



1. Find the equation of the circle passing through points (3, 4), (3, 2), and (1, 4)

Sol: Let A (3, 4), B (3, 2), and C (1, 4) Let the required eq'n of the circle be

$$S \equiv x^2 + y^2 + 2gx + 2fy + c = 0 \dots (*)$$

A (3, 4) lies on (*)

$$\Rightarrow (3)^2 + (4)^2 + 2g(3) + 2f(4) + c = 0$$

$$9 + 16 + 6g + 8f + c = 0$$

$$6g + 8f + c + 25 = 0 \dots \dots \dots (1)$$

B (3, 2) lies on (*)

$$\Rightarrow (3)^2 + (2)^2 + 2g(3) + 2f(2) + c = 0$$

$$9 + 4 + 6g + 4f + c = 0$$

$$6g + 4f + c + 13 = 0 \dots \dots \dots (2)$$

C (1, 4) lies on (*)

$$\Rightarrow (1)^2 + (4)^2 + 2g(1) + 2f(4) + c = 0$$

$$1 + 16 + 2g + 8f + c = 0$$

$$2g + 8f + c + 17 = 0 \dots \dots \dots (3)$$

Solving eq'n (1) & (2)

$$6g + 8f + c + 25 = 0$$

$$6g + 4f + c + 13 = 0$$

$$0 + 4f + 12 = 0 \quad \{ \div \text{ by } 4 \}$$

$$\Rightarrow f = -3 \dots \dots \dots (4)$$



Solving eq'n (2) & (3)

$$6g + 4f + c + 13 = 0$$

$$\underline{2g + 8f + c + 17 = 0}$$

$$\underline{4g - 4f - 4 = 0} \quad \{ \div \text{ by } 4 \}$$

$$\Rightarrow g - f - 1 = 0 \dots \dots \dots (5)$$

Solving eq'n (4) & (5)

$$\Rightarrow g - (-3) - 1 = 0$$

$$\Rightarrow g + 3 - 1 = 0$$

$$\Rightarrow g + 2 = 0$$

$$g = -2$$

Sub 'g' and 'f' values in eq'n (1)

$$\Rightarrow 6(-2) + 8(-3) + c + 25 = 0$$

$$\Rightarrow -12 - 24 + 25 + c = 0 \Rightarrow c = 11$$

sub the values of 'g', 'f' and 'c' in (*)

$$x^2 + y^2 + 2(-2)x + 2(-3)y + (11) = 0$$

$$\therefore x^2 + y^2 - 4x - 6y + 11 = 0$$



2. Show that the points $(1, 1)$, $(-6, 0)$, $(-2, 2)$ and $(-2, -8)$ are concyclic.

Sol: Let A $(1, 1)$, B $(-6, 0)$, C $(-2, 2)$ and D $(-2, -8)$

Let the required eq'n of the circle be

$$S \equiv x^2 + y^2 + 2gx + 2fy + c = 0 \dots (*)$$

A $(1, 1)$ lies on $(*)$

$$\Rightarrow (1)^2 + (1)^2 + 2g(1) + 2f(1) + c = 0$$

$$1 + 1 + 2g + 2f + c = 0$$

$$2g + 2f + c + 2 = 0 \dots \dots \dots (1)$$

B $(-6, 0)$ lies on $(*)$

$$\Rightarrow (-6)^2 + (0)^2 + 2g(-6) + 2f(0) + c = 0$$

$$36 + 0 - 12g + 0 + c = 0$$

$$-12g + 0 + c + 36 = 0 \dots \dots \dots (2)$$

C $(-2, 2)$ lies on $(*)$

$$\Rightarrow (-2)^2 + (2)^2 + 2g(-2) + 2f(4) + c = 0$$

$$4 + 4 - 4g + 4f + c = 0$$

$$-4g + 4f + c + 8 = 0 \dots \dots \dots (3)$$

Solving eq'n (1) & (2)

$$2g + 2f + c + 2 = 0$$

$$-12g + 0 + c + 36 = 0$$

$$\underline{14g + 2f - 34 = 0} \quad \{\div \text{ by } 2\}$$

$$\Rightarrow 7g + f - 17 = 0 \dots \dots \dots (4)$$



Solving eq'n (2) & (3)

$$-12g + 0 + c + 36 = 0$$

$$\underline{-4g + 4f + c + 8 = 0}$$

$$\underline{-8g - 4f + 28 = 0} \quad \{\div \text{ by } -4\}$$

$$\Rightarrow 2g + f - 7 = 0 \dots \dots \dots (5)$$

Solving eq'n (4) & (5)

$$7 \quad 1 \quad -17 \quad 7$$

$$2 \quad 1 \quad -7 \quad 2$$

$$(g, f) = \left[\frac{-7+17}{7-2}, \frac{-34+4}{7-2} \right] = \left[\frac{10}{5}, \frac{15}{5} \right] = (2, 3)$$

Sub 'g' and 'f' values in eq'n (1)

$$\Rightarrow 2(2) + 2(3) + c + 2 = 0$$

$$\Rightarrow 4 + 6 + 2 + c = 0 \Rightarrow c = -12$$

sub the values of 'g', 'f' and 'c' in $(*)$

$$x^2 + y^2 + 2(2)x + 2(3)y + (-12) = 0$$

$$\therefore x^2 + y^2 + 4x + 6y - 12 = 0$$

Since D $(-2, -8)$ also lies on $(*)$

$$\Rightarrow (-2)^2 + (-8)^2 + 4(-2) + 6(-8) - 12$$

$$= 4 + 64 - 8 - 48 - 12$$

$$= 68 - 68 = 0 \quad \therefore \text{Given points are concyclic.}$$



3. Show that the points (1, 2), (3, -4), (5, -6) and (19, 8) are concyclic.

Sol: Let A (1, 2), B (3, -4), C (5, -6) and D (19, 8)

Let the required eq'n of the circle be

$$S \equiv x^2 + y^2 + 2gx + 2fy + c = 0 \dots (*)$$

A (1, 2) lies on (*)

$$\Rightarrow (1)^2 + (2)^2 + 2g(1) + 2f(2) + c = 0$$

$$1 + 4 + 2g + 4f + c = 0$$

$$2g + 4f + c + 5 = 0 \dots \dots \dots (1)$$

B (3, -4) lies on (*)

$$\Rightarrow (3)^2 + (-4)^2 + 2g(3) + 2f(-4) + c = 0$$

$$9 + 16 + 6g - 8f + c = 0$$

$$6g - 8f + c + 25 = 0 \dots \dots \dots (2)$$

C (5, -6) lies on (*)

$$\Rightarrow (5)^2 + (-6)^2 + 2g(5) + 2f(-6) + c = 0$$

$$25 + 36 + 10g - 12f + c = 0$$

$$10g - 12f + c + 61 = 0 \dots \dots \dots (3)$$

Solving eq'n (1) & (2)

$$2g + 4f + c + 5 = 0$$

$$6g - 8f + c + 25 = 0$$

$$\underline{-4g + 12f - 20 = 0} \quad \{\div \text{ by } -4\}$$

$$\Rightarrow 1g - 3f + 5 = 0 \dots \dots \dots (4)$$

Solving eq'n (2) & (3)

$$6g - 8f + c + 25 = 0$$

$$\underline{10g - 12f + c + 61 = 0}$$

$$\underline{-4g + 4f - 36 = 0} \quad \{\div \text{ by } -4\}$$

$$\Rightarrow 1g - f + 9 = 0 \dots \dots \dots (5)$$

Solving eq'n (4) & (5)

$$\begin{array}{cccc} 1 & -3 & 5 & 1 \\ 1 & -1 & 9 & 1 \end{array}$$

$$\begin{array}{cccc} 1 & -3 & 5 & 1 \\ 1 & -1 & 9 & 1 \end{array}$$

$$(g, f) = \left[\frac{-27+5}{-1+3}, \frac{5-9}{-1+3} \right] = \left[\frac{-22}{2}, \frac{-4}{2} \right] = (-11, -2)$$

Sub 'g' and 'f' values in eq'n (1)

$$\Rightarrow 2(-11) + 4(-2) + c + 5 = 0$$

$$\Rightarrow -22 - 8 + 5 + c = 0 \Rightarrow c = 25$$

sub the values of 'g', 'f' and 'c' in (*)

$$x^2 + y^2 + 2(-11)x + 2(-2)y + (25) = 0$$

$$\therefore x^2 + y^2 - 22x - 4y + 25 = 0$$

Since D (19, 8) also lies on (*)

$$\Rightarrow (19)^2 + (8)^2 - 22(19) - 4(8) + 25 = 0$$

$$= 361 + 64 - 418 - 32 + 25 = 0$$

$450 - 450 = 0 \therefore$ Given points are concyclic.



4. If $(2, 0)$, $(0, 1)$, $(4, 5)$ and $(0, c)$ are concyclic, and then find the value of c .

Sol: Let A $(2, 0)$, B $(0, 1)$, C $(4, 5)$ and D $(0, c)$

Let the required eq'n of the circle be

$$S \equiv x^2 + y^2 + 2gx + 2fy + k = 0 \dots (*)$$

A $(2, 0)$ lies on $(*)$

$$\Rightarrow (2)^2 + (0)^2 + 2g(2) + 2f(0) + k = 0$$

$$4 + 0 + 4g + 0 + k = 0$$

$$4g + 0 + k + 4 = 0 \dots \dots \dots (1)$$

B $(0, 1)$ lies on $(*)$

$$\Rightarrow (0)^2 + (1)^2 + 2g(0) + 2f(1) + k = 0$$

$$0 + 1 + 0 + 2f + k = 0$$

$$0 + 2f + k + 1 = 0 \dots \dots \dots (2)$$

C $(4, 5)$ lies on $(*)$

$$\Rightarrow (4)^2 + (5)^2 + 2g(4) + 2f(5) + c = 0$$

$$16 + 25 + 8g + 10f + k = 0$$

$$8g + 10f + k + 41 = 0 \dots \dots \dots (3)$$

Solving eq'n (1) & (2)

$$4g + 0 + k + 4 = 0$$

$$0 + 2f + k + 1 = 0$$

$$\underline{4g - 2f + 3 = 0}$$

$$\Rightarrow \underline{\underline{4g - 2f + 3 = 0}} \dots \dots \dots (4)$$



Solving eq'n (2) & (3)

$$0 + 2f + k + 1 = 0$$

$$\underline{8g + 10f + k + 41 = 0}$$

$$\underline{-8g - 8f - 40 = 0} \quad \{ \text{div by } -8 \} \Rightarrow g + f + 5 = 0 \dots \dots \dots (5)$$

Solving eq'n (4) & (5)

$$\begin{array}{cccc} 4 & -2 & 3 & 4 \\ 1 & 1 & 5 & 1 \end{array}$$

$$(g, f) = \left[\frac{-10-3}{4+2}, \frac{3-20}{4+2} \right] = \left[-\frac{13}{6}, -\frac{17}{6} \right]$$

Sub 'g' and 'f' values in eq'n (1) $\Rightarrow 4(-\frac{13}{6}) + k + 4 = 0$

$$\Rightarrow k = \frac{26}{3} - 4 = \frac{26-12}{3} = \frac{14}{3}$$

sub the values of 'g', 'f' and 'k' in $(*)$

$$x^2 + y^2 + 2\left(-\frac{13}{6}\right)x + 2\left(-\frac{17}{6}\right)y + \left(\frac{14}{3}\right) = 0$$

$$x^2 + y^2 - \left(\frac{13}{3}\right)x - \left(\frac{17}{3}\right)y + \frac{14}{3} = 0$$

Since D $(0, c)$ also lies on $(*)$

$$\Rightarrow (0)^2 + (c)^2 - \left(\frac{13}{3}\right)(0) - \left(\frac{17}{3}\right)(c) + \frac{14}{3} = 0$$

$$\Rightarrow (c)^2 - \left(\frac{17}{3}\right)(c) + \frac{14}{3} = 0 \Rightarrow 3c^2 - 17c + 14 = 0$$

$$\Rightarrow 3c^2 - 3c - 14c + 14 = 0$$

$$\Rightarrow 3c[c-1] - 14[c-1] = 0$$

$$\Rightarrow [c-1](3c-14) = 0$$

$$\Rightarrow [c-1] = 0 \text{ or } (3c-14) = 0 \quad c = 1 \text{ or } c = \frac{14}{3}$$



5. Find the equation of the circle whose centre lies on X-axis and passing through $(-2, 3), (4, 5)$

Sol: Let A $(-2, 3)$, B $(4, 5)$

Let the required eq'n of the circle be

$$S \equiv x^2 + y^2 + 2gx + 2fy + c = 0 \dots (*)$$

A $(-2, 3)$ lies on $(*)$

$$\Rightarrow (-2)^2 + (3)^2 + 2g(-2) + 2f(3) + c = 0$$

$$4 + 9 - 4g + 6f + c = 0$$

$$-4g + 6f + c + 13 = 0 \dots \dots \dots (1)$$

B $(4, 5)$ lies on $(*)$

$$\Rightarrow (4)^2 + (5)^2 + 2g(4) + 2f(5) + c = 0$$

$$16 + 25 + 8g + 10f + c = 0$$

$$8g + 10f + c + 41 = 0 \dots \dots \dots (2)$$

Solving eq'n (1) & (2)

$$-4g + 6f + c + 13 = 0$$

$$\underline{8g + 10f + c + 41 = 0}$$

$$\underline{-12g - 4f - 28 = 0} \quad \{\div \text{ by } -4\}$$

$$\Rightarrow 3g + f + 7 = 0 \dots \dots \dots (3)$$



Given centre $(-g, -f)$ lies on X – axis

\Rightarrow Y – coordinate is zero i.e., $f = 0$

sub $f = 0$ in eq'n (3)

$$\Rightarrow 3g + 0 + 7 = 0$$

$$\Rightarrow 3g = -7$$

$$\Rightarrow g = -\frac{7}{3}$$

Sub 'g' and 'f' values in eq'n (1)

$$\Rightarrow -4\left(-\frac{7}{3}\right) + 6(0) + c + 13 = 0$$

$$\Rightarrow \frac{28}{3} + 13 + c = 0 \Rightarrow c = \frac{-28-39}{3} = -\frac{67}{3}$$

sub the values of 'g', 'f' and 'c' in $(*)$

$$x^2 + y^2 + 2\left(-\frac{7}{3}\right)x + 2(0)y + \left(-\frac{67}{3}\right) = 0$$

$$\therefore 3(x^2 + y^2) - 14x - 67 = 0$$



6. Find the equation of the circle passing through points (4, 1), (6, 5) and whose centre lies on the line $4x + 3y - 24 = 0$.

Sol: Let A (4, 1), B (6, 5)

Let the required eq'n of the circle be

$$S \equiv x^2 + y^2 + 2gx + 2fy + c = 0 \dots \dots (*)$$

A (4, 1) lies on (*)

$$\Rightarrow (4)^2 + (1)^2 + 2g(4) + 2f(1) + c = 0$$

$$16 + 1 + 8g + 2f + c = 0$$

$$8g + 2f + c + 17 = 0 \dots \dots \dots (1)$$

B (6, 5) lies on (*)

$$\Rightarrow (6)^2 + (5)^2 + 2g(6) + 2f(5) + c = 0$$

$$36 + 25 + 12g + 10f + c = 0$$

$$12g + 10f + c + 61 = 0 \dots \dots \dots (2)$$

Solving eq'n (1) & (2)

$$8g + 2f + c + 17 = 0$$

$$\underline{12g + 10f + c + 61 = 0}$$

$$\underline{-4g - 8f - 44 = 0} \quad \{\div \text{ by } -4\}$$

$$\Rightarrow \underline{1g + 2f + 11 = 0} \dots \dots \dots (3)$$

Given centre $(-g, -f)$ lies on $4x + 3y - 24 = 0$

$$\Rightarrow 4(-g) + 3(-f) - 24 = 0$$

$$\Rightarrow \underline{4g + 3f + 24 = 0} \dots \dots \dots (4)$$



Solving eq'n (3) & (4)

$$\begin{array}{cccc} 1 & 2 & 11 & 1 \\ 4 & 3 & 24 & 4 \end{array}$$

$$(g, f) = \left[\frac{48-33}{3-8}, \frac{44-24}{3-8} \right] = \left[-\frac{15}{5}, -\frac{20}{5} \right] \\ = (-3, -4)$$

Sub 'g' and 'f' values in eq'n (1)

$$\Rightarrow 8(-3) + 2(-4) + c + 17 = 0$$

$$\Rightarrow -24 - 8 + 17 + c = 0 \Rightarrow c = 15$$

sub the values of 'g', 'f' and 'c' in (*)

$$x^2 + y^2 + 2(-3)x + 2(-4)y + (15) = 0$$

$$\therefore x^2 + y^2 - 6x - 8y + 15 = 0.$$



1. Find the equation of the circle passing through points (3, 4), (3, 2), and (1, 4)

Sol: Let A (3, 4), B (3, 2), and C (1, 4)

The required eq'n of the circle passing through A (x_1, y_1) , B (x_2, y_2) is

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + k \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

$$\Rightarrow (x - 3)(x - 3) + (y - 4)(y - 2) + k \begin{vmatrix} x & y & 1 \\ 3 & 4 & 1 \\ 3 & 2 & 1 \end{vmatrix} = 0$$

$$\Rightarrow x^2 - 3x - 3x + 9 + y^2 - 2y - 4y + 8 + k[x(4 - 2) - y(3 - 3)] + 16 - 12 = 0$$

$$\Rightarrow x^2 - 6x + 9 + y^2 - 6y + 8 + k[x(2) - y(0) + 1(-6)] = 0$$

$$\Rightarrow x^2 + y^2 - 6x - 6y + 17 + k[2x - 6] = 0 \dots \dots \dots (1)$$

C(1, 4) passing through ..(1)

$$\Rightarrow 1^2 + 4^2 - 6(1) - 6(4) + 17 + k[2(1) - 6] = 0$$

$$\Rightarrow 34 - 30 - 4k = 0$$

$$4 = 4k \quad \Rightarrow k = 1 \text{ sub } k = 1 \text{ in (1)}$$

$$\Rightarrow x^2 + y^2 - 6x - 6y + 17 + 1[2x - 6] = 0$$

$$\Rightarrow x^2 + y^2 - 4x - 6y + 11 = 0$$



2. Show that the points (1, 1), (-6, 0), (-2, 2) and (-2, -8) are concyclic.

Sol: Let A (1, 1), B (-6, 0), C (-2, 2) and D (-2, -8)

The required eq'n of the circle passing through A (x_1, y_1) , B (x_2, y_2) is

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + k \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

$$\Rightarrow (x - 1)(x + 6) + (y - 1)(y - 0) + k \begin{vmatrix} x & y & 1 \\ 1 & 1 & 1 \\ -6 & 0 & 1 \end{vmatrix} = 0$$

$$\Rightarrow x^2 + 6x - 1x - 6 + y^2 - 1y + k[x(1 - 0) - y(1 + 6) + 1(0 + 6)] = 0$$

$$\Rightarrow x^2 + 5x - 6 + y^2 - y + k[x(1) - y(7) + 1(6)] = 0$$

$$\Rightarrow x^2 + y^2 + 5x - y - 6 + k[x - 7y + 6] = 0 \dots \dots \dots (1)$$

C(-2, 2) passing through ..(1)

$$\Rightarrow (-2)^2 + 2^2 + 5(-2) - 1(2) - 6 + k[-2 - 7(2) + 6] = 0$$

$$\Rightarrow 4 + 4 - 10 - 2 - 6 + k[-16 + 6] = 0$$

$$-10 = 10k \quad \Rightarrow k = -1, \text{ sub } k = -1 \text{ in (1)}$$

$$\Rightarrow x^2 + y^2 + 5x - y - 6 - 1[x - 7y + 6] = 0$$

$$\Rightarrow x^2 + y^2 + 4x + 6y - 12 = 0 \dots \dots \dots (2)$$

verify D(-2, -8)

$$\Rightarrow (-2)^2 + (-8)^2 + 4(-2) + 6(-8) - 12$$

$$\Rightarrow 4 + 64 - 8 - 48 - 12$$

$$\Rightarrow 68 - 68 = 0 \quad \therefore \text{Given four points are concyclic.}$$



3. Show that the points $(1, 2)$, $(3, -4)$, $(5, -6)$ and $(19, 8)$ are concyclic.

Sol: Let A $(1, 2)$, B $(3, -4)$, C $(5, -6)$ and D $(19, 8)$

The required eq'n of the circle passing through A (x_1, y_1) , B (x_2, y_2) is

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + k \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

$$\Rightarrow (x - 1)(x - 3) + (y - 2)(y + 4) + k \begin{vmatrix} x & y & 1 \\ 1 & 2 & 1 \\ 3 & -4 & 1 \end{vmatrix} = 0$$

$$\Rightarrow x^2 - 3x - 1x + 3 + y^2 + 4y - 2y - 8 + k[x(2+4) - y(1-3)] +$$

$$\Rightarrow x^2 - 4x + y^2 + 2y - 5 + k[x(6) - y(-2) + 1(-10)] = 0$$

$$\Rightarrow x^2 + y^2 - 4x + 2y - 5 + k[6x + 2y - 10] = 0 \dots \dots \dots (1)$$

C $(5, -6)$ passing through ..(1)

$$\Rightarrow (5)^2 + (-6)^2 - 4(5) + 2(-6) - 5 + k[6(5) + 2(-6) - 10] = 0$$

$$\Rightarrow 25 + 36 - 20 - 12 - 5 + k[30 - 12 - 10] = 0$$

$$24 = -8k \Rightarrow k = -3, \text{ sub } k = -3 \text{ in (1)}$$

$$\Rightarrow x^2 + y^2 - 4x + 2y - 5 - 3[6x + 2y - 10] = 0$$

$$\Rightarrow x^2 + y^2 - 4x + 2y - 5 - 18x - 6y + 30 = 0$$

$$\Rightarrow x^2 + y^2 - 22x - 4y + 25 = 0 \dots \dots \dots (2)$$

$$\text{verify } D(19, 8) \Rightarrow (19)^2 + (8)^2 - 22(19) - 4(8) + 25$$

$$\Rightarrow 361 + 64 - 418 - 32 + 25$$

$$\Rightarrow 450 - 450 = 0 \therefore \text{Given four points are concyclic.}$$



4. If $(2, 0)$, $(0, 1)$, $(4, 5)$ and $(0, c)$ are concyclic, and then find the value of c .

Sol: Let A $(2, 0)$, B $(0, 1)$, C $(4, 5)$ and D $(0, c)$

The required eq'n of the circle passing through A (x_1, y_1) , B (x_2, y_2) is

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + k \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

$$\Rightarrow (x - 2)(x - 0) + (y - 0)(y - 1) + k \begin{vmatrix} x & y & 1 \\ 2 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} = 0$$

$$\Rightarrow x^2 - 2x + y^2 - y + k[x(0-1) - y(2-0) + 1(2-0)] = 0$$

$$\Rightarrow x^2 - 2x + y^2 - y + k[x(-1) - y(2) + 1(2)] = 0$$

$$\Rightarrow x^2 + y^2 - 2x - y + k[-x - 2y + 2] = 0 \dots \dots \dots (1)$$

C $(4, 5)$ passing through ..(1)

$$\Rightarrow (4)^2 + (5)^2 - 2(4) - 1(5) + k[-1(4) - 2(5) + 2] = 0$$

$$\Rightarrow 16 + 25 - 8 - 5 + k[-4 - 10 + 2] = 0$$

$$28 = 12k \Rightarrow k = \frac{7}{3}, \text{ sub } k = \frac{7}{3} \text{ in (1)}$$

$$\Rightarrow (x^2 + y^2 - 2x - y) + \frac{7}{3}[-x - 2y + 2] = 0$$

$$\Rightarrow 3(x^2 + y^2 - 2x - y) - 7x - 14y + 14 = 0 = 0$$

$$\Rightarrow 3x^2 + 3y^2 - 13x - 17y + 14 = 0 \dots \dots \dots (2)$$

since given four points are concyclic.

D $(0, c)$ lies on (2)

$$\Rightarrow 3(0)^2 + 3(c)^2 - 13(0) - 17(c) + 14 = 0$$

$$\begin{aligned} \Rightarrow 3c^2 - 17c + 14 &= 0 \\ \Rightarrow 3c^2 - 3c - 14c + 14 &= 0 \\ \Rightarrow 3c[c - 1] - 14[c - 1] &= 0 \\ \Rightarrow [c - 1](3c - 14) &= 0 \\ \Rightarrow [c - 1] = 0 \text{ or } (3c - 14) &= 0 \\ c = 1 \text{ or } c &= \frac{14}{3} \end{aligned}$$



5. Find the equation of the circle whose centre lies on X-axis and passing through (-2, 3), (4, 5)

Sol: Let A (-2, 3), B (4, 5)

The required eq'n of the circle passing through A (x_1, y_1) , B (x_2, y_2) is

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + k \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

$$\Rightarrow (x + 2)(x - 4) + (y - 3)(y - 5) + k \begin{vmatrix} x & y & 1 \\ -2 & 3 & 1 \\ 4 & 5 & 1 \end{vmatrix} = 0$$

$$\Rightarrow x^2 - 4x + 2x - 8 + y^2 - 5y - 3y + 15 + k[x(3 - 5) - y(-2 - 4)] + 1 - 10 - 12 = 0$$

$$\Rightarrow x^2 - 2x + y^2 - 8y + 7 + k[x(-2) - y(-6) + 1(-22)] = 0$$

$$\Rightarrow x^2 + y^2 - 2x - 8y + 7 + k[-2x + 6y - 22] = 0 \dots \dots \dots (1)$$

$$x^2 + y^2 - 2x - 2kx - 8y + 6ky + 7 - 22k = 0$$

$$x^2 + y^2 - 2x[1 + k] - 2y[4 - 3k] + 7 - 22k = 0$$

$$\text{centre } (-g, -f) = [1 + k, 4 - 3k]$$

$$\text{Given centre lies on } x - \text{axis} \Rightarrow y = 0$$

$$\Rightarrow 4 - 3k = 0 \Rightarrow 3k = 4 \Rightarrow k = \frac{4}{3}$$

$$\text{sub } k = \frac{4}{3} \text{ in (1)}$$

$$\Rightarrow (x^2 + y^2 - 2x - 8y + 7) + \frac{4}{3}[-2x + 6y - 22] = 0$$

$$\Rightarrow 3(x^2 + y^2 - 2x - 8y + 7) - 8x + 24y - 88 = 0$$

$$\Rightarrow 3x^2 + 3y^2 - 6x - 24y + 21 - 8x + 24y - 88 = 0$$

$$\Rightarrow 3x^2 + 3y^2 - 14x - 67 = 0 \dots \dots \dots (2)$$



6. find the equation of the circle passes through the points (4, 1), (6, 5) and whose centre lies on $4x + 3y - 24 = 0$.

The required eq'n of the circle passing through A (x_1, y_1) , B (x_2, y_2) is

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + k \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

$$\Rightarrow (x - 4)(x - 6) + (y - 1)(y - 5) + k \begin{vmatrix} x & y & 1 \\ 4 & 1 & 1 \\ 6 & 5 & 1 \end{vmatrix} = 0$$

$$\Rightarrow x^2 - 4x - 6x + 24 + y^2 - 5y - y + 5 + k[x(1 - 5) - y(4 - 6)] + 120 - 6 = 0$$

$$\Rightarrow x^2 - 10x + y^2 - 6y + 29 + k[x(-4) - y(-2) + 1(14)] = 0$$

$$\Rightarrow x^2 + y^2 - 10x - 6y + 29 + k[-4x + 2y + 14] = 0 \dots \dots \dots (1)$$

$$x^2 + y^2 - 10x - 4kx - 6y + 2ky + 29 + 14k = 0$$

$$x^2 + y^2 - 2x[5 + 2k] - 2y[3 - k] + 29 + 14k = 0$$

$$\text{centre } (-g, -f) = [5 + 2k, 3 - k]$$

$$\text{Given centre lies on } 4x + 3y = 24$$

$$\Rightarrow 4(5 + 2k) + 3(3 - k) = 24$$

$$\Rightarrow 20 + 8k + 9 - 3k = 24 \Rightarrow 5k = 24 - 29$$

$$\Rightarrow k = -\frac{5}{5} = -1, \text{sub } k = -1 \text{ in (1)}$$

$$\Rightarrow (x^2 + y^2 - 10x - 6y + 29) - 1[-4x + 2y + 14] = 0$$

$$x^2 + y^2 - 6x - 8y + 15 = 0 \dots \dots \dots (2)$$

**MODEL:-2**

7. Show that the circles $x^2 + y^2 - 6x - 2y + 1 = 0$, $x^2 + y^2 + 2x - 8y + 13 = 0$ touch each other. Find the point of contact and eq'n of tangent at point of contact.

