

$$\Rightarrow r = 2\sqrt{2}.$$

$$\Rightarrow PQ = 2\sqrt{2}.$$

4. A straight line through  $Q(2, 3)$  makes an angle  $\frac{3\pi}{4}$  with negative direction of the x-axis. If the straight line intersects the line  $x + y - 7 = 0$  at P, find the distance PQ.

Sol: Since the line through Q makes an angle  $\frac{3\pi}{4}$  with the negative direction of X-axis,

$$\theta = \pi - \frac{3\pi}{4} = \frac{\pi}{4}.$$

$$\text{Coordinates of } P = (x_1 + r \cos \theta, y_1 + r \sin \theta)$$

$$= \left( 2 + r \cos \frac{\pi}{4}, 3 + r \sin \frac{\pi}{4} \right)$$

$$= \left( 2 + \frac{r}{\sqrt{2}}, 3 + \frac{r}{\sqrt{2}} \right)$$

since P lies on the line  $x + y - 7 = 0$ ,

$$2 + \frac{r}{\sqrt{2}} + 3 + \frac{r}{\sqrt{2}} - 7 = 0.$$

$$\Rightarrow \frac{2r}{\sqrt{2}} = 2$$

$$\Rightarrow r = \sqrt{2}$$

$$\Rightarrow PQ = \sqrt{2} \text{ units.}$$

5. Find the value of  $k$  if the lines  $2x - 3y + k = 0$ ,  $3x - 4y - 13 = 0$  and  $8x - 11y - 33 = 0$  are concurrent.

**Sol:** Given lines are  $2x - 3y + k = 0 \dots (1)$

$$3x - 4y - 13 = 0 \dots (2)$$

$$8x - 11y - 33 = 0 \dots (3)$$

solving (2), (3) by cross multiplication rule,

$$\begin{array}{r} -4 & -13 & 3 & -4 \\ -11 & -33 & 8 & -11 \end{array}$$

$$\frac{x}{132 - 143} = \frac{y}{-104 + 99} = \frac{1}{-33 + 32}$$

$$\Rightarrow \frac{x}{-11} = \frac{y}{-5} = \frac{1}{-1}$$

$$(x, y) = (11, 5).$$

so, the point of intersection of (2), (3) is (11, 5).

since the given lines are concurrent, (11, 5) should lie on  $2x - 3y + k = 0$ .

$$\Rightarrow 2(11) - 3(5) + k = 0$$

$$\Rightarrow 22 - 15 + k = 0 \Rightarrow k = -7.$$

6. If the straight lines  $ax + by + c = 0$ ,  $bx + cy + a = 0$  and  $cx + ay + b = 0$  are concurrent, then prove that  $a^3 + b^3 + c^3 = 3abc$ .

**Sol:** Given lines are  $ax + by + c = 0 \dots (1)$

$$bx + cy + a = 0 \dots (2)$$

$$cx + ay + b = 0 \dots (3)$$

solving (1) & (2) by the cross multiplication rule,

$$\frac{x}{ab - c^2} = \frac{y}{bc - a^2} = \frac{1}{ac - b^2} \quad \begin{array}{rrrr} b & c & a & b \\ c & a & b & c \end{array}$$

$$\therefore (x, y) = \left( \frac{ab - c^2}{ac - b^2}, \frac{bc - a^2}{ac - b^2} \right)$$

since the given three lines are concurrent, so

$$c\left(\frac{ab - c^2}{ac - b^2}\right) + a\left(\frac{bc - a^2}{ac - b^2}\right) + b = 0$$

$$\Rightarrow c(ab - c^2) + a(bc - a^2) + b(ac - b^2) = 0.$$

$$\Rightarrow abc - c^3 + abc - a^3 + abc - b^3 = 0.$$

$$\Rightarrow 3abc - a^3 - b^3 - c^3 = 0.$$

$$a^3 + b^3 + c^3 = 3abc.$$

$$\text{OR } \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0$$

$$\Rightarrow a(bc - a^2) - b(b^2 - ac) + c(ab - c^2) = 0.$$

$$\Rightarrow abc - a^3 - b^3 + abc + abc - c^3 = 0.$$

$$\Rightarrow 3abc - a^3 - b^3 - c^3 = 0$$

$$a^3 + b^3 + c^3 = 3abc.$$

7. If  $3a + 2b + 4c = 0$ , then show that the equation  $ax + by + c = 0$  represents a family of concurrent straight lines and find the point of concurrency.

**Sol:**  $3a + 2b + 4c = 0$

$$\Rightarrow 4c = -3a - 2b$$

$$ax + by + c = 0 \quad x 4$$

$$\Rightarrow 4ax + 4by + 4c = 0$$

$$\Rightarrow 4ax + 4by - 3a - 2b = 0$$

$$\Rightarrow a(4x - 3) + b(4y - 2) = 0.$$

Which is in the form  $l_1 L_1 + l_2 L_2 = 0$ . So it represents a family of concurrent lines.

Point of concurrency is the point of intersection of the lines  $4x - 3 = 0, 4y - 2 = 0$ .

$$\Rightarrow x = \frac{3}{4}, y = \frac{2}{4} = \frac{1}{2}.$$

$$\text{Coordinates of point of concurrence} = \left( \frac{3}{4}; \frac{1}{2} \right).$$

8. Find the values of  $k$ , if the angle between the straight lines  $kx + y + 9 = 0$  and  $3x - y + 4 = 0$  is  $\frac{\pi}{4}$ .

**Sol:** Given that  $\frac{\pi}{4}$  is the angle between the lines  $kx + y + 9 = 0$  and  $3x - y + 4 = 0$ .

$$\cos q = \frac{|a_1 a_2 + b_1 b_2|}{\sqrt{a_1^2 + b_1^2} \sqrt{a_2^2 + b_2^2}}$$

$$\cos \frac{\pi}{4} = \frac{|k(3) + 1(-1)|}{\sqrt{k^2 + 1} \sqrt{9 + 1}}$$

$$\frac{1}{\sqrt{2}} = \frac{|3k - 1|}{\sqrt{k^2 + 1} \sqrt{10}}$$

Squaring and cross multiplying

$$5(k^2 + 1) = (3k - 1)^2$$

$$5k^2 + 5 = 9k^2 - 6k + 1$$

$$\Rightarrow 4k^2 - 6k - 4 = 0$$

$$\Rightarrow 2k^2 - 3k - 2 = 0$$

$$\Rightarrow 2k^2 - 4k + k - 2 = 0$$

$$\Rightarrow 2k(k - 2) + 1(k - 2) = 0$$

$$\Rightarrow (k - 2)(2k + 1) = 0$$

$$k = 2, \frac{-1}{2}$$

9. Show that the lines  $x - 7y - 22 = 0$ ,  $3x + 4y + 9 = 0$  and  $7x + y - 54 = 0$  form a right angled isosceles triangle.

Sol: Let ABC be the given triangle with the equations of sides AB, BC, CA

$$\cos A = \frac{|a_1 a_2 + b_1 b_2|}{\sqrt{a_1^2 + b_1^2} \sqrt{a_2^2 + b_2^2}}$$

$$= \frac{|1(7) - 7(1)|}{\sqrt{1+49} \sqrt{49+1}} = 0. \quad A = 90^\circ.$$

$$\cos B = \frac{|1(3) + (-7)(4)|}{\sqrt{9+16} \sqrt{1+49}}$$

$$= \frac{|3 - 28|}{\sqrt{25} \sqrt{50}} = \frac{25}{5(5\sqrt{2})} = \frac{1}{\sqrt{2}}$$

$$B = 45^\circ$$

$$C = 180^\circ - (A + B)$$

$$= 180^\circ - (90^\circ + 45^\circ) = 45^\circ$$

Hence ABC is a right angled isosceles triangle.

10. Find the equations of the lines passing through the point  $(-3, 2)$  and making an angle  $45^\circ$  with the line  $3x - y + 4 = 0$ .

Sol: Let the slope of the required line be  $m$ .

Given that this line makes angle  $45^\circ$  with the line  $3x - y + 4 = 0$  whose slope is  $m_1 = \frac{-3}{-1} = 3$ .

$$\tan 45^\circ = \frac{|m_1 - m_2|}{1 + m_1 m_2}$$

$$1 = \frac{|3 - m|}{1 + 3m}$$

$$(1 + 3m)^2 = (3 - m)^2$$

$$\Rightarrow 1 + 6m + 9m^2 = 9 - 6m + m^2$$

$$\Rightarrow 8m^2 + 12m - 8 = 0$$

$$\Rightarrow 2m^2 + 3m - 2 = 0$$

$$\Rightarrow 2m^2 + 4m - m - 2 = 0$$

$$\Rightarrow 2m(m + 2) - 1(m + 2) = 0$$

$$m = \frac{1}{2} \text{ or } -2$$

Equation of the line passing through  $(-3, 2)$  and

having slope  $\frac{1}{2}$  is  $y - 2 = \frac{1}{2}(x + 3)$

$$2y - 4 = x + 3 \Rightarrow x - 2y + 7 = 0$$

Equation of the line passing through  $(-3, 2)$  and having slope  $-2$  is  $y - 2 = -2(x + 3)$

$$2x + y + 4 = 0.$$

11. Find the equation of the straight line making equal intercepts on the coordinate axes and passing through the point of intersection of the lines  $2x - 5y + 1 = 0$  and  $x - 3y - 4 = 0$ .

Sol: Equations of given two lines are  $2x - 5y + 1 = 0$  ---(1)

$$x - 3y - 4 = 0$$
 ---(2)

solving (1), (2), we get  $2x - 5y + 1 = 0$  ----- (1)

$$\underline{2x - 6y - 8 = 0} \quad \text{----- (2) } \times 2$$

$$y + 9 = 0$$

$$\Rightarrow y = -9$$

$$\text{From (1), } 2x + 45 + 1 = 0$$

$$2x = -46 \quad \text{P } x = -23$$

The point of intersection of two lines is  $(-23, -9)$ .

Equation of the line which makes equal intercepts

$$\text{on the axes is } \frac{x}{a} + \frac{y}{a} = 1 \quad \Rightarrow \quad x + y = a$$

since it passes through  $(-23, -9)$

$$-23 - 9 = a \Rightarrow a = -32$$

Hence the required equation of the line is  $x + y = -32$ .

$$x + y + 32 = 0.$$

12. Find the equation of the straight line perpendicular to the line  $5x - 2y = 7$  and passing through the point of intersection of the lines  $2x + 3y = 1$  &  $3x + 4y = 6$ .

Sol: Solving

$$2x + 3y = 1 \quad \text{----- (1)}$$

$$3x + 4y = 6 \quad \text{----- (2)}$$

$$6x + 9y = 3 \quad \text{----- (1) } \times 3$$

$$\underline{6x + 8y = 12} \quad \text{----- (2) } \times 2$$

on subtraction  $y = -9$

$$2x + 3(-9) = 1$$

$$2x = 28$$

$$x = 14$$

Point of intersection of (1), (2) is  $(14, -9)$ .