Sol: eq'n s of the given circles

$$S \equiv x^2 + y^2 - 6x - 2y + 1 = 0 \dots\dots (1)$$

$$S' \equiv x^2 + y^2 + 2x - 8y + 13 = 0 \dots\dots (2)$$

centres (-g, -f): $C_1(3, 1)$, $C_2(-1, 4)$

$$\text{radius}(r) = \sqrt{g^2 + f^2 - c}$$

$$r_1 = \sqrt{3^2 + 1^2 - 1} = \sqrt{3^2} = 3$$

$$r_2 = \sqrt{1^2 + 4^2 - 13} = \sqrt{17 - 13} = \sqrt{4} = 2$$

$$C_1C_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$= \sqrt{(-1 - 3)^2 + (4 - 1)^2}$$

$$= \sqrt{(-4)^2 + (3)^2} = \sqrt{25} = 5$$

$$C_1C_2 = r_1 + r_2$$

the circles touch each othe externally.

the point of contact p divides C_1C_2 Internally in the ratio $r_1:r_2 = 3:2 = m:n$

$$P = \left[\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n} \right] = \left(\frac{3(-1)+2(3)}{3+2}, \frac{3(4)+2(1)}{3+2} \right)$$

$$= \left(\frac{-3+6}{5}, \frac{12+2}{5} \right) = \left(\frac{3}{5}, \frac{14}{5} \right)$$

eq'n of common tangent at p is $s - s' = 0$

$$x^2 + y^2 - 6x - 2y + 1 = 0$$

$$x^2 + y^2 + 2x - 8y + 13 = 0$$

$$-8x + 6y - 12 = 0 \quad \{\div -2\} \quad \therefore 4x - 3y + 6 = 0.$$



8. Show that the circles $x^2 + y^2 - 4x - 6y - 12 = 0$, $x^2 + y^2 + 6x + 18y + 26 = 0$ Touch each other. Find the point of contact and eq'n of tangent at point of contact.

Sol: eq'n s of the given circles

$$S \equiv x^2 + y^2 - 4x - 6y - 12 = 0 \dots\dots (1)$$

$$S' \equiv x^2 + y^2 + 6x + 18y + 26 = 0 \dots\dots (2)$$

centres (-g, -f): $C_1(2, 3)$, $C_2(-3, -9)$

$$\text{radius}(r) = \sqrt{g^2 + f^2 - c}$$

$$r_1 = \sqrt{2^2 + 3^2 + 12} = \sqrt{4 + 9 + 12} = 5$$

$$r_2 = \sqrt{3^2 + 9^2 - 26} = \sqrt{9 + 81 - 26} = \sqrt{64} = 8$$

$$C_1C_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$= \sqrt{(-3 - 2)^2 + (9 - 3)^2}$$

$$= \sqrt{(-5)^2 + (-12)^2} = \sqrt{25 + 144} = \sqrt{169} = 13$$

$C_1C_2 = r_1 + r_2$ **the circles touch each othe externally.**

the point of contact p divides C_1C_2 Internally in the ratio $r_1:r_2 = 5:8 = m:n$

$$P = \left[\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n} \right] = \left(\frac{5(-3)+8(2)}{5+8}, \frac{5(-9)+8(3)}{5+8} \right)$$

$$= \left(\frac{-15+16}{13}, \frac{-45+16}{5} \right) = \left(\frac{1}{13}, \frac{-29}{13} \right)$$

eq'n of common tangent at p is $s - s' = 0$

$$x^2 + y^2 - 4x - 6y - 12 = 0$$

$$x^2 + y^2 + 6x + 18y + 26 = 0$$

$$-10x - 246y - 38 = 0 \quad \{\div -2\}$$

$$\therefore 5x + 12y + 19 = 0.$$



9. Show that the circles $x^2 + y^2 - 6x - 9y + 13 = 0$, $x^2 + y^2 - 2x - 16y = 0$ Touch each other. Find the point of contact and eq'n of tangent at point of contact.

Sol: eq'n s of the given circles

$$S \equiv x^2 + y^2 - 6x - 9y - 13 = 0 \dots\dots(1)$$

$$S' \equiv x^2 + y^2 - 2x - 16y = 0 \dots\dots(2)$$

centres $(-g, -f)$: $C_1\left(3, \frac{9}{2}\right)$, $C_2(1, 8)$

$$\text{radius}(r) = \sqrt{g^2 + f^2 - c}$$

$$r_1 = \sqrt{3^2 + \left(\frac{9}{2}\right)^2 - 13} = \sqrt{9 + \frac{81}{4} - 13} \\ = \sqrt{\frac{36+81-52}{4}} = \frac{\sqrt{65}}{2}$$

$$r_2 = \sqrt{1^2 + 8^2 - 0} = \sqrt{1 + 64} = \sqrt{65}$$

$$C_1C_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(1 - 3)^2 + \left(8 - \frac{9}{2}\right)^2}$$

$$= \sqrt{(-2)^2 + \left(-\frac{7}{2}\right)^2} = \sqrt{4 + \frac{49}{4}} = \frac{\sqrt{65}}{2}$$

$C_1C_2 = |r_1 - r_2|$ the circles touch each other internally.

the point of contact p divides C_1C_2 externally in the ratio $r_1:r_2 = \frac{\sqrt{65}}{2}:\sqrt{65} = 1:2 = m:n$

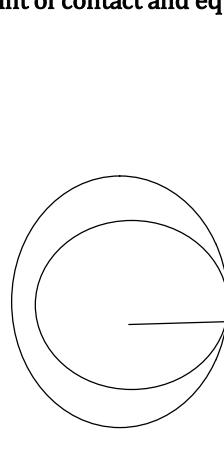
$$P = \left[\frac{mx_2 - nx_1}{m-n}, \frac{my_2 - ny_1}{m-n} \right] = \left(\frac{1(1)-2(3)}{1-3}, \frac{1(8)-2\left(\frac{9}{2}\right)}{1-3} \right) = \left(\frac{1-6}{-1}, \frac{8-9}{-1} \right) = (5, 1)$$

eq'n of common tangent at p is $s - s' = 0$

$$x^2 + y^2 - 6x - 9y + 13 = 0$$

$$x^2 + y^2 - 2x - 16y = 0$$

$$-4x + 7y + 13 = 0 \quad \{\div -\} \quad \therefore 4x - 7y - 13 = 0.$$



10. S.T the circles $x^2 + y^2 - 4x - 6y - 12 = 0$, $5(x^2 + y^2) - 8x - 14y - 32 = 0$ Touch each other. Find the point of contact and eq'n of tangent at point of contact.

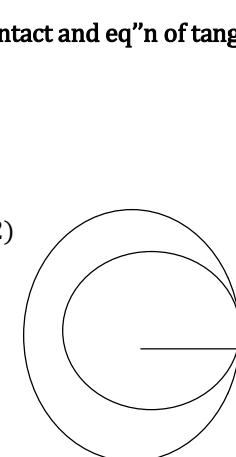
Sol: eq'n s of the given circles

$$S \equiv x^2 + y^2 - 4x - 6y - 12 = 0 \dots\dots(1)$$

$$S' \equiv x^2 + y^2 - \frac{8}{5}x - \frac{14}{5}y - 32 = 0 \dots\dots(2)$$

centres $(-g, -f)$: $C_1(2, 3)$, $C_2\left(\frac{4}{5}, \frac{7}{5}\right)$

$$\text{radius}(r) = \sqrt{g^2 + f^2 - c}$$



$$r_1 = \sqrt{2^2 + 3^2 + 12} = \sqrt{4 + 9 + 12} = 5$$

$$r_2 = \sqrt{\left(\frac{4}{5}\right)^2 + \left(\frac{7}{5}\right)^2 + \frac{32}{5}} = \sqrt{\frac{16+49+160}{25}} = \sqrt{\frac{225}{25}} = \sqrt{9} = 3$$

$$C_1C_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$= \sqrt{\left(\frac{4}{5} - 2\right)^2 + \left(\frac{7}{5} - 3\right)^2}$$

$$= \sqrt{\left(-\frac{6}{5}\right)^2 + \left(-\frac{8}{5}\right)^2} = \sqrt{\frac{36+64}{25}} = \sqrt{\frac{100}{25}} = \sqrt{4} = 2$$

$C_1C_2 = |r_1 - r_2|$ the circles touch each other internally.

the point of contact p divides C_1C_2 externally in the ratio

$$r_1:r_2 = 5:3 = m:n$$

$$P = \left[\frac{mx_2 - nx_1}{m-n}, \frac{my_2 - ny_1}{m-n} \right] = \left(\frac{\frac{5}{5}(4)-3(2)}{5-3}, \frac{\frac{5}{5}(7)-3(3)}{5-3} \right)$$

$$= \left(\frac{4-6}{-2}, \frac{7-9}{-2} \right) = \left(-\frac{2}{-2}, \frac{-2}{-2} \right) = (1, 1)$$



eq'n of common tangent at p is $5s - s' = 0$

$$5x^2 + 5y^2 - 20x - 30y - 60 = 0$$

$$\underline{5x^2 + 5y^2 - 8x - 14y - 32 = 0}$$

$$-12x - 16y - 28 = 0 \quad \{\div -4\}$$

$$\therefore 3x + 3y + 7 = 0.$$

11. Find the transverse common tangents of the circles $x^2 + y^2 - 4x - 10y + 28 = 0$, $x^2 + y^2 + 4x - 6y + 4 = 0$.

Sol: eq'n s of the given circles

$$S \equiv x^2 + y^2 - 4x - 10y + 28 = 0 \dots\dots (1) \quad S' \equiv x^2 + y^2 + 4x - 6y + 4 = 0 \dots\dots (2)$$

centres (-g, -f): $C_1(2, 5)$,

$C_2(-2, 3)$

$$\text{radius}(r) = \sqrt{g^2 + f^2 - c}$$

$$r_1 = \sqrt{2^2 + 5^2 - 28} = \sqrt{4 + 25 - 28} = 1, r_2 = \sqrt{2^2 + 3^2 - 4} = \sqrt{4 + 9 - 4} = \sqrt{9} = 3$$

$$C_1C_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$= \sqrt{(-2 - 2)^2 + (3 - 5)^2}$$

$$= \sqrt{(-4)^2 + (-2)^2} = \sqrt{16 + 4} = \sqrt{20}$$

$C_1C_2 > r_1 + r_2$ There exists two transverse common tangents

Internal centre of similitude divides C_1C_2 internally in the ratio

$$r_1:r_2 = 1:3 = m:n$$

$$P = \left[\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n} \right] = \left(\frac{1(-2) + 3(2)}{1+3}, \frac{1(3) + 3(5)}{1+3} \right)$$

$$= \left(\frac{-2+6}{4}, \frac{3+15}{4} \right) = \left(\frac{4}{4}, \frac{18}{4} \right) = (1, \frac{9}{2})$$



let m be the slope of common tangent

$$\text{Eq'n of tangent is } (y - y_1)m = (x - x_1) \quad P(1, \frac{9}{2})$$

$$\Rightarrow \left(y - \frac{9}{2} \right) = m(x - 1) \dots\dots\dots (1)$$

$$\Rightarrow \frac{(2y-9)}{2} = m(x-1) \Rightarrow 2mx - 2y - 2m + 9 = 0$$

condition for tangent $r = \perp r$ distance

$r_1 = \perp$ lr distance from $C_1(2, 5)$ to (1)

$$d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}} \Rightarrow \frac{|2m(2) - 2(5) - 2m + 9|}{\sqrt{(2m)^2 + (2)^2}} = 1$$

$$\Rightarrow \frac{|2m-1|}{\sqrt{4m^2+4}} = 1 \quad \text{S.O.B.S} \Rightarrow (2m-1)^2 = 4m^2 + 4$$

$$\Rightarrow 4m^2 + 1 - 4m = 4m^2 + 4 \quad \left\{ m = \infty = \frac{1}{0} \right\}$$

$$\Rightarrow -3 = 4m \text{ or } m = -\frac{3}{4}$$

Case(1) $m = 0$ sub in (1)

$$\left(y - \frac{9}{2} \right) = \frac{1}{0}(x - 1)$$

$$x - 1 = 0$$

Case(ii) $m = -\frac{3}{4}$ sub in (1)

$$\left(y - \frac{9}{2} \right) = -\frac{3}{4}(x - 1)$$

$$\Rightarrow 4y - 18 = -3x + 3$$

$$3x + 4y - 21 = 0.$$



12. Find the direct common tangents of the circles $x^2 + y^2 + 22x - 4y - 100 = 0$

$$100 = 0,$$

$$x^2 + y^2 - 22x + 4y + 100 = 0.$$

Sol: eq'n s of the given circles

$$S \equiv x^2 + y^2 + 22x - 4y - 100 = 0 \dots\dots (1) \quad S' \equiv x^2 + y^2 - 22x + 4y + 100 = 0 \dots\dots (2)$$

centres ($-g, -f$): $C_1(-11, 2)$,

$C_2(11, -2)$

$$\text{radius}(r) = \sqrt{g^2 + f^2 - c}$$

$$r_1 = \sqrt{11^2 + 2^2 + 100} = \sqrt{121 + 4 + 100}$$

$$= \sqrt{225} = 15$$

$$r_2 = \sqrt{11^2 + 4^2 - 100} = \sqrt{121 + 16 - 100}$$

$$= \sqrt{25} = 5$$

$$C_1C_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$= \sqrt{(11 + 11)^2 + (-2 - 2)^2}$$

$$= \sqrt{(22)^2 + (-4)^2} = \sqrt{484 + 16} = \sqrt{500}$$

$C_1C_2 > r_1 + r_2$ There exists two direct common tangents

External centre of similitude divides C_1C_2 internally in the ratio

$$r_1:r_2 = 15:3 = 3:1 = m:n$$

$$P = \left[\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n} \right] = \left(\frac{3(11) - 1(-11)}{3-1}, \frac{3(-2) - 1(2)}{3-1} \right)$$

$$= \left(\frac{33+11}{2}, \frac{-6-2}{2} \right) = \left(\frac{44}{2}, \frac{-8}{2} \right) = (22, -4)$$

let m be the slope of common tangent

Eq'n of tangent is $(y - y_1)m = (x - x_1)$ p(22, -4)

$$\Rightarrow (y + 4) = m(x - 22) \dots\dots (1)$$

$$\Rightarrow (y + 4) = mx - 22m$$



$$\Rightarrow mx - y - 22m - 4 = 0$$

condition for tangent $r_2 = \text{perpendicular distance from } C_2 \text{ to (1)}$

$$d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}} = \frac{|m(11) - 1(-2) - 22m - 4|}{\sqrt{(m)^2 + (-1)^2}} = 5$$

$$\Rightarrow \frac{|-11m - 4|}{\sqrt{m^2 + 1}} = 5 \quad \text{S.O.B.S}$$

$$\Rightarrow (11m + 4)^2 = 25(m^2 + 1)$$

$$\Rightarrow 121m^2 + 4 + 44m = 25m^2 + 25$$

$$\Rightarrow 96m^2 + 44m - 21 = 0$$

$$\Rightarrow 96m^2 + 72m - 28m - 21 = 0$$

$$\Rightarrow 24m(4m + 3) - 7(4m + 3) = 0$$

$$\Rightarrow (24m - 7)(4m + 3) = 0$$

$$(24m - 7) = 0 \text{ and } (4m + 3) = 0$$

$$m = \frac{7}{24}, m = -\frac{3}{4}$$

\therefore eq'n of tangents are

$$\text{case (i)} m = \frac{7}{24}$$

$$(y + 4) = \frac{7}{24}(x - 22)$$

$$\Rightarrow 24y + 96 = 7x - 156$$

$$7x - 24y - 250 = 0.$$

$$\text{Case (ii)} m = -\frac{3}{4}$$

$$(y + 4) = -\frac{3}{4}(x - 22)$$

$$\Rightarrow 4y + 16 = -3x + 66$$

$$3x + 4y - 50 = 0.$$

$$3x + 4y - 50 = 0 \text{ and } 7x - 24y - 250 = 0.$$

**MODEL-3**

11. Find the equation of the circles which touch

$2x-3y+1=0$ at $(1, 1)$ and having radius $\sqrt{13}$.

Sol: given two circles touch the line

$$2x - 3y + 1 = 0 \dots \dots \dots (1)$$

At $P(1, 1)$ given radius (r) = $\sqrt{13}$

The centre's C_1, C_2 lies on a line \perp lar to (1) at a distance r from

$$P(x_1, y_1) = (1, 1)$$

$$\text{Slope of (1)} = -\left(\frac{2}{-3}\right) = \frac{2}{3}$$

$$\text{Slope of line } C_1 C_2 = \frac{-3}{2}$$

$$m = \tan\theta = \frac{-3}{2}$$

$$\Rightarrow \sin\theta = \frac{3}{\sqrt{13}} \text{ and } \cos\theta = \frac{-2}{\sqrt{13}}$$

The centres C_1, C_2 can be written as

$$a = x_1 \pm r\cos\theta, b = y_1 \pm r\sin\theta$$

$$= \left(1 \pm \sqrt{13} \left[\frac{-2}{\sqrt{13}}\right], 1 \pm \sqrt{13} \left[\frac{3}{\sqrt{13}}\right]\right) = \{1 \pm (-2), 1 \pm 3\}$$

$$= (1-2, 1+3) \text{ and } (1+2, 1-3)$$

$$= (-1, 4) \text{ and } (3, -2)$$

Eq'n of the circle with $C_1(-1, 4)$

and $r = \sqrt{13}$

$$(x - a)^2 + (y - b)^2 = r^2$$

$$\Rightarrow (x + 1)^2 + (y - 4)^2 = \sqrt{13}^2$$

$$\Rightarrow x^2 + y^2 + 2x - 8y + 4 = 0$$

Eq'n of the circle with $C_2(3, -2)$

and $r = \sqrt{13}$

$$(x - a)^2 + (y - b)^2 = r^2$$

$$\Rightarrow (x - 3)^2 + (y + 2)^2 = \sqrt{13}^2 \Rightarrow x^2 + y^2 - 6x + 4y = 0.$$



12. Show that the poles of tangents to the circle $x^2 + y^2 = a^2$ w.r.to the circle

$$(x + a)^2 + y^2 = 2a^2 \text{ lie on } y^2 + 4ax = 0.$$

Sol: Given circles

$$x^2 + y^2 = r^2 \dots \dots (1)$$

$$x^2 + y^2 + 2ax - a^2 = 0 \dots \dots (2)$$

Let $P(x_1, y_1)$ be the pole of the tangents to the circle (1) w. r. t the circle (2)

Now the polar of p w.r.t to S=0 is $S_1=0$

$$\Rightarrow xx_1 + yy_1 + a(x+x_1) - a^2 = 0$$

$$\Rightarrow x(x_1 + a) + yy_1 + (ax_1 - a^2) = 0 \dots \dots (1)$$

the tangential condition

r = perpendicular distance from (1)

$$d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

$$a = \frac{|0+0+ax_1 - a^2|}{\sqrt{(x_1+a)^2 + y_1^2}}$$

$$\Rightarrow a\sqrt{(x_1 + a)^2 + y_1^2} = a(x_1 - a) \text{ S.O.B}$$

$$\Rightarrow (x_1 + a)^2 + y_1^2 = (x_1 - a)^2$$

$$\Rightarrow x_1^2 + a^2 + 2ax_1 + y_1^2 = x_1^2 + a^2 - 2ax_1$$

$$\Rightarrow y_1^2 + 4ax_1 = 0$$

the pole of the tangents to the circle (1) w. r. t the circle (2) lie on the curve

$$y^2 + 4ax = 0$$



13. If θ_1, θ_2 are the angles of inclination of tangents through a point p to the circle $x^2+y^2=a^2$, then find locus of p when $\cot\theta_1 + \cot\theta_2 = k$.

Sol: given equation of the circle $x^2+y^2=a^2$ (1)

: Let P(x_1, y_1) be the any point on the locus

Equation of tangent through p with slope 'm' is

$$y=mx \pm a\sqrt{1+m^2}$$

This passes through P(x_1, y_1)

$$\Rightarrow Y_1=mx_1 \pm a\sqrt{1+m^2}$$

$$\Rightarrow Y_1-mx_1 = \pm a\sqrt{1+m^2} \quad \text{S.O.B}$$

$$\Rightarrow (Y_1 - mx_1)^2 = (\pm a\sqrt{1+m^2})^2$$

$$\Rightarrow y_1^2 + m^2 x_1^2 - 2mx_1 y_1 = a^2(1+m^2)$$

$$\Rightarrow y_1^2 + m^2 x_1^2 - 2mx_1 y_1 = a^2 + a^2 m^2$$

$$\Rightarrow (x_1^2 - a^2)m^2 - (2x_1 y_1)m + (y_1^2 - a^2) = 0 \{ax^2 + bx + c = 0\}$$

Where m_1, m_2 be the slopes of the tangents which make angles

θ_1, θ_2 with X – axis

$$m_1=\tan\theta_1, m_2=\tan\theta_2 \left\{ m_1 + m_2 = -\frac{b}{a}, m_1 m_2 = \frac{c}{a} \right\}$$

Given $\cot\theta_1 + \cot\theta_2 = k$.

$$\Rightarrow \frac{1}{\tan\theta_1} + \frac{1}{\tan\theta_2} = k$$

$$\Rightarrow \frac{1}{m_1} + \frac{1}{m_2} = k$$

$$\Rightarrow \frac{m_1+m_2}{m_1 m_2} = k$$

$$\Rightarrow m_1 + m_2 = km_1 m_2$$

$$\Rightarrow \left[\frac{2x_1 y_1}{x_1^2 - a^2} \right] = k \left[\frac{y_1^2 - a^2}{x_1^2 - a^2} \right]$$

$$\Rightarrow 2x_1 y_1 = k(y_1^2 - a^2)$$

\therefore The equation of locus of P(x_1, y_1) is

$$2xy = k(y^2 - a^2)$$



14. Prove that the combined equation of the pair of tangents drawn from an external point p (x_1, y_1) to the circle S=0 is $S_1^2 = SS_{11}$.

Sol: Suppose that the tangents drawn from P to the circle S=0 touch the circle at A and B the equation of AB is $S_1 = 0$.

$$\Rightarrow xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0 \dots (1)$$

Let Q (x_2, y_2) be any point on these tangents. Now the locus of Q will be the equation of pair of tangents drawn from P.

The segment PQ is divided by the line AB in the ratio $-S_{11}:S_{12}$ or $S_{11}:S_{12}$

$$\frac{PB}{QB} = \left| \frac{S_{11}}{S_{12}} \right| \dots\dots (2)$$

$$\text{But } PB = \sqrt{S_{11}} \text{ and } QB = \sqrt{S_{22}}$$

$$\therefore \frac{PB}{QB} = \frac{\sqrt{S_{11}}}{\sqrt{S_{22}}} \dots\dots (3)$$

$$\text{From (2), (3) we get } \frac{\sqrt{S_{11}}}{\sqrt{S_{22}}} = \frac{S_{11}}{S_{12}} \quad \text{S.O.B}$$

$$\Rightarrow \frac{S_{11}}{S_{22}} = \frac{S_{11}^2}{S_{12}^2}$$

$$\Rightarrow S_{11} S_{22} = S_{12}^2$$

Hence the equation of the locus of

$$Q(x_2, y_2) \text{ is } S_1^2 = SS_{11}.$$

$\{\because (x_2, y_2)$ replaced by $(x, y)\}$



15. Find the pair of tangents drawn from $(1, 3)$ to the circle $x^2 + y^2 - 2x + 4y - 11 = 0$ and also find the angle b/w them.

Sol: Given eq'n of the circle

$$S \equiv x^2 + y^2 - 2x + 4y - 11 = 0$$

$$P(1, 3) = (x_1, y_1)$$

$$S_1 = xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c$$

$$S_1 = 1x + 3y - 1(x + 1) + 2(y + 3) - 11$$

$$S_1 = x + 3y - x - 1 + 2y + 6 - 11$$

$$S_1 = 5y - 6$$

$$S_{11} = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$$

$$= (1)^2 + (3)^2 - 2(1) + 4(3) - 11$$

$$S_{11} = 9$$

Eq'n of pair of tangents is $S_1^2 = SS_{11}$

$$(5y - 6)^2 = (x^2 + y^2 - 2x + 4y - 11)(9)$$

$$\Rightarrow 25y^2 + 36 - 60y = 9x^2 + 9y^2 - 18x + 36y - 99$$

$$9x^2 - 16y^2 - 18x + 96y - 135 = 0$$

$$\text{Required angle } \cos \theta = \frac{|a+b|}{\sqrt{(a-b)^2+(2h)^2}}$$

$$\cos \theta = \frac{|9-16|}{\sqrt{(9+16)^2+(0)^2}}$$

$$\cos \theta = \frac{|7|}{25}$$

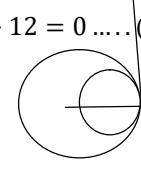


16. Find the eq'n of the circle which touches the circle $x^2 + y^2 - 4x + 6y - 12 = 0$ at $(-1, 1)$ internally with radius 2.

Sol: $S \equiv x^2 + y^2 - 4x + 6y - 12 = 0 \dots\dots (1)$

centres $(-g, -f)$: $C_1(2, -3)$

$$\text{radius}(r) = \sqrt{g^2 + f^2 - c}$$



$$r_1 = \sqrt{2^2 + 3^2 + 12} = \sqrt{4 + 9 + 12}$$

$$= \sqrt{25} = 5$$

let C_2 be the centre and r_2 be the radius of the required circle.

Since two circles touch internally, we have

$$C_1C_2 = r_1 - r_2 = 5 - 2 = 3$$

Hence C_2 divides C_2P in the ratio 3:2 internally, where $P(-1, 1)$

$$C_2 = \left[\frac{3(-1)+2(2)}{3+2}, \frac{3(1)+2(-3)}{3+2} \right] = \left(\frac{1}{5}, -\frac{3}{5} \right)$$

eq'n of the required circle is

$$(x - a)^2 + (y - b)^2 = (r)^2$$

$$\Rightarrow (x - \frac{1}{5})^2 + (y + \frac{3}{5})^2 = (2)^2$$

$$\Rightarrow x^2 + \frac{1}{25} - \frac{2x}{5} + y^2 + \frac{9}{25} + \frac{6y}{5} = 4$$

$$\Rightarrow 25x^2 + 1 - 10x + 25y^2 + 9 + 30y = 100$$

$$\Rightarrow 5x^2 + 5y^2 - 2x + 6y - 18 = 0$$



17. Find the eq'n of the circle which touches the circle $x^2 + y^2 - 2x - 4y - 20 = 0$ at (5, 5) externally with radius 5.

Sol: $S \equiv x^2 + y^2 - 2x - 4y - 20 = 0 \dots\dots (1)$

centres (-g, -f): $C_1 (1, 2)$

$$\text{radius}(r) = \sqrt{g^2 + f^2 - c}$$

$$r_1 = \sqrt{1^2 + 2^2 + 20} = \sqrt{25} = 5$$

let $C_2 (x, y)$ be the centre and r_2 be the radius of the required circle.

Since two circles touch externally, we have

$$C_1 C_2 = r_1 + r_2 = 5 + 5 = 10 \text{ hence } P \text{ is the midpoint of } C_1 C_2$$

$$\text{Where } P(5, 5) = \left[\frac{1+a}{2}, \frac{2+b}{2} \right]$$

$$\Rightarrow \frac{1+a}{2} = 5 \text{ and } \frac{2+b}{2} = 5$$

$$1 + a = 10 \text{ and } 2 + b = 10$$

$$\Rightarrow a = 9 \text{ and } b = 8$$

eq'n of the required circle is

$$(x - a)^2 + (y - b)^2 = (r)^2$$

$$\Rightarrow (x - 9)^2 + (y - 8)^2 = (5)^2$$

$$\Rightarrow x^2 + 81 - 18x + y^2 + 64 - 16y = 25$$

$$\Rightarrow x^2 + y^2 - 18x - 16y + 120 = 0$$



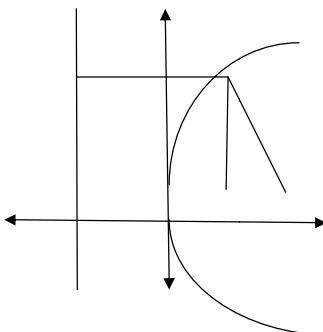
1. Equation of a parabola in standard form

Proof:

- Let S be the focus, l=0 be the directrix of the parabola.
 - Let P (x_1, y_1) be a point on the parabola.
 - Let M, Z be the projections of the P, S on l=0 respectively.
 - Let N be the projection of p on SZ.
 - Let A be the midpoint of SZ, $\Rightarrow SA = AZ$
- Let SA=AZ=a, A is called vertex of the parabola.
- Take AS as X-axis and AY as Y-axis

$$\Rightarrow A(0,0), S(a,0) \text{ and } P(x_1, y_1)$$

$$PM = NZ = NA + AZ = |x_1 + a|$$



From the definition of the parabola $\frac{SP}{PM} = e = 1$

S.O.B

$$SP^2 = PM^2$$

$$(x_1 - a)^2 + (y_1 - 0)^2 = (x_1 + a)^2$$

$$\Rightarrow x_1^2 + a^2 - 2ax_1 + y_1^2 = x_1^2 + a^2 + 2ax_1$$

$$\Rightarrow y_1^2 = 4ax_1$$

*∴ Thus the equation of the parabola in standard form
is $y^2 = 4ax$*



1. Find the coordinates of the vertex and focus, the equation of the directrix and axis of the parabola $y^2 - x + 4y + 5 = 0$.

Sol: the given equation is $y^2 - x + 4y + 5 = 0$

$$\Rightarrow y^2 + 4y = x - 5$$

$$\Rightarrow y^2 + 4y + 2^2 = x - 5 + 2^2$$

$$\Rightarrow (y + 2)^2 = (x - 1)$$

$$\Rightarrow [y - (-2)]^2 = (x - 1) \quad \text{Comparing with } [y - k]^2 = 4a(x - h)$$

We get, $4a=1 \Rightarrow a=1/4$ and $(h, k) = (1, -2)$

i. Vertex $(1, -2)$ (ii) Focus $(h+a, k) = (1 + \frac{1}{4}, -2) = (\frac{5}{4}, -2)$

(iii) Equation of the directrix is $x = h-a$

$$\Rightarrow x = 1 - \frac{1}{4} = \frac{3}{4} \Rightarrow 4x - 3 = 0$$

ii. Length of the latus rectum is $4a=1$.

iii. Eq'n of axis $y - \beta = 0 \Rightarrow y + 2 = 0$

2. Find the coordinates of the vertex and focus, the equation of the directrix and axis of the parabola $x^2 - 2x + 4y - 3 = 0$.

Sol: the given equation is $x^2 - 2x + 4y - 3 = 0$

$$\Rightarrow x^2 - 2x = -4y + 3$$

$$\Rightarrow x^2 - 2x + 1^2 = -4y + 3 + 1^2$$

$$\Rightarrow (x - 1)^2 = -4(y - 1) \Rightarrow [x - 1]^2 = -4(y - 1)$$

Comparing with $[x - h]^2 = 4a(y - k)$

We get, $4a=4 \Rightarrow a=1$ and $(h, k) = (1, 1)$

1. Vertex $(1, 1)$

2. Focus $(h, k-a) = (1, 1 - 1) = (1, 0)$

3. Equation of the directrix is

$$y = k + a \Rightarrow y = 1 + 1 = 2 \Rightarrow y - 2 = 0 = 0$$

4. Length of the latus rectum is $4a=4$.

5. Eq'n of axis $x - \alpha = 0 \Rightarrow x - 1 = 0$



4. Find the equation of the parabola whose axis is parallel to the y-axis and passing through the points (4, 5), (-2, 11), (-4, 21).

Sol: The equation of the parabola whose axis is parallel to the y-axis

$$y = ax^2 + bx + c \dots (*)$$

(4, 5) lies on (*)

$$\Rightarrow 5 = a(4)^2 + b(4) + c$$

$$\Rightarrow 5 = 16a + 4b + c \dots \dots \quad (1)$$

(-2, 11) lies on (*)

$$\Rightarrow 11 = a(-2)^2 + b(-2) + c$$

$$\Rightarrow 11 = 4a - 2b + c \dots \dots \quad (2)$$

(-4, 21) lies on (*)

$$\Rightarrow 21 = a(-4)^2 + b(-4) + c$$

$$\Rightarrow 21 = 16a - 4b + c \dots \dots \quad (3)$$

Solving (1) & (2)

$$5 = 16a + 4b + c$$

$$\underline{21 = 16a - 4b + c}$$

$$\underline{-6 = 12a + 6b \dots \dots (4)}$$

solving (1) & (3)

$$5 = 16a + 4b + c \quad 11 = 4a - 2b + c$$

$$\underline{-1 = 12a - 8b \dots \dots (5)}$$

$$\Rightarrow b = -2$$

Sub b = -2 in (4)

$$\Rightarrow -6 = 12a - 12$$

$$\Rightarrow 12a = -6 + 12 = 6$$

$$\Rightarrow a = \frac{6}{12} = \frac{1}{2}$$

Sub a, b in (1)

$$\Rightarrow 5 = \frac{16}{2} - 8 + c \Rightarrow c = 5$$

Substituting the values of a,b and c in (*) we get the required equation of the parabola

$$y = \frac{1}{2}x^2 + (-2)x + 5$$

$$\Rightarrow x^2 - 2y - 4x + 10 = 0.$$



5. Find the equation of the parabola whose axis is parallel to the x-axis and passing through the points (-2, 1), (1, 2), (-1, 3).

Sol: Sol: The equation of the parabola whose axis is parallel to the y-axis

$$x = ay^2 + by + c \dots (*)$$

(-2, 1) lies on (*)

$$\Rightarrow -2 = a(1)^2 + b(1) + c$$

$$\Rightarrow -2 = a + b + c \dots \dots \quad (1)$$

(1, 2) lies on (*)

$$\Rightarrow 1 = a(2)^2 + b(2) + c$$

$$\Rightarrow 1 = 4a + 2b + c \dots \dots \quad (2)$$

(-1, 3) lies on (*)

$$\Rightarrow -1 = a(3)^2 + b(3) + c$$

$$\Rightarrow -1 = 9a + 3b + c \dots \dots \quad (3)$$

Solving (1) & (2)

$$-2 = a + b + c$$

$$\underline{-1 = 9a + 3b + c}$$

$$\underline{-3 = -8a - 2b \dots \dots (4)}$$

solving (1) & (3)

$$-2 = a + b + c \quad 1 = 4a + 2b + c$$

$$\underline{-1 = -8a - 2b \dots \dots (5)}$$

Solving (4) & (5)

$$\begin{array}{cccc} -3 & -1 & 3 & -3 \\ -8 & -2 & 1 & -8 \end{array}$$

$$(a, b) = \left[\frac{-1+6}{6-8}, \frac{-24+3}{6-8} \right] = \left[\frac{5}{-2}, \frac{-21}{-2} \right] = \left[-\frac{5}{2}, \frac{21}{2} \right]$$

Substituting the values of a,b in (1) we get

$$\Rightarrow \left(-\frac{5}{2} \right) + \left(\frac{21}{2} \right) + c = -2$$

$$\Rightarrow 8 + c = -2$$

$$\therefore c = -10$$

Substituting the values of a,b and c in (*) we get the required equation of the parabola

$$y = -\frac{5}{2}x^2 + \left(-\frac{21}{2} \right)x - 10$$

$$\Rightarrow 5x^2 + 2y + 21x + 10 = 0.$$



6. Show that the equation of common tangents to the circle $x^2 + y^2 = 2a^2$ and the parabola $y^2 = 8ax$ are $y = \pm(x + 2a)$.

Sol:

Given equation of the circle $x^2 + y^2 = 2a^2$... (1)

$$[x^2 + y^2 = (\sqrt{2}a)^2, \quad r = \sqrt{2}a]$$

Parabola $y^2 = 8ax$ (2)

$$[y^2 = 4a'x, \quad a' = 2a]$$

Let 'm' be the slope of common tangent.

Equation of tangent to (1),

$$y = mx \pm \sqrt{2a^2(1+m^2)} \dots \dots \dots (3)$$

$$\left[\begin{array}{l} y = mx \pm r \sqrt{1+m^2} \\ \text{is eq'n of tangent to } x^2 + y^2 = r^2 \end{array} \right]$$

Equation of tangent to (2),

$$y = mx + \frac{2a}{m} \dots \dots (4)$$

$$\left[\begin{array}{l} \therefore y = mx + \frac{a'}{m} \text{ is eq'n of} \\ \text{tangent to } y^2 = 4a'x \end{array} \right]$$

(3), (4) Represents same line

$$\pm \sqrt{2a^2(1+m^2)} = \frac{2a}{m}$$

squaring on both sides

$$\Rightarrow 2a^2(1+m^2) = \frac{4a^2}{m^2}$$

$$\Rightarrow m^2(1+m^2) = 2$$

$$\Rightarrow m^2 + m^4 - 2 = 0$$

$$\Rightarrow m^4 + 2m^2 - m^2 - 2 = 0$$

$$\Rightarrow m^2(m^2 + 2) - 1(m^2 + 2) = 0$$

$$\Rightarrow (m^2 - 1)(m^2 + 2) = 0$$

$$\Rightarrow m^2 - 1 = 0, \text{ or } m^2 + 2 = 0$$

$\Rightarrow m^2 = 1$ or $m^2 = -2$ is not possible

$\therefore m = \pm 1$ sub in (4)

$$y = \pm mx + \frac{2a}{\pm 1}$$

$$y = \pm(x + 2a).$$



7. Find the equation of the parabola whose focus is $(-2, 3)$ and directrix is the line $2x + 3y - 4 = 0$. Also find the length of the latus rectum and the equation of the axis of the parabola.

Sol:

Given S $(-2, 3)$

$$\text{Eq'n of directrix } l = 2x + 3y - 4 = 0.$$

Let P (x_1, y_1) be a point on the parabola.

Draw a perpendicular PM on the to the line L=0.

$$\Rightarrow \frac{SP}{PM} = 1 \Rightarrow SP = PM \left\{ \perp \text{ lar dist} = \frac{ax_1+by_1+c}{\sqrt{a^2+b^2}} \right\}$$

$$\sqrt{(x_1 + 2)^2 + (y_1 - 3)^2} = \left| \frac{2x_1 + 3y_1 - 4}{\sqrt{2^2 + 3^2}} \right| \quad \text{S.O.B}$$

$$\Rightarrow (x_1 + 2)^2 + (y_1 - 3)^2 = \frac{(2x_1 + 3y_1 - 4)^2}{13}$$

$$\Rightarrow 13[x_1^2 + 4 + 4x_1 + y_1^2 + 9 - 6y_1]$$

$$= [4x_1^2 + 9y_1^2 + 16 + 12x_1y_1 - 24y_1 - 16x_1]$$

$$\Rightarrow 13x_1^2 + 52 + 52x_1 + 13y_1^2 + 117 - 78y_1$$

$$-4x_1^2 - 9y_1^2 - 16 - 12x_1y_1 + 24y_1 + 16x_1 = 0$$

$$\Rightarrow 9x_1^2 - 12x_1y_1 + 4y_1^2 + 68x_1 - 54y_1 + 153 = 0$$

\therefore Locus of P is

$9x^2 - 12xy + 4y^2 + 68x - 54y + 153 = 0$ is the eq'n of the parabola.

Length of latus rectum = $4a = 2|2a| = 2|sz|$

$= 2[\perp \text{ lar dist from } S(-2, 3) \text{ to } l = 0]$

$$= 2 \left| \frac{2(-2) + 3(3) - 4}{\sqrt{2^2 + 3^2}} \right| = 2 \left| \frac{-4 + 9 - 4}{\sqrt{4 + 9}} \right| = 2 \left| \frac{1}{\sqrt{13}} \right| = \left| \frac{2}{\sqrt{13}} \right|$$

Eq'n of the axis is the line \perp lar to the directrix

and passing through S $(-2, 3)$ is

$$b(x - x_1) - a(y - y_1) = 0$$

$$\Rightarrow 3(x + 2) - 2(y - 3) = 0$$

$$\Rightarrow 3x + 6 - 2y + 6 = 0 \quad \therefore 3x - 2y + 12 = 0$$



8. Prove that the area of the triangle inscribed in the parabola $y^2 = 4ax$ is $\frac{1}{8a} |(y_1 - y_2)(y_2 - y_3)(y_3 - y_1)|$ Sq.unts where y_1, y_2, y_3 are ordinates of its vertices.

Sol: Given eq'n of the parabola $y^2 = 4ax \dots (1)$

$$Let A(x_1, y_1) = (at_1^2, 2at_1);$$

$$B(x_2, y_2) = (at_2^2, 2at_2)$$

$$C(x_3, y_3) = (at_3^2, 2at_3)$$

$$A \text{ lies on (1)} \Rightarrow y_1^2 = 4ax_1 \Rightarrow x_1 = \frac{y_1^2}{4a}$$

ΔABC In scribed in the parabola

$$= \frac{1}{2} \begin{vmatrix} x_1 - x_2 & y_1 - y_2 \\ x_2 - x_3 & y_2 - y_3 \end{vmatrix}$$

$$= \frac{1}{2} \begin{vmatrix} \left(\frac{y_1^2 - y_2^2}{4a}\right) & y_1 - y_2 \\ \left(\frac{y_2^2 - y_3^2}{4a}\right) & y_2 - y_3 \end{vmatrix}$$

$$= \frac{1}{8a} \begin{vmatrix} (y_1 - y_2)(y_1 + y_2) & (y_1 - y_2) \\ (y_2 - y_3)(y_2 + y_3) & (y_2 - y_3) \end{vmatrix}$$

$$= \frac{(y_1 - y_2)(y_1 - y_2)}{8a} \begin{vmatrix} (y_1 + y_2) & 1 \\ (y_2 + y_3) & 1 \end{vmatrix}$$

$$= \frac{(y_1 - y_2)(y_1 - y_2)}{8a} |y_1 + y_2 - y_2 - y_3|$$

$$\frac{1}{8a} |(y_1 - y_2)(y_2 - y_3)(y_3 - y_1)| \text{ Sq.unts}$$



9. Prove that the area of the triangle formed by the tangents at $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ to the Parabola $y^2 = 4ax$ is $\frac{1}{16a} |(y_1 - y_2)(y_2 - y_3)(y_3 - y_1)|$ Sq.unts where y_1, y_2, y_3 are ordinates of its vertices.

Sol: Given eq'n of the parabola $y^2 = 4ax \dots (1)$

$$Let D(x_1, y_1) = (at_1^2, 2at_1);$$

$$E(x_2, y_2) = (at_2^2, 2at_2)$$

$$F(x_3, y_3) = (at_3^2, 2at_3)$$

Point of intersections of tangents are

$$A = [at_1 t_2, a(t_1 + t_2)];$$

$$B = [at_2 t_3, a(t_2 + t_3)];$$

$$C = [at_3 t_1, a(t_3 + t_1)];$$

ΔABC In scribed in the parabola

$$= \frac{1}{2} \begin{vmatrix} x_1 - x_2 & y_1 - y_2 \\ x_2 - x_3 & y_2 - y_3 \end{vmatrix}$$

$$= \frac{1}{2} \begin{vmatrix} at_1 t_2 - at_2 t_3 & at_1 + at_2 - at_2 - at_3 \\ at_2 t_3 - at_3 t_1 & at_2 + at_2 - at_3 - at_1 \end{vmatrix}$$

$$= \frac{a^2}{2} \begin{vmatrix} t_2(t_1 - t_3) & (t_1 - t_3) \\ t_3(t_2 - t_1) & (t_2 - t_1) \end{vmatrix}$$

$$= \frac{a^2(t_1 - t_3)(t_2 - t_1)}{2} \begin{vmatrix} t_2 & 1 \\ t_3 & 1 \end{vmatrix} = \frac{a^2}{2} (t_1 - t_3)(t_2 - t_1) |t_2 - t_3|$$

$$= \frac{a^2}{2} |(t_1 - t_2)(t_2 - t_3)(t_3 - t_1)| \text{ Sq.unts}$$

$$y_1 = 2at_1 \Rightarrow t_1 = \frac{y_1}{2a}$$

$$= \frac{a^2}{2} \left| \frac{(y_1 - y_2)}{2a} \cdot \frac{(y_2 - y_3)}{2a} \cdot \frac{(y_3 - y_1)}{2a} \right| \text{ Sq.unts}$$

$$= \frac{1}{16a} |(y_1 - y_2)(y_2 - y_3)(y_3 - y_1)| \text{ Sq.unts.}$$



10. If a normal chord at point 't' on the parabola $y^2 = 4ax$ subtends a right angle at vertex, then prove that $t = \pm\sqrt{2}$

Sol: Given eq'n of the parabola $y^2 = 4ax$ (1)

Eq'n of the normal at 't' on the parabola is

$$y + xt = 2at + at^3 \Rightarrow \frac{y+xt}{2at+a^3} = 1 \dots \dots \dots (2)$$

homogenising (1) Using (2) we get,

$$y^2 = 4ax \quad (1)$$

$$\Rightarrow y^2 = 4ax\left(\frac{y+xt}{2at+a^3}\right)$$

$$\Rightarrow y^2 = 4x\left(\frac{y+xt}{2t+t^3}\right)$$

$$\Rightarrow y^2(2t + t^3) = 4x(y + xt)$$

$$\Rightarrow y^2(2t + t^3) = 4xy + 4x^2t$$

$$\Rightarrow 4tx^2 - y^2(2t + t^3) + 4xy = 0$$

Subtends a right angle coeff x^2 – coeff $y^2 = 0$

$$\Rightarrow 4t - 2t - t^3 = 0$$

$$\Rightarrow 2t - t^3 = 0$$

$$\Rightarrow t(2 - t^2) = 0$$

$$\Rightarrow t = 0 \text{ and } t^2 = 2$$

$$t = \pm\sqrt{2}$$





SOME STANDARD ELEMENTARY INTEGRALS

$$1. \frac{d}{dx}(c) = 0 \Rightarrow \int(0) dx = c$$

$$2. \frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = \frac{1}{n+1} \cdot (n+1)x^n = x^n \Rightarrow \int x^n dx = \frac{x^{n+1}}{n+1} + c.$$

$$3. \frac{d}{dx}(x) = 1 \Rightarrow \int 1 dx = x + c.$$

$$4. \frac{d}{dx}(kx) = k \Rightarrow \int k dx = kx + c.$$

$$5. \frac{d}{dx}(\log|x|) = \frac{1}{x} \Rightarrow \int \frac{1}{x} dx = \log|x| + c.$$

$$6. \frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}} \Rightarrow \int \frac{1}{\sqrt{x}} dx = 2\sqrt{x} + c.$$

$$7. \int \frac{|x|}{x} dx = |x| + c$$

$$8. \frac{d}{dx}(e^x) = e^x \Rightarrow \int e^x dx = e^x + c.$$

$$9. \frac{d}{dx}\left(\frac{a^x}{\log a}\right) = \frac{a^x}{\log a}(\log a) = a^x \Rightarrow \int a^x dx = \frac{a^x}{\log a} + c.$$

$$10. \frac{d}{dx}(\cos x) = -\sin x \Rightarrow \int \sin x dx = -\cos x + c.$$

$$11. \frac{d}{dx}(\sin x) = \cos x \\ \Rightarrow \int \cos x dx = \sin x + c.$$



$$12. \frac{d}{dx}(\tan x) = \sec^2 x \\ \Rightarrow \int \sec^2 x dx = \tan x + c.$$

$$13. \frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x \\ \Rightarrow \int \operatorname{cosec}^2 x dx = -\cot x + c.$$

$$14. \frac{d}{dx}(\sec x) = \sec x \cdot \tan x \\ \Rightarrow \int \sec x \cdot \tan x dx = \sec x + c.$$

$$15. \frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cdot \cot x \\ \Rightarrow \int \operatorname{cosec} x \cdot \cot x dx = -\operatorname{cosec} x + c.$$

$$16. \frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} \\ \Rightarrow \int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + c.$$

$$17. \frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}} \\ \Rightarrow \int -\frac{1}{\sqrt{1-x^2}} dx = \cos^{-1} x + c.$$

$$18. \frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2} \\ \Rightarrow \int \frac{1}{1+x^2} dx = \tan^{-1} x + c.$$



Model-1: [Type - I]

$$1. \int \frac{x+1}{x^2+3x+12} dx \quad [\because \int \frac{px+q}{ax^2+bx+c} dx \text{ OR } \int \frac{L}{Q} dx]$$

Sol:

{ L = AQ' + B }

Part-1

$$\text{Let } x+1 = [A(x^2 + 3x + 12)' + B] \dots \dots \dots (1)$$

$$\Rightarrow x+1 = A(2x+3) + B$$

$$\Rightarrow x+1 = 2Ax + 3A + B$$

equating co-efficient of x & consts on B.S

$$2A=1 \Rightarrow A=\frac{1}{2}; \quad 3A+B=1 \Rightarrow B=1-3\left(\frac{1}{2}\right) \Rightarrow B=-\frac{1}{2}$$

Part-2

$$\text{Now } I = \int \frac{x+1}{x^2+3x+12} dx = \int \frac{[A(x^2+3x+12)' + B]}{x^2+3x+12} dx$$

$$= A \int \frac{(x^2+3x+12)'}{x^2+3x+12} dx + B \int \frac{1}{x^2+3x+12} dx$$

$\therefore \int \frac{f'(x)}{f(x)} dx = \log|f(x)| + c$

$$= \frac{1}{2} \log|x^2 + 3x + 12| + B I_1 \text{ (consider) ... (2)}$$



Part-3

$$I_1 = \int \frac{1}{x^2+3x+12} dx$$

$$= \int \frac{1}{[x^2+3x+\left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 + 12]} dx$$

$$\text{add and sub } \left(\frac{x-\text{co-e}}{2}\right)^2 = \left(\frac{3}{2}\right)^2$$

$$= \int \frac{1}{\left(x+\frac{3}{2}\right)^2 - \frac{9}{4} + 12} dx$$

$$= \int \frac{1}{\left(x+\frac{3}{2}\right)^2 + \frac{39}{4}} dx$$

$$= \int \frac{1}{\left(x+\frac{3}{2}\right)^2 + \left(\frac{\sqrt{39}}{2}\right)^2} dx \quad \boxed{\therefore \int \frac{1}{X^2 + a^2} dx = \frac{1}{a} \tan^{-1} \left[\frac{x}{a} \right] + c}$$

$$= \frac{1}{\frac{\sqrt{39}}{2}} \tan^{-1} \left[\frac{x+\frac{3}{2}}{\frac{\sqrt{39}}{2}} \right]$$

$$= \frac{2}{\sqrt{39}} \tan^{-1} \left[\frac{2x+3}{\sqrt{39}} \right] + c$$

$$\int \frac{x+1}{x^2+x+1} dx = \frac{1}{2} \log|x^2 + x + 1| + \frac{1}{2} \frac{1}{\frac{\sqrt{3}}{2}} \tan^{-1} \left[\frac{2x+1}{\sqrt{3}} \right] + c$$

$$I = \frac{1}{2} \log|x^2 + x + 1| + \frac{1}{\sqrt{3}} \tan^{-1} \left[\frac{2x+1}{\sqrt{3}} \right] + c$$



[Type - II]

2. $\int \frac{2x+5}{\sqrt{x^2-2x+10}} dx \left[\because \int \frac{px+q}{\sqrt{ax^2+bx+c}} dx OR \int \frac{L}{\sqrt{Q}} dx \right]$
 $\{ L = AQ' + B \}$

Part-1

let $2x+5 = [A(x^2 - 2x + 10)' + B] eq'n ... (1)$

$$2x+5 = A(2x-2) + B$$

$$2x+5 = 2Ax - 2A + B$$

equating co-efficient of x & consts on B.S

$$2A=2 \Rightarrow A=1 \quad -2A+B=5 \Rightarrow B=5+(2A) \Rightarrow B=7$$

Part-2

Now $I = \int \frac{2x+5}{\sqrt{x^2-2x+10}} dx = \int \frac{[A(x^2-2x+10)' + B]}{\sqrt{x^2-2x+10}} dx$

$$= 1 \int \frac{(x^2-2x+10)'}{\sqrt{x^2-2x+10}} dx + B \int \frac{1}{\sqrt{x^2-2x+10}} dx$$

$$\boxed{\int \frac{f'(x)}{\sqrt{f(x)}} dx = 2\sqrt{f(x)} + C}$$

$$= 2\sqrt{x^2 - 2x + 10} + B I_1 \text{ (consider) ... (2)}$$



$$I_1 = \int \frac{1}{\sqrt{x^2-2x+10}} dx$$

$$= \int \frac{1}{\sqrt{x^2-2x+(1)^2-(1)^2+10}} dx$$

{ add and sub $\left(\frac{x-c_0e}{2}\right)^2 = \left(\frac{1}{2}\right)^2 \}$

$$= \int \frac{1}{\sqrt{(x-1)^2+9}} dx$$

$$= \int \frac{1}{\sqrt{(x-1)^2+3^2}} dx$$

$$= \sinh^{-1} \left[\frac{x-1}{3} \right] + C$$

$$\therefore \int \frac{1}{\sqrt{a^2+x^2}} dx = \sinh^{-1} \left[\frac{x}{a} \right] + C$$

$$\therefore \int \frac{2x+5}{\sqrt{x^2-2x+10}} dx = 2\sqrt{x^2 - 2x + 10} + 7 \sinh^{-1} \left[\frac{x-1}{3} \right] + C$$



$$3. \int \sqrt{\frac{5-x}{x-2}} dx = \int \frac{5-x}{\sqrt{(x-2)(5-x)}} dx = \int \frac{5-x}{\sqrt{-x^2+7x-10}} dx = \\ \left[\therefore \int \frac{px+q}{\sqrt{ax^2+bx+c}} dx \text{ OR } \int \frac{L}{\sqrt{Q}} dx \right]$$

Sol: $\{ L = AQ' + B \}$

Part-1

Let $5-x = [A(-x^2+7x-10)' + B] eq'n$

$$5-x = A(-2x+7) + B$$

$$5-x = -2Ax + 7A + B$$

equating co-efficient of x & consts on B.S

$$-2A = -1 \Rightarrow A = \frac{1}{2}; \quad 7A + B = 5 \Rightarrow B = 5 - 7A \Rightarrow B = 5 - \frac{7}{2} = \frac{3}{2}$$