Now, equation of the line perpendicular to the line  $5x - 2y = 7$  is in the form  $2x + 5y + k = 0$ . since it passes through  $(14, -9)$ ,

$$2(14) + 5(-9) + k = 0$$

$$\Rightarrow 28 - 45 + k = 0$$

$$\Rightarrow k = 17.$$

Hence, the required equation of the line is

$$2x + 5y + 17 = 0.$$

**13. Find the equation straight line parallel to the line  $3x + 4y = 7$  and the point of intersection of  $x - 2y - 3 = 0$  and  $x + 3y - 6 = 0$ .**

**Sol:** Given lines  $x - 2y - 3 = 0$  and  $x + 3y - 6 = 0$  equation of any lines passing through point of intersection of lines is  $L_1 + KL_2 = 0$  then  
 $x - 2y - 3 + k(x + 3y - 6) = 0 \dots\dots\dots(1)$   
 $(1 + k)x + (-2 + 3k)y + (-3 - 6k) = 0 \dots\dots\dots(2)$

then equation (2) is parallel to  $3x + 4y - 7 = 0$ .

$$\text{then } \frac{a_1}{a_2} = \frac{b_1}{b_2} \Rightarrow \frac{3}{1+k} = \frac{4}{-2+3k}$$

$$\Rightarrow 3(-2+3k) = 4(1+k)$$

$$\Rightarrow -6 + 9k = 4 + 4k$$

$$\Rightarrow 5k = 10$$

$$\Rightarrow k = 2$$

now, sub k in (1) then equation of required line is

$$3x + 4y - 15 = 0.$$

**14. Find the equation of the line perpendicular to the line  $3x + 4y + 6 = 0$  and making an intercept -4 on the x-axis.**

**Sol:** Equation of the line perpendicular to  $3x + 4y + 6 = 0$  is of the form  $4x - 3y + k = 0 \dots\dots\dots(1)$

$$4x - 3y = -k$$

$$\frac{x}{-k/4} + \frac{y}{k/3} = 1$$

$$\text{Given that x-intercept} = \frac{-k}{4} = -4$$

$$\Rightarrow k = 16.$$

$$\text{Required equation of the line is } 4x - 3y + 16 = 0.$$

**15. Transform the equation  $\frac{x}{a} + \frac{y}{b} = 1$  into the normal form where  $a > 0$ ,  $b > 0$ . If the perpendicular distance of the straight line from the origin is  $p$ , deduce that**

$$\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2}.$$

A: Given equation of the line  $\frac{x}{a} + \frac{y}{b} = 1$

$$\Rightarrow bx + ay = ab$$

$$\text{dividing throughout by } \sqrt{b^2 + a^2}$$

$$x\left(\frac{b}{\sqrt{b^2 + a^2}}\right) + y\left(\frac{a}{\sqrt{a^2 + b^2}}\right) = \frac{ab}{\sqrt{a^2 + b^2}}$$

which is in the normal form  $x \cos \alpha + y \sin \alpha = p$

$$\text{where } a = \tan^{-1} \frac{a}{b}, p = \frac{ab}{\sqrt{a^2 + b^2}}$$

$$\Rightarrow p^2 = \frac{a^2 b^2}{a^2 + b^2}$$

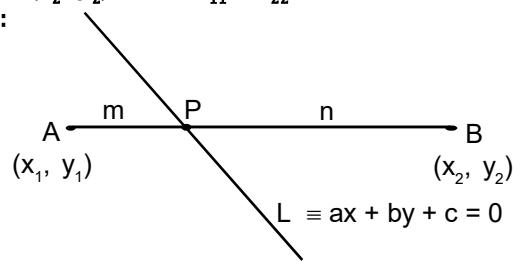
$$\Rightarrow \frac{1}{p^2} = \frac{a^2 + b^2}{a^2 b^2}$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{b^2} + \frac{1}{a^2}$$

$$\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2}$$

**16. Prove that the ratio in which the straight line  $L \equiv ax + by + c = 0$  divides the line segment joining the points  $A(x_1, y_1)$  and  $B(x_2, y_2)$  is  $-L_{11} : L_{22}$ .**

**Sol:**



Suppose the straight line  $L \equiv ax + by + c = 0$  divides the line segment  $\overline{AB}$  in the ratio  $m:n$  at P.

The coordinates of P =  $\left( \frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n} \right)$

since P lies on  $ax + by + c = 0$ , we get

$$a\left(\frac{mx_2 + nx_1}{m+n}\right) + b\left(\frac{my_2 + ny_1}{m+n}\right) + c = 0$$

$$\Rightarrow a(mx_2 + nx_1) + b(my_2 + ny_1) + c(m+n) = 0.$$

$$\Rightarrow m(ax_2 + by_2 + c) + n(ax_1 + by_1 + c) = 0.$$

$$\Rightarrow mL_{22} = -nL_{11}$$

$$\Rightarrow \frac{m}{n} = \frac{-L_{11}}{L_{22}}$$

Hence  $m:n = -L_{11} : L_{22}$ .

- 17.** Find the equations of the straight lines passing through (1, 3) and (i) parallel to (ii) perpendicular to the line passing through the points (3, -5) and (-6, 1).

**Sol:** Let A(3, -5), B(-6, 1) be the given points.

$$m = \text{slope of AB} = \frac{1+5}{-6-3} = \frac{6}{-9} = \frac{-2}{3}$$

i) Equation of the line parallel to AB and passing through (1, 3) with slope  $\frac{-2}{3}$

$$y - y_1 = m(x - x_1)$$

$$y - 3 = \frac{-2}{3}(x - 1)$$

$$\Rightarrow 3y - 9 = -2x + 2$$

$$\Rightarrow 2x + 3y - 11 = 0$$

ii) Equation of the line perpendicular to AB and passing through (1, 3) with slope  $m = \frac{3}{2}$

$$y - 3 = \frac{3}{2}(x - 1)$$

$$\Rightarrow 2y - 6 = 3x - 3$$

$$\Rightarrow 3x - 2y + 3 = 0$$

Equations of the required lines are

$$2x + 3y - 11 = 0. \quad 3x - 2y + 3 = 0.$$

- 18.** Find the equation of the straight line passing through the points (-1, 2) and (5, -1) and also find the area of the triangle formed by it with the axes of coordinates.

**Sol:** Let P(-1, 2), Q(5, -1) be the given points.  
Equation of PQ is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1} (x - x_1)$$

$$y - 2 = \frac{-1-2}{5+1} (x + 1)$$

$$y - 2 = \frac{-3}{6} (x + 1)$$

$$y - 2 = \frac{-1}{2} (x + 1)$$

$$2y - 4 = -x - 1.$$

$$x + 2y - 3 = 0$$

$$\text{Area of triangle OAB} = \frac{c^2}{2|ab|}$$

$$= \frac{9}{2|(1)(2)|} = \frac{9}{4} \text{ sq. units.}$$

- 19.** If  $x - 3y - 5 = 0$  is the perpendicular bisector of the line segment joining the points A, B. If A = (-1, -3), find co-ordinates of B.

**Sol:** Let B(h, k) is the image of A in the line

$$x - 3y - 5 = 0$$

Let (h, k) is the image of A( $x_1, y_1$ ) show the line  $ax + by + c = 0$ , then

$$\frac{h - x_1}{a} = \frac{k - y_1}{b} = \frac{-2(ax_1 + by_1 + c)}{a^2 + b^2}$$

$$\frac{h+1}{1} = \frac{k+3}{-3} = \frac{-2(-1+9-5)}{10}$$

$$\frac{h+1}{1} = \frac{k+3}{-3} = \frac{-6}{10}$$

$$\frac{h+1}{1} = \frac{k+3}{-3} = \frac{-3}{5}$$

$$\frac{h+1}{1} = \frac{-3}{5} \Rightarrow \frac{k+3}{-3} = \frac{-3}{5}$$

$$5h + 5 = -3. \quad 5k + 15 = 9$$

$$5h = -8 \quad 5k = -6$$

$$h = \frac{-8}{5} \quad k = \frac{-6}{5}$$

The coordinates of B =  $\left(\frac{-8}{5}, \frac{-6}{5}\right)$ .

1. Show that  $f(x) = \begin{cases} \frac{\cos ax - \cos bx}{x^2} & \text{if } x \neq 0 \\ \frac{1}{2}(b^2 - a^2) & \text{if } x = 0 \end{cases}$  is continuous at 0.

**Sol:** Given that  $f(0) = \frac{1}{2} (b^2 - a^2)$

$$\begin{aligned} \lim_{x \rightarrow 0} f(x) &= \lim_{x \rightarrow 0} \frac{\cos ax - \cos bx}{x^2} \\ &= \lim_{x \rightarrow 0} \frac{-2 \sin\left(\frac{a+b}{2}\right)x}{x} \cdot \frac{\sin\left(\frac{a-b}{2}\right)x}{x} \\ &= -2 \lim_{\left(\frac{a+b}{2}\right)x \rightarrow 0} \frac{\sin\left(\frac{a+b}{2}\right)x}{\left(\frac{a+b}{2}\right)x} \lim_{\left(\frac{a-b}{2}\right)x \rightarrow 0} \frac{\sin\left(\frac{a-b}{2}\right)x}{\left(\frac{a-b}{2}\right)x} \\ &\cdot \left(\frac{a+b}{2}\right) \left(\frac{a-b}{2}\right) \\ &= -2(1)(1) \left(\frac{a^2 - b^2}{4}\right) = \frac{b^2 - a^2}{2} \\ \therefore f(x) &\text{ is continuous at } x = 0. \end{aligned}$$

2. Check the continuity of  $f$  given by

$$f(x) = \begin{cases} \frac{x^2 - 9}{x^2 - 2x - 3}, & \text{if } 0 < x < 5 \text{ and } x \neq 3 \\ 1.5, & \text{if } x = 3 \end{cases}$$

at the point 3.

**Sol:** Given that  $f(3) = 1.5$

$$\begin{aligned} \lim_{x \rightarrow 3} f(x) &= \lim_{x \rightarrow 3} \frac{x^2 - 9}{x^2 - 2x - 3} \\ &= \lim_{x \rightarrow 3} \frac{(x-3)(x+3)}{(x-3)(x+1)} \\ &= \lim_{x \rightarrow 3} \frac{x+3}{x+1} \\ &= \frac{3+3}{3+1} \\ &= \frac{6}{4} = \frac{3}{2} = 1.5 \end{aligned}$$

Here  $\lim_{x \rightarrow 3} f(x) = f(3)$   
so  $f(x)$  is continuous at  $x = 3$ .

3. Check the continuity of the function  $f(x)$  at

$$x = 2 \text{ if } f(x) = \begin{cases} \frac{1}{2}(x^2 - 4) & \text{if } 0 < x < 2 \\ 0 & \text{if } x = 2 \\ 2 - 8x^{-3} & \text{if } x > 2 \end{cases}$$

**Sol:** Given that  $f(2) = 0$

$$\begin{aligned} \lim_{x \rightarrow 2^-} f(x) &= \lim_{x \rightarrow 2} \frac{1}{2}(x^2 - 4) \\ &= \frac{1}{2}(4 - 4) \\ &= 0 \\ \lim_{x \rightarrow 2^+} f(x) &= \lim_{x \rightarrow 2} \left(2 - \frac{8}{x^3}\right) \\ &= 2 - \frac{8}{2^3} \\ &= 2 - \frac{8}{8} \\ &= 2 - 1 \\ &= 1. \end{aligned}$$

Here  $\lim_{x \rightarrow 2^-} f(x) \neq \lim_{x \rightarrow 2^+} f(x)$   
so  $f(x)$  is discontinuous at  $x = 2$ .