Part-2

$$\text{Now } I = \int \frac{5-x}{\sqrt{-x^2+7x-10}} dx = \int \frac{[A(-x^2+7x-10)' + B]}{\sqrt{-x^2+7x-10}} dx$$

$$= \frac{1}{2} \int \frac{(-x^2+7x-10)'}{\sqrt{-x^2+7x-10}} dx + \frac{3}{2} \int \frac{1}{\sqrt{-x^2+7x-10}} dx$$

$$\therefore \int \frac{f'(x)}{\sqrt{f(x)}} dx = 2\sqrt{f(x)} + C$$

$$= \frac{1}{2} 2\sqrt{-x^2+7x-10} + BI_1 \text{ (consider) ... (2)}$$



$$I_1 = \int \frac{1}{\sqrt{-x^2+7x-10}} dx = \int \frac{1}{\sqrt{-[x^2-7x+10]}} dx \\ \{ \text{ add and sub } \left(\frac{x-7}{2} \right)^2 = \left(\frac{7}{2} \right)^2 \}$$

$$= \int \frac{1}{\sqrt{-[x^2-7x+\left(\frac{7}{2}\right)^2+10-\left(\frac{7}{2}\right)^2]}} dx$$

$$= \int \frac{1}{\sqrt{-\left[\left(x-\frac{7}{2}\right)^2+10-\frac{49}{4}\right]}} dx$$

$$= \int \frac{1}{\sqrt{-\left[\left(x-\frac{7}{2}\right)^2-\frac{9}{4}\right]}} dx$$

$$= \int \frac{1}{\sqrt{\left(\frac{3}{2}\right)^2 - \left(x-\frac{7}{2}\right)^2}} dx$$

$$= \sin^{-1} \left| \frac{x-\frac{7}{2}}{\frac{3}{2}} \right| + C$$

$$= \sin^{-1} \left| \frac{2x-7}{3} \right| + C$$

$$\int \sqrt{\frac{5-x}{x-2}} dx = \sqrt{-x^2+7x-10} + \frac{3}{2} \sin^{-1} \left| \frac{2x-7}{3} \right| + C$$

$$\therefore \int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1} \left| \frac{x}{a} \right| + C$$



$$4. \int \frac{x+1}{\sqrt{x^2-x+1}} dx \left[\because \int \frac{px+q}{\sqrt{ax^2+bx+c}} dx \right] \left[\because \int \frac{L}{\sqrt{Q}} dx \right]$$

Sol: Rule: $L = AQ' + B$

LET $x+1 = [A(x^2 - x + 1)' + B] \text{ eq'n}$

PART-1

$$x+1 = A(2x-1) + B$$

$$x+1 = 2Ax - A + B$$

equating co-efficients of x & consts on R.H.S

$$2A=1 \Rightarrow A=\frac{1}{2}; \quad -A+B=1 \Rightarrow B=1+\left(\frac{1}{2}\right) \Rightarrow B=\frac{3}{2}$$

$$I = \int \frac{x+1}{\sqrt{x^2-x+1}} dx$$

PART-2

$$= \int \frac{\left[\frac{1}{2}(x^2-x+1)' + \frac{3}{2}\right]}{\sqrt{x^2-x+1}} dx$$

$$= \frac{1}{2} \int \frac{(x^2-x+1)'}{\sqrt{x^2-x+1}} dx - \frac{1}{2} \int \frac{1}{\sqrt{x^2-x+1}} dx$$

$$\therefore \int \frac{f'(x)}{\sqrt{f(x)}} dx = 2\sqrt{f(x)} + c$$

$$= \frac{1}{2} 2\sqrt{x^2-x+1} + \frac{3}{2} \int \frac{1}{\sqrt{x^2-x+1}} dx$$



Consider:

$$\begin{aligned} I_1 &= \int \frac{1}{\sqrt{x^2-x+1}} dx \\ &= \int \frac{1}{\sqrt{x^2-x+\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2+1}} dx \\ &= \int \frac{1}{\sqrt{\left(x-\frac{1}{2}\right)^2-\frac{1}{4}+1}} dx \\ &= \int \frac{1}{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}} dx \\ &= \int \frac{1}{\left(x-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} dx \end{aligned}$$

$$\begin{aligned} &= \sinh^{-1} \left[\frac{x-\frac{1}{2}}{\frac{\sqrt{3}}{2}} \right] + c \\ &= \sinh^{-1} \left[\frac{2x-1}{\sqrt{3}} \right] + c \end{aligned}$$

$$\int \frac{x+1}{\sqrt{x^2-x+1}} dx = \sqrt{x^2-x+1} + \frac{3}{2} \sinh^{-1} \left[\frac{2x-1}{\sqrt{3}} \right] + c$$

$$\therefore \int \frac{1}{\sqrt{a^2+x^2}} dx = \sinh^{-1} \left[\frac{x}{a} \right] + c$$



$$[Type - III] \quad \therefore \int (px + q)\sqrt{ax^2 + bx + c} dx$$

$$5. \int (6x + 5)\sqrt{6 - 2x^2 + x} dx$$

sol:

$$L = A Q' + B$$

$$6x + 5 = [A(6 - 2x^2 + x)' + B] \text{ eq'n(1)}$$

$$\Rightarrow 6x + 5 = A(-4x + 1) + B$$

$$\Rightarrow 6x + 5 = -4Ax + A + B$$

equating co-efficients of x & consts on B.S

$$-4A = 6 \Rightarrow A = \frac{-3}{2} \quad A + B = 5 \Rightarrow B = 5 - \left(\frac{-3}{2}\right) \Rightarrow B = \frac{13}{2}$$

Sub A, B in eq'n (1)

$$I = \int (6x + 5)\sqrt{6 - 2x^2 + x} dx$$

$$= \int \left[\frac{-3}{2}(6 - 2x^2 + x)' + \frac{13}{2}\right] \sqrt{6 - 2x^2 + x} dx$$

$$= -\frac{3}{2} \int (6 - 2x^2 + x)' \sqrt{6 - 2x^2 + x} dx + \frac{13}{2} \sqrt{6 - 2x^2 + x} dx$$

$$\int \{f(x)\}^{1/2} f'(x) dx = \frac{\{f(x)\}^{\frac{3}{2}}}{\frac{3}{2}} + C.$$

$$= -\frac{3}{2} \frac{(6 - 2x^2 + x)^{\frac{3}{2}}}{\frac{3}{2}} + \frac{13}{2} \sqrt{6 - 2x^2 + x} dx$$



$$\text{Consider: } \int \sqrt{6 - 2x^2 + x} dx$$

$$= \int \sqrt{-2(x^2 - \frac{x}{2} - 3)} dx \quad \text{add and sub } \left(\frac{x - \text{coff}}{2}\right)^2 = \left(\frac{1}{4}\right)^2$$

$$= \int \sqrt{-2 \left[x^2 - \frac{x}{2} + \left(\frac{1}{4}\right)^2 - \left(\frac{1}{4}\right)^2 - 3\right]} dx$$

$$= \int \sqrt{-2 \left[\left(x - \frac{1}{4}\right)^2 - \frac{1}{16} - 3\right]} dx$$

$$= \int \sqrt{-2 \left[\left(x - \frac{1}{4}\right)^2 - \frac{49}{16}\right]} dx$$

$$= \int \sqrt{-2 \left[\left(x - \frac{1}{4}\right)^2 - \left(\frac{7}{4}\right)^2\right]} dx$$

$$= \int \sqrt{2} \sqrt{\left(\frac{7}{4}\right)^2 - \left(x - \frac{1}{4}\right)^2} dx$$

$$\therefore \int \sqrt{a^2 - x^2} dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \left(\frac{x}{a}\right) + C$$

$$= \sqrt{2} \frac{\left(x - \frac{1}{4}\right)}{2} \sqrt{\left(\frac{7}{4}\right)^2 - \left(x - \frac{1}{4}\right)^2} + \frac{49/16}{2} \sin^{-1} \left(\frac{x - \frac{1}{4}}{7/4}\right)$$

$$= \sqrt{2} \frac{(4x-1)}{8} \sqrt{\left(\frac{7}{4}\right)^2 - \left(x - \frac{1}{4}\right)^2} + \frac{49/16}{2} \sin^{-1} \left(\frac{4x-1}{7}\right) + C$$

I =

$$-(6 - 2x^2 + x)^{\frac{3}{2}} +$$

$$\frac{13}{2} \sqrt{2} \left\{ \frac{(4x-1)}{8} \sqrt{\left(\frac{7}{4}\right)^2 - \left(x - \frac{1}{4}\right)^2} + \frac{49/16}{2} \sin^{-1} \left(\frac{4x-1}{7}\right) \right\}$$



$$6. \int (3x - 2)\sqrt{2x^2 - x + 1} dx$$

Rule: $(px + q) = A(ax^2 + bx + c)' + B$

$$sol: 3x - 2 = [A(2x^2 - x + 1)' + B] eq'n(1)$$

$$\Rightarrow 3x - 2 = A(4x - 1) + B$$

$$\Rightarrow 3x - 2 = 4Ax - A + B$$

equating co-efficients of x & constants on B.S

$$4A=3 \Rightarrow A=\frac{3}{4}; -A+B=-2 \Rightarrow B=-2+\left(\frac{3}{4}\right) \Rightarrow B=\frac{-5}{4}$$

$$I=\int (3x - 2)\sqrt{2x^2 - x + 1} dx$$

$$= \int \left[\frac{3}{4}(2x^2 - x + 1)' - \frac{5}{4} \right] \sqrt{2x^2 - x + 1} dx$$

$$= \frac{3}{4} \int (2x^2 - x + 1)' \sqrt{2x^2 - x + 1} dx - \frac{5}{4} \int \sqrt{2x^2 - x + 1} dx$$

$$\int \{f(x)\}^n \cdot f'(x) dx = \frac{\{f(x)\}^{n+1}}{n+1} + c.$$

$$= \frac{3}{4} \frac{(2x^2 - x + 1)^{\frac{3}{2}}}{\frac{3}{2}} - \frac{5}{4} \int \sqrt{2x^2 - x + 1} dx$$



Consider: $\int \sqrt{2x^2 - x + 1} dx$

$$= \int \sqrt{2(x^2 - \frac{x}{2} + \frac{1}{2})} dx$$

$$= \int \sqrt{2[x^2 - \frac{x}{2} + (\frac{1}{4})^2 + \frac{1}{2} - (\frac{1}{4})^2]} dx$$

$$= \int \sqrt{2[(x - \frac{1}{4})^2 + \frac{1}{2} - \frac{1}{16}]} dx$$

$$= \int \sqrt{2[(x - \frac{1}{4})^2 + \frac{7}{16}]} dx$$

$$= \int \sqrt{2[(x - \frac{1}{4})^2 + (\frac{\sqrt{7}}{4})^2]} dx$$

$$\therefore \int \sqrt{a^2 + x^2} dx = \frac{x}{2} \sqrt{a^2 + x^2} + \frac{a^2}{2} \sinh^{-1} \left(\frac{x}{a} \right) + c$$

$$= \frac{(x - \frac{1}{4})}{2} \sqrt{(x - \frac{1}{4})^2 + \frac{7}{16}} + \frac{7/16}{2} \sinh^{-1} \left(\frac{x - \frac{1}{4}}{\sqrt{7}/4} \right)$$

$$= \frac{(4x-1)}{8} \sqrt{(x - \frac{1}{4})^2 + \frac{7}{16}} + \frac{7/16}{2} \sinh^{-1} \left(\frac{4x-1}{\sqrt{7}} \right)$$

$$I = \frac{3}{4} \frac{(2x^2 - x + 1)^{\frac{3}{2}}}{\frac{3}{2}} - \frac{5\sqrt{2}}{4} \left\{ \frac{(4x-1)}{8} \sqrt{(x - \frac{1}{4})^2 + (\frac{\sqrt{7}}{4})^2} + \frac{7}{2} \sinh^{-1} \left(\frac{4x-1}{\sqrt{7}} \right) \right\} + c$$

$$= \frac{(2x^2 - x + 1)^{\frac{3}{2}}}{2} - \left\{ \frac{5}{32} (4x-1) \sqrt{2x^2 - x + 1} - \frac{35}{64\sqrt{2}} \sinh^{-1} \left(\frac{4x-1}{\sqrt{7}} \right) \right\} + c$$



[Type - IV] $\int \frac{1}{(px+q)\sqrt{ax^2+bx+c}} dx$

7. $\int \frac{1}{(1+x)\sqrt{3+2x-x^2}} dx$

sol: Put $(1+x) = \frac{1}{t} \Rightarrow$

$$t = \frac{1}{1+x}$$

diff . w.r.t x \Rightarrow

$$dx = -\frac{1}{t^2} dt$$

and $x = \frac{1}{t} - 1 \Rightarrow$

$$\int \frac{1}{(1+x)\sqrt{3+2x-x^2}} dx$$

$$x = \frac{1-t}{t}$$

$$\int \frac{1}{\frac{1}{t}\sqrt{3+2(\frac{1-t}{t})-(\frac{1-t}{t})^2}} \left(-\frac{1}{t^2} dt\right)$$

$$= - \int \frac{1}{\frac{1}{t}\sqrt{\frac{3t^2+2t(1-t)-(1+t^2-2t)}{t^2}}} \left(\frac{1}{t^2} dt\right)$$

$$= - \int \frac{1}{\frac{1}{t^2}\sqrt{3t^2+2t-2t^2-1-t^2+2t}} \left(\frac{1}{t^2} dt\right)$$

$$= - \int \frac{1}{\sqrt{4t-1}} dt$$

$$= - \frac{2\sqrt{4t-1}}{4}$$

$$= - \frac{1}{2} \sqrt{4\left(\frac{1}{1+x}\right) - 1}$$

$$= - \frac{1}{2} \sqrt{\frac{4-1-x}{1+x}} = - \frac{1}{2} \sqrt{\frac{3-x}{1+x}} + C$$

$$\int \frac{1}{\sqrt{ax+b}} dx = \frac{2\sqrt{ax+b}}{a} + c$$



8. $\int \frac{1}{(1-x)\sqrt{3-2x-x^2}} dx$

sol: Put $(1-x) = \frac{1}{t} \Rightarrow t = \frac{1}{1-x}$

diff . w.r.t x \Rightarrow

$$-dx = -\frac{1}{t^2} dt$$

and $-x = \frac{1}{t} - 1 \Rightarrow$

$$= \int \frac{1}{(1-x)\sqrt{3-2x-x^2}} dx$$

$$x = \frac{t-1}{t}$$

$$= \int \frac{1}{\frac{1}{t}\sqrt{3-2(\frac{t-1}{t})-(\frac{t-1}{t})^2}} \left(\frac{1}{t^2} dt\right)$$

$$= \int \frac{1}{\frac{1}{t}\sqrt{\frac{3t^2-2t(t-1)-(1+t^2-2t)}{t^2}}} \left(\frac{1}{t^2} dt\right)$$

$$= \int \frac{1}{\frac{1}{t^2}\sqrt{3t^2-2t^2+2t-1-t^2+2t}} \left(\frac{1}{t^2} dt\right)$$

$$\int \frac{1}{\sqrt{ax+b}} dx = \frac{2\sqrt{ax+b}}{a} + c$$

$$= \int \frac{1}{\sqrt{4t-1}} dt = \frac{2\sqrt{4t-1}}{4}$$

$$= \frac{1}{2} \sqrt{4\left(\frac{1}{1-x}\right) - 1}$$

$$= \frac{1}{2} \sqrt{\frac{4-1+x}{1-x}} = \frac{1}{2} \sqrt{\frac{3+x}{1-x}} + C$$



Model-2: integration of functions which are rational in
 $\sin x$ and $\cos x$

I. If the integral of the form

$$\int \frac{1}{a+b\cos x} dx \text{ or } \int \frac{1}{a+b\sin x} dx \text{ or } \int \frac{1}{a\cos x+b\sin x+c} dx$$

[Type - I]

$$1) \int \frac{1}{5+4\cos x} dx$$

$$sol: \int \frac{1}{5+4\cos x} dx$$

$$= \int \frac{1}{\left[5+4\left(\frac{1-t^2}{1+t^2}\right)\right]} \left(\frac{2dt}{1+t^2}\right)$$

$$\text{Let } t = \tan\left(\frac{x}{2}\right);$$

$$\sin x = \frac{2t}{1+t^2}$$

$$dx = \frac{2dt}{1+t^2};$$

$$\cos x = \frac{1-t^2}{1+t^2}$$

$$= \int \frac{1}{\left[\frac{5+5t^2+4-4t^2}{1+t^2}\right]} \left(\frac{2dt}{1+t^2}\right)$$

$$= 2 \int \frac{1}{[t^2 + 9]} dt$$

$$= 2 \int \frac{1}{[t^2 + 3^2]} dt$$

$$\therefore \int \frac{1}{a^2+x^2} dx = \frac{1}{a} \tan^{-1} \left[\frac{x}{a} \right] + C$$

$$= \frac{2}{3} \tan^{-1} \left[\frac{t}{3} \right] + C$$

$$= \frac{2}{3} \tan^{-1} \left[\frac{\tan\left(\frac{x}{2}\right)}{3} \right] + C .$$



$$2) \int \frac{1}{4+5\sin x} dx$$

$$sol: \int \frac{1}{4+5\sin x} dx = \int \frac{1}{\left[4+5\left(\frac{2t}{1+t^2}\right)\right]} \left(\frac{2dt}{1+t^2}\right)$$

$$= \int \frac{1}{\left[\frac{4+4t^2+10t}{1+t^2}\right]} \left(\frac{2dt}{1+t^2}\right) = \frac{2}{4} \int \frac{1}{\left[t^2 + \frac{10t}{4} + 1\right]} dt$$

$$= \frac{1}{2} \int \frac{1}{\left[t^2 + \frac{5t}{4} + \left(\frac{5}{4}\right)^2 - \left(\frac{5}{4}\right)^2 + 1\right]} dt$$

$$= \frac{1}{2} \int \frac{1}{\left[\left(t + \frac{5}{4}\right)^2 - \frac{25}{16} + 1\right]} dt = \frac{1}{2} \int \frac{1}{\left[\left(t + \frac{5}{4}\right)^2 - \frac{9}{16}\right]} dt$$

$$= \frac{1}{2} \int \frac{1}{\left[\left(t + \frac{5}{4}\right)^2 - \left(\frac{3}{4}\right)^2\right]} dt$$

$$\therefore \int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + C$$

$$= \frac{1}{2} \cdot \frac{1}{2\left(\frac{3}{4}\right)} \log \left| \frac{\left(t + \frac{5}{4}\right) - \frac{3}{4}}{\left(t + \frac{5}{4}\right) + \frac{3}{4}} \right| + C$$

$$= \frac{1}{3} \log \left| \frac{4t+5-3}{4t+5+3} \right|$$

$$= \frac{1}{3} \log \left| \frac{4t+2}{4t+8} \right| + C$$

$$= \frac{1}{3} \log \left| \frac{2(2t+1)}{2(4t+2)} \right| + C$$

$$= \frac{1}{3} \log \left| \frac{2 \tan\left(\frac{x}{2}\right) + 1}{2 \left[\tan \frac{x}{2} + 2 \right]} \right| + C$$



$$3) \int \frac{1}{1+\sin x + \cos x} dx$$

$$\text{Sol: } I = \int \frac{1}{\cos x + \sin x + 1} dx$$

$$= \int \frac{1}{\left[\frac{(1-t^2)}{1+t^2} + \frac{2t}{1+t^2} + 1\right]} \frac{2dt}{1+t^2}$$

$$= \int \frac{1}{\frac{1-t^2+2t+1+t^2}{1+t^2}} \left(\frac{2dt}{1+t^2}\right)$$

$$= \int \frac{1}{[2+2t]} 2dt$$

$$= \int \frac{2}{2[1+t]} dt$$

$$= \int \frac{1}{[1+t]} dt$$

$$= \log|1+t| + C$$

$$I = \log \left| 1 + \tan \left(\frac{x}{2} \right) \right| + C$$

$$\text{Let } t = \tan \left(\frac{x}{2} \right);$$

$$dx = \frac{2dt}{1+t^2};$$

$$\cos x = \frac{1-t^2}{1+t^2}$$

$$\sin x = \frac{2t}{1+t^2}$$

$$\therefore \int \frac{1}{x} dx = \log|x| + C$$



$$4) \int \frac{1}{4\cos x + 3\sin x} dx$$

$$\text{sol: } \int \frac{1}{4\cos x + 3\sin x} dx$$

$$= \int \frac{1}{\left[4\left(\frac{1-t^2}{1+t^2}\right) + 3\left(\frac{2t}{1+t^2}\right)\right]} \frac{2dt}{1+t^2}$$

$$= 2 \int \frac{1}{\left[\frac{4-4t^2+6t}{1+t^2}\right]} \frac{dt}{1+t^2}$$

$$= 2 \int \frac{1}{[-4(t^2 - \frac{6}{4}t - 1)]} dt = -\frac{1}{2} \int \frac{1}{[(t^2 - \frac{3}{2}t - 1)]} dt$$

$$= -\frac{1}{2} \int \frac{1}{\left[t^2 - \frac{3}{2}t + \left(\frac{-3}{4}\right)^2 - \left(\frac{-3}{4}\right)^2 - 1\right]} dt$$

$$= -\frac{1}{2} \int \frac{1}{\left[\left(t - \frac{3}{4}\right)^2 - \frac{9}{16} - 1\right]} dt$$

$$= -\frac{1}{2} \int \frac{dt}{\left[\left(t - \frac{3}{4}\right)^2 - \left(\frac{5}{4}\right)^2\right]}$$

$$= \frac{1}{2} \int \frac{dt}{\left[\left(\frac{5}{4}\right)^2 - \left(t - \frac{3}{4}\right)^2\right]}$$

$$\text{add \& sub } \left(\frac{t \text{ coeff}}{2}\right)^2 = \left(\frac{-3}{2 \times 2}\right)^2 = \left(\frac{-3}{4}\right)^2$$

$$\therefore \int \frac{1}{a^2 - x^2} dx = \frac{1}{2a} \log \left| \frac{a+x}{a-x} \right| + C$$

$$= -\frac{1}{2} \frac{1}{2\left(\frac{5}{4}\right)} \log \left| \frac{\frac{5}{4} + \left(t - \frac{3}{4}\right)}{\frac{5}{4} - \left(t - \frac{3}{4}\right)} \right| - \frac{1}{5} \log \left| \frac{5+4t-3}{5-4t+3} \right| + C$$

$$= \frac{1}{5} \log \left| \frac{2+4t}{8-4t} \right| = \frac{1}{5} \log \left| \frac{1+2t}{4-2t} \right| + C$$

$$= \frac{1}{5} \log \left| \frac{1+2\tan \left(\frac{x}{2} \right)}{4-2\tan \left(\frac{x}{2} \right)} \right| + C$$



$$5) \int \frac{1}{\sin x + \sqrt{3} \cos x} dx$$

$$\text{Sol: } \int \frac{1}{\sin x + \sqrt{3} \cos x} dx$$

$$\begin{aligned} &= \int \frac{1}{\left[\frac{2t}{1+t^2} + \sqrt{3} \left(\frac{1-t^2}{1+t^2} \right) \right]} \left(\frac{2dt}{1+t^2} \right) \\ &= 2 \int \frac{1}{\left[\frac{2t + \sqrt{3} - \sqrt{3}t^2}{1+t^2} \right]} \left(\frac{dt}{1+t^2} \right) \\ &= 2 \int -\frac{1}{\sqrt{3} \left[t^2 - \frac{2}{\sqrt{3}}t - 1 \right]} dt \end{aligned}$$

add&sub

$$\left(\frac{t \text{coeff}}{2} \right)^2 = \left(\frac{-2}{\sqrt{3} \times 2} \right)^2 = \left(\frac{-1}{\sqrt{3}} \right)^2$$

$$= -\frac{2}{\sqrt{3}} \int \frac{1}{\left[t^2 - \frac{2}{\sqrt{3}}t + \left(\frac{-1}{\sqrt{3}} \right)^2 - \left(\frac{-1}{\sqrt{3}} \right)^2 - 1 \right]} dt$$

$$= -\frac{2}{\sqrt{3}} \int \frac{1}{\left[\left(t - \frac{1}{\sqrt{3}} \right)^2 - \frac{1}{3} - 1 \right]} dt$$

$$= -\frac{2}{\sqrt{3}} \int \frac{1}{\left[\left(t - \frac{1}{\sqrt{3}} \right)^2 - \frac{4}{3} \right]} dt$$

$$= \frac{2}{\sqrt{3}} \int \frac{1}{\left[\left(\frac{2}{\sqrt{3}} \right)^2 - \left(t - \frac{1}{\sqrt{3}} \right)^2 \right]} dt$$

$$\therefore \int \frac{1}{a^2 - x^2} dx = \frac{1}{2a} \log |a+x| + C$$

$$= \frac{2}{\sqrt{3}} \frac{1}{2 \cdot \left(\frac{2}{\sqrt{3}} \right)} \log \left| \frac{\frac{2}{\sqrt{3}} + \left(t - \frac{1}{\sqrt{3}} \right)}{\frac{2}{\sqrt{3}} - \left(t - \frac{1}{\sqrt{3}} \right)} \right| + C$$

$$= \frac{1}{2} \log \left| \frac{\frac{1}{\sqrt{3}} + t}{\frac{3}{\sqrt{3}} - t} \right| + C$$

$$= \frac{1}{2} \log \left| \frac{1 + \sqrt{3} \tan\left(\frac{x}{2}\right)}{3 - \sqrt{3} \tan\left(\frac{x}{2}\right)} \right| + C$$

$$\text{Let } t = \tan\left(\frac{x}{2}\right)$$

$$dx = \frac{2dt}{1+t^2}$$

$$\sin x = \frac{2t}{1+t^2}$$

$$\cos x = \frac{1-t^2}{1+t^2}$$



$$6) \int \frac{1}{3 \cos x + 4 \sin x + 6} dx$$

$$\text{Sol: } \int \frac{1}{3 \cos x + 4 \sin x + 6} dx$$

$$= \int \frac{1}{\left[3 \left(\frac{1-t^2}{1+t^2} \right) + 4 \left(\frac{2t}{1+t^2} \right) + 6 \right]} \left(\frac{dt}{1+t^2} \right)$$

$$= 2 \int \frac{1}{\left[\frac{3-3t^2+8t+6+6t^2}{1+t^2} \right]} \left(\frac{dt}{1+t^2} \right)$$

$$= 2 \int \frac{1}{[3t^2+8t+9]} dt$$

$$= 2 \int \frac{1}{3[t^2+\frac{8}{3}t+3]} dt$$

$$\left(\frac{t \text{coeff}}{2} \right)^2 = \left(\frac{8}{3 \times 2} \right)^2 = \left(\frac{4}{3} \right)^2$$

$$= \frac{2}{3} \int \frac{1}{\left[t^2 + \frac{8}{3}t + \left(\frac{4}{3} \right)^2 + 3 - \left(\frac{4}{3} \right)^2 \right]} dt$$

$$= \frac{2}{3} \int \frac{1}{\left[\left(t + \frac{4}{3} \right)^2 + 3 - \frac{16}{9} \right]} dt$$

$$= \frac{2}{3} \int \frac{1}{\left[\left(t + \frac{4}{3} \right)^2 + \frac{11}{9} \right]} dt$$

$$= \frac{2}{3} \int \frac{1}{\left[\left(t + \frac{4}{3} \right)^2 + \left(\frac{\sqrt{11}}{3} \right)^2 \right]} dt$$

$$\therefore \int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \left[\frac{x}{a} \right] + C$$

$$= \frac{2}{3} \frac{1}{\sqrt{11}} \tan^{-1} \left[\frac{t + \frac{4}{3}}{\sqrt{11}} \right] + C$$

$$= \frac{2}{\sqrt{11}} \tan^{-1} \left[\frac{3t+4}{\sqrt{11}} \right] + C$$

$$= \frac{2}{\sqrt{11}} \tan^{-1} \left[\frac{3 \tan\left(\frac{x}{2}\right) + 4}{\sqrt{11}} \right] + C.$$



$$7) \int \frac{1}{5+4\cos 2x} dx$$

$$sol: \int \frac{1}{5+4\cos 2x} dx$$

$$= \int \frac{1}{\left[5+4\left(\frac{1-t^2}{1+t^2}\right)\right]} \left(\frac{dt}{1+t^2}\right)$$

$$= \int \frac{1}{\left[\frac{5+5t^2+4-4t^2}{1+t^2}\right]} \left(\frac{dt}{1+t^2}\right) = \int \frac{1}{[t^2+9]} dt$$

$$= \int \frac{1}{[t^2+3^2]} dt$$

$$= \frac{1}{3} \tan^{-1} \left[\frac{t}{3} \right] + C$$

$$= \frac{2}{3} \tan^{-1} \left[\frac{\tan(x)}{3} \right] + C$$

Let $t = \tan(x)$

$$dx = \frac{dt}{1+t^2}$$

$$\sin 2x = \frac{2t}{1+t^2}$$

$$\cos 2x = \frac{1-t^2}{1+t^2}$$



$$8) \int \frac{1}{2-3\cos 2x} dx$$

$$sol: \int \frac{1}{2-3\cos 2x} dx$$

$$= \int \frac{1}{\left[2-3\left(\frac{1-t^2}{1+t^2}\right)\right]} \left(\frac{dt}{1+t^2}\right)$$

$$= \int \frac{1}{\left[\frac{2+2t^2-3+3t^2}{1+t^2}\right]} \left(\frac{dt}{1+t^2}\right) = \int \frac{1}{[5t^2-1]} dt$$

$$= \int \frac{1}{[(\sqrt{5}t)^2-1^2]} dt$$

$$= \frac{1}{2\sqrt{5}} \log \left| \frac{\sqrt{5}t-1}{\sqrt{5}t+1} \right|$$

$$= \frac{1}{2\sqrt{5}} \log \left| \frac{\sqrt{5} \tan\left(\frac{x}{2}\right)-1}{\sqrt{5} \tan\left(\frac{x}{2}\right)+1} \right| + c$$