4. Find the real constant  $a, b$  so that the function 'f' given by

$$f(x) = f(x) = \begin{cases} \sin x & \text{if } x \leq 0 \\ x^2 + a & \text{if } 0 < x < 1 \\ bx + 3 & \text{if } 1 \leq x \leq 3 \\ -3 & \text{if } x > 3 \end{cases} \text{ On R.}$$

**Sol:** Given  $f(x)$  is continuous at  $x = 0$

$$\begin{aligned} \therefore \lim_{x \rightarrow 0^-} f(x) &= \lim_{x \rightarrow 0^+} f(x) \\ \lim_{x \rightarrow 0^-} \sin x &= \lim_{x \rightarrow 0^+} (x^2 + a) \\ 0 &= 0 + a \\ a &= 0 \end{aligned}$$

If  $f(x)$  is continuous at  $x = 3$

$$\begin{aligned} \therefore \lim_{x \rightarrow 3^-} f(x) &= \lim_{x \rightarrow 3^+} f(x) \\ \lim_{x \rightarrow 3^-} (bx + 3) &= \lim_{x \rightarrow 3^+} (-3) \\ 3b + 3 &= -3 \\ 3b &= -6 \\ b &= -2 \end{aligned}$$

5. If  $f$  is given by  $f(x) = \begin{cases} k^2x - k & \text{if } x \geq 1 \\ 2 & \text{if } x < 1 \end{cases}$  is a continuous function of  $\mathbb{R}$ , then find the values of  $k$ .

$$\text{Sol: LHL} = \lim_{x \rightarrow 1^-} f(x)$$

$$= \lim_{x \rightarrow 1^-} (2) = 2$$

$$\text{RHL} = \lim_{x \rightarrow 1^+} f(x)$$

$$= \lim_{x \rightarrow 1^+} (k^2x - k) \\ = k^2 - k$$

Given that  $f(x)$  is continuous at  $x = 1$

$$\Rightarrow \text{LHL} = \text{RHL}$$

$$\Rightarrow 2 = k^2 - k$$

$$\Rightarrow k^2 - k - 2 = 0$$

$$\Rightarrow k^2 - 2k + k - 2 = 0$$

$$\Rightarrow k(k - 2) + 1(k - 2) = 0$$

$$\Rightarrow (k + 1)(k - 2) = 0$$

$$\Rightarrow k = -1 \text{ or } k = 2.$$

6. Show that  $f(x) = \begin{cases} \frac{\sin 2x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$  is discontinuous at  $x = 0$ .

**Sol:** Given that  $f(0) = 1$

$$\begin{aligned} \lim_{x \rightarrow 0} f(x) &= \lim_{x \rightarrow 0} \frac{\sin 2x}{x} \\ &= \lim_{x \rightarrow 0} \frac{\sin 2x}{2x} \cdot 2 \\ &= 1 \cdot 2 \\ &= 2 \end{aligned}$$

$$\therefore \lim_{x \rightarrow 0} f(x) \neq f(0)$$

so  $f(x)$  is discontinuous at  $x = 0$ .

7. Check the continuity of the function  $f$  given

below at 1 and 2.  $f(x) = \begin{cases} x+1 & \text{if } x \leq 1 \\ 2x & \text{if } 1 < x < 2 \\ 1+x^2 & \text{if } x \geq 2 \end{cases}$

**Sol:** At  $x = 1$ :

$$\text{LHL} \lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} (x + 1) = 1 + 1 = 2.$$

$$\text{RHL} \lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} 2x = 2(1) = 2$$

$$\lim_{x \rightarrow 1} f(x) = 2 = f(1)$$

$\therefore f$  is continuous at  $x = 1$

At  $x = 2$ :

$$\text{LHL} \lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} 2x = 2(2) = 4$$

$$\text{RHL} \lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} (1 + x^2) = 1 + 4 = 5$$

$$\lim_{x \rightarrow 2^-} f(x) \neq \lim_{x \rightarrow 2^+} f(x)$$

$\lim_{x \rightarrow 2} f(x)$  does not exist

$\therefore f$  is not continuous at  $x = 2$ .

8. Check the continuity of 'f' given by

$$f(x) = \begin{cases} 4 - x^2 & \text{if } x \leq 0 \\ x - 5 & \text{if } 0 < x \leq 1 \\ 4x^2 - 9 & \text{if } 1 < x < 2 \\ 3x + 4 & \text{if } x \geq 2 \end{cases} \text{ at the points } 0, 1 \text{ and } 2.$$

**Sol:** Case (i): At  $x = 0$

$$\begin{aligned} \text{LHL} &= \lim_{x \rightarrow 0^-} f(x) & \text{RHL} &= \lim_{x \rightarrow 0^+} f(x) \\ &= \lim_{x \rightarrow 0^-} (4 - x^2) & &= \lim_{x \rightarrow 0^+} (x - 5) \\ &= 4 - 0 = 4 & &= 0 - 5 = -5 \end{aligned}$$

$$\text{LHL} \neq \text{RHL}$$

$\therefore f(x)$  is not continuous at  $x = 0$ .

Case (ii) : At  $x = 1$

$$\begin{aligned} \text{LHL} &= \lim_{x \rightarrow 1^-} f(x) & \text{RHL} &= \lim_{x \rightarrow 1^+} f(x) \\ &= \lim_{x \rightarrow 1^-} (x - 5) & &= \lim_{x \rightarrow 1^+} (4x^2 - 9) \\ &= 1 - 5 & &= 4 - 9 \\ &= -4 & &= -5 \end{aligned}$$

$$\text{LHL} \neq \text{RHL}$$

$\therefore f(x)$  is not continuous at  $x = 1$ .

Case (iii): At  $x = 2$

$$\begin{aligned} \text{LHL} &= \lim_{x \rightarrow 2^-} f(x) & \text{RHL} &= \lim_{x \rightarrow 2^+} f(x) \\ &= \lim_{x \rightarrow 2^-} (4x^2 - 9) & &= \lim_{x \rightarrow 2^+} (3x + 4) \\ &= 4(4) - 9 & &= 3(2) + 4 \\ &= 16 - 9 & &= 6 + 4 \\ &= 7 & &= 10 \end{aligned}$$

$$\text{LHL} \neq \text{RHL}$$

$\therefore f(x)$  is not continuous at  $x = 2$ .

9. Compute  $\lim_{x \rightarrow \infty} \frac{x^2 - \sin x}{x^2 - 2}$ .

Sol:  $-1 \leq \sin x \leq 1$ , for  $x \neq 0$

$$\Rightarrow -1 \leq -\sin x \leq 1$$

$$x^2 - 1 \leq x^2 - \sin x \leq x^2 + 1$$

$$\Rightarrow \frac{x^2 - 1}{x^2 - 2} \leq \frac{x^2 - \sin x}{x^2 - 2} \leq \frac{x^2 + 1}{x^2 - 2}$$

$$\Rightarrow \lim_{x \rightarrow \infty} \frac{x^2 - 1}{x^2 - 2} \leq \lim_{x \rightarrow \infty} \frac{x^2 - \sin x}{x^2 - 2} \leq \lim_{x \rightarrow \infty} \frac{x^2 + 1}{x^2 - 2}$$

$$\Rightarrow \lim_{x \rightarrow \infty} \frac{1 - \frac{1}{x^2}}{1 - \frac{2}{x^2}} \leq \lim_{x \rightarrow \infty} \frac{x^2 - \sin x}{x^2 - 2} \leq \lim_{x \rightarrow \infty} \frac{1 + \frac{1}{x^2}}{1 - \frac{2}{x^2}}$$

$$\Rightarrow \frac{1-0}{1-0} \leq \lim_{x \rightarrow \infty} \frac{x^2 - \sin x}{x^2 - 2} \leq \frac{1+0}{1-0}$$

$$\Rightarrow 1 \leq \lim_{x \rightarrow \infty} \frac{x^2 - \sin x}{x^2 - 2} \leq 1. \therefore \lim_{x \rightarrow \infty} \frac{x^2 - \sin x}{x^2 - 2} = 1.$$

10. Evaluate  $\lim_{x \rightarrow 0} \frac{\cos ax - \cos bx}{x^2}$ .

Sol:  $\lim_{x \rightarrow 0} \frac{\cos ax - \cos bx}{x^2}$

$$\cos C - \cos D = -2 \sin\left(\frac{C+D}{2}\right) \sin\left(\frac{C-D}{2}\right)$$

$$= \lim_{x \rightarrow 0} \frac{-2 \sin\left(\frac{ax+bx}{2}\right) \sin\left(\frac{ax-bx}{2}\right)}{x^2}$$

$$= -2 \lim_{x \rightarrow 0} \frac{\sin\left(\frac{a+b}{2}\right)x}{x} \cdot \frac{\sin\left(\frac{a-b}{2}\right)x}{x}$$

$$= -2 \lim_{\left(\frac{a+b}{2}\right)x \rightarrow 0} \frac{\sin\left(\frac{a+b}{2}\right)x}{\left(\frac{a+b}{2}\right)x}.$$

$$\lim_{\left(\frac{a-b}{2}\right)x \rightarrow 0} \frac{\sin\left(\frac{a-b}{2}\right)x}{\left(\frac{a-b}{2}\right)x} \left(\frac{a+b}{2}\right) \left(\frac{a-b}{2}\right)$$

$$= -2(1)(1) \left(\frac{a^2 - b^2}{4}\right) \quad \text{As } x \rightarrow 0, \left(\frac{a+b}{2}\right)x \rightarrow 0, \\ \left(\frac{a-b}{2}\right)x \rightarrow 0$$

$$= \frac{b^2 - a^2}{2} \quad \therefore \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1.$$

11. Compute  $\lim_{x \rightarrow a} \frac{x \sin a - a \sin x}{x - a}$ .

Sol:  $\lim_{x \rightarrow a} \frac{x \sin a - a \sin x}{x - a}$

$$= \lim_{x \rightarrow a} \frac{x \sin a - a \sin a + a \sin a - a \sin x}{x - a}$$

$$= \lim_{x \rightarrow a} \frac{(x-a)\sin a - a(\sin x - \sin a)}{x - a}$$

$$= \lim_{x \rightarrow a} \frac{(x-a)\sin a}{x-a} - a \lim_{x \rightarrow a} \frac{2\cos\left(\frac{x+a}{2}\right)\sin\left(\frac{x-a}{2}\right)}{x-a}$$

$$= \sin a - 2a \lim_{x \rightarrow a} \cos\left(\frac{x+a}{2}\right) \cdot (x-a) \rightarrow 0 \frac{\sin\left(\frac{x-a}{2}\right)}{\left(\frac{x-a}{2}\right)(2)}$$

$$= \sin a - 2a \cos a \left(\frac{1}{2}\right)$$

$$= \sin a - a \cos a.$$

12. Compute:  $\lim_{x \rightarrow 2} \frac{2x^2 - 7x - 4}{(2x-1)(\sqrt{x}-2)}$ .

Sol:  $\lim_{x \rightarrow 2} \frac{2x^2 - 7x - 4}{(2x-1)(\sqrt{x}-2)}$

$$= \frac{2(2)^2 - 7(2) - 4}{(2(2)-1)(\sqrt{2}-2)}$$

$$= \frac{(8-14-4)}{3(-1)(2-\sqrt{2})} x \frac{2+\sqrt{2}}{(2+\sqrt{2})}$$

$$= \frac{-10(2+\sqrt{2})}{-3(4-2)} = \frac{-10(2+\sqrt{2})}{6}$$

$$= \frac{-5(2+\sqrt{2})}{3}$$

13. Evaluate  $\lim_{x \rightarrow a} \left( \frac{\sqrt{a+2x} - \sqrt{3x}}{\sqrt{3a+x} - 2\sqrt{x}} \right)$ .

$$\begin{aligned} \text{Sol: } & \lim_{x \rightarrow a} \left( \frac{\sqrt{a+2x} - \sqrt{3x}}{\sqrt{3a+x} - 2\sqrt{x}} \right) \\ &= \lim_{x \rightarrow a} \frac{(\sqrt{a+2x} - \sqrt{3x})(\sqrt{a+2x} + \sqrt{3x})(\sqrt{3a+x} - 2\sqrt{x})}{(\sqrt{3a+x} - 2\sqrt{x})(\sqrt{a+2x} + \sqrt{3x})(\sqrt{a+x} + 2\sqrt{x})} \\ &= \lim_{x \rightarrow a} \frac{(a+2x-3x)(\sqrt{3a+x} + 2\sqrt{x})}{(3a+x-4x)(\sqrt{a+2x} + 3\sqrt{x})} \\ &= \lim_{x \rightarrow a} \frac{(a-x)(\sqrt{3a+x} + 2\sqrt{x})}{3(a-x)(\sqrt{a+2x} + 3\sqrt{x})} \\ &= \frac{1}{3} \frac{(\sqrt{3a+a} + 2\sqrt{a})}{(\sqrt{a+2a} + 3\sqrt{a})} \\ &= \frac{2\sqrt{a} + 2\sqrt{a}}{3(\sqrt{3a} + \sqrt{3a})} = \frac{4\sqrt{a}}{6\sqrt{3}\sqrt{a}} = \frac{2}{3\sqrt{3}} \end{aligned}$$