[Type - II]

$$9) \int \frac{2\cos x + 3\sin x}{4\cos x + 5\sin x} dx$$

Sol:

$$= A [4\cos x + 5\sin x]' + B [4\cos x + 5\sin x]$$

$$\Rightarrow 2\cos x + 3\sin x$$

$$= A [-4\sin x + 5\cos x] + B [4\cos x + 5\sin x]$$

$$\Rightarrow 2\cos x + 3\sin x$$

$$= -4A\sin x + 5A\cos x + 4B\cos x + 5B\sin x$$

$$\Rightarrow 2\cos x + 3\sin x$$

$$= \sin x(-4A + 5B) + \cos x(5A + 4B)$$

Equating the co-efficient of
 $\cos x \Rightarrow 5A + 4B = 2;$

$$\Rightarrow 5A + 4B - 2 = 0 \dots (1)$$

$\sin x \Rightarrow -4A + 5B = 3;$

$$\Rightarrow -4A + 5B - 3 = 0 \dots (2)$$

solving (1) & (2)

$$\begin{array}{cccc} 5 & 4 & -2 & 5 \\ -4 & 5 & -3 & -4 \end{array}$$

$$(A, B) = \left[\frac{-12 - (-10)}{25 - (-16)}, \frac{8 - (-15)}{25 - (-16)} \right]$$

$$= \left[\frac{-12 + 10}{41}, \frac{8 + 15}{41} \right] = \left[\frac{-2}{41}, \frac{23}{41} \right]$$



$$I = \int \frac{2\cos x + 3\sin x}{4\cos x + 5\sin x} dx$$

$$= \int \frac{\frac{-2}{41}[4\cos x + 5\sin x]' + \frac{23}{41}[4\cos x + 5\sin x]}{4\cos x + 5\sin x} dx$$

$$= -\frac{2}{41} \int \frac{(4\cos x + 5\sin x)'}{(4\cos x + 5\sin x)} + \frac{23}{41} \int \frac{(4\cos x + 5\sin x)}{(4\cos x + 5\sin x)} dx$$

$$\therefore \int \frac{f'(x)}{f(x)} dx = \log|f(x)| + c$$

$$\therefore \int 1 dx = x + c$$

$$I = -\frac{2}{41} \log|(4\cos x + 5\sin x)| + \frac{23}{41} x + c.$$

$$10) \quad \int \frac{9\cos x - \sin x}{4\sin x + 5\cos x} dx$$

sol:

$$\text{Rule: } Nr = A \frac{d}{dx}(Dr) + B(Dr)$$

$$9\cos x - \sin x = A [4\sin x + 5\cos x]' + B [4\sin x + 5\cos x]$$

$$\Rightarrow 9\cos x - \sin x = A [4\cos x - 5\sin x] + B [4\sin x + 5\cos x]$$

$$\Rightarrow 9\cos x - \sin x = 4A\cos x - 5A\sin x + 4B\sin x + 5B\cos x$$

$$\Rightarrow 9\cos x - \sin x = \sin x(-5A + 4B) + \cos x(4A + 5B)$$



Equating the co-efficient of

$$\cos x \Rightarrow 4A + 5B = 9$$

$$\Rightarrow 4A + 5B - 9 = 0 \dots\dots (1)$$

$$\sin x \Rightarrow -5A + 4B = -1;$$

$$\Rightarrow -5A + 4B + 1 = 0 \dots\dots (2)$$

solving (1) & (2)

$$\begin{array}{cccc} 4 & 5 & -9 & 4 \\ -5 & 4 & 1 & -5 \end{array}$$

$$(A, B) = \left[\frac{5 - (-36)}{16 - (-25)}, \frac{45 - 4}{16 - (-25)} \right]$$

$$= \left[\frac{41}{41}, \frac{41}{41} \right] = [1, 1]$$

$$9\cos x - \sin x$$

$$= 1 [4\sin x + 5\cos x]' + 1 [4\sin x + 5\cos x]$$

$$\int \frac{9\cos x - \sin x}{4\sin x + 5\cos x} dx$$

$$= \int \frac{1(4\sin x + 5\cos x)' + 1[4\sin x + 5\cos x]}{4\sin x + 5\cos x} dx$$

$$= 1 \int \frac{(4\sin x + 5\cos x)'}{(4\sin x + 5\cos x)} dx + 1 \int \frac{(4\sin x + 5\cos x)}{(4\sin x + 5\cos x)} dx$$

$$\therefore \int \frac{f'(x)}{f(x)} dx = \log|f(x)| + c$$

$$\therefore \int 1 dx = x + c$$

$$I = \log|(4\sin x + 5\cos x)| + x + c.$$



[Type - III = I + II]

11) $\int \frac{\cos x + 3\sin x + 7}{\cos x + \sin x + 1} dx$

sol.

Consider:

$$I_1 = \int \frac{1}{\cos x + \sin x + 1} dx$$

$$= \int \frac{1}{\left[\left(\frac{1-t^2}{1+t^2} \right) + \left(\frac{2t}{1+t^2} \right) + 1 \right]} \frac{2dt}{1+t^2}$$

$$= \int \frac{1}{\frac{1-t^2+2t+1+t^2}{1+t^2}} \left(\frac{2dt}{1+t^2} \right)$$

$$= \int \frac{1}{[2+2t]} 2dt$$

$$= \int \frac{2}{2[1+t]} dt$$

$$= \int \frac{1}{[1+t]} dt$$

$$= \log|1+t| + C$$

$$I_1 = \log \left| 1 + \tan \left(\frac{x}{2} \right) \right| + C$$

$$\text{Let } t = \tan \left(\frac{x}{2} \right);$$

$$dx = \frac{2dt}{1+t^2};$$

$$\cos x = \frac{1-t^2}{1+t^2}$$

$$\sin x = \frac{2t}{1+t^2}$$

$$\therefore \int \frac{1}{x} dx = \log|x|$$



$$\text{Rule: } Nr = A \frac{d}{dx} (Dr) + B(Dr) + k$$

$$\begin{aligned} \cos x + 3\sin x + 7 &= A [\cos x + \sin x + 1]' + B [\cos x + \sin x + 1] + k \\ &= A [-\sin x + \cos x] + B [\cos x + \sin x + 1] + k \\ &= -A\sin x + A\cos x + B\cos x + B\sin x + B + k \end{aligned}$$

$$\Rightarrow \cos x + 3\sin x + 7 = \sin x(-A + B) + \cos x(A + B) + (B + k)$$

Equating the co-efficient of

$$\cos x \Rightarrow A + B = 1;$$

$$\Rightarrow A + B - 1 = 0 \dots \dots (1)$$

$$\sin x \Rightarrow -A + B = 3;$$

$$\Rightarrow -A + B - 3 = 0 \dots \dots (2)$$

$$\text{And } B + k = 7 \dots \dots (3)$$

solving (1) And (2)

$$\begin{array}{cccc} 1 & 1 & -1 & 1 \\ -1 & 1 & -3 & -1 \end{array}$$

$$(A, B) = \left[\frac{-3 - (-1)}{1 - (-1)}, \frac{1 - (-3)}{1 - (-1)} \right]$$

$$= \left[\frac{-2}{2}, \frac{4}{2} \right]$$

$= [-1, 2]$ Sub the value of B=2 in (3)

$$k = 7 - 2 = 5$$



$$I = \int \frac{\cos x + 3\sin x + 7}{\cos x + \sin x + 1} dx$$

$$= \int \frac{-1 [\cos x + \sin x + 1]' + 2 [\cos x + \sin x + 1] + 5}{\cos x + \sin x + 1} dx$$

$$\begin{aligned} &= -1 \int \frac{(\cos x + \sin x + 1)'}{(\cos x + \sin x + 1)} + 2 \int \frac{(\cos x + \sin x + 1) dx}{(\cos x + \sin x + 1)} \\ &\quad + \int \frac{5}{\cos x + \sin x + 1} dx \end{aligned}$$

$$\therefore \int \frac{f'(x)}{f(x)} dx = \log|f(x)| + c$$

$$\therefore \int 1 dx = x + c$$

$$= -1 \log|(\cos x + \sin x + 1)| + 2x + 5 \log \left| 1 + \tan \left(\frac{x}{2} \right) \right| + C$$

{from I₁}

$$I = -1 \log|(\cos x + \sin x + 1)| + 2x + 5 \log \left| 1 + \tan \left(\frac{x}{2} \right) \right| + C$$



12) $\int \frac{2\sin x + 3\cos x + 4}{3\sin x + 4\cos x + 5} dx$
sol:

Consider:

$$I_1 = \int \frac{1}{3\sin x + 4\cos x + 5} dx$$

$$= \int \frac{1}{[3(\frac{2t}{1+t^2}) + 4(\frac{1-t^2}{1+t^2}) + 5(\frac{1+t^2}{1+t^2})]} \left(\frac{2dt}{1+t^2} \right)$$

$$= \int \frac{1}{[\frac{6t+4-4t^2+5+5t^2}{1+t^2}]} \left(\frac{2dt}{1+t^2} \right)$$

$$= \int \frac{1}{[\frac{t^2+6t+9}{1+t^2}]} \frac{2dt}{1+t^2}$$

$$= \int \frac{1}{(t+3)^2} 2dt$$

Let $t = \tan\left(\frac{x}{2}\right);$

$$dx = \frac{2dt}{1+t^2};$$

$$\cos x = \frac{1-t^2}{1+t^2}$$

$$\sin x = \frac{2t}{1+t^2}$$

$$\therefore \int \frac{1}{x^2} dx = -\frac{1}{x} + C$$

$$= 2 \frac{-1}{[t+3]}$$

$$= \frac{-2}{[\tan(\frac{x}{2})+3]}$$



Rule: $Nr = A \frac{d}{dx}(Dr) + B(Dr) + k$

$$Nr = 2\sin x + 3\cos x + 4$$

$$= A [3\sin x + 4\cos x + 5]' + B [3\sin x + 4\cos x + 5] + k$$

$$= A [3\cos x - 4\sin x] + B [3\sin x + 4\cos x + 5] + k$$

$$= 3A\cos x - 4A\sin x + 3B\sin x + 4B\cos x + 5B + k$$

$$= \sin x(-4A + 3B) + \cos x(3A + 4B) + (5B + k)$$

Equating the co-efficient of

$$\cos x \Rightarrow 3A + 4B = 3;$$

$$\Rightarrow 3A + 4B - 3 = 0 \dots\dots (1)$$

$$\sin x \Rightarrow -4A + 3B = 2;$$

$$\Rightarrow -4A + 3B - 2 = 0 \dots\dots (2)$$

$$5B + k = 4 \dots\dots (3)$$

solving eq'n (1)&(2)

$$\begin{array}{rrrr} 1 & x & y \\ 3 & 4 & -3 & 3 \\ -4 & 3 & -2 & -4 \end{array} (A, B) = \left[\frac{-8-(-9)}{9-(-1)}, \frac{12-(-6)}{9-(-1)} \right] = \left[\frac{1}{25}, \frac{18}{25} \right]$$

$$\text{from (3)} \Rightarrow 5B + k = 4$$

$$\Rightarrow k = 4 - 5 \left(\frac{18}{25} \right)$$

$$\Rightarrow k = 4 - \left(\frac{18}{5} \right) = \frac{20-18}{5} = \frac{2}{5}$$



$$I = \frac{1}{25} [3\sin x + 4\cos x + 5]' + \frac{18}{25} [3\sin x + 4\cos x + 5] + \frac{2}{5}$$

$$I = \int \frac{\frac{1}{25}[3\sin x + 4\cos x + 5]' + \frac{18}{25}[3\sin x + 4\cos x + 5] + \frac{2}{5}}{3\sin x + 4\cos x + 5} dx$$

$$\begin{aligned} &= \frac{1}{25} \int \frac{(3\sin x + 4\cos x + 5)'}{(3\sin x + 4\cos x + 5)} dx + \frac{18}{25} \int \frac{(3\sin x + 4\cos x + 5)dx}{(3\sin x + 4\cos x + 5)} \\ &\quad + \frac{2}{5} \int \frac{1}{3\sin x + 4\cos x + 5} dx \end{aligned}$$

$$\int \frac{f'(x)}{f(x)} dx = \log|f(x)| + C$$

$$\therefore \int 1 dx = x + C$$

$$= \frac{1}{25} \log |(3\sin x + 4\cos x + 5)| + \frac{18}{25} x + \frac{2}{5} \frac{-2}{\tan(\frac{x}{2}) + 3}$$

{From I₁}

$$I = \frac{1}{25} \log |3\sin x + 4\cos x + 5| + \frac{18}{25} x + \frac{-\frac{4}{5}}{\tan(\frac{x}{2}) + 3} + C$$



Model-3: REDUCTION FORMULAE

1) If $I_n = \int \sin^n x dx$, then show that

$$I_n = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} I_{n-2} \text{ and hence find } I_4.$$

$$\text{Sol: } I_n = \int \sin^n x dx = \int \sin^{n-1} x \cdot \sin x dx$$

$$\text{Here } U = \sin^{n-1} x \Rightarrow U' = (n-1) \sin^{n-2} x (\cos x)$$

$$V = \sin x \Rightarrow \int \sin x dx = -\cos x + C$$

By using integration by parts

$$\int (UV) dx = U \int V dx - \int [U' \left(\int V dx \right)] dx$$

$$\begin{aligned} I_n &= \sin^{n-1} x \cdot (-\cos x) \\ &\quad - \int (n-1) \sin^{n-2} x (\cos x) (-\cos x) dx \end{aligned}$$

$$I_n = -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x (1 - \sin^2 x) dx$$

$$= \dots + (n-1) \int \sin^{n-2} x dx - (n-1) \int \sin^n x dx$$

$$I_n = -\sin^{n-1} x \cos x + (n-1) I_{n-2} - (n-1) I_n$$

$$I_n + (n-1) I_n = -\sin^{n-1} x \cos x + (n-1) I_{n-2}$$

$$I_n (1 + n - 1) = -\sin^{n-1} x \cos x + (n-1) I_{n-2}$$

$$I_n (n) = -\sin^{n-1} x \cos x + (n-1) I_{n-2}$$

$$I_n = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} I_{n-2}$$



Now

$$\begin{aligned} I_4 &= -\frac{\sin^{4-1} x \cos x}{4} + \frac{4-1}{4} I_{4-2} \\ &= -\frac{\sin^3 x \cos x}{4} + \frac{3}{4} I_2 \\ &= -\frac{\sin^3 x \cos x}{4} + \frac{3}{4} \left[-\frac{\sin^{2-1} x \cos x}{2} + \frac{2-1}{2} I_{2-2} \right] \\ &= -\frac{\sin^3 x \cos x}{4} - \frac{3 \sin^1 x \cos x}{8} + \frac{3}{8} I_0 \end{aligned}$$

$$[I_0 = \int \sin^0 x dx = \int 1 dx = x + c]$$

$$I_4 = -\frac{\sin^3 x \cos x}{4} - \frac{3}{8} \sin x \cos x + \frac{3}{8} x + c.$$

2) If $I_n = \int \cos^n x dx$, then show that

$$I_n = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} I_{n-2} \text{ and hence find } I_5, I_4.$$

$$\text{Sol: } I_n = \int \cos^n x dx = \int \cos^{n-1} x \cdot \cos x dx$$

$$\text{Here } U = \cos^{n-1} x \Rightarrow U' = (n-1) \cos^{n-2} x (-\sin x)$$

$$V = \cos x \Rightarrow \int \cos x dx = \sin x + c$$

By using integration by parts

$$\int (UV) dx = U \int V dx - \int [U' \int V dx] dx$$

$$\begin{aligned} I_n &= \cos^{n-1} x \cdot (\sin x) \\ &\quad - \int (n-1) \cos^{n-2} x (-\sin x) (\sin x) dx \end{aligned}$$

$$I_n = \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x (1 - \cos^2 x) dx$$



$$= \dots + (n-1) \int \cos^{n-2} x dx - (n-1) \int \cos^n x dx$$

$$I_n = \cos^{n-1} x \sin x + (n-1) I_{n-2} - (n-1) I_n$$

$$I_n + (n-1) I_n = \cos^{n-1} x \sin x + (n-1) I_{n-2}$$

$$I_n (1 + n - 1) = \cos^{n-1} x \sin x + (n-1) I_{n-2}$$

$$I_n (n) = \cos^{n-1} x \sin x + (n-1) I_{n-2}$$

$$I_n = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} I_{n-2}$$

Now

$$\begin{aligned} I_4 &= \frac{\cos^{4-1} x \sin x}{4} + \frac{4-1}{4} I_{4-2} \\ &= \frac{\cos^{4-1} x \sin x}{4} + \frac{3}{4} I_2 \\ &= \frac{\cos^3 x \sin x}{4} + \frac{3}{4} \left[-\frac{\cos^{2-1} x \sin x}{2} + \frac{2-1}{2} I_{2-2} \right] \\ &= \frac{\cos^3 x \sin x}{4} + \frac{3}{4} \cos^2 x \sin x + \frac{3}{8} I_0 \end{aligned}$$

$$[I_0 = \int \cos^0 x dx = \int 1 dx = x + c]$$

$$I_4 = \frac{\cos^3 x \sin x}{4} + \frac{3}{8} \cos x \sin x + \frac{3}{8} x + c.$$



3) Find the reduction formula for $\int \tan^n x dx$ and hence

find $\int \tan^6 x dx$.

Sol: $I_n = \int \tan^n x dx$

$$= \int \tan^{n-2} x (\tan^2 x) dx$$

$$= \int \tan^{n-2} x (\sec^2 x - 1) dx$$

$$= \int [\tan^{n-2} x \sec^2 x - \tan^{n-2} x] dx$$

$$= \int \tan^{n-2} x \sec^2 x dx - \int \tan^{n-2} x dx$$

let $\tan x = t$ Diff w.r.t.'x'

$$\sec^2 x dx = dt$$

$$I_n = \int t^{n-2} dt - I_{n-2}$$

$$I_n = \frac{t^{n-2+1}}{n-2+1} - I_{n-2}$$

$$I_n = \frac{t^{n-1}}{n-1} - I_{n-2}$$

$$I_n = \frac{\tan^{n-1} x}{n-1} - I_{n-2}$$

case(1): if n is even, I_n end with I_0 .

$$\{I_0 = \int \tan^0 x dx = \int 1 dx = x + C\}$$

case(2): if n is odd, I_n end with I_1 .

$$\{I_1 = \int \tan^1 x dx = \log|\sec x| + C\}$$

$$\text{Now } I_6 = \frac{\tan^{6-1} x}{6-1} - I_{6-2}$$



$$= \frac{\tan^5 x}{5} - I_4 = \frac{\tan^5 x}{5} - \left[\frac{\tan^{4-1} x}{4-1} - I_{4-2} \right]$$

$$= \frac{\tan^5 x}{5} - \frac{\tan^3 x}{3} + I_2$$

$$= \frac{\tan^5 x}{5} - \frac{\tan^3 x}{3} + \left[\frac{\tan^{2-1} x}{2-1} - I_{2-2} \right]$$

$$= \frac{\tan^5 x}{5} - \frac{\tan^3 x}{3} + \frac{\tan^1 x}{1} - I_0$$

$$= \frac{\tan^5 x}{5} - \frac{\tan^3 x}{3} + \frac{\tan^1 x}{1} - x + C$$

4) Find the reduction formula for $\int \cot^n x dx$ and hence

find $\int \cot^4 x dx$.

Sol: $I_n = \int \cot^n x dx$

$$= \int \cot^{n-2} x (\cot^2 x) dx$$

$$= \int \cot^{n-2} x (\cosec^2 x - 1) dx$$

$$= \int [\cot^{n-2} x \cosec^2 x - \cot^{n-2} x] dx$$

$$= \int \cot^{n-2} x \cosec^2 x dx - \int \cot^{n-2} x dx$$

let $\cot x = t$ Diff w.r.t.'x'

$$-\cosec^2 x dx = dt$$

$$I_n = - \int t^{n-2} dt - I_{n-2}$$



$$I_n = -\frac{t^{n-2+1}}{n-2+1} - I_{n-2}$$

$$I_n = -\frac{t^{n-1}}{n-1} - I_{n-2}$$

$$I_n = -\frac{\cot^{n-1}x}{n-1} - I_{n-2}$$

case(1): if n is even, I_n end with I_0 .

$$\{I_0 = \int \cot^0 x dx = \int 1 dx = x + C\}$$

case(2): if n is odd, I_n end with I_1 .

$$\{I_1 = \int \cot^1 x dx = \log|\sin x| + C\}$$

$$\text{Now } I_4 = -\frac{\cot^{4-1}x}{4-1} - I_{4-2}$$

$$= -\frac{\cot^3x}{3} - I_2 = -\frac{\cot^3x}{3} - \left[-\frac{\cot^{2-1}x}{2-1} - I_{2-2} \right]$$

$$= -\frac{\cot^3x}{3} + \frac{\cot^1x}{1} + I_0$$

$$= -\frac{\cot^3x}{3} + \cot x + x + C$$



5) If $I_n = \int \sec^n x dx$, then show that

$$I_n = -\frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} I_{n-2}$$

$$\text{Sol: } I_n = \int \sec^n x dx = \int \sec^{n-2} x \sec^2 x dx$$

$$U = \sec^{n-2} x \Rightarrow u' = (n-2) \sec^{n-3} x (\sec x \tan x)$$

$$V = \sec^2 x \Rightarrow \int \sec^2 x dx = \tan x + c$$

By using integration by parts

$$\int (uv) dx = U \int V dx - \int [U' \int V dx] dx$$

$$I_n = \sec^{n-2} x \int \sec^2 x dx \\ - \int \{(\sec^{n-2} x)' \int \sec^2 x dx\} dx$$

$$= \sec^{n-2} x \tan x - \int \{(n-2) \sec^{n-3} x \sec x \tan x dx . (\tan x)\} dx$$

$$= \sec^{n-2} x \tan x - (n-2) \int \sec^{n-2} x (\sec^2 x - 1) dx$$

$$='' - (n-2) \int (\sec^{n-2} x \sec^2 x - \sec^{n-2} x) dx$$

$$='' - (n-2) \int \sec^n x dx + (n-2) \int \sec^{n-2} x dx$$

$$I_n = \sec^{n-2} x \tan x - (n-2) I_n + (n-2) I_{n-2}$$

$$I_n + (n-2) I_n = \sec^{n-2} x \tan x + (n-2) I_{n-2}$$

$$I_n (1 + n - 2) = \sec^{n-2} x \tan x + (n-2) I_{n-2}$$



$$I_n(n-1) = \sec^{n-2}x \tan x + (n-2)I_{n-2}$$

$$I_n = \frac{\sec^{n-2}x \cdot \tan x}{n-1} + \frac{n-2}{n-1} I_{n-2}$$

$$\text{Now } I_5 = \frac{\sec^{5-2}x \cdot \tan x}{5-1} + \frac{5-2}{5-1} I_{5-2}$$

$$I_6 = \frac{\sec^3 x \cdot \tan x}{4} + \frac{3}{4} I_3$$

$$I_5 = \frac{\sec^3 x \cdot \tan x}{4} + \frac{3}{4} \left\{ \frac{\sec^1 x \cdot \tan x}{2} + \frac{1}{2} I_0 \right\}$$

$$\{I_0 = \int 1 dx = x + c\}$$

$$I_5 = \frac{\sec^3 x \cdot \tan x}{4} + \frac{3 \sec^1 x \cdot \tan x}{8} + \frac{3}{8} x + c$$

6) If $I_n = \int \sec^n x dx$, then show that

$$I_n = -\frac{\cosec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} I_{n-2}$$

$$\text{Sol: } I_n = \int \cosec^n x dx = \int \cosec^{n-2} x \cosec^2 x dx$$

$$U = \cosec^{n-2} x$$

$$\Rightarrow u' = (n-2) \cosec^{n-3} x (-\cosec x \cot x)$$

$$V = \cosec^2 x \Rightarrow \int \cosec^2 x dx = -\cot x + c$$

By using integration by parts

$$\boxed{\int (uv) dx = u \int v dx - \int [u' \int v dx] dx}$$

$$= -\cosec^{n-2} x \cot x - \int \{(n-2) \cosec^{n-3} x \cosec x \cot x dx . \\ (\tan x)\} dx$$



$$= -\cosec^{n-2} x \cot x - (n-2) \int \cosec^{n-2} x \cot^2 x dx$$

$$= -\cosec^{n-2} x \cot x - (n-2) \int \cosec^{n-2} x (\cosec^2 x - 1) dx$$

$$='' - (n-2) \int (\cosec^{n-2} x \cosec^2 x - \cosec^{n-2} x) dx$$

$$='' - (n-2) \int \cosec^n x dx + (n-2) \int \cosec^{n-2} x dx$$

$$I_n = -\cosec^{n-2} x \cot x - (n-2)I_n + (n-2)I_{n-2}$$

$$I_n + (n-2)I_n = -\cosec^{n-2} x \cot x + (n-2)I_{n-2}$$

$$I_n(1+n-2) = -\cosec^{n-2} x \cot x + (n-2)I_{n-2}$$

$$I_n(n-1) = -\cosec^{n-2} x \cot x + (n-2)I_{n-2}$$

$$I_n = \frac{-\cos^{n-2} x \cot x}{n-1} + \frac{n-2}{n-1} I_{n-2}$$

$$\text{Now } I_5 = -\frac{\cosec^{5-2} x \cdot \cot x}{5-1} + \frac{5-2}{5-1} I_{5-2}$$

$$I_6 = -\frac{\cosec^3 x \cdot \cot x}{4} + \frac{3}{4} I_3$$

$$I_5 = -\frac{\cosec^3 x \cdot \cot x}{4} + \frac{3}{4} \left\{ \frac{-\cosec^1 x \cdot \cot x}{2} + \frac{1}{2} I_0 \right\}$$

$$\{I_0 = \int 1 dx = x + c\}$$

$$I_5 = -\frac{\cosec^3 x \cdot \cot x}{4} - \frac{3}{8} \frac{\cosec^1 x \cdot \cot x}{1} + \frac{3}{8} x + c$$



7) If $\int \sin^m x \cos^n x dx$, then show that

$$I_{m,n} = \frac{\cos^{n-1} x \sin^{m+1} x}{m+n} + \frac{n-1}{m+n} I_{m,n-2}$$

$$\text{Sol: } I_n = \int \sin^m x \cos^n x dx$$

$$\begin{aligned} &= \int \sin^m x \cos^{n-1} x \cos x dx \\ &= \int (\cos^{n-1} x) (\sin^m x \cos x) dx \end{aligned}$$

$$\text{Here } U = (\cos^{n-1} x) \Rightarrow U' = (n-1) \cos^{n-2} x (-\sin x)$$

$$V = \sin^m x \cos x \Rightarrow \int \sin^m x \cos x dx$$

$$\begin{aligned} \text{If } f(x) &= \sin^m x, f'(x) = \cos x \\ \Rightarrow V' &= \frac{\sin^{m+1} x}{m+1} \end{aligned}$$