14. Compute  $\lim_{x \rightarrow 0} \frac{x(e^x - 1)}{1 - \cos x}$ .

$$\text{Sol: } \lim_{x \rightarrow 0} \frac{x(e^x - 1)}{1 - \cos x}$$

$$\begin{aligned} &= \lim_{x \rightarrow 0} \frac{x \left( \frac{e^x - 1}{x} \right) x}{2 \sin^2 \left( \frac{x}{2} \right)} \\ &= \frac{\lim_{x \rightarrow 0} \left( \frac{e^x - 1}{x} \right) x^2}{2 \sin^2 \left( \frac{x}{2} \right)} = \frac{1}{(1) \frac{1}{2}} = 2 \end{aligned}$$

$$\lim_{\frac{x}{2} \rightarrow 0} \frac{\left( \frac{x}{2} \right)^2}{\frac{4}{2}}$$

16. Find  $\lim_{x \rightarrow 0} \frac{\sqrt[3]{1+x} - \sqrt[3]{1-x}}{x}$ .

$$\begin{aligned} \text{Sol: } & \lim_{x \rightarrow 0} \frac{\sqrt[3]{1+x} - \sqrt[3]{1-x}}{x} \\ &= \lim_{x \rightarrow 0} \frac{\frac{1}{3}(1+x)^{\frac{1}{3}} - 1 - \frac{1}{3}(1-x)^{\frac{1}{3}} + 1}{x} \\ &= \lim_{x \rightarrow 0} \frac{\frac{1}{3}(1+x)^{\frac{1}{3}} - 1}{x} - \frac{\frac{1}{3}(1-x)^{\frac{1}{3}} - 1}{x} \\ &= \lim_{1+x \rightarrow 1} \frac{\frac{1}{3}(1+x)^{\frac{1}{3}} - 1}{(1+x)-1} + \lim_{1-x \rightarrow 1} \frac{\frac{1}{3}(1-x)^{\frac{1}{3}} - 1}{(1-x)-1} \\ &= \frac{1}{3} \cdot 1^{\frac{1}{3}-1} + \frac{1}{3} \cdot 1^{\frac{1}{3}-1} \\ &\left[ \because \lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} = n.a^{n-1} \right] \\ &= \frac{1}{3} + \frac{1}{3} = \frac{2}{3}. \end{aligned}$$

1. Find the derivatives of the following functions from the first principles

- i)  $\sin 2x$
- ii)  $\tan 2x$
- iii)  $\sec 3x$
- iv)  $\cos ax$

Sol: (i)  $f(x) = \sin 2x$

From the first principle,

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\sin 2(x+h) - \sin 2x}{h}$$

$$\therefore \sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right) \sin\left(\frac{C-D}{2}\right)$$

$$= \lim_{h \rightarrow 0} \frac{2\cos\left(\frac{2x+2h+2x}{2}\right) \sin\left(\frac{2x+2h-2x}{2}\right)}{h}$$

$$= \lim_{h \rightarrow 0} 2 \cos(2x+h) \frac{\sin h}{h}$$

$$= 2 \cos(2x+0) (1) \because \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1.$$

$$= 2 \cos 2x$$

ii)  $f(x) = \tan 2x$

From the first principle,

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\tan 2(x+h) - \tan 2x}{h}$$

$$= \lim_{h \rightarrow 0} \frac{1}{h} \left[ \frac{\sin(2x+2h)}{\cos(2x+2h)} - \frac{\sin 2x}{\cos 2x} \right]$$

$$= \lim_{h \rightarrow 0} \frac{1}{h} \left[ \frac{\sin(2x+2h)\cos 2x - \cos(2x+2h)\sin 2x}{\cos(2x+2h)\cos 2x} \right]$$

$$= \lim_{h \rightarrow 0} \frac{\sin(2x+2h) - \sin 2x}{h \cdot \cos(2x+2h)\cos 2x}$$

$$= \lim_{h \rightarrow 0} \frac{\sin 2h}{h} \cdot \frac{1}{\cos(2x+2h)\cos 2x}$$

$$= \lim_{h \rightarrow 0} 2 \frac{\sin 2h}{2h} \lim_{h \rightarrow 0} \frac{1}{\cos(2x+2h)\cos 2x}$$

$$\therefore \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1.$$

$$= 2(1) \cdot \frac{1}{\cos(2x+0)\cos 2x}$$

$$= 2 \sec^2(2x).$$

(iii)  $f(x) = \sec 3x$

From the first principle,

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\sec 3(x+h) - \sec 3x}{h}$$

$$= \lim_{h \rightarrow 0} \frac{1}{h} \left[ \frac{1}{\cos(3x+3h)} - \frac{1}{\cos 3x} \right]$$

$$= \lim_{h \rightarrow 0} \frac{1}{h} \frac{\cos 3x - \cos(3x+3h)}{\cos(3x+3h)\cos 3x}$$

$$\therefore \cos C - \cos D = -2 \sin\left(\frac{C+D}{2}\right) \sin\left(\frac{C-D}{2}\right)$$

$$= \lim_{h \rightarrow 0} \frac{1}{h} \frac{-2 \sin\left(\frac{3x+3x+3h}{2}\right) \sin\left(\frac{3x-3x-3h}{2}\right)}{\cos(3x+3h)\cos 3x}$$

$$= \lim_{h \rightarrow 0} \frac{-2 \sin\left(\frac{3x+3h}{2}\right)}{\cos(3x+3h)\cos 3x} \cdot \frac{\sin\left(\frac{-3h}{2}\right)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{-2 \sin\left(\frac{3x+3h}{2}\right)}{\cos(3x+3h)\cos 3x}$$

$$\stackrel{h \rightarrow 0}{=} \frac{\sin\left(\frac{-3h}{2}\right)}{\left(\frac{-3h}{2}\right)} \left(\frac{-3}{2}\right)$$

$$= \frac{3 \sin 3x}{\cos 3x} \cdot \frac{1}{\cos 3x}$$

$$= 3 \tan 3x \cdot \sec 3x.$$

(iv)  $f(x) = \cos ax$

From the first principle,  
 $f(x+h) = \cos a(x+h)$

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\cos a(x+h) - \cos ax}{h} \\ &= \lim_{h \rightarrow 0} \frac{-2 \sin\left(\frac{ax+ah+ax}{2}\right) \sin\left(\frac{ax+ah-ax}{2}\right)}{h} \\ &= \lim_{h \rightarrow 0} \frac{-2 \sin\left(\frac{2ax+ah}{2}\right) \sin\left(\frac{ah}{2}\right)}{h} \\ &= \lim_{h \rightarrow 0} -2 \sin\left(ax + \frac{ah}{2}\right) \frac{ah}{2} \lim_{h \rightarrow 0} \frac{\sin\left(\frac{ah}{2}\right)}{\left(\frac{ah}{2}\right)} \cdot \left(\frac{a}{2}\right) \\ &= -\sin ax (1)(1) \\ &= -a \sin ax. \end{aligned}$$

2. Find the derivatives of the following functions using the definition

- (i)  $\cos^2 x$       (ii)  $\sqrt{x+1}$       (iii)  $\log x$   
 (iv)  $x \sin x$       (v)  $x^3$

Sol:

i)  $f(x) = \cos^2 x$

By the definition

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ f'(x) &= \lim_{h \rightarrow 0} \frac{\cos^2(x+h) - \cos^2 x}{h} \\ &\quad \because \cos^2 B - \cos^2 A = \sin(A+B)\sin(A-B) \\ &= \lim_{h \rightarrow 0} \frac{\sin[x+(x+h)]\sin[x-(x+h)]}{h} \\ &= \lim_{h \rightarrow 0} \frac{\sin(2x+h)\sin(-h)}{h} \\ &= -\lim_{h \rightarrow 0} \sin(2x+h) \lim_{h \rightarrow 0} \frac{\sin h}{h} \\ &= -\sin(2x+0)(1) \\ &= -\sin 2x. \end{aligned}$$

ii)  $f(x) = \sqrt{x+1}$

By the definition

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\sqrt{x+h+1} - \sqrt{x+1}}{h} \\ &= \lim_{h \rightarrow 0} \frac{\sqrt{x+h+1} - \sqrt{x+1}}{h} \cdot \frac{\sqrt{x+h+1} + \sqrt{x+1}}{\sqrt{x+h+1} + \sqrt{x+1}} \\ &= \lim_{h \rightarrow 0} \frac{x+h+1 - (x+1)}{h[\sqrt{x+h+1} + \sqrt{x+1}]} \\ &= \lim_{h \rightarrow 0} \frac{h}{h[\sqrt{x+h+1} + \sqrt{x+1}]} \\ &= \lim_{h \rightarrow 0} \frac{1}{\sqrt{x+1+h} + \sqrt{x+1}} \\ &= \frac{1}{\sqrt{x+0+1} + \sqrt{x+1}} = \frac{1}{2\sqrt{x+1}} \end{aligned}$$

iii)  $f(x) = \log_e x$

By the definition

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\log_e(x+h) - \log_e x}{h} \\ &\quad \because \log_e a - \log_e b = \log_e \left(\frac{a}{b}\right) \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \log_e \left(\frac{x+h}{x}\right) \\ &= \lim_{h \rightarrow 0} \frac{1}{x} \frac{1}{h} \log_e \left(1 + \frac{h}{x}\right) \\ &= \lim_{h \rightarrow 0} \frac{1}{x} \log_e \left(1 + \frac{h}{x}\right)^{\frac{1}{h}} \\ &= \lim_{h \rightarrow 0} \frac{1}{x} \log_e \left(1 + \frac{h}{x}\right)^{\frac{x}{h}} \\ &= \frac{1}{x} \lim_{\frac{h}{x} \rightarrow 0} \log_e \left(1 + \frac{h}{x}\right)^{\frac{x}{h}} \quad \because x \lim_{h \rightarrow 0} (1+h)^{1/h} = e \\ &= \frac{1}{x} \log_e e \\ &= \frac{1}{x}(1) = \frac{1}{x} \quad \therefore \log_e e = 1. \end{aligned}$$

iv)  $f(x) = x \sin x$ .

$$f(x+h) = (x+h) \sin(x+h)$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{(x+h) \sin(x+h) - x \sin x}{h}$$

$$= \lim_{h \rightarrow 0} \frac{x(\sin(x+h) - \sin x) + h \sin(x+h)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{x \cdot 2 \cos\left(\frac{x+h+x}{2}\right) \sin\left(\frac{x+h-x}{2}\right) + h \sin(x+h)}{h}$$

$$= \lim_{h \rightarrow 0} x \cdot 2 \cos\left(\frac{2x+h}{2}\right) \lim_{h \rightarrow 0} \left(\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right) \cdot \frac{1}{2} + \lim_{h \rightarrow 0} \frac{h \sin(x+h)}{h}$$

$$= x \cos x + \sin x.$$

3. If  $\sin y = x \cdot \sin(a+y)$  prove that

$$\frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}.$$

Sol: Given that  $\sin y = x \sin(a+y)$

$$\Rightarrow x = \frac{\sin y}{\sin(a+y)}$$

$$\frac{d}{dx}(u/v) = \frac{v \cdot u' - u \cdot v'}{v^2}$$

differentiating with respect to 'y'

$$\frac{dx}{dy} = \frac{\sin(a+y) \cos y - \sin y \cos(a+y)}{\sin^2(a+y)}$$

$$\therefore \sin A \cos B - \cos A \sin B = \sin(A - B)$$

$$= \frac{\sin(a+y-y)}{\sin^2(a+y)}$$

$$= \frac{\sin a}{\sin^2(a+y)}$$

$$\therefore \frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)} = \frac{\sin^2(a+y)}{\sin a}.$$

4. (i) If  $x^y = e^{x-y}$ , prove that  $\frac{dy}{dx} = \frac{\log x}{(1+\log x)^2}$ .

Sol: Given:  $x^y = e^{x-y}$

Taking logarithms on both sides,

$$y \log x = (x-y) \log e \quad \therefore \log e = 1$$

$$\Rightarrow y \log x = x - y$$

$$\Rightarrow y \log x + y = x$$

$$\Rightarrow y (\log x + 1) = x$$

$$\Rightarrow y = \frac{x}{\log x + 1}$$

Differentiating w.r.t. using quotient rule,

$$\frac{d}{dx}(u/v) = \frac{v \cdot u' - u \cdot v'}{v^2}$$

$$\frac{dy}{dx} = \frac{(\log x + 1) \cdot 1 - x \cdot \left(\frac{1}{x}\right)}{(\log x + 1)^2}$$

$$= \frac{\log x + 1 - 1}{(\log x + 1)^2}$$

$$\therefore \frac{dy}{dx} = \frac{\log x}{(\log x + 1)^2}.$$

(ii). If  $y = x^y$ , show that

$$\frac{dy}{dx} = \frac{y^2}{x(1-y \log x)} = \frac{y^2}{x(1-\log y)}.$$

Sol: Given :  $y = x^y$  taking logarithms on both sides,

$$\log y = y \log x$$

Differentiating w.r.t. x,

$$\frac{1}{y} \frac{dy}{dx} = y \cdot \frac{1}{x} + \log x \cdot \frac{dy}{dx}$$

$$\frac{dy}{dx} \left[ \frac{1}{y} - \log x \right] = \frac{y}{x}$$

$$\frac{dy}{dx} \left[ \frac{1-y \log x}{y} \right] = \frac{y}{x}$$

$$\therefore \frac{dy}{dx} = \frac{y^2}{x(1-y \log x)} = \frac{y^2}{x(1-\log y)}$$

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(iii). If  $x^y = y^x$ , then  $\frac{dy}{dx} = \frac{y(x \log y - y)}{x(y \log x - x)}$ .

**Sol:** Given that  $x^y = y^x$   
taking logarithms on both sides,

$$y \cdot \log x = x \log y$$

differentiating w.r.t x,

$$\frac{dy}{dx} \cdot \log x + y \cdot \frac{1}{x} = 1 \cdot \log y + x \cdot \frac{1}{y} \frac{dy}{dx}$$

$$\frac{dy}{dx} \left[ \log x - \frac{x}{y} \right] = \log y - \frac{y}{x}$$

$$\frac{dy}{dx} \left[ \frac{y \log x - x}{y} \right] = \frac{x \log y - y}{x}$$

$$\frac{dy}{dx} = \frac{y(x \log y - y)}{x(y \log x - x)}.$$

(iv). If  $x^{\log y} = \log x$ , then prove that

$$\frac{dy}{dx} = \frac{y}{x} \left( \frac{1 - \log x \log y}{(\log x)^2} \right).$$

**Sol:** Given that  $x^{\log y} = \log x$   
taking logarithms on both sides,  
 $\log y \cdot \log x = \log \log x$   
differentiating w.r.t. x, we get

$$\frac{1}{y} \frac{dy}{dx} \log x + \log y \cdot \frac{1}{x} = \frac{1}{\log x} \cdot \frac{1}{x}$$

$$\frac{1}{y} \log x \cdot \frac{dy}{dx} = \frac{1}{x \log x} - \frac{\log y}{x}$$

$$= \frac{1 - \log x \log y}{x \log x}$$

$$\therefore \frac{dy}{dx} = \frac{y}{x} \left[ \frac{1 - \log x \log y}{(\log x)^2} \right].$$

5. (i) If  $y = \tan^{-1} \left( \frac{2x}{1-x^2} \right) + \tan^{-1} \left( \frac{3x-x^3}{1-3x^2} \right) - \tan^{-1} \left( \frac{4x-4x^3}{1-6x^2+x^4} \right)$  then prove that  $\frac{dy}{dx} = \frac{1}{1+x^2}$ .

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**Sol:** Given:  $y = \tan^{-1} \left( \frac{2x}{1-x^2} \right) + \tan^{-1} \left( \frac{3x-x^3}{1-3x^2} \right) - \tan^{-1} \left( \frac{4x-4x^3}{1-6x^2+x^4} \right)$

$$1 \left( \frac{4x-4x^3}{1-6x^2+x^4} \right)$$

$$\text{Put } x = \tan \theta \Rightarrow \theta = \tan^{-1} x$$

$$y = \tan^{-1} \left( \frac{2 \tan \theta}{1-\tan^2 \theta} \right) + \tan^{-1} \left( \frac{3 \tan \theta - \tan^3 \theta}{1-3 \tan^2 \theta} \right)$$

$$- \tan^{-1} \left( \frac{4 \tan \theta - 4 \tan^3 \theta}{1-6 \tan^2 \theta + \tan^4 \theta} \right)$$

$$\Rightarrow y = \tan^{-1}(\tan 2\theta) + \tan^{-1}(\tan 3\theta) - \tan^{-1}(\tan 4\theta)$$

$$\Rightarrow y = 2\theta + 3\theta - 4\theta$$

$$\Rightarrow y = \theta$$

$$\Rightarrow y = \tan^{-1} x$$

Differentiating w.r.t. x,

$$\frac{dy}{dx} = \frac{1}{1+x^2}.$$

(ii). Differentiate  $f(x) = \tan^{-1} \left( \frac{2x}{1-x^2} \right)$  w.r.t.

$$g(x) = \sin^{-1} \left( \frac{2x}{1+x^2} \right).$$

**Sol:** Let  $y = f(x) = \tan^{-1} \left( \frac{2x}{1-x^2} \right)$

$$z = g(x) = \sin^{-1} \left( \frac{2x}{1+x^2} \right)$$

$$\text{Put } x = \tan \theta \Rightarrow \theta = \tan^{-1} x$$

$$\Rightarrow y = \tan^{-1} \left( \frac{2 \tan \theta}{1-\tan^2 \theta} \right) \quad z = \sin^{-1} \left( \frac{2 \tan \theta}{1+\tan^2 \theta} \right)$$

$$\Rightarrow y = \tan^{-1}(\tan 2\theta) \quad z = \sin^{-1} \sin 2\theta$$

$$\Rightarrow y = 2\theta$$

$$\Rightarrow y = 2 \tan^{-1} x \quad z = 2 \tan^{-1} x$$

Now, derivative of  $f(x)$  w.r.t.  $g(x)$

$$= \frac{dy}{dz} = \frac{\frac{d}{dx}(y)}{\frac{d}{dx}(z)}$$

$$= \frac{\frac{d}{dx}(2 \tan^{-1} x)}{\frac{d}{dx}(2 \tan^{-1} x)} = \frac{\left( \frac{2}{1+x^2} \right)}{\left( \frac{2}{1+x^2} \right)} = 1.$$

(iii). Differentiate  $f(x)$  w.r.t.  $g(x)$  if

$$f(x) = \sec^{-1} \left( \frac{1}{2x^2 - 1} \right), g(x) = \sqrt{1-x^2}.$$

Sol: Let  $y = f(x) = \sec^{-1} \left( \frac{1}{2x^2 - 1} \right)$ ,  $z = g(x) = \sqrt{1-x^2}$

$$\text{Put } x = \cos \theta \Rightarrow \theta = \cos^{-1} x$$

$$y = \sec^{-1} \left( \frac{1}{2\cos^2 \theta - 1} \right)$$

$$\Rightarrow y = \sec^{-1} \left( \frac{1}{\cos 2\theta} \right)$$

$$\Rightarrow y = \sec^{-1} \sec 2\theta$$

$$\Rightarrow y = 2\theta$$

$$\Rightarrow y = 2 \cos^{-1} x, z = \sqrt{1-x^2}$$

$\therefore$  Derivative of  $f(x)$  w.r.t.  $g(x)$

$$= \frac{dy}{dz} = \frac{\frac{d}{dx}(y)}{\frac{d}{dx}(z)}$$

$$= \frac{\frac{d}{dx}(2 \cos^{-1} x)}{\frac{d}{dx}(\sqrt{1-x^2})}$$

$$= \frac{\frac{-2}{\sqrt{1-x^2}}}{\frac{1}{2\sqrt{1-x^2}}(-2x)}$$

$$= \frac{2}{x}.$$

(iv). Find the derivative of  $f(x) = \tan^{-1}$

$$\left( \frac{\sqrt{1+x^2} - 1}{x} \right) \text{ w.r.t. } g(x) = \tan^{-1} x.$$

Sol: Let  $y = f(x) = \tan^{-1} \frac{\sqrt{1+x^2} - 1}{x}$ ,  $z = g(x) = \tan^{-1} x$ .

$$\text{Put } x = \tan \theta \Rightarrow \theta = \tan^{-1} x$$

$$\Rightarrow y = \tan^{-1} \left[ \frac{\sqrt{1+x^2} - 1}{x} \right]$$

$$= \tan^{-1} \left[ \frac{\sqrt{1+\tan^2 \theta} - 1}{\tan \theta} \right]$$

$$= \tan^{-1} \left[ \frac{\sec \theta - 1}{\tan \theta} \right]$$

$$= \tan^{-1} \left[ \frac{\frac{1}{\cos \theta} - 1}{\frac{\sin \theta}{\cos \theta}} \right]$$

$$= \tan^{-1} \left[ \frac{1 - \cos \theta}{\sin \theta} \right]$$

$$= \tan^{-1} \left[ \frac{2 \sin^2 \left( \frac{\theta}{2} \right)}{2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}} \right]$$

$$= \tan^{-1} \tan \frac{\theta}{2} = \frac{\theta}{2}$$

$$y = \frac{1}{2} \tan^{-1} x, z = \tan^{-1} x$$

$\therefore$  Derivative of  $f(x)$  w.r.t.  $g(x)$

$$= \frac{dy}{dz} = \frac{\frac{d}{dx}(y)}{\frac{d}{dx}(z)}$$

$$= \frac{\frac{d}{dx} \left( \frac{1}{2} \tan^{-1} x \right)}{\frac{d}{dx} (\tan^{-1} x)}$$

$$= \frac{\frac{1}{2} \cdot \frac{1}{1+x^2}}{\frac{1}{1+x^2}} = \frac{1}{2}.$$

6.(i) If  $x = 3 \cos t - 2 \cos^3 t$ ,  $y = 3 \sin t - 2 \sin^3 t$ , then

find  $\frac{dy}{dx}$ .

Sol: Given that  $x = 3 \cos t - 2 \cos^3 t$ ,  $y = 3 \sin t - 2 \sin^3 t$   
By parametric differentiation,

$$\frac{dy}{dx} = \frac{\left( \frac{dy}{dt} \right)}{\left( \frac{dx}{dt} \right)}$$

$$= \frac{\frac{d}{dt}(3 \sin t - 2 \sin^3 t)}{\frac{d}{dt}(3 \sin t - 2 \cos^3 t)}$$

$$= \frac{3 \cos t - 2(3 \sin^2 t) \frac{d}{dt}(\sin t)}{3(-\sin t) - 2(3 \cos^2 t) \frac{d}{dt}(\cos t)}$$

$$= \frac{3 \cos t - 6 \sin^2 t (\cos t)}{-3 \sin t - 6 \cos^2 t (-\sin t)}$$

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$$\begin{aligned}
 &= \frac{3\cos t - 6\sin^2 t \cos t}{6\sin t \cos^2 t - 3\sin t} \\
 &= \frac{3\cos t(1 - 2\sin^2 t)}{3\sin t(2\cos^2 t - 1)} \\
 &= \frac{\cot t \cancel{\cos 2t}}{\cancel{\cot 2t}} \\
 &= \cot t.
 \end{aligned}$$

(ii).  $\frac{dy}{dx} = ? ; x = a(\cos t + t \sin t), y = a(\sin t - t \cos t)$ .