By using integration by parts

$$\int (uv) dx = U \int V dx - \int [U' \int V dx] dx$$

$$\begin{aligned} I_{m,n} &= \cos^{n-1} x \cdot \int \sin^m x \cos x dx \\ &\quad - \int \{(\cos^{n-1} x)' \int \sin^m x \cos x dx\} dx \\ &= \cos^{n-1} x \frac{\sin^{m+1} x}{m+1} \\ &\quad - \int (n-1) \cos^{n-2} x (-\sin x) \frac{\sin^{m+1} x}{m+1} dx \\ &= '' + \frac{(n-1)}{m+1} \int \cos^{n-2} x \sin^{m+2} x dx \end{aligned}$$



$$= '' + \frac{(n-1)}{m+1} \int \cos^{n-2} x \sin^m x \sin^2 x dx$$

$$= '' + \frac{(n-1)}{m+1} \int \cos^{n-2} x \sin^m x (1 - \cos^2 x) dx$$

$$\begin{aligned} &= '' + \frac{(n-1)}{m+1} \int \cos^{n-2} x \sin^m x dx \\ &\quad - \frac{(n-1)}{m+1} \int \cos^{n-2} x \sin^m x \cos^2 x dx \end{aligned}$$

$$\begin{aligned} &= '' + \frac{(n-1)}{m+1} \int \cos^{n-2} x \sin^m x dx \\ &\quad - \frac{(n-1)}{m+1} \int \cos^n x \sin^m x dx \end{aligned}$$

$$I_{m,n} = \cos^{n-1} x \frac{\sin^{m+1} x}{m+1} - \frac{(n-1)}{m+1} I_{m,n-2} - \frac{(n-1)}{m+1} I_{m,n}$$

$$I_{m,n} + \frac{(n-1)}{m+1} I_{m,n} = \cos^{n-1} x \frac{\sin^{m+1} x}{m+1} + \frac{(n-1)}{m+1} I_{m,n-2}$$

$$I_{m,n} (1 + \frac{n-1}{m+1}) = \cos^{n-1} x \frac{\sin^{m+1} x}{m+1} + \frac{(n-1)}{m+1} I_{m,n-2}$$

$$I_{m,n} (\frac{m+1+n-1}{m+1}) = \cos^{n-1} x \frac{\sin^{m+1} x}{m+1} + \frac{(n-1)}{m+1} I_{m,n-2}$$

$$I_{m,n} (\frac{m+n}{m+1}) = \cos^{n-1} x \frac{\sin^{m+1} x}{m+1} + \frac{(n-1)}{m+1} I_{m,n-2}$$

$$I_{m,n} = \cos^{n-1} x \frac{\sin^{m+1} x}{m+n} + \frac{(n-1)}{m+n} I_{m,n-2}$$

**Model-4:** Integration by using partial fraction

$$1) \int \frac{2x+3}{(x+3)(x^2+4)} dx$$

Sol:

$$\text{Let } \frac{2x+3}{(x+3)(x^2+4)} = \frac{A}{x+3} + \frac{Bx+C}{x^2+4}$$

$$= \frac{A(x^2 + 4) + (Bx + C)(x + 3)}{(x + 3)(x^2 + 4)}$$

$$\Rightarrow (2x+3) = A(x^2 + 4) + (Bx + C)(x + 3) \dots (1)$$

$$\text{Put } x=-3 \Rightarrow A = \frac{-3}{13} \text{ and}$$

equating the co-effiting of x^2

$$\Rightarrow A+B=0 \Rightarrow B = \frac{3}{13}$$

$$\text{Constant terms: } 4A+3C=3$$

$$\Rightarrow 3C = 3 + 4\left(\frac{3}{13}\right)$$

$$\Rightarrow 3C = \frac{39+12}{13} = \frac{51}{13}$$

$$\therefore C = \frac{17}{13}$$

$$\int \frac{2x+3}{(x+3)(x^2+4)} dx = \int \left[\frac{\frac{-3}{13}}{(x+3)} + \frac{\frac{3x+17}{13}}{x^2+4} \right] dx$$

$$= -\frac{3}{13} \int \frac{1}{x+3} dx + \frac{3}{13} \int \frac{x}{x^2+4} dx + \frac{17}{13} \int \frac{1}{x^2+4} dx$$

$$\boxed{\int \frac{1}{x} dx = \log|x| + c}$$

$$= -\frac{3}{13} \log|(x+3)| + \frac{3}{13} \left(\frac{1}{2}\right) \int \frac{2x}{x^2+2^2} dx + \frac{17}{13} \int \frac{1}{x^2+2^2} dx$$



$$\therefore \int \frac{f'(x)}{f(x)} dx = \log|f(x)| + c$$

$$\therefore \int \frac{1}{a^2+x^2} dx = \frac{1}{a} \tan^{-1} \left[\frac{x}{a} \right]$$

$$= -\frac{3}{13} \log|(x+3)| + \frac{3}{26} \log|x^2+2^2| + \frac{17}{13} \cdot \frac{1}{2} \tan^{-1} \left[\frac{x}{2} \right] + c$$

$$= -\frac{3}{13} \log|(x+3)| + \frac{3}{26} \log|x^2+2^2| + \left(\frac{17}{26}\right) \tan^{-1} \left[\frac{x}{2} \right] + c$$

$$2) \int \frac{\sin x \cos x}{\cos^2 x + 3 \cos x + 2} dx$$

Sol: let $\cos x = t \Rightarrow -\sin x dx = dt$

$$\int \frac{\sin x \cos x}{\cos^2 x + 3 \cos x + 2} dx$$

$$= \int \frac{t}{t^2+3t+2} (-dt)$$

$$= - \int \frac{t}{(t+1)(t+2)} (dt)$$

$$\text{let } \frac{t}{(t+1)(t+2)} = \frac{A}{(t+1)} + \frac{B}{(t+2)}$$

$$\Rightarrow t = A(t+2) + B(t+1) \dots (1)$$

$$\text{put } t = -1 \Rightarrow -1 = A(-1+2) + B(-1+1)$$

$$\Rightarrow A = -1$$

$$\text{put } t = -2 \Rightarrow -2 = A(-2+2) + B(-2+1)$$

$$\Rightarrow B = 2$$

$$-\int \frac{t}{(t+1)(t+2)} (dt)$$



$$= - \int \left[\frac{-1}{(t+1)} + \frac{2}{(t+2)} \right] dt$$

$$= \int \frac{1}{(t+1)} dt - 2 \int \frac{1}{(t+2)} dt$$

$$= \log|t+1| - 2 \log|t+2| + c$$

$$= \log|\cos x + 1| - 2 \log|\cos x + 2| + c$$

Integration by parts

$$\int (uv) dx = U \int V dx - \int [U' \int V dx] dx$$

3) Show that $\int \sqrt{a^2 + x^2} dx = \frac{x}{2} \sqrt{a^2 + x^2} + \frac{a^2}{2} \sinh^{-1} \left(\frac{x}{a} \right) + c$

Sol:

$$\text{Let } I = \int \sqrt{a^2 + x^2} dx$$

$$= \int 1 \cdot \sqrt{a^2 + x^2} dx$$

$$\text{here } u = \sqrt{a^2 + x^2} \Rightarrow u' = \frac{1}{2\sqrt{a^2+x^2}}(0+2x) = \frac{x}{\sqrt{a^2+x^2}}$$

$$v = 1 \Rightarrow \int 1 dx = x + c$$

Using integration by parts

$$I = \sqrt{a^2 + x^2} \cdot x - \int \frac{x}{\sqrt{a^2 + x^2}} \cdot x dx$$

$$\Rightarrow I = \sqrt{a^2 + x^2} \cdot x - \int \frac{x^2}{\sqrt{a^2 + x^2}} dx$$



$$\Rightarrow I = \sqrt{a^2 + x^2} \cdot x - \int \frac{a^2 + x^2 - a^2}{\sqrt{a^2 + x^2}} dx$$

$$\Rightarrow I = \sqrt{a^2 + x^2} \cdot x - \int \frac{a^2 + x^2}{\sqrt{a^2 + x^2}} dx + \int \frac{a^2}{\sqrt{a^2 + x^2}} dx$$

$$\Rightarrow I = \sqrt{a^2 + x^2} \cdot x - \int \sqrt{a^2 + x^2} dx + a^2 \int \frac{1}{\sqrt{a^2 + x^2}} dx$$

$$\Rightarrow I = x \sqrt{a^2 + x^2} - I + a^2 \sinh^{-1} \left(\frac{x}{a} \right) + c$$

$$\Rightarrow 2I = x \sqrt{a^2 + x^2} + a^2 \sinh^{-1} \left(\frac{x}{a} \right) + c$$

$$\Rightarrow I = \frac{x}{2} \sqrt{a^2 + x^2} + \frac{a^2}{2} \sinh^{-1} \left(\frac{x}{a} \right) + c$$

$$\therefore \int \sqrt{a^2 + x^2} dx = \frac{x}{2} \sqrt{a^2 + x^2} + \frac{a^2}{2} \sinh^{-1} \left(\frac{x}{a} \right) + c.$$

4) Using integration by parts, evaluate $\int e^x \cos x dx$.

Sol

$$\text{let } I = \int e^x \cdot \cos x dx$$

$$\text{here } u = \cos x \Rightarrow u' = -\sin x$$

$$v = e^x \Rightarrow \int e^x dx = e^x + c$$

Using integration by parts

$$\int(uv)dx = U \int Vdx - \int[U' \int Vdx]dx$$

$$I = \cos x e^x - \int(-\sin x) e^x dx$$

$$I = \cos x e^x + \int \sin x e^x dx$$

again by using integration by parts

$$\text{here } u = \sin x \Rightarrow u' = \cos x$$

$$v = e^x \Rightarrow \int e^x dx = e^x + c$$

$$I = \cos x e^x + \sin x e^x - \int \cos x e^x dx$$

$$I = \cos x e^x + \sin x e^x - I$$

$$2I = e^x(\cos x + \sin x)$$

$$\therefore I = \frac{e^x}{2}(\cos x + \sin x) + c$$



$$1. \int_0^{\pi/4} \frac{\sin x + \cos x}{9+16\sin 2x} dx$$

sol: let

$$\sin x - \cos x = t$$

diff . w.r.t 'x'

$$(\cos x + \sin x)dx = dt$$

$$L.L: x = 0 \Rightarrow t = \sin 0 - \cos 0 = 0 - 1 = -1$$

$$U.L: x = \frac{\pi}{4} \Rightarrow t = \sin\left(\frac{\pi}{4}\right) - \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = 0$$

And $\sin x - \cos x = t$ S.O.B

$$\Rightarrow (\sin x - \cos x)^2 = t^2$$

$$\Rightarrow \sin^2 x + \cos^2 x - 2\sin x \cos x = t^2$$

$$\Rightarrow 1 - \sin 2x = t^2$$

$$\Rightarrow 1 - t^2 = \sin 2x$$

$$I = \int_0^{\pi/4} \frac{\sin x + \cos x}{9+16\sin 2x} dx$$

$$= \int_{-1}^0 \frac{1}{9+1(1-t^2)} dt$$

$$= \int_{-1}^0 \frac{1}{9+16-16t^2} dt$$

$$= \int_{-1}^0 \frac{1}{25-16} dt$$

$$= \int_{-1}^0 \frac{1}{(5)^2-(4t)^2} dt$$

$$\therefore \int \frac{1}{a^2-x^2} dx = \frac{1}{2a} \log \left| \frac{a+x}{a-x} \right| + C$$



$$= \frac{\left[\frac{1}{2[5]} \log \left| \frac{5+4t}{4-5t} \right| \right]_0^{-1}}{4}$$

$$= \frac{1}{40} \left[\log \left| \frac{5+0}{5-0} \right| - \log \left| \frac{5-4}{5+4} \right| \right]$$

$$= \frac{1}{40} \left[\log 1 - \log \frac{1}{9} \right]$$

$$\log 1 = 0$$

$$= \frac{1}{40} [0 - \log 3^{-2}]$$

$$= \frac{1}{40} [2 \log 3]$$

$$= \frac{1}{20} \log 3$$



$$2. \int_0^1 \frac{\log(1+x)}{(1+x^2)} dx$$

sol: let $x = \tan\theta$

$$dx = \sec^2\theta d\theta$$

diff . w.r.t 'x'

$$L.L: x = 0 \Rightarrow \theta = 0$$

$$U.L: x = 1 \Rightarrow \theta = \frac{\pi}{4}$$

$$\int_0^1 \log \frac{(1+x)}{(1+x^2)} dx$$

$$= \int_0^{\pi/4} \frac{\log(1+\tan\theta)}{(1+\tan^2\theta)} \cdot \sec^2\theta d\theta$$

$$I = \int_0^{\pi/4} \log(1 + \tan\theta) d\theta \dots \dots \dots (1)$$

$$\therefore \int_0^a f(x) dx = \int_0^a f(a-x) dx$$

$$I = \int_0^{\pi/4} \log \left(1 + \tan \left[\frac{\pi}{4} - \theta \right] \right) d\theta$$

$$I = \int_0^{\pi/4} \log \left(1 + \frac{1-\tan\theta}{1+\tan\theta} \right) d\theta$$

$$I = \int_0^{\pi/4} \log \left(\frac{1+\tan\theta+1-\tan\theta}{1+\tan\theta} \right) d\theta$$



$$I = \int_0^{\pi/4} \log \left(\frac{2}{1+\tan\theta} \right) d\theta$$

$$\log(a/b) = \log a - \log b$$

$$I = \int_0^{\pi/4} \log 2 d\theta - \int_0^{\pi/4} \log(1 + \tan\theta) d\theta$$

$$I + I = \int_0^{\pi/4} \log 2 d\theta$$

$$2I = \log 2 \int_0^{\pi/4} (1) d\theta$$

$$2I = \log 2 [\theta]_0^{\pi/4}$$

$$2I = \log 2 \left[\frac{\pi}{4} - 0 \right]$$

$$I = \frac{\pi}{8} \log 2$$



$$3. \int_0^{\pi} \frac{x \cdot \sin x}{1 + \sin x} dx$$

Sol:

$$I = \int_0^{\pi} \frac{x \cdot \sin x}{1 + \sin x} dx$$

$$\therefore \int_0^a f(x) dx = \int_0^a f(a-x) dx$$

$$I = \int_0^{\pi} \frac{(\pi-x) \cdot \sin(\pi-x)}{1 + \sin(\pi-x)} dx$$

$$I = \int_0^{\pi} \frac{(\pi-x) \cdot \sin x}{1 + \sin x} dx$$

$$I = \int_0^{\pi} \frac{\pi \sin x}{1 + \sin x} dx - \int_0^{\pi} \frac{x \sin x}{1 + \sin x} dx$$

$$I + I = \pi \int_0^{\pi} \frac{\sin x}{1 + \sin x} dx$$

$$2I = \pi \int_0^{\pi} \frac{\sin x(1 - \sin x)}{(1 + \sin x)(1 - \sin x)} dx$$

$$2I = \pi \int_0^{\pi} \frac{\sin x - \sin^2 x}{(1 - \sin^2 x)} dx$$

$$2I = \pi \int_0^{\pi} \frac{\sin x - \sin^2 x}{(\cos^2 x)} dx$$



$$2I = \pi \int_0^{\pi} \left[\frac{\sin x}{\cos x} \cdot \frac{1}{\cos x} - \frac{\sin^2 x}{\cos^2 x} \right] dx$$

$$2I = \pi \int_0^{\pi} [\tan x \cdot \sec x - \tan^2 x] dx$$

$$2I = \pi \int_0^{\pi} \tan x \cdot \sec x \cdot dx - \pi \int_0^{\pi} \tan^2 x \cdot dx$$

$$\int \tan x \cdot \sec x \cdot dx = \sec x + c$$

$$2I = \pi[\sec x]_0^{\pi} - \pi \int_0^{\pi} (\sec^2 x - 1) dx$$

$$\int \sec^2 x dx = \tan x + c$$

$$\int 1 \cdot dx + c$$

$$2I = \pi[\sec \pi - \sec 0] - \pi[\tan x]_0^{\pi} - \pi[x]_0^{\pi}$$

$$2I = \pi[-1 - 1] - \pi[0 - 0] + \pi[\pi - 0]$$

$$2I = -2\pi + \pi^2$$

$$I = \frac{\pi^2}{2} - \pi$$



$$4. \int_0^\pi \frac{x}{1+\sin x} dx$$

Sol:

$$I = \int_0^\pi \frac{x}{1+\sin x} dx$$

$$\therefore \int_0^a f(x) dx = \int_0^a f(a-x) dx$$

$$I = \int_0^\pi \frac{(\pi-x)}{1+\sin(\pi-x)} dx$$

$$I = \int_0^\pi \frac{(\pi-x)}{1+\sin x} dx$$

$$I = \int_0^\pi \frac{\pi}{1+\sin x} dx - \int_0^\pi \frac{x}{1+\sin x} dx$$

$$I + I = \pi \int_0^\pi \frac{1}{1+\sin x} dx$$

$$2I = \pi \int_0^\pi \frac{(1-\sin x)}{(1+\sin x)(1-\sin x)} dx$$

$$2I = \pi \int_0^\pi \frac{1-\sin x}{(1-\sin^2 x)} dx$$

$$2I = \pi \int_0^\pi \frac{1-\sin x}{(\cos^2 x)} dx$$



$$2I = \pi \int_0^\pi \left[\frac{1}{\cos^2 x} - \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x} \right] dx$$

$$2I = \pi \int_0^\pi [\sec^2 x - \tan x \cdot \sec x] dx$$

$$\int \tan x \cdot \sec x \cdot dx = \sec x + c$$

$$\int \sec^2 x dx = \tan x + c$$

$$2I = \pi [\tan x]_0^\pi - \pi [\sec x]_0^\pi$$

$$2I = \pi [\tan \pi - \tan 0] - \pi [\sec \pi - \sec 0]$$

$$2I = \pi [0 - 0] - \pi [-1 - 1]$$

$$2I = 2\pi$$

$$I = \pi$$



$$5. \int_0^\pi \frac{x \sin x}{1 + \cos^2 x} dx$$

Sol:

$$I = \int_0^\pi \frac{x \sin x}{1 + \cos^2 x} dx$$

$$\therefore \int_0^a f(x) dx = \int_0^a f(a-x) dx$$

$$I = \int_0^\pi \frac{(\pi-x) \sin(\pi-x)}{1 + \cos^2(\pi-x)} dx$$

$$I = \int_0^\pi \frac{(\pi-x) \sin x}{1 + \cos^2 x} dx$$

$$I = \int_0^\pi \frac{\pi \sin x}{1 + \cos^2 x} dx - \int_0^\pi \frac{x \sin x}{1 + \cos^2 x} dx$$

$$I + I = \pi \int_0^\pi \frac{\sin x}{1 + \cos^2 x} dx$$

$$2I = \pi \int_0^\pi \frac{\sin x}{1 + \cos^2 x} dx$$

let $\cos x = t \Rightarrow -\sin x dx = dt$

or

$$\sin x dx = -dt$$



$$L.L: x = 0 \Rightarrow t = 1; \quad U.L: x = \pi \Rightarrow t = -1$$

$$\Rightarrow 2I = \pi \int_1^{-1} \frac{-1}{1+t^2} dt$$

$$\Rightarrow 2I = \pi \int_{-1}^1 \frac{1}{1+t^2} dt$$

$$2I = \pi [\tan^{-1} t]$$

$$2I = \pi [\tan^{-1}(1) - \tan^{-1}(-1)]$$

$$2I = \pi \left[\frac{\pi}{4} + \frac{\pi}{4} \right]$$

$$2I = \pi \left[2 \cdot \frac{\pi}{4} \right] = \pi \left[\frac{\pi}{2} \right]$$

$$I = \frac{\pi^2}{4}$$



$$6. \int_0^{\pi} \frac{x \sin^3 x}{1 + \cos^2 x} dx$$

Sol:

$$I = \int_0^{\pi} \frac{x \sin^3 x}{1 + \cos^2 x} dx$$

$$\therefore \int_0^a f(x) dx = \int_0^a f(a-x) dx$$

$$I = \int_0^{\pi} \frac{(\pi-x) \sin^3 (\pi-x)}{1 + \cos^2 (\pi-x)} dx$$

$$I = \int_0^{\pi} \frac{(\pi-x) \sin^3 x}{1 + \cos^2 x} dx$$

$$I = \int_0^{\pi} \frac{\pi \sin^3 x}{1 + \cos^2 x} dx - \int_0^{\pi} \frac{x \sin^3 x}{1 + \cos^2 x} dx$$

$$I + I = \pi \int_0^{\pi} \frac{\sin^3 x}{1 + \cos^2 x} dx$$

$$= \pi \int_0^{\pi} \frac{\sin^3 x}{1 + \cos^2 x} dx$$

let $\cos x = t \Rightarrow -\sin x dx = dt$

or $\sin x dx = -dt$

$$\sin^2 x = 1 - \cos^2 x = 1 - t^2$$

$$\sin^3 x = -(1 - t^2) dt$$



$$L.L: x = 0 \Rightarrow t = 1; \quad U.L: x = \pi \Rightarrow t = -1$$

$$\Rightarrow 2I = \pi \int_1^{-1} \frac{-(1-t^2)}{1+t^2} dt$$

$$\Rightarrow 2I = \pi \int_1^{-1} \frac{t^2-1}{1+t^2} dt$$

$$= \pi \int_1^{-1} \frac{t^2+1-2}{1+t^2} dt$$

$$\Rightarrow 2I = \pi \int_1^{-1} \left[1 - \frac{2}{1+t^2} \right] dt$$

$$2I = \pi [t - 2\tan^{-1} t]_1^{-1}$$

$$2I = \pi[-1 - 1] - 2\pi[\tan^{-1}(-1) - \tan^{-1}(1)]$$

$$2I = -2\pi + 2\pi \left[\frac{\pi}{4} + \frac{\pi}{4} \right]$$

$$2I = -2\pi + 2\pi \left[\frac{\pi}{2} \right]$$

$$I = -\pi + \frac{\pi^2}{2}$$

$$= \frac{\pi}{2}(\pi - 2)$$



$$7. \int_0^{\pi/2} \frac{x}{\cos x + \sin x} dx$$

sol:

$$I = \int_0^{\pi/2} \frac{x}{\cos x + \sin x} dx$$

$$\therefore \int_0^a f(x) dx = \int_0^a f(a-x) dx$$

$$I = \int_0^{\pi/2} \frac{(\frac{\pi}{2}-x)}{\cos(\frac{\pi}{2}-x) + \sin(\frac{\pi}{2}-x)} dx$$

$$I = \int_0^{\pi/2} \frac{(\frac{\pi}{2}-x)}{\sin x + \cos x} dx$$

$$= \int_0^{\pi/2} \frac{\frac{\pi}{2}}{\cos x + \sin x} dx - \int_0^{\pi/2} \frac{x}{\cos x + \sin x} dx$$

$$I + I = \frac{\pi}{2} \int_0^{\pi/2} \frac{1}{\cos x + \sin x} dx$$

$$2I = \frac{\pi}{2} \int_0^{\pi/2} \frac{1}{\cos x + \sin x} dx$$

$$2I = \frac{\pi}{2} \int_0^1 \frac{1}{\left(\frac{1-t^2}{1+t^2}\right)^2 + \left(\frac{2t}{1+t^2}\right)^2} \left(\frac{2dt}{1+t^2}\right)$$

$$2I = \frac{\pi}{2} \int_0^1 \frac{1}{\left[\frac{1-t^2+2t}{1+t^2}\right]} \left(\frac{2dt}{1+t^2}\right)$$

$$2I = \frac{\pi}{2} 2 \int_0^1 \frac{1}{-(t^2-2t-1)} dt$$

Let $t = \tan\left(\frac{x}{2}\right);$
 $dx = \frac{2d}{1+t^2};$
 $\cos x = \frac{1-t^2}{1+t^2}$
 $\sin x = \frac{2t}{1+t^2}$

$$I = \frac{\pi}{2} \int_0^1 \frac{1}{-[t^2-2t+(1)^2-(1)^2-1]} dt$$

$$I = \frac{\pi}{2} \int_0^1 \frac{1}{-[t^2-2t+(\sqrt{2})^2]} dt$$

$$I = \frac{\pi}{2} \int_0^1 \frac{1}{[(\sqrt{2})^2-(t-1)^2]} dt$$

$$\therefore \int \frac{1}{a^2-x^2} dx = \frac{1}{2a} \log \left| \frac{a+x}{a-x} \right| + C$$

$$2I = \frac{\pi}{2} \frac{1}{2(\sqrt{2})} \left[\log \left| \frac{\sqrt{2}+t-1}{\sqrt{2}-t+1} \right| \Big|_0^1 \right]$$

$$I = \frac{\pi}{2} \frac{1}{2\sqrt{2}} \left[\log \left| \frac{\sqrt{2}+1-1}{\sqrt{2}-1+1} \right| - \log \left| \frac{\sqrt{2}-1}{\sqrt{2}+1} \right| \right]$$

$$I = \frac{\pi}{2} \frac{1}{2\sqrt{2}} \left[\log |1| + \log \left| \frac{\sqrt{2}+1}{\sqrt{2}-1} \right| \right]$$

$$I = \frac{\pi}{2} \frac{1}{2\sqrt{2}} \log \left| \frac{\sqrt{2}+1}{\sqrt{2}-1} \right|$$

$$I = \frac{\pi}{2} \frac{1}{2\sqrt{2}} \log \left| \frac{(\sqrt{2}+1)(\sqrt{2}+1)}{(\sqrt{2}-1)(\sqrt{2}+1)} \right|$$

$$I = \frac{\pi}{2} \frac{1}{2\sqrt{2}} \log \frac{(\sqrt{2}+1)^2}{\sqrt{2}^2 - 1^2}$$

$$I = \frac{2}{2\sqrt{2}} \log \frac{(\sqrt{2}+1)}{2-1}$$

$$I = \frac{\pi}{2\sqrt{2}} \log(\sqrt{2} + 1)$$



$$8. \int_0^{\pi/2} \frac{\sin^2 x}{\cos x + \sin x} dx$$

sol:

$$I = \int_0^{\pi/2} \frac{\sin^2 x}{\cos x + \sin x} dx \dots (1)$$

$$\therefore \int_0^a f(x) dx = \int_0^a f(a-x) dx$$

$$I = \int_0^{\pi/2} \frac{\sin^2(\frac{\pi}{2}-x)}{\cos(\frac{\pi}{2}-x)+\sin(\frac{\pi}{2}-x)} dx$$

$$I = \int_0^{\pi/2} \frac{\cos^2 x}{\sin x + \cos x} dx \dots (2)$$

Adding (1) & (2)

$$= \int_0^{\pi/2} \frac{\sin^2 x}{\cos x + \sin x} dx + \int_0^{\pi/2} \frac{\cos^2 x}{\cos x + \sin x} dx$$

$$I + I = \int_0^{\pi/2} \frac{\sin^2 x + \cos^2 x}{\cos x + \sin x} dx$$

$$2I = \int_0^{\pi/2} \frac{1}{\cos x + \sin x} dx$$

$$2I = \int_0^1 \frac{1}{\left(\frac{1-t^2}{1+t^2}\right) + \left(\frac{2t}{1+t^2}\right)} \left(\frac{2d}{1+t^2}\right)$$

$$2I = \int_0^1 \frac{1}{\left[\frac{1-t^2+2t}{1+t^2}\right]} \left(\frac{2d}{1+t^2}\right)$$

$$\text{Let } t = \tan\left(\frac{x}{2}\right);$$

$$dx = \frac{2d}{1+t^2};$$

$$\cos x = \frac{1-t^2}{1+t^2}$$

$$\sin x = \frac{2t}{1+t^2}$$

$$2I = 2 \int_0^1 \frac{1}{-(t^2-2t-1)} dt$$



$$I = \int_0^1 \frac{1}{-[t^2-2t+(1)^2-(1)^2-1]} dt$$

$$I = \int_0^1 \frac{1}{-[(t-1)^2-(\sqrt{2})^2]} dt$$

$$I = \int_0^1 \frac{1}{[(\sqrt{2})^2-(t-1)^2]} dt$$

$$\therefore \int \frac{1}{a^2-x^2} dx = \frac{1}{2a} \log \left| \frac{a+x}{a-x} \right| + c$$

$$2I = \frac{1}{2(\sqrt{2})} \left[\log \left| \frac{\sqrt{2}+t-1}{\sqrt{2}-t+1} \right| \Big|_0^1 \right]$$

$$I = \frac{1}{2\sqrt{2}} \left[\log \left| \frac{\sqrt{2}+1-1}{\sqrt{2}-1+1} \right| - \log \left| \frac{\sqrt{2}-1}{\sqrt{2}+1} \right| \right]$$

$$I = \frac{1}{2\sqrt{2}} \left[\log |1| + \log \left| \frac{\sqrt{2}+1}{\sqrt{2}-1} \right| \right]$$

$$I = \frac{1}{2\sqrt{2}} \log \left| \frac{\sqrt{2}+1}{\sqrt{2}-1} \right|$$

$$I = \frac{1}{2\sqrt{2}} \log \left| \frac{(\sqrt{2}+1)(\sqrt{2}+1)}{(\sqrt{2}-1)(\sqrt{2}+1)} \right|$$

$$I = \frac{1}{2\sqrt{2}} \log \frac{(\sqrt{2}+1)^2}{\sqrt{2}^2 - 1^2}$$

$$I = \frac{2}{2\sqrt{2}} \log \frac{(\sqrt{2}+1)}{2-1}$$

$$I = \frac{1}{\sqrt{2}} \log(\sqrt{2} + 1)$$



$$9. \int_3^7 \sqrt{\frac{7-x}{x-3}} dx.$$

Sol:

$$\begin{aligned} \text{let } x &= 3\cos^2\theta + 7\sin^2\theta \\ dx &= 8\sin\theta\cos\theta.d\theta \end{aligned}$$

$$\text{L.L: } x = 3 \Rightarrow \theta = 0 \quad \text{U.L: } x = 7 \Rightarrow \theta = \frac{\pi}{2}$$

$$\begin{aligned} 7 - x &= 7 - 3\cos^2\theta - 7\sin^2\theta \\ &= 7(1 - \sin^2\theta) - 3\cos^2\theta \\ &= 7(\cos^2\theta) - 3\cos^2\theta \\ &= 4\cos^2\theta \end{aligned}$$

$$\begin{aligned} x - 3 &= 3\cos^2\theta + 7\sin^2\theta - 3 \\ &= 7\sin^2\theta - 3(1 - \cos^2\theta) \\ &= 7\sin^2\theta - 3(\sin^2\theta) \\ &= 4\sin^2\theta \end{aligned}$$

$$\begin{aligned} \int_3^7 \sqrt{\frac{7-x}{x-3}} dx \\ &= \int_0^{\pi/2} \sqrt{\frac{4\cos^2\theta}{4\sin^2\theta}} 8\sin\theta\cos\theta d\theta \end{aligned}$$

$$\begin{aligned} &= 8 \int_0^{\pi/2} \cos^2\theta d\theta \\ \int_0^{\pi/2} \cos^n x dx &= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{1}{2} \end{aligned}$$

$$= 8 \cdot \left(\frac{1}{2}\right) \frac{\pi}{2} = 2\pi$$



$$10. \int_4^9 \frac{1}{\sqrt{(9-x)(x-4)}} dx.$$

Sol:

$$\begin{aligned} \text{let } x &= 4\cos^2\theta + 9\sin^2\theta \\ dx &= 10\sin\theta\cos\theta.d\theta \end{aligned}$$

$$\text{L.L: } x = 4 \Rightarrow \theta = 0$$

$$\begin{aligned} \text{U.L: } x &= 9 \Rightarrow \theta = \frac{\pi}{2} \\ 9 - x &= 9 - 4\cos^2\theta - 9\sin^2\theta \\ &= 9(1 - \sin^2\theta) - 3\cos^2\theta \\ &= 9(\cos^2\theta) - 4\cos^2\theta \\ &= 5\cos^2\theta \end{aligned}$$

$$\begin{aligned} x - 4 &= 4\cos^2\theta + 9\sin^2\theta - 4 \\ &= 9\sin^2\theta - 4(1 - \cos^2\theta) \\ &= 9\sin^2\theta - 4(\sin^2\theta) \\ &= 5\sin^2\theta \end{aligned}$$

$$\begin{aligned} \int_4^9 \frac{1}{\sqrt{(9-x)(x-4)}} dx. \\ &= \int_0^{\pi/2} \frac{1}{\sqrt{5\cos^2\theta 5\sin^2\theta}} 10\sin\theta\cos\theta d\theta \end{aligned}$$

$$== \int_0^{\pi/2} \frac{1}{5\sin\theta\cos\theta} 10\sin\theta\cos\theta d\theta$$

$$I = 2 \int_0^{\pi/2} 1 dx$$

$$= 2[x]_0^{\pi/2} = \frac{2\pi}{2} = \pi$$



11. Evaluate $\int_a^b \sqrt{(x-a)(x-b)} dx$

$$\text{Sol: } \int_a^b \sqrt{-(x-a)(x-b)} dx$$

$$I = \int_a^b \sqrt{-[x^2 - (a+b)x + ab]} dx$$

$$= \int_a^b \sqrt{-\left[x^2 - (a+b)x + \frac{[a+b]^2}{4} - \frac{[a+b]^2}{4} + ab\right]} dx$$

$$= \int_a^b \sqrt{-\left[\left[x - \frac{a+b}{2}\right]^2 - \frac{[b-a]^2}{4}\right]} dx$$

$$= \int_a^b \sqrt{\left[\frac{[b-a]^2}{2}\right] - \left[\left[x - \frac{a+b}{2}\right]^2\right]} dx$$

$$\therefore \int \sqrt{a^2 - x^2} dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \left(\frac{x}{a} \right) + C$$

$$I = \left[\frac{\left[x - \frac{a+b}{2}\right]}{2} \sqrt{(a-x)(b-x)} + \frac{[b-a]^2}{4 \cdot 2} \sin^{-1} \left(\frac{\left(x - \frac{a+b}{2}\right)}{\frac{b-a}{2}} \right) \right]_a^b$$

$$= \left[\frac{\left[b - \frac{a+b}{2}\right]}{2} \sqrt{(a-b)(b-x)} + \frac{[b-a]^2}{4 \cdot 2} \sin^{-1} \left(\frac{\left(b - \frac{a+b}{2}\right)}{\frac{b-a}{2}} \right) \right]$$



$$- \left[\frac{\left[a - \frac{a+b}{2}\right]}{2} \sqrt{(a-a)(b-a)} + \frac{[b-a]^2}{4 \cdot 2} \sin^{-1} \left(\frac{\left(a - \frac{a+b}{2}\right)}{\frac{b-a}{2}} \right) \right]$$

$$= \left[\frac{[b-a]^2}{8} \sin^{-1} \left(\frac{\frac{b-a}{2}}{\frac{b-a}{2}} \right) \right] - \left[\frac{[b-a]^2}{8} \sin^{-1} \left(\frac{\frac{a-b}{2}}{\frac{b-a}{2}} \right) \right]$$

$$= \left[\frac{[b-a]^2}{8} \sin^{-1} 1 \right] - \left[\frac{[b-a]^2}{8} \sin^{-1}(-1) \right]$$

$$= \left[\frac{[b-a]^2}{8} \left(\frac{\pi}{2} \right) \right] + \left[\frac{[b-a]^2}{8} \left(\frac{\pi}{2} \right) \right] = \frac{\pi [b-a]^2}{8}$$



12. $\int_0^\pi x \cdot \sin^7 x \cos^6 x \, dx$

Sol:

$$I = \int_0^\pi x \cdot \sin^7 x \cos^6 x \, dx$$

$$\therefore \int_0^a f(x)dx = \int_0^a f(a-x)dx$$

$$I = \int_0^\pi (\pi - x) \sin^7(\pi - x) \cos^6(\pi - x) \, dx$$

$$I = \int_0^\pi (\pi - x) \sin^7 x \cos^6 x \, dx$$

$$I = \int_0^\pi \pi \sin^7 x \cos^6 x \, dx - \int_0^\pi x \sin^7 x \cos^6 x \, dx$$

$$I = \int_0^\pi \pi \sin^7 x \cos^6 x \, dx - I$$

$$I + I = \pi \int_0^\pi \sin^7 x \cos^6 x \, dx$$

$$\therefore \int_0^{2a} f(x)dx = 2 \int_0^a f(x)dx$$

$$2I = 2\pi \int_0^{\pi/2} \sin^7 x \cos^6 x \, dx$$

$$I = \pi \int_0^{\pi/2} \sin^7 x \cos^6 x \, dx$$

$$\int_0^{\pi/2} \cos^n x \sin^m x \, dx = \frac{(n-1)(n-3)(n-5)\dots(m-1)(m-3)\dots}{(m+n)(m+n-2)(m+n-4)\dots}$$

$$I = \pi \frac{6.4.2.5.3.1}{13.11.9.7.5.3.1}$$

$$I = \pi \frac{6.4.2}{13.11.9.7} = \frac{16\pi}{3003}$$



13. Find the area enclosed by the curves

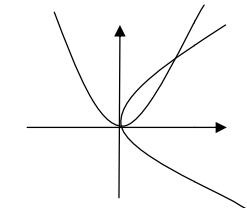
$$y^2 = 4ax \text{ and } x^2 = 4by.$$

Sol:

Given eq'n

$$y^2 = 4ax \Rightarrow y = \sqrt{4ax} \dots (1)$$

$$x^2 = 4by \Rightarrow y = \frac{x^2}{4b} \dots \dots (2)$$



solving (1) and (2)

$$\sqrt{4ax} = \frac{x^2}{4b} \text{ S.O.B}$$

$$\Rightarrow 4ax = \frac{x^4}{16b^2}$$

$$\Rightarrow 64ab^2x = x^4 \Rightarrow 64ab^2x - x^4 = 0$$

$$\Rightarrow x(64ab^2 - x^3) = 0$$

$$x = 0 \text{ or } x^3 = 64ab^2$$

$$\Rightarrow x = 4a^{1/3}b^{2/3} = u$$

$$\text{Required Area} = \int_0^u [(1) - (2)] \, dx$$

$$= \int_0^u \left[\sqrt{4ax} - \frac{x^2}{4b} \right] dx$$

$$= \int_0^u \left[2\sqrt{ax}^{1/2} - \frac{x^2}{4b} \right] dx$$

$$= \left[\frac{2\sqrt{a}x^{3/2}}{3/2} - \frac{x^3}{4b^3} \right]_0^u$$

$$= \frac{4}{3}\sqrt{a}[u^{3/2} - 0^2] - \frac{1}{12}[u^3 - 0^3]$$

$$= \frac{4}{3}\sqrt{a} \left[\left[4a^{\frac{1}{3}}b^{\frac{2}{3}} \right]^{\frac{3}{2}} - 0^2 \right] - \frac{1}{12} \left[[4a^{1/3}b^{2/3}]^3 - 0^3 \right]$$

$$= \frac{4}{3}[8ab] - \frac{1}{12b}[64ab^2]$$

$$= \frac{32ab}{3} - \frac{16ab}{3} = \frac{16ab}{3} \text{ sq. units}$$

14. Find the area enclosed by the curves

$$y = 4 - 2x \text{ and } y = x^2 - 5x.$$

Sol:

Given eq'n

$$y = 4 - 2x \dots \dots (1)$$

$$y = x^2 - 5x \dots (1)$$

solving (1) and (2)

$$x^2 - 5x = 4 - 2x$$

$$\Rightarrow x^2 - 3x - 4 = 0$$

$$\Rightarrow x^2 + 1x - 4x - 4 = 0$$

$$\Rightarrow x(x+1) - 4(x+1) = 0$$

$$x = 4 \text{ or } x = -1$$

$$\begin{aligned} \text{Required Area} &= \int_{-1}^4 [(1) - (2)] dx \\ &= \int_{-1}^4 [4 - 2x - x^2 + 5x] dx \\ &= \int_{-1}^4 [4 + 3x - x^2] dx \\ &= \left[4x + \frac{3x^2}{2} - \frac{x^3}{3} \right]_{-1}^4 \end{aligned}$$

$$\begin{aligned} &= 4[4 + 1] + \frac{3}{2}[4^2 - (-1)^2] - \frac{1}{3}[4^3 - (-1)^3] \\ &= 20 + \frac{3}{2}(16 - 1) - \frac{1}{3}(64 + 1) \end{aligned}$$

$$= 20 + \frac{45}{2} - \frac{65}{3}$$

$$\begin{aligned} &= \frac{120 + 135 - 135}{6} \\ &= \frac{125}{6} \text{ sq. units} \end{aligned}$$



15. Find the area enclosed by the curves

$$y^2 = 4x \text{ and } y^2 = 4(4 - x).$$

Sol:

$$\text{Given eq'n } y^2 = 4x \Rightarrow y = \sqrt{4x} \dots (1)$$

$$y^2 = 4(4 - x) \Rightarrow y = \sqrt{4(4 - x)} \dots (2)$$

solving (1)and (2)

$$4x = 4(4 - x)$$

$$\Rightarrow x = 4 - x$$

$$\Rightarrow 2x = 4$$

$$x = 2$$

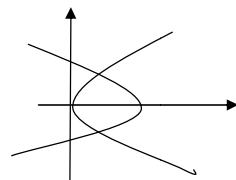
Sub x=2 in (1)

$$y^2 = 4x = 4(2) = 8$$

$$y = \sqrt{8} = \pm 2\sqrt{2}$$

Two parabolas are symmetric about X-axis

$$\text{Required Area} = 2 \left[\int_0^2 (1) dx + \int_2^4 (2) dx \right]$$



$$= 2 \left[\int_0^2 \sqrt{4x} dx + \int_2^4 \sqrt{4(4-x)} dx \right]$$

$$= 2 \left[2 \int_0^2 x^{1/2} dx + 2 \int_2^4 (4-x)^{1/2} dx \right]$$

$$= 4 \left[\frac{x^{3/2}}{3/2} \right]_0^2 + 4 \left[\frac{(4-x)^{3/2}}{-3/2} \right]_2^4$$

$$= \frac{8}{3} [2^{3/2} - 0^{3/2}] - \frac{8}{3} [(4-4)^{3/2} - (4-2)^{3/2}]$$

$$= \frac{8}{3} [2\sqrt{2}] + \frac{8}{3} [2\sqrt{2}] = \frac{16\sqrt{2}}{2} + \frac{16\sqrt{2}}{2}$$

$$= \frac{32\sqrt{2}}{2} \text{ sq. units}$$



16. Find the area enclosed by the curves

$$y = 2 - x^2 \text{ and } y = x^2.$$

Sol:

$$\text{Given eq'n } y = 2 - x^2 \dots (1)$$

$$y = x^2 \dots (2)$$

solving (1)and (2)

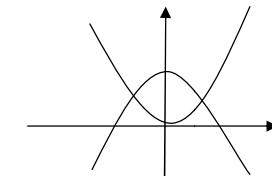
$$2 - x^2 = x^2$$

$$2 = 2x^2$$

$$x^2 = 1$$

$$x = \pm 1$$

$$\therefore x = 1 \text{ or } x = -1$$



$$\text{Required Area} = \int_{-1}^1 [(1) - (2)] dx$$

$$= \int_{-1}^1 [2 - x^2 - x^2] dx$$

$$= \int_{-1}^1 [2 - 2x^2] dx$$

$$= \left[2x - 2 \frac{x^3}{3} \right]_{-1}^1$$

$$= 2[1 + 1] - \frac{2}{3}[(1)^3 - (-1)^3]$$

$$= 4 - \frac{2}{3}(1 + 1)$$

$$= 4 - \frac{4}{3} = \frac{12-4}{3}$$

$$= \frac{8}{3} \text{ sq. unit}$$

17. Show that the area of the region bounded by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ is } \pi ab \text{ also deduce the area of the circle}$$

$$x^2 + y^2 = a^2.$$

Sol: Given eq'n of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \dots \dots \dots (1)$

$$\Rightarrow \frac{y^2}{b^2} = 1 - \frac{x^2}{a^2} \Rightarrow \frac{y^2}{b^2} = \frac{1}{a^2} [a^2 - x^2]$$

$$\Rightarrow y^2 = \frac{b^2}{a^2} [a^2 - x^2] \Rightarrow y = \pm \frac{b}{a} \sqrt{a^2 - x^2}$$

Ellipse is symmetric about both the axes. Required area
=4 area of shaded region

$$\text{Area} = \int_0^a y dx = 4 \int_0^a \frac{b}{a} \sqrt{a^2 - x^2} dx$$

$$= 4 \frac{b}{a} \int_0^a \sqrt{a^2 - x^2} dx$$

$$\therefore \int \sqrt{a^2 - x^2} dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \left(\frac{x}{a} \right) + c$$

$$= \frac{4b}{a} \left[\frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \left(\frac{x}{a} \right) \right]_0^a$$

$$= \frac{4b}{a} \left[\frac{a}{2} \sqrt{a^2 - a^2} + \frac{a^2}{2} \sin^{-1} \left(\frac{a}{a} \right) - 0 - 0 \right]$$

$$= \frac{4b a^2}{a} \sin^{-1}(1)$$

$$= \frac{2ab\pi}{2} = \pi ab \text{ sq units}$$

If $a=b$ the ellipse becomes a circle

\therefore Area of the circle $x^2 + y^2 = a^2$.
is $\pi a \cdot a = \pi a^2 \text{ sq . units}$



1. Solve $(x^3 - 3xy^2)dx + (3x^2y - y^3)dy = 0$.

Sol: $(x^3 - 3xy^2)dx + (3x^2y - y^3)dy = 0$.

$$\Rightarrow \frac{dy}{dx} = -\frac{(x^3 - 3xy^2)}{(3x^2y - y^3)} \dots\dots (1)$$

this is homogenous D.E

$$\text{let } y = vx \Rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx}$$

$$\text{Eq'n (1)} \Rightarrow v + x \frac{dv}{dx} = -\frac{x^3 - 3x(vx)^2}{3x^2(vx) - (vx)^3}$$

$$\Rightarrow v + x \frac{dv}{dx} = -\frac{x^3(1-3v^2)}{x^3(3v-v^3)}$$

$$\Rightarrow v + x \frac{dv}{dx} = -\left(\frac{1-3v^2}{3v-v^3}\right)$$

$$\Rightarrow x \frac{dv}{dx} = \frac{-1+3v^2}{3v-v^3} - \frac{v}{1}$$

$$\Rightarrow x \frac{dv}{dx} = \frac{-1+3v^2-3v^2+v^4}{3v-v^3}$$

$$\Rightarrow x \frac{dv}{dx} = \left(\frac{v^4-1}{3v-v^3}\right)$$

$$\Rightarrow \int \frac{3v-v^3}{v^4-1} dv = \int \frac{1}{x} dx$$

By partial fraction



$$\Rightarrow \int \frac{3v-v^3}{(v-1)(v+1)(v^2+1)} dv = \int \frac{1}{x} dx$$

$$\left[\frac{1}{2(v+1)} + \frac{1}{2(v-1)} - \frac{2v}{(v^2+1)} \right] dv = \int \frac{1}{x} dx$$

$$\Rightarrow \frac{1}{2} \log|v+1| + \frac{1}{2} \log|v-1| - \log|v^2+1| = \log x + \log c$$

$$\Rightarrow \log \left| \frac{\sqrt{v+1}\sqrt{v-1}}{v^2+1} \right| = \log(cx)$$

$$\Rightarrow \frac{\sqrt{v+1}\sqrt{v-1}}{v^2+1} = cx \quad \Rightarrow \frac{\sqrt{v^2-1}}{v^2+1} = cx$$

$$\Rightarrow \frac{v^2-1}{(v^2+1)^2} = (cx)^2$$

$$\Rightarrow \frac{(y^2-x^2)}{x^2} = c^2 x^2 \frac{(y^2+x^2)^2}{x^4}$$

$$\Rightarrow (y^2 - x^2) = (y^2 + x^2)^2$$

Which is required general solution.



2. Given the solution of

$x \sin^2\left(\frac{y}{x}\right) dx = y dx - x dy$ Which passes through the point $(1, \frac{\pi}{4})$

$$\text{Sol: } x \sin^2\left(\frac{y}{x}\right) dx = y dx - x dy$$

$$\Rightarrow x dy = y dx - x \sin^2\left(\frac{y}{x}\right) dx$$

$$\Rightarrow x dy = \left[y - x \sin^2\left(\frac{y}{x}\right)\right] dx \Rightarrow dy = \left[\frac{y}{x} - \frac{x \sin^2\left(\frac{y}{x}\right)}{x}\right] dx$$

$$\Rightarrow \frac{dy}{dx} = \left[\frac{y}{x} - \sin^2\left(\frac{y}{x}\right)\right] \dots \dots (1)$$

$$\text{let } y = vx \Rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx}$$

$$\Rightarrow v + x \frac{dv}{dx} = v - \sin^2(v)$$

$$\Rightarrow x \frac{dv}{dx} = -\sin^2(v) \Rightarrow \int \frac{1}{\sin^2(v)} dv = - \int \frac{1}{x} dx$$

$$\Rightarrow \int \csc^2 v dv = - \int \frac{1}{x} dx$$

$$\Rightarrow -\cot v = -\log x + c \Rightarrow -\cot\left(\frac{y}{x}\right) = -\log x + c$$

this is passing Through the point $(1, \frac{\pi}{4})$

$$\Rightarrow -\cot\left(\frac{\pi}{4}\right) = -\log 1 + c \Rightarrow -1 = 0 + c \Rightarrow c = -1$$

$$\therefore -\cot\left(\frac{y}{x}\right) = -\log x - 1$$



3. solve the differential equation

$$\frac{dy}{dx} = \frac{x-y+3}{2x-2y+5}$$

$$\text{Sol: } \frac{dy}{dx} = \frac{x-y+3}{2x-2y+5} \quad \left[\frac{a}{a'} = \frac{b}{b'} \right]$$

this is non-homogeneous D.E of case(2)

$$\frac{dy}{dx} = \frac{x-y+3}{2(x-y)+5} \dots \dots (1)$$

$$\text{let } (x-y) = v \Rightarrow 1 - \frac{dy}{dx} = \frac{dv}{dx} \Rightarrow \frac{dy}{dx} = 1 - \frac{dv}{dx}$$

Now eq'n (1) becomes

$$\Rightarrow 1 - \frac{dv}{dx} = \frac{v+3}{2v+5} \Rightarrow 1 - \frac{v+3}{2v+5} = \frac{dv}{dx}$$

$$\Rightarrow \frac{2v+5-v-3}{2v+5} = \frac{dv}{dx}$$

$$\Rightarrow \frac{v+2}{2v+5} = \frac{dv}{dx}$$

$$\Rightarrow \int 1 dx = \int \frac{2v+5}{v+2} dv$$

$$\Rightarrow \int 1 dx = \int \frac{2v+4+1}{v+2} dv$$

$$\Rightarrow \int 1 dx = \int \frac{2(v+2)+1}{v+2} dv$$

$$\Rightarrow \int 1 dx = \int \left(\frac{2(v+2)}{v+2} + \frac{1}{v+2} \right) dv$$

$$\Rightarrow \int 1 dx = \int \left(2 + \frac{1}{v+2} \right) dv$$

$$\Rightarrow x = 2v + \log(v+2) + c$$

$$\Rightarrow x = 2(x-y) + \log(x-y+2) + c$$

$$\therefore x - 2y + \log(x-y+2) = c$$

**4. solve the differential equation**

$$(2x + y + 1)dx + (4x + 2y - 1)dy = 0.$$

$$\text{Sol: } \frac{dy}{dx} = \frac{-(2x+y+1)}{4x+2y-1} \left[\frac{a}{a'} = \frac{b}{b'} \right]$$

this is non-homogeneous D.E of case(2)

$$\frac{dy}{dx} = \frac{-(2x+y+1)}{4x+2y-1} \dots\dots(1)$$

$$\text{let } (2x + y) = v \Rightarrow 2 + \frac{dy}{dx} = \frac{dv}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{dv}{dx} - 2$$

$$\text{Now eq'n (1) becomes } \Rightarrow \frac{dv}{dx} - 2 = \frac{-(v+1)}{2v-1}$$

$$\Rightarrow \frac{dv}{dx} = \frac{-v-1}{2v-1} + 2$$

$$\Rightarrow \frac{dv}{dx} = \frac{-v-1+4v-2}{2v-1}$$

$$\Rightarrow \frac{dv}{dx} = \frac{3v-3}{2v-1}$$

$$\Rightarrow \int \frac{2v-1}{v-1} dv = \int 3dx$$

$$\Rightarrow \int \frac{2(v-1)+1}{v-1} dv = \int 3dx$$

$$\Rightarrow \int \left(\frac{2(v-1)}{v-1} + \frac{1}{v-1} \right) dv = \int 3dx$$

$$\Rightarrow \int 3dx = \int \left(2 + \frac{1}{v-1} \right) dv$$

$$\Rightarrow 3x = 2v + \log(v-1) + c$$

$$\Rightarrow 3x = 2(2x + y) + \log(2x + y - 1) + c$$

$$\Rightarrow 3x = 4x + 2y + \log(2x + y - 1) + c$$

$$\therefore x + 2y + \log(2x + y - 1) = c$$

**5. solve $\frac{dy}{dx} = \frac{2x+y+3}{2y+x+1}$.**

$$\text{Sol: Given eq'n } \frac{dy}{dx} = \frac{2x+y+3}{2y+x+1} [a/a' \neq b/b']$$

this is non-homogeneous D.E of case(3)

put $x = X + h$ and $y = Y + k$

$$\frac{dy}{dx} = \frac{2x+y+3}{2y+x+1} \Leftrightarrow \frac{dY}{dX} = \frac{2(X+h)+(Y+k)+3}{2(Y+k)+(X+h)+1}$$

$$\Rightarrow \frac{dY}{dX} = \frac{2X+Y+(2h+k+3)}{X+2Y+(h+2k+1)} \dots (*) \text{ now choose } h \text{ and } k \text{ such that}$$

$$2h + k + 3 = 0 \dots (1) \text{ and } h + 2k + 1 = 0 \dots (2)$$

solving (1) & (2)

$$\begin{array}{cccc} 2 & 1 & 3 & 2 \end{array}$$

$$\begin{array}{cccc} 1 & 2 & 1 & 1 \end{array}$$

$$(h, k) = \left[\frac{1-6}{4-1}, \frac{3-2}{4-1} \right] = \left[-\frac{5}{3}, \frac{1}{3} \right]$$

Hence (*) becomes

$$\Rightarrow \frac{dY}{dX} = \frac{2X+Y}{X+2Y} \text{ is a homogeneous equation.}$$

$$\text{put } y = VX \Rightarrow \frac{dY}{dX} = V + \frac{XdV}{dX}$$

$$\Rightarrow V + \frac{XdV}{dX} = \frac{2X+VX}{X+2VX}$$

$$\Rightarrow V + \frac{XdV}{dX} = \frac{X(2+V)}{X(1+2V)}$$

$$\Rightarrow V + \frac{XdV}{dX} = \frac{(2+V)}{(1+2V)}$$



$$\Rightarrow \frac{X dV}{dX} = \frac{(2+V)}{(1+2V)} - \frac{V}{1}$$

$$\Rightarrow \frac{X dV}{dX} = \frac{2+V-V-2V^2}{(1+2V)} \Rightarrow \frac{X dV}{dX} = \frac{2-2V^2}{(1+2V)}$$

$$\Rightarrow \frac{X dV}{dX} = \frac{2(1-V^2)}{(1+2V)}$$

$$\Rightarrow \left[-\frac{1}{2(1+V)} + \frac{3}{2(1-V)} \right] dV = 2 \frac{dX}{X}$$

$$\Rightarrow -\frac{1}{2} \int \frac{1}{(1+V)} dV + \frac{3}{2} \int \frac{1}{(1-V)} dV = 2 \int \frac{1}{X} dX$$

$$\Rightarrow -\frac{1}{2} \log|1+V| - \frac{3}{2} \log|1-V| = 2 \log X + \log C$$

$$\Rightarrow \log|1+V| + 3 \log|1-V| = -4 \log X + \log C$$

$$\Rightarrow \log(1+V)(1-V)^3 = \log(C/X^4)$$

$$\Rightarrow (1+V)(1-V)^3 = \frac{C}{X^4} \quad \{V=\frac{Y}{X}\}$$

$$\Rightarrow \left(1 + \frac{Y}{X}\right) \left(1 - \frac{Y}{X}\right)^3 = \frac{C}{X^4}$$

$$\Rightarrow \frac{(X+Y)(X-Y)^3}{X^4} = \frac{C}{X^4}$$

$$\Rightarrow (X+Y)(X-Y)^3 = C$$

$$(h, k) = \left[-\frac{5}{3}, \frac{1}{3}\right] \quad X = x + \frac{5}{3}, Y = y - \frac{1}{3}$$

$$\Rightarrow \left(x + \frac{5}{3} + y - \frac{1}{3}\right) \left(x + \frac{5}{3} - y + \frac{1}{3}\right)^3 = C$$

$$\Rightarrow \left(x + y + \frac{4}{3}\right) (x - y + 2)^3 = C$$

$$(3x + 3y + 4)(x - y + 2)^3 = 3C$$

This is the required solution.