**Sol:** Given that  $x = a(\cos t + t \sin t)$ ,  $y = a(\sin t - t \cos t)$   
since  $x$  and  $y$  are functions of  $t$ , by parametric differentiation,

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} \\
 &= \frac{\frac{d}{dt}[a(\sin t - t \cos t)]}{\frac{d}{dt}[a(\cos t + t \sin t)]} \\
 &= \frac{a[\cos t - \{t \cdot (-\sin t) + \cos t \cdot (1)\}]}{a[-\sin t + \{t \cdot \cos t + \sin t \cdot 1\}]} \\
 &= \frac{\cos t + t \sin t - \cos t}{-\sin t + t \cos t + \sin t} \\
 &= \frac{t \sin t}{t \cos t} \\
 &= \tan t.
 \end{aligned}$$

(iii). If  $x = a \left[ \cos t + \log \left( \tan \frac{t}{2} \right) \right]$ ,  $y = a \sin t$ ,

find  $\frac{dy}{dx}$ .

**Sol:** Given :  $x = a \left[ \cos t + \log \left( \tan \frac{t}{2} \right) \right]$ ,  $y = a \sin t$

Since  $x$  and  $y$  are functions of  $t$ , by parametric differentiation,

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{\frac{d}{dt}(a \sin t)}{\frac{d}{dt}\left\{a \left[ \cos t + \log \tan \frac{t}{2} \right]\right\}} \\
 &= \frac{a \cos t}{a \left[ -\sin t + \frac{1}{\tan \frac{t}{2}} \cdot \sec^2 \frac{t}{2} \cdot \frac{1}{2} \right]}
 \end{aligned}$$

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$$\begin{aligned}
 &\text{cost} \\
 &= \frac{\cos \frac{t}{2}}{\sin \frac{t}{2} \cdot \frac{1}{2 \cos^2 \frac{t}{2}}} - \sin t \\
 &\text{cost} \\
 &= \frac{1}{2 \sin \frac{t}{2} \cos \frac{t}{2}} - \sin t \\
 &\text{cost} \\
 &= \frac{1}{\sin t} - \sin t \\
 &= \frac{\text{cost}}{\left(\frac{1 - \sin^2 t}{\sin t}\right)} = \frac{\cos \sin t}{\cos^2 t} = \tan t.
 \end{aligned}$$

7. If  $x = a(t - \sin t)$ ,  $y = a(1 + \cos t)$ , find  $\frac{d^2y}{dx^2}$ .

**Sol:** Given that  $x = a(t - \sin t)$ ,  $y = a(1 + \cos t)$   
since  $x$  and  $y$  are functions of  $t$ ,  
by parametric differentiation,

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} \\
 &= \frac{\frac{d}{dt}[a(1 + \cos t)]}{\frac{d}{dt}[a(t - \sin t)]} \\
 &= \frac{a(-\sin t)}{a(1 - \cos t)} \\
 &= \frac{-2 \sin \frac{t}{2} \cos \frac{t}{2}}{2 \sin^2 \frac{t}{2}} \\
 &= -\cot \frac{t}{2}
 \end{aligned}$$

since  $\frac{dy}{dx}$  is a function of  $t$ , thus

$$\begin{aligned}
 \frac{d^2y}{dx^2} &= \frac{d}{dt} \left( \frac{dy}{dx} \right) \cdot \frac{dt}{dx} \\
 &= \frac{d}{dt} \left( -\cot \frac{t}{2} \right) \cdot \frac{1}{a(1 - \cos t)} \\
 &= \operatorname{cosec}^2 \frac{t}{2} \cdot \frac{1}{2} \cdot \frac{1}{a \cdot 2 \sin^2 \frac{t}{2}} \\
 &= \frac{1}{4a} \operatorname{cosec}^4 \left( \frac{t}{2} \right).
 \end{aligned}$$

**DIFFERENTIATION**

8. If  $y = ax^{n+1} + bx^n$ , then show that

$$x^2y'' = n(n+1)y.$$

**Sol:** Given  $y = ax^{n+1} + bx^n$   
differentiating w.r.t. x successively for 2 times,

$$y' = a(n+1)x^n + b(-n)x^{n-1}$$

$$y'' = a(n+1)n.x^{n-1} + b(-n)[- (n+1)]x^{n-2}$$

multiplying bothsides by  $x^2$ ,

$$\begin{aligned} x^2y'' &= n(n+1)[a \cdot x^{n-1} \cdot x^2 + b \cdot x^{n-2} \cdot x^2] \\ &= n(n+1)[a \cdot x^{n+1} + b \cdot x^n] \\ &= n(n+1)y. \end{aligned}$$

9. If  $ay^4 = (x+b)^5$  then show that,  $5yy'' = (y')^2$

$$\text{sol: } ay^4 = (x+b)^5 \Rightarrow y^4 = \frac{(x+b)^5}{a}$$

$$\Rightarrow y = \frac{(x+b)^{\frac{5}{4}}}{a^{\frac{1}{4}}}$$

diff w.r.t to x

$$y_1 = \frac{1}{a^{\frac{1}{4}}} \frac{5}{4} (x+b)^{\left(\frac{5}{4}-1\right)} = \frac{5}{4a^{\frac{1}{4}}} (x+b)^{\frac{1}{4}}$$

again diff w.r.t to x

$$\Rightarrow y_2 = \frac{5}{4a^{\frac{1}{4}}} \frac{1}{4} (x+b)^{\left(\frac{1}{4}-1\right)} = \frac{5}{16a^{\frac{1}{4}}} (x+b)^{-\frac{3}{4}}$$

$$\text{LHS} \Rightarrow 5yy'' = 5 \cdot \frac{(x+b)^{\frac{5}{4}}}{a^{\frac{1}{4}}} \cdot \frac{5}{16a^{\frac{1}{4}}} (x+b)^{-\frac{3}{4}}$$

$$\Rightarrow \frac{25}{16a^{\frac{1}{2}}} (x+b)^{\frac{3}{4}}$$

$$\Rightarrow \frac{5}{4a^{\frac{1}{4}}} (x+b)^{\frac{1}{4}} \Rightarrow (y')^2$$

10. If  $ax^2 + 2hxy + by^2 = 1$ , prove that

$$\frac{d^2y}{dx^2} = \frac{h^2 - ab}{(hx + by)^3}.$$

**Sol:**  $ax^2 + 2hxy + by^2 = 1$   
differentiating w.r.t. x,

$$a \cdot 2x + 2h \left[ x \cdot \frac{dy}{dx} + y \cdot 1 \right] + b \cdot 2y \frac{dy}{dx} = 0.$$

$$2 \frac{dy}{dx} (hx + by) = -2(ax + hy)$$

$$\frac{dy}{dx} = \frac{-(ax + hy)}{hx + by} \quad \frac{d}{dx}(u/v) = \frac{v \cdot u' - u \cdot v'}{v^2}$$

differentiating w.r.t. x,

$$\frac{d^2y}{dx^2} = \frac{-(hx + by) \left( a + h \frac{dy}{dx} \right) + (ax + hy) \left( h + b \frac{dy}{dx} \right)}{(hx + by)^2}$$

$$= \frac{-(hx + by) \left[ a - h \frac{(ax + hy)}{hx + by} \right] + (ax + hy) \left[ h - b \frac{(ax + hy)}{hx + by} \right]}{(hx + by)^2}$$

$$= \frac{-(hx + by)[ahx + aby - ahx - h^2y] + (ax + hy)[h^2x + bhy - abx - bhy]}{(hx + by)(hx + by)^2}$$

$$= \frac{(h^2 - ab)y(hx + by) + (h^2 - ab)x(ax + hy)}{(hx + by)^3}$$

$$= \frac{(h^2 - ab)[hxy + by^2 + ax^2 + hxy]}{(hx + by)^3}$$

$$= \frac{(h^2 - ab)[ax^2 + 2hxy + by^2]}{(hx + by)^3}$$

$$= \frac{(h^2 - ab)(1)}{(hx + by)^3} \quad \therefore ax^2 + 2hxy + by^2 = 1$$

$$\therefore \frac{d^2y}{dx^2} = \frac{h^2 - ab}{(hx + by)^3}.$$

- 1. The distance time formula for the motion of a particle along a straight line is  $s = t^3 - 9t^2 + 24t - 18$ . Find when and where the velocity is zero.**

**Sol:** Given :  $s = t^3 - 9t^2 + 24t - 18$   
differentiating w.r.t. 't',

$$\frac{ds}{dt} = 3t^2 - 9(2t) + 24$$

$$\frac{ds}{dt} = 3t^2 - 18t + 24$$

Now, velocity = 0

$$\Rightarrow 3t^2 - 18t + 24 = 0 \Rightarrow t^2 - 6t + 8 = 0$$

$$\Rightarrow (t - 2)(t - 4) = 0. \Rightarrow t = 2, 4.$$

$$\begin{aligned} \text{when } t = 2, \text{ distance } s &= 2^3 - 9(2^2) + 24(2) - 18 \\ &= 8 - 36 + 48 - 18 \\ &= 2 \text{ units.} \end{aligned}$$

$$\begin{aligned} \text{when } t = 4, s &= 4^3 - 9(4^2) + 24(4) - 18 \\ &= 64 - 144 + 96 - 18 \\ &= -2 \text{ units} \end{aligned}$$

so, the particle comes to rest at  $t = 2$  sec and 4sec and the particle is at a distance of 2units in either direction from the starting point.

- 2. A particle is moving in a straight line so that after  $t$  seconds its distance is  $s$  (in cms) from a fixed point on the line is given by  $s = f(t) = 8t + t^3$ . Find (i) the velocity at time  $t = 2$  sec (ii) the initial velocity (iii) acceleration at  $t = 2$  sec.**

**Sol:** The distance  $s$  and time  $t$  are connected by the relation

$$s = f(t) = 8t + t^3 \dots\dots\dots (1)$$

$$\therefore \text{velocity } v = \frac{ds}{dt} = 8 + 3t^2 \dots\dots\dots (2)$$

and the acceleration is given by

$$a = \frac{d^2s}{dt^2} = 6t$$

$$\text{i) The velocity at } t = 2 \text{ is } v_{(t=2)} = 8 + 3(4) = 20 \text{ cm/sec.}$$

$$\text{ii) The initial velocity } (t = 0) \text{ is } v_{(t=0)} = 8 \text{ cm/sec.}$$

$$\text{iii) The acceleration at } t = 2 \text{ is } a_{(t=2)} = 6(2) = 12 \text{ cm/sec}^2.$$

- 3. The displacement of a particle travelling in a straight line in  $t$  sec. is given by  $s = 45t + 11t^2 - t^3$ . Find the time when the particle come rest.**

**Sol:**  $s = 45t + 11t^2 - t^3$

$$v = \frac{ds}{dt} = 45 + 22t - 3t^2$$

Given the particle comes to rest

$$\Rightarrow v = 0$$

$$\Rightarrow 45 + 22t - 3t^2 = 0$$

$$\Rightarrow 3t^2 - 22t - 45 = 0$$

$$\Rightarrow 3t^2 - 27t + 5t - 45 = 0$$

$$\Rightarrow 3t(t - 9) + 5(t - 9) = 0$$

$$\Rightarrow (3t + 5)(t - 9) = 0$$

$$t = 9 \text{ (or) } -5/3 \quad (\text{But } t \geq 0)$$

$$\therefore t = 9 \text{ sec.}$$

- 4. A particle is moving along a line according to  $s = f(t) = 4t^3 - 3t^2 + 5t - 1$  where  $s$  is measured in meters and  $t$  is measured in seconds. Find the velocity and acceleration at time  $t$ . At what time the acceleration is zero.**

**Sol:** Given  $f(t) = 4t^3 - 3t^2 + 5t - 1$ ,  
The velocity at time  $t$  is

$$v = \frac{ds}{dt} = 12t^2 - 6t + 5$$

The acceleration at time  $t$  is

$$a = \frac{d^2s}{dt^2} = 24t - 6.$$

The acceleration is 0 if  $24t - 6 = 0$

$$\text{i.e., } t = \frac{1}{4}$$

The acceleration of the particle is zero at

$$t = \frac{1}{4} \text{ sec.}$$

5. A point P is moving on a curve  $y = 2x^2$ . The x-coordinate of P is increasing at the rate of 4 units per second. Find the rate at which the y-coordinate is increasing when the point is at (2, 8).

**Sol:** Given  $y = 2x^2$ ,  $\frac{dx}{dt} = 4$  units/sec

$$\text{At } P = (2, 8), \frac{dy}{dt} = ?$$

differentiating  $y = 2x^2$  w.r.t. 't',

$$\Rightarrow \frac{dy}{dt} = 2(2x) \frac{dx}{dt}$$

$$\Rightarrow \left( \frac{dy}{dt} \right)_{(2,8)} = 4(2)(4) = 32 \text{ units/sec}$$

6. The volume of a cube is increasing at a rate of 9 cubic centimeters per second. How fast is the surface area increasing when the length of the edge is 10 centimetres ?

**Sol:** Let  $x$  be the length of the edge of the cube,  $V$  be its volume and  $S$  be its surface area.

Then,  $V = x^3$  and  $S = 6x^2$ .

Given that rate of change of volume is 9 cm<sup>3</sup>/sec.

$$\text{Therefore, } \frac{dV}{dt} = 9 \text{ cm}^3/\text{sec.}$$

$$v = x^3$$

Now differentiating  $V$  w.r.t. t

$$\frac{dV}{dt} = 3x^2 \frac{dx}{dt}$$

$$\Rightarrow 9 = 3x^2 \frac{dx}{dt}$$

$$\Rightarrow \frac{dx}{dt} = \frac{3}{x^2}.$$

$$S = 6x^2$$

Differentiating  $S$  w.r.t. t

$$\frac{ds}{dt} = 12x \frac{dx}{dt}$$

$$= 12x \frac{3}{x^2}$$

$$= \frac{36}{x}$$

Given  $x = 10$  cm,

$$\frac{ds}{dt} = \frac{36}{10} = 3.6 \text{ cm}^2/\text{sec}$$

7. The radius of circle increasing at the rate of 0.7 cm/sec. What is the rate of increasing of its circumference.

**Sol:** Let 'r' be the radius and 'S' be the circumference of the circle.

$$S = 2\pi r, \frac{dr}{dt} = 0.7 \text{ cm/sec.}$$

Differentiating w.r.t. t

$$\begin{aligned} \frac{dS}{dt} &= 2\pi \cdot \frac{dr}{dt} \\ &= 2\pi(0.7) \\ &= 1.4\pi \text{ cm/sec.} \end{aligned}$$

8. A stone is dropped into a quiet lake and ripples move in circles at the speed of 5cm/sec. At the instant when the radius of a circular ripple is 8 cm., how fast is the enclosed area increases?

**Sol:** Let 'r' be the radius and 'A' be the area of circle.

$$\text{given } \frac{dr}{dt} = 5 \text{ cm/sec}$$

$$A = \pi r^2$$

Differentiating w.r.t. t

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

$$\text{Given } r = 8 \text{ cm}$$

$$\begin{aligned} \frac{dA}{dt} &= 2\pi(8)(5) \\ &= 80\pi \text{ sq.cm/sec} \end{aligned}$$

9. The radius of an air bubble is increasing at the rate of  $\frac{1}{2}$  cm/sec. At what rate is the volume of the bubble increasing when the radius is 1 cm?

**Sol:** Let 'V' be the volume and 'r' be the radius of bubble.

$$\frac{dr}{dt} = \frac{1}{2} \text{ cm / sec}$$

$$V = \frac{4}{3} \pi r^3$$

Differentiating  $V$  w.r.t. t

$$\frac{dV}{dt} = \frac{4}{3} \pi \cdot 3r^2 \frac{dr}{dt}$$

$$= 4\pi(1)^2 \frac{1}{2}$$

$$= 2\pi \text{ c.c/sec}$$

10. A balloon, which always remains spherical on inflation, is being inflated by pumping in 900 cubic centimeters of gas per second. Find the rate at which the radius of balloon increase when the radius is 15 cm.

**Sol:** Let 'r' be the radius, 'V' be the volume of sphere.

$$\therefore \frac{dV}{dt} = 900 \text{ cc/sec}$$

V = volume of sphere

$$\Rightarrow V = \frac{4}{3} \pi r^3$$

$$\frac{dV}{dt} = \frac{4}{3} \pi \cdot 3r^2 \frac{dr}{dt}$$

$$900 = 4\pi(15)^2 \cdot \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{900}{900\pi}$$

$$= \frac{1}{\pi} \text{ cm/sec.}$$

11. A container is in the shape of an inverted cone has height 8m and radius 6m at the top. If it is filled with water at the rate of  $2 \text{ m}^3/\text{minute}$ , how fast is the height of water changing when the level is 4m?

**Sol:** Let 'h' be the height, 'r' be the radius of cone  $\Delta PAB$ ,  $\Delta PCD$  are similar  $\Delta$ 's

$$\therefore \frac{8}{6} = \frac{h}{r}$$

$$\Rightarrow r = \frac{6h}{8} = \frac{3}{4}h \quad \dots \dots \dots (1)$$

Let 'V' be the volume of cone

$$\therefore V = \frac{1}{3} \pi r^2 h$$

$$V = \frac{\pi}{3} \left[ \frac{3}{4}h \right]^2 h$$

$$V = \frac{\pi}{3} \cdot \frac{9}{16} h^3$$

$$V = \frac{3\pi}{16} h^3$$

Differentiating w.r.t.t

$$\frac{dv}{dt} = \frac{3\pi}{16} \cdot 3h^2 \frac{dh}{dt}$$

Given  $h = 4\text{m}$ ,  $\frac{dv}{dt} = 2\text{m}^3/\text{min.}$

$$2 = \frac{9\pi}{16} \cdot (4)^2 \frac{dh}{dt}$$

$$\Rightarrow \frac{dh}{dt} = \frac{2}{9\pi} \text{ m / minute}$$

12. Suppose we have a rectangular aquarium with dimensions of length 8m, width 4m and height 3m. Suppose we are filling the tank with water at the rate of  $0.4 \text{ m}^3/\text{sec}$ . How fast is the height of water changing when the water level is 2.5m?

**Sol:** Let ' $\ell$ ' be the length, 'b' be the width and 'h' be the height of cuboid.

$$\therefore \ell = 8\text{m}$$

$$b = 4\text{m}$$

$$h = 3\text{m},$$

Let 'V' be the volume,

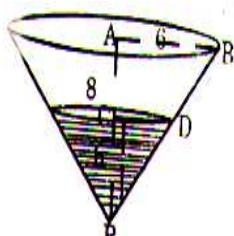
$$V = \ell \cdot b \cdot h, \quad \text{given } \frac{dV}{dt} = 0.4 \text{ m}^3/\text{sec}$$

Differentiating w.r.t.t

$$\frac{dV}{dt} = \ell \cdot b \cdot \frac{dh}{dt}$$

$$0.4 = 8 \times 4 \cdot \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{0.4}{32} = \frac{1}{80} \text{ m/sec.}$$



**1. Find the equations of the tangent and the normal to the curve  $y^4 = ax^3$  at  $(a, a)$ .**

**Sol:** Equation of the given curve is  $y^4 = ax^3$  P(a, a)  
Differentiating w.r.t.x.

$$4y^2 \frac{dy}{dx} = 3ax^2$$

$$\frac{dy}{dx} = \frac{3ax^2}{4y^2}$$

$$\left. \frac{dy}{dx} \right|_{(a,a)} = \frac{3a^3}{4a^3} = \frac{3}{4}$$

$$\text{slope of tangent at } P(m) = \frac{3}{4}$$

$\therefore$  Equation of the tangent at P is  $y - y_1 = m(x - x_1)$

$$y - a = \frac{3}{4}(x - a)$$

$$4y - 4a = 3x - 3a.$$

$$3x - 4y + a = 0.$$

$\therefore$  Equation of the normal at P is  $y - y_1 = \frac{-1}{m}(x - x_1)$

$$y - a = \frac{-4}{3}(x - a)$$

$$3y - 3a = -4x + 4a$$

$$4x + 3y - 7a = 0.$$

**2. Find the length of normal and subnormal at a point on the curve  $y = \frac{a}{2} \left( e^{\frac{x}{a}} + e^{-\frac{x}{a}} \right)$ .**

**Sol:** Equation of the curve is

$$y = \frac{a}{2} \left( e^{\frac{x}{a}} + e^{-\frac{x}{a}} \right)$$

$$y = a \left( \frac{e^{\frac{x}{a}} + e^{-\frac{x}{a}}}{2} \right)$$

$$y = \operatorname{acosh} \left( \frac{x}{a} \right)$$

$$\frac{dy}{dx} = a \sinh \left( \frac{x}{a} \right). \frac{1}{a} = \sinh \left( \frac{x}{a} \right)$$

$$\text{slope of the tangent at } P(x, y) = \sinh \left( \frac{x}{a} \right) = m$$

$$\therefore \text{Length of the normal} = \left| y \sqrt{1 + m^2} \right|$$

$$= \left| a \cosh \left( \frac{x}{a} \right) \sqrt{1 + \sinh^2 \left( \frac{x}{a} \right)} \right|$$

$$= \left| a \cosh \left( \frac{x}{a} \right) \cosh \left( \frac{x}{a} \right) \right|$$

$$= \left| a \cosh^2 \left( \frac{x}{a} \right) \right|$$

$$= a \cosh^2 \left( \frac{x}{a} \right)$$

$$\therefore \text{Length of the sub-normal} = |y_1 m|$$

$$= \left| a \cosh \left( \frac{x}{a} \right) \sinh \left( \frac{x}{a} \right) \right|$$

$$= \left| \frac{a}{2} \left( 2 \sinh \frac{x}{a} \cosh \frac{x}{a} \right) \right|$$

$$= \left| \frac{a}{2} \sinh \frac{2x}{a} \right|$$

**3. Find the equation of tangent and normal to the curve  $y = x^3 + 4x^2$  at (-1, 3).**

**Sol:** Equation of the curve is  $y = x^3 + 4x^2$  P(-1, 3)  
Differentiating w.r.t. x

$$\frac{dy}{dx} = 3x^2 + 8x$$

$$\left. \frac{dy}{dx} \right|_{(-1,3)} = 3(-1)^2 + 8(-1) = 3 - 8 = -5$$

$$m = -5.$$

$\therefore$  Equation of the tangent at P is

$$\Rightarrow y - 3 = -5(x + 1)$$

$$\Rightarrow y - 3 = -5x - 5$$

$$\Rightarrow 5x + y + 2 = 0$$

$\therefore$  Equation of the normal at P is

$$\Rightarrow y - 3 = \frac{1}{5}(x + 1)$$

$$\Rightarrow 5y - 15 = x + 1$$

$$\Rightarrow x - 5y + 16 = 0.$$

**4. Show that tangent at  $P(x_1, y_1)$  on the curve**

$$\sqrt{x} + \sqrt{y} = \sqrt{a} \text{ is } y \cdot y_1^{-1/2} + x \cdot x_1^{-1/2} = a^{1/2}.$$

**Sol:** Given equation of curve is  $\sqrt{x} + \sqrt{y} = \sqrt{a}$  ---- (1)

since  $P(x_1, y_1)$  lies on the curve,  $\sqrt{x_1} + \sqrt{y_1} = \sqrt{a}$  ----(2)

differentiating  $\sqrt{x} + \sqrt{y} = \sqrt{a}$  with respect to x,

$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}} \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-\sqrt{y}}{\sqrt{x}}$$

Slope of the tangent at P is  $\frac{-\sqrt{y_1}}{\sqrt{x_1}}$

Equation of tangent at P is

$$y - y_1 = \frac{-\sqrt{y_1}}{\sqrt{x_1}} (x - x_1)$$

$$\Rightarrow \frac{y}{\sqrt{y_1}} - \frac{y_1}{\sqrt{y_1}} = \frac{-x}{\sqrt{x_1}} + \frac{x_1}{\sqrt{x_1}}$$

$$\Rightarrow \frac{x}{\sqrt{x_1}} + \frac{y}{\sqrt{y_1}} = \sqrt{x_1} + \sqrt{y_1}$$

$$\Rightarrow x \cdot x_1^{-1/2} + y \cdot y_1^{-1/2} = \sqrt{a} \text{ from (2)}$$

$$\Rightarrow y \cdot y_1^{-1/2} + x \cdot x_1^{-1/2} = a^{1/2}.$$

**5. At a point  $P(x_1, y_1)$  on the curve  $x^3 + y^3 = 3axy$ , show that the equation of the tangent at P is**

$$(x_1^2 - ay_1)x + (y_1^2 - ax_1)y = ax_1y_1.$$

**Sol:** Given equation of the curve is  $x^3 + y^3 = 3axy$  ----(1)

since  $P(x_1, y_1)$  lies on the given curve, we get

$$x_1^3 + y_1^3 = 3ax_1y_1$$

differentiating (1) w.r.t. x,

$$3x^2 + 3y^2 \frac{dy}{dx} = 3a \left[ 1 \cdot y + x \cdot \frac{dy}{dx} \right]$$

$$x^2 + y^2 \frac{dy}{dx} = ay + ax \frac{dy}{dx}$$

$$\frac{dy}{dx} [y^2 - ax] = ay - x^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{ay - x^2}{y^2 - ax}$$

$$\text{Slope of the tangent at P} = \frac{ay_1 - x_1^2}{y_1^2 - ax_1}$$

Equation of tangent at P is

$$y - y_1 = \frac{-(x_1^2 - ay_1)}{y_1^2 - ax_1} (x - x_1)$$

$$\Rightarrow y(y_1^2 - ax_1) - y_1^3 + ax_1y_1 = -x(x_1^2 - ay_1) + x_1^3 - ax_1y_1$$

$$\Rightarrow (x_1^2 - ay_1)x + (y_1^2 - ax_1)y = x_1^3 + y_1^3 - 2ax_1y_1$$

=  $3ax_1y_1 - 2ax_1y_1$  from (2)

$$\therefore (x_1^2 - ay_1)x + (y_1^2 - ax_1)y = ax_1y_1$$

**6. Show that the tangent at any point  $\theta$  on the curve  $x = c \sec \theta, y = c \tan \theta$  is  $y \sin \theta = x - c \sec \theta$ .**

**Sol:** Given  $x = c \sec \theta$ ,

Differentiating w.r.t.  $\theta$ .

$$\frac{dx}{d\theta} = a \sec \theta \tan \theta.$$

Given by  $y = c \tan \theta$ .

Differentiating w.r.t.  $\theta$ .

$$\frac{dy}{d\theta} = a \sec^2 \theta.$$

$$\frac{dy}{dx} = \frac{\left( \frac{dy}{d\theta} \right)}{\left( \frac{dx}{d\theta} \right)} \frac{a \sec^2 \theta}{a \sec \theta \tan \theta} = \frac{\sec \theta}{\tan \theta} = \frac{1}{\sin \theta}$$

$$\text{slope of tangent (m)} = \frac{1}{\sin \theta}.$$

The equation of the tangent is

$$y - y_1 = m(x - x_1)$$

$$y - c \tan \theta = \frac{1}{\sin \theta} (x - c \sec \theta)$$

$$y \sin \theta - c \tan \theta \cdot \sin \theta = x - c \sec \theta.$$

$$y \sin \theta = x - c \sec \theta + c \tan \theta \sin \theta.$$

$$y \sin \theta = x - \frac{c}{\cos \theta} + c \frac{\sin \theta}{\cos \theta} \cdot \sin \theta.$$

$$y \sin \theta = x - \frac{c}{\cos \theta} (1 - \sin^2 \theta)$$

$$y \sin \theta = x - c \frac{\cos^2 \theta}{\cos \theta}$$

$$y \sin \theta = x - c \cos \theta.$$

7. Show that the curves  $6x^2 - 5x + 2y = 0$  and  $4x^2 + 8y^2 = 3$  touch each other at  $\left(\frac{1}{2}, \frac{1}{2}\right)$ .

**Sol:** Equation of the first curve is

$$6x^2 - 5x + 2y = 0$$

Differentiating w.r.t.x

$$12x - 5 + 2\frac{dy}{dx} = 0.$$

$$\frac{dy}{dx} = \frac{5 - 12x}{2}$$

$$\left(\frac{dy}{dx}\right)_{\left(\frac{1}{2}, \frac{1}{2}\right)} = \frac{5 - 12\left(\frac{1}{2}\right)}{2}$$

$$m_1 = \frac{5 - 6}{2} = \frac{-1}{2}$$

Equation of the second curve is  $4x^2 + 8y^2 = 3$ .

Differentiating w.r.t.x

$$8x + 16y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-8x}{16y} = -\frac{x}{2y}$$

$$\left(\frac{dy}{dx}\right)_{\left(\frac{1}{2}, \frac{1}{2}\right)} = \frac{\frac{-1}{2}}{2\left(\frac{1}{2}\right)}$$

$$m_2 = -\frac{1}{2}$$

$$m_1 = m_2$$

The given curves touch each other at  $\left(\frac{1}{2}, \frac{1}{2}\right)$ .

8. Show that the area of the triangle formed by the tangent at any point on the curve  $xy = c(c \neq 0)$  with the coordinate axes is constant.

**Sol:** Let  $P(x_1, y_1)$  be a point on the curve  $xy = c$ . Differentiating w.r.t.x

$$x\frac{dy}{dx} + y(1) = 0$$

$$\frac{dy}{dx} = -\frac{y}{x}$$

$$\left.\frac{dy}{dx}\right|_{P(x_1, y_1)} = -\frac{y_1}{x_1}$$

$$m = -\frac{y_1}{x_1}$$

Equation of the tangent at  $P(x_1, y_1)$  is

$$y - y_1 = \frac{-y_1}{x_1}(x - x_1)$$

$$x_1y - x_1y_1 = -y_1x + x_1y_1$$

$$y_1x + x_1y = 2x_1y_1$$

$$\frac{x}{2x_1} + \frac{y}{2y_1} = 1$$

The area of the triangle formed by the tangent and coordinate axes is

$$= \frac{1}{2}|(2x_1)(2y_1)|$$

$$= 2x_1y_1 \quad (\because x_1y_1 = c)$$

$$= 2c$$

$$= \text{a constant.}$$

9. Find the value of k, so that the length of the subnormal at any point on the curve  $y = a^{1-k} x^k$  is a constant.

**Sol:** Equation of the given curve is  $y = a^{1-k} x^k$ .

Let  $P(x_1, y_1)$  is on a curve.

Differentiating w.r.t. x

$$\left.\frac{dy}{dx}\right|_{P(x,y)} = a^{1-k} kx^{k-1}$$

$$m = ka^{1-k} x^{k-1}$$

$$\text{Length of the subnormal} = |y_1 m|$$

$$= |y_1 a^{1-k} x^{k-1}|$$

$$= |a^{1-k} x^k \cdot ka^{1-k} x^{k-1}|$$

$$= |ka^{2-2k} x^{2k-1}|$$

In order to make these value is a constant, we should have  $2k - 1 = 0$ .

$$\Rightarrow k = \frac{1}{2}.$$

10. Show that at any point  $(x, y)$  on the curve  $y = be^{x/a}$ , the length of the subtangent is constant and the length of the sub-normal

$$\text{is } \frac{y^2}{a}.$$

**Sol:** Equation of the curve is  $y = be^{x/a}$   
Let  $P(x_1, y_1)$  is on a curve  
Differentiating w.r.t.x

$$\frac{dy}{dx} = b \cdot e^{x/a} \cdot \frac{1}{a} = \frac{y}{a}$$

$$\left. \frac{dy}{dx} \right|_P = \frac{y_1}{a}$$

$$\text{Length of the sub-tangent} = \left| \frac{y_1}{m} \right|$$

$$= \left| \frac{y_1}{\left( \frac{y_1}{a} \right)} \right| = a = \text{constant}$$

$$\text{Length of the sub-normal} = |y_1 m|$$

$$= \left| y_1 \cdot \frac{y_1}{a} \right| = \frac{y_1^2}{a}.$$

11. Determine the intervals in which

$$f(x) = \frac{2}{(x-1)^2} + 18x \quad \forall x \in \mathbb{R} - \{0\} \text{ is strictly increasing and decreasing.}$$

**Sol:** Given that  $f(x) = \frac{2}{(x-1)^2} + 18x$ .

Differentiating it w.r.t. x

$$f'(x) = \frac{-1}{(x-1)^3} \cdot 2 + 18 \text{ and } f'(x) = 0$$

$$\Rightarrow \frac{2}{(x-1)^2} = 18$$

$$\Rightarrow (x-1)^2 = 1/9.$$

$$x-1 = \pm \frac{1}{3}$$

$$x-1 = \frac{1}{3}, x-1 = -\frac{1}{3}$$

$$x = \frac{1}{3} + 1, x = 1 - \frac{1}{3}$$

$$x = \frac{4}{3}, x = \frac{2}{3}.$$

The derivative of  $f(x)$  can be expressed as

$$f'(x) = \frac{18}{(x-1)^2} \cdot \left( x - \frac{2}{3} \right) \left( x - \frac{4}{3} \right)$$

Interval	Sign of $f'(x)$
$\left( -\infty, \frac{2}{3} \right)$	Positive
$\left( \frac{2}{3}, \frac{4}{3} \right)$	Negative
$\left( \frac{4}{3}, \infty \right)$	Positive

$\therefore$  The given function  $f(x)$  is strictly increasing

on  $\left( -\infty, \frac{2}{3} \right)$  and  $\left( \frac{4}{3}, \infty \right)$  and it is strictly

12. Find the tangent and normal to the curve  $y = 2e^{-x/3}$  at the point where the curve meets the y - axis.

**Sol:** Equation of the curve is  $y = 2e^{-x/3}$ .

The point of intersection of the curve and y-axis i.e.,  $x = 0 \Rightarrow y = 2$

$$\therefore P(0, 2)$$

Differentiating it w.r.t. x

$$\frac{dy}{dx} = 2e^{-x/3} \left( -\frac{1}{3} \right)$$

$$\left. \frac{dy}{dx} \right|_{P(0,2)} = \frac{-2}{3} e^{\frac{-0}{3}}$$

$$m = \frac{-2}{3} (1) = \frac{-2}{3}$$

$\therefore$  Equation of the tangent at P is

$$\Rightarrow y - 2 = \frac{2}{3} (x - 0)$$

$$\Rightarrow 3y - 6 = -2x$$

$$\Rightarrow 2x + 3y - 6 = 0$$

Equation of the normal at P is

$$\Rightarrow y - 2 = \frac{3}{2} (x - 0)$$

$$\Rightarrow 2y - 4 = 3x$$

$$\Rightarrow 3x - 2y + 4 = 0$$