$$6. \text{ solve } \frac{dy}{dx} = \frac{3y-7x+7}{3x-7y-3}$$

Sol: Given eq "n" $\frac{dy}{dx} = \frac{3y-7x+7}{3x-7y-3} [a/a' \neq b/b']$

this is non-homogeneous D.E of case(3)

put $x = X + h$ and $y = Y + k$

$$\frac{dy}{dx} = \frac{3y-7x+7}{3x-7y-3}$$

$$\Rightarrow \frac{dY}{dX} = \frac{3(Y+k)-7(X+h)+7}{3(X+h)-7(Y+k)-3}$$

$$\Rightarrow \frac{dY}{dX} = \frac{-7X+3Y+(-7h+3k+7)}{3X-7Y+(3h-7k-3)} \dots (*)$$

now choose h and k such that

$$-7h + 3k + 7 = 0 \dots (1) \text{ and } 3h - 7k - 3 = 0 \dots (2)$$

solving (1)&(2)

$$\begin{array}{cccc} -7 & 3 & 7 & -7 \end{array}$$

$$\begin{array}{cccc} 3 & -7 & -3 & 3 \end{array}$$

$$(h, k) = \left[\frac{-9+49}{49-9}, \frac{21-21}{49-9} \right] = [1, 0]$$

Hence (*) becomes

$\Rightarrow \frac{dY}{dX} = \frac{-7X+3Y}{3X-7Y}$ is a homogeneous equation.

$$\text{put } y = VX \Rightarrow \frac{dY}{dX} = V + \frac{X dV}{dX}$$

$$\Rightarrow V + \frac{X dV}{dX} = \frac{-7X+3V}{3X-7VX}$$

$$\Rightarrow V + \frac{X dV}{dX} = \frac{X(-7+3V)}{X(3-7V)}$$



$$\Rightarrow V + \frac{X dV}{dX} = \frac{(-7+3V)}{(3-7V)}$$

$$\Rightarrow \frac{X dV}{dX} = \frac{(-7+3V)}{(3-7V)} - \frac{V}{1}$$

$$\Rightarrow \frac{X dV}{dX} = \frac{-7+3V-3V+7V^2}{(3-7V)}$$

$$\Rightarrow \frac{X dV}{dX} = \frac{-7+7V^2}{3-7V} \Rightarrow \frac{X dV}{dX} = \frac{-7(V^2-1)}{3-7V}$$

$$\Rightarrow \int \frac{3-7V}{V^2-1} dV = -7 \frac{dX}{X}$$

$$\Rightarrow \left[-\frac{1}{2(1+V)} + \frac{3}{2(1-V)} \right] dV = -7 \frac{dX}{X}$$

$$\Rightarrow -\frac{1}{2} \int \frac{1}{(1+V)} dV + \frac{3}{2} \int \frac{1}{(1-V)} dV = 2 \int \frac{1}{X} dX$$

$$\Rightarrow -\frac{1}{2} \log|1+V| - \frac{3}{2} \log|1-V| = 2 \log X + \log C$$

$$\Rightarrow \log|1+V| + 3 \log|1-V| = -4 \log X + \log C$$

$$\Rightarrow \log(1+V)(1-V)^3 = \log(C/X^4)$$

$$\Rightarrow (1+V)(1-V)^3 = \frac{C}{X^4} \quad \{V = \frac{Y}{X}\}$$

$$\Rightarrow \left(1 + \frac{Y}{X}\right) \left(1 - \frac{Y}{X}\right)^3 = \frac{C}{X^4}$$

$$\Rightarrow \frac{(X+Y)(X-Y)^3}{X^4} = \frac{C}{X^4} \Rightarrow (X+Y)(X-Y)^3 = C$$

$$(h, k) = \left[-\frac{5}{3}, \frac{1}{3}\right] \quad X = x + \frac{5}{3}, Y = y - \frac{1}{3}$$

$$\Rightarrow \left(x + \frac{5}{3} + y - \frac{1}{3}\right) \left(x + \frac{5}{3} - y + \frac{1}{3}\right)^3 = C$$

$$(3x + 3y + 4)(x + -y + 2)^3 = 3C$$

This is the required solution.



v. solve the differential equation

$$\frac{dy}{dx} = \frac{x+2y+3}{2x+3y+4}$$

Sol: Given eq'n $\frac{dy}{dx} = \frac{x+2y+3}{2x+3y+4}$ [a/a' ≠ b/b']

this is non-homogeneous D.E of case(3)

put $x = X + h$ and $y = Y + k$

$$\frac{dy}{dx} = \frac{x+2y+3}{2x+3y+4}$$

$$\Rightarrow \frac{dY}{dX} = \frac{(X+h)+2(Y+k)+3}{2(X+h)+(Y+k)+4}$$

$$\Rightarrow \frac{dY}{dX} = \frac{X+2Y+(h+2k+3)}{2X+3Y+(2h+3k+4)} \dots (*)$$

now choose h and k such that

$$h + 2k + 3 = 0 \dots (1)$$

and

$$2h + 3k + 4 = 0 \dots (2)$$

solving (1)& (2)

$$\begin{matrix} 1 & 2 & 3 & 1 \\ 2 & 3 & 4 & 2 \end{matrix}$$

$$(h, k) = \left[\frac{8-9}{3-4}, \frac{6-4}{3-4}\right] = [1, -2]$$

Hence (*) becomes

$\Rightarrow \frac{dY}{dX} = \frac{X+2Y}{2X+3Y}$ is a homogeneous equation.

$$\text{put } y = VX \Rightarrow \frac{dY}{dX} = V + \frac{X dV}{dX}$$

$$\Rightarrow V + \frac{X dV}{dX} = \frac{X+2VX}{2X+3VX}$$



$$\Rightarrow V + \frac{X dV}{dX} = \frac{X(1+2V)}{X(2+3V)}$$

$$\Rightarrow V + \frac{X dV}{dX} = \frac{(1+2V)}{(2+3V)}$$

$$\Rightarrow \frac{X dV}{dX} = \frac{(1+2V)}{(2+3V)} - \frac{V}{1}$$

$$\Rightarrow \frac{X dV}{dX} = \frac{1+2V-2V-3V^2}{(2+3V)}$$

$$\Rightarrow \frac{X dV}{dX} = \frac{1-3V^2}{2+3V}$$

$$\Rightarrow \int \frac{2+3V}{1-3V^2} dV = \frac{dX}{X}$$

$$\Rightarrow \left[\frac{2}{1-3V^2} + \frac{3V}{1-3V^2} \right] dV = \frac{dX}{X}$$

$$\Rightarrow \frac{2}{3} \int \frac{1}{\left[\frac{1}{\sqrt{3}} \right]^2 - V^2} dV - \frac{1}{2} \int \frac{6V}{1-3V^2} dV = \int \frac{1}{X} dX$$

$$\Rightarrow \frac{2}{3} \cdot \frac{1}{2(\frac{1}{\sqrt{3}})} \log \left| \frac{\frac{1}{\sqrt{3}} + V}{\frac{1}{\sqrt{3}} - V} \right| - \frac{1}{2} \log |1 - 3V^2| = \log CX$$

$$\Rightarrow \frac{1}{(\sqrt{3})} \log \left| \frac{1+\sqrt{3}V}{1-\sqrt{3}V} \right| - \frac{1}{2} \log |1 - 3V^2| = \log CX$$

$$\Rightarrow \frac{1}{(\sqrt{3})} \log \left| \frac{1+\sqrt{3}\left(\frac{Y}{X}\right)}{1-\sqrt{3}\left(\frac{Y}{X}\right)} \right| - \frac{1}{2} \log \left| 1 - 3 \left(\frac{Y}{X} \right)^2 \right| = \log CX$$

$$\Rightarrow \frac{1}{(\sqrt{3})} \log \left| \frac{X+\sqrt{3}Y}{X-\sqrt{3}Y} \right| - \frac{1}{2} \log \left| \frac{X^2-3Y^2}{X^2} \right| = \log CX$$

$$X = x + 1 \text{ and } Y = y + 2$$

$$\Rightarrow \frac{1}{(\sqrt{3})} \log \left| \frac{x+1 + \sqrt{3}(y-2)}{x+1 - \sqrt{3}(y-2)} \right| - \frac{1}{2} \log \left| \frac{(x+1)^2 - 3(y-2)^2}{(x+1)^2} \right| = \log CX$$

1. If the abscissa of points A, B are the roots of the equation

$x^2 + 2ax - b^2 = 0$ and the ordinates of A, B are the roots of $x^2 + 2px - q^2 = 0$, then find the equation of a circle for which AB as a diameter.

Sol: Let A(x_1, y_1) and B(x_2, y_2)

Given that

x_1, x_2 be the roots of $x^2 + 2ax - b^2 = 0$

$$\Rightarrow (x - x_1)(x - x_2) = x^2 + 2ax - b^2$$

And

Given that

y_1, y_2 be the roots of $y^2 + 2py - q^2 = 0$

$$\Rightarrow (y - y_1)(y - y_2) = y^2 + 2py - q^2$$

Equation of the circle with AB as a diameter is

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$$

$$\Rightarrow (x^2 + 2ax - b^2) + (y^2 + 2py - q^2) = 0$$

$\Rightarrow x^2 + y^2 + 2ax + 2py - b^2 - q^2 = 0$ Is the required eq'n of the circle.

2. If a point P is moving such that the lengths of tangents from P to the circle $x^2 + y^2 - 6x - 4y - 12 = 0$, $x^2 + y^2 + 6x + 18y + 26 = 0$ are in the ratio 2:3 the find the equation of the locus of p.

Sol: let P(x_1, y_1) be any point on the locus

$$\text{Given that } \frac{\sqrt{S_{11}}}{\sqrt{S'_{11}}} = \frac{2}{3} \Rightarrow 3\sqrt{S_{11}} = 2\sqrt{S'_{11}} \quad \text{S.O.B}$$

$$\Rightarrow 9(S_{11}) = 4(S'_{11})$$

$$\begin{aligned} \Rightarrow 9(x_1^2 + y_1^2 - 6x_1 - 4y_1 - 12) \\ = 4(x_1^2 + y_1^2 + 6x_1 + 18y_1 + 26) \end{aligned}$$

$$\begin{aligned} \Rightarrow (9x_1^2 + 9y_1^2 - 54x_1 - 36y_1 - 108) - 4x_1^2 \\ - 4y_1^2 - 24x_1 - 72y_1 - 104 = 0 \end{aligned}$$

$$\Rightarrow 5x_1^2 + 5y_1^2 - 78x_1 - 108y_1 - 212 = 0$$

\therefore the equation of locus of p is

$$5x^2 + 5y^2 - 78x - 108y - 212 = 0.$$



3. Find the pole of $3x + 4y - 45 = 0$ with respect to

$$x^2 + y^2 - 6x - 8y + 5 = 0$$

Sol: given equation of the circle

$$x^2 + y^2 - 6x - 8y + 5 = 0 \dots\dots(1)$$

$$\text{Centre } (3, 4) \text{ and } r = \sqrt{(-3)^2 + (-4)^2 - 5} = \sqrt{20}$$

Given line $3x + 4y - 45 = 0$ here $l = 3, m = 4$ & $n = -45$

$$\text{The pole} = \left(-g + \frac{lr^2}{lg+mf-n}, -f + \frac{mr^2}{lg+mf} \right)$$

$$= \left(3 + \frac{3(20)}{3(-3)+4(-4)+45}, 4 + \frac{4(20)}{3(-3)+4(-4)+45} \right)$$

$$= \left(3 + \frac{3(20)}{20}, 4 + \frac{4(20)}{20} \right) = (3 + 3, 4 + 4) = (6, 8)$$

(h/w) Find the pole of $x + y + 2 = 0$ with respect to

$$x^2 + y^2 - 4x - 6y - 12 = 0$$

4. Find the value of k if $kx + 3y - 1 = 0$,

$2x + y + 5 = 0$ are conjugate lines with respect to the circle

$$x^2 + y^2 - 2x - 4y - 4 = 0.$$

Sol: Given equation of the circle

$$x^2 + y^2 - 2x - 4y - 4 = 0 \dots\dots(1)$$

$$\text{Centre } (1, 2) \text{ and } r = \sqrt{(1)^2 + (2)^2 + 4} = \sqrt{9} = 3$$

Given line $2x + y + 5 = 0$ here $l = 2, m = 1$ and $n = 5$

$$\text{The pole} = \left(-g + \frac{lr^2}{lg+mf-n}, -f + \frac{mr^2}{lg+mf} \right)$$

$$= \left(1 + \frac{2(9)}{2(-1)+1(-2)-5}, 2 + \frac{1(9)}{2(-1)+1(-2)-5} \right)$$

$$= \left(1 + \frac{2(9)}{-9}, 2 + \frac{1(9)}{-9} \right)$$

$$= (1 - 2, 2 - 1) = (-1, 1)$$

$(-1, 1)$ lies on $kx + 3y - 1 = 0$

$$\Rightarrow -k + 3 - 1 = 0 \Rightarrow k = 2.$$

5. Find the equations of the tangents to the circle

$x^2 + y^2 + 2x - 2y - 3 = 0$ which are perpendicular to
 $3x - y + 4 = 0$.

Sol: given equation of the circle $x^2 + y^2 + 2x - 2y - 3 = 0 \dots\dots(1)$

Centre $(-1, 1)$ and radius $(r) = \sqrt{(1)^2 + (-1)^2 + 3} = \sqrt{5}$

The given line $3x - y + 4 = 0 \dots\dots(2)$

$$\text{Slope}(m) = -\frac{a}{b} = -\frac{3}{-1} = 3 \text{ and } \perp^{\text{lar}} \text{slope}(m) = -\frac{1}{3} \Rightarrow m^2 = \frac{1}{9}$$

Eq'n of tangent to S=0 & \perp^{lar} to (2)

$$\text{is } (y - y_1) = m(x - x_1) \pm r\sqrt{1 + m^2}$$

$$\Rightarrow (y - 1) = -\frac{1}{3}(x + 1) \pm \sqrt{5}\sqrt{1 + \left(\frac{1}{3}\right)^2}$$

$$\Rightarrow (y - 1) = -\frac{(x+1)}{3} \pm \frac{\sqrt{5}\sqrt{10}}{3}$$

$$\Rightarrow 3(y - 1) = -(x + 1) \pm 5\sqrt{2}$$

$$\Rightarrow x + 1 + 3y - 3 \pm 5\sqrt{2} = 0$$

Hence required eq'n of tangents are

$$x + 3y - 2 \pm 5\sqrt{2} = 0.$$

**6. Find the equations of the tangents to the circle**

$x^2 + y^2 - 4x + 6y - 12 = 0$ which are parallel to $x + y - 8 = 0$.

Sol: given equation of the circle

$$S \equiv x^2 + y^2 - 4x + 6y - 12 = 0$$

Centre $(2, -3)$ and radius $(r) = \sqrt{(-2)^2 + (3)^2 + 12}$

$$= \sqrt{4 + 9 + 12} = \sqrt{25} = 5$$

The given line $x + y - 8 = 0 \dots\dots(1)$

$$\text{Slope}(m) = -\frac{a}{b} = -\frac{1}{1} = -1$$

Eq'n of tangent to S=0 & parallel to (1)

$$\text{is } (y - y_1) = m(x - x_1) \pm r\sqrt{1 + m^2}$$

$$\Rightarrow (y + 3) = -1(x - 2) \pm 5\sqrt{1 + 1}$$

$$\Rightarrow x - 2 + y + 3 \pm 5\sqrt{2} = 0$$

Hence required eq'n of tangents are $x + y + 1 \pm 5\sqrt{2} = 0$.

7. Find the equation of the tangent to

$$x^2 + y^2 - 2x + 4y = 0 \text{ at } (3, -1).$$

Also find the equation of tangent parallel to it.

Sol: given equation of the circle

$$x^2 + y^2 - 2x + 4y = 0 \dots\dots(1)$$

Centre $(1, -2)$ and radius $(r) = \sqrt{(-1)^2 + (2)^2 + 0} = \sqrt{5}$

The equation of tangent at $(3, -1)$ is

$$S_1 = xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$$

$$\Rightarrow x(3) + y(-1) - 1(x + 3) + 2(y - 1) = 0$$

$$\Rightarrow 3x - y - x - 3 + 2y - 2 = 0$$

$$\Rightarrow 2x + y - 5 = 0 \dots\dots(2)$$

here slope (m) = -2

Required eq'n of the tangent to (1) and it is parallel to (2) is

$$(y - y_1) = m(x - x_1) \pm r\sqrt{1 + m^2}$$

$$\Rightarrow (y + 2) = -2(x - 1) \pm \sqrt{5}\sqrt{1 + (-2)^2}$$

$$\Rightarrow (y + 2) = -2(x - 1) \pm \sqrt{5}\sqrt{5}$$

$$\Rightarrow (y + 2) = -2(x - 1) \pm 5$$

$$\Rightarrow y + 2 = -2x + 2 \pm 5$$

$$\therefore 2x + y \pm 5 = 0.$$



8. Show that the tangent at $(-1, 2)$ of the circle

$x^2 + y^2 - 4x - 8y + 7 = 0$ touches the circle

$x^2 + y^2 + 4x + 6y = 0$ and also find its point of contact.

Sol: equation of the tangent at $(-1, 2)$ to the circle

$$x^2 + y^2 - 4x - 8y + 7 = 0 \text{ is}$$

$$S_1 = xx_1 + yy_1 + \frac{2g(x+x_1)}{2} + \frac{2f(y+y_1)}{2} + c = 0$$

$$\Rightarrow x(-1) + y(2) - 2(x-1) - 4(y+2) + 7 = 0$$

$$\Rightarrow -3x - 2y + 1 = 0$$

$$\Rightarrow 3x + 2y - 1 = 0 \dots (1)$$

For the circle $x^2 + y^2 + 4x + 6y = 0$ centre $(-2, -3)$,

$$r = \sqrt{(2)^2 + (3)^2 - 0} = \sqrt{13}$$

\perp Distance from centre $(-2, -3)$ to given line (1)

$$= \frac{|3(-2)+2(-3)-1|}{\sqrt{(3)^2+(2)^2}} = \frac{|-6-6-1|}{\sqrt{13}} = \frac{13}{\sqrt{13}} =$$

$\sqrt{13}$ so the line (1) also touches the 2nd circle.

let (h, k) be the required point of contact.

so it is the foot of the \perp from the centre $(-2, -3)$

$$\frac{h-x_1}{a} = \frac{k-y_1}{b} = -\frac{(ax_1+by_1+c)}{a^2+b^2}$$

$$\Rightarrow \frac{h+2}{3} = \frac{k+3}{2} = -\frac{[3(-2)+2(-3)-1]}{3^2+2^2}$$

$$\Rightarrow \frac{h+2}{3} = \frac{k+3}{2} = -\frac{(-13)}{13} = 1$$

$$\Rightarrow \frac{h+2}{3} = 1 \text{ and } \Rightarrow \frac{k+3}{2} = 1$$

$$h+2 = 3 \text{ and } k+3 = 2$$

$$h = 3-2 \text{ and } k = 2-3$$

$$h = 1, k = -1$$

Coordinate of point of contact = $(1, -1)$.



9. Find the equations of normal to the circle

$$x^2 + y^2 - 4x + 6y + 11 = 0 \text{ at } (3, 2)$$

also find the other point where normal meets the circle.

Sol: given equation of the circle

$$x^2 + y^2 - 4x - 6y + 11 = 0 \dots (1)$$

$$\text{Centre C } (2, 3) = (-g, -f)$$

$$\text{Given point A } (3, 2) = (x_1, y_1)$$



The equation of the normal is

$$(x - x_1)(y_1 + f) - (y - y_1)(x_1 + g) = 0$$

$$\Rightarrow (x-3)(2-3) - (y-2)(3-2) = 0$$

$$\Rightarrow -x + 3 - y + 2 = 0 \Rightarrow x + y - 5 = 0.$$

centre of the circle is mid point of A and B

$$\left[\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2} \right] = (2, 3)$$

$$\Rightarrow \left[\frac{3+a}{2}, \frac{2+b}{2} \right] = (2, 3)$$

$$\Rightarrow \frac{3+a}{2} = 2 \text{ and } \frac{2+b}{2} = 3$$

$$\Rightarrow 3+a = 4 \text{ and } 2+b = 6$$

$$\Rightarrow a = 4-3 \text{ and } b = 6-2$$

$$B(a, b) = (1, 4)$$

10. Find the mid point of the chord intercepted by

$x^2 + y^2 - 2x - 10y + 1 = 0$ on the line $x - 2y + 7 = 0$. Also find the length of the chord.

Sol: circle $x^2 + y^2 - 2x - 10y + 1 = 0$ centre $(1, 5)$,

$$r = \sqrt{(1)^2 + (5)^2 - 1} = 5$$

\perp Distance from centre $(1, 5)$ to given line $x - 2y + 7 = 0$

$$= \frac{|1(1)-2(5)+7|}{\sqrt{(1)^2+(2)^2}} = \frac{|1-10+7|}{\sqrt{5}} = \frac{2}{\sqrt{5}}$$



length of chord intercepted by the circle is

$$2\sqrt{r^2 - d^2} = 2\sqrt{25 - \frac{4}{5}} = 2\sqrt{\frac{125-4}{5}}$$

$$= 2\sqrt{\frac{121}{5}} = \frac{2(11)}{\sqrt{5}} = \frac{22}{\sqrt{5}} \text{ units}$$

let (h, k) be the required mid point.

so it is the foot of the \perp from the centre $(1, 5)$

$$\frac{h-x_1}{a} = \frac{k-y_1}{b} = -\frac{(ax_1+by_1+c)}{a^2+b^2}$$

$$\Rightarrow \frac{h-1}{1} = \frac{k-5}{-2} = -\frac{[1(1)-2(5)+7]}{1^2+2^2}$$

$$\Rightarrow \frac{h-1}{1} = \frac{k-5}{-2} = -\frac{(-2)}{5} = \frac{2}{5}$$

$$\Rightarrow \frac{h-1}{1} = \frac{2}{5} \text{ and } \Rightarrow \frac{k-5}{-2} = \frac{2}{5}$$

$$5h - 5 = 2 \text{ and } 5k - 25 = -4$$

$$5h = 2 + 5 = 7 \text{ and } 5k = -4 + 25$$

$$h = \frac{7}{5}, k = \frac{21}{5}$$

11. Find the length of chord intercepted by the circle

$x^2 + y^2 - 8x - 2y - 8 = 0$ on the line $x + y + 1 = 0$.

Sol: given equation of the circle

$$x^2 + y^2 - 8x - 2y - 8 = 0 \dots (1)$$

$$\text{Centre } (4, 1) \text{ and } r = \sqrt{(-4)^2 + (-1)^2 + 8} = \sqrt{25} = 5$$

Given line $x + y + 1 = 0$

\perp Distance from centre $(-2, -3)$ to given line (1)

$$= \frac{|1(4)+1(1)+1|}{\sqrt{(1)^2+(1)^2}} = \frac{|4+1+1|}{\sqrt{2}} = \frac{6}{\sqrt{2}} = 3\sqrt{2} = \sqrt{18}$$



length of chord intercepted by the circle is

$$2\sqrt{r^2 - d^2} = 2\sqrt{25 - 18} = 2\sqrt{7} \text{ units}$$

Find the length of chord intercepted by the circle

$x^2 + y^2 - x + 3y - 2 = 0$ on the line $y = x - 3$. [Ans: $2\sqrt{26}$]

12. Find the equation of the circle with centre $(-2, 3)$ cutting a chord length 2 units on $3x + 4y + 4 = 0$.

Sol: given centre $C(-2, 3)$

Given equation of the chord $3x + 4y + 4 = 0 \dots (1)$

$d = \perp$ Distance from centre $C(-2, 3)$ to given line (1)

$$d = \frac{|3(-2)+4(3)+4|}{\sqrt{(3)^2+(4)^2}} = \frac{|-6+12+4|}{\sqrt{25}} = \frac{10}{5} = 2$$

Given length of chord $2\sqrt{r^2 - d^2} = 2$

$$\Rightarrow \sqrt{r^2 - d^2} = 1$$

$$\Rightarrow r^2 - d^2 = 1 \text{ (d = 2)}$$

$$\Rightarrow r^2 - 4 = 1$$

$$\therefore r^2 = 5$$

Required eq'n of the circle is

$$(x - a)^2 + (y - b)^2 = r^2$$

$$\Rightarrow (x + 2)^2 + (y - 3)^2 = 5$$

$$x^2 + y^2 + 4x - 6y + 8 = 0$$

13. Find the equation of pair of tangents drawn from $(0,0)$ to $x^2 + y^2 + 10x + 10y + 40 = 0$

Sol: given equation of the circle

$$x^2 + y^2 + 10x + 10y + 40 = 0 \dots\dots(1), P(x_1, y_1) = (0, 0)$$

$$S_1 = xx_1 + yy_1 + \frac{2g(x+x_1)}{2} + \frac{2f(y+y_1)}{2} + c = 0$$

$$S_1 \equiv x(0) + y(0) + 5(x+0) + 5(y+0) + 40$$

$$S_1 \equiv 5x + 5y + 40$$

$$S_{11} \equiv 02 + 02 + 10(0) + 10(0) + 40 = 40$$

$$\text{eq}'\text{n of the pair of tangents } S_1^2 = SS_{11}$$

$$(5x + 5y + 40)^2 = (x^2 + y^2 + 10x + 10y + 40)(40)$$

$$25(x+y+8)^2 = (x^2 + y^2 + 10x + 10y + 40)(40)$$

$$\Rightarrow 5\{x^2 + y^2 + 64 + 2xy + 16y + 16x\} = \{8x^2 + 8y^2 + 80x + 80y + 320\}$$

$$\Rightarrow \{8x^2 + 8y^2 + 80x + 80y + 320 - 5x^2 - 5y^2 - 320 - 10xy - 80y - 80x = 0$$

$$\Rightarrow 3x^2 - 10xy + 3y^2 = 0$$

13. Find the condition that the tangents drawn from the exterior point $(0,0)$ to the circle $S = x^2 + y^2 + 2gx + 2fy + c = 0$ are perpendicular to each other.

Sol: given equation of the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0 \dots(1)$$

$$r = \sqrt{g^2 + f^2 - c}, \text{ length of tangent} = \sqrt{S_{11}}$$

$$\text{if } \theta \text{ is angle } \frac{b}{w} \text{ the}$$

pair of tangents drawn from

$(0,0)$ to $S=0$ is

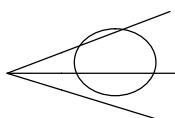
$$S_{11} = 0^2 + 0^2 + 2g(0) + 2f(0) + c = c$$

$$\text{Then } \tan \frac{\theta}{2} = \frac{r}{\sqrt{S_{11}}} \quad [\theta = 90^\circ]$$

$$\Rightarrow \tan \frac{90^\circ}{2} = \frac{\sqrt{g^2 + f^2 - c}}{\sqrt{c}} \Rightarrow \tan 45^\circ = \frac{\sqrt{g^2 + f^2 - c}}{\sqrt{c}}$$

$$1 = \frac{\sqrt{g^2 + f^2 - c}}{\sqrt{c}} \text{ S.O.B and cross multiplying} \Rightarrow c = g^2 + f^2 - c$$

$$\therefore 2c = g^2 + f^2$$



14. Find the area of the triangle formed by the normal at $(3, -4)$ to the circle $x^2 + y^2 - 22x - 4y + 25 = 0$ with the coordinate axes.

Sol: given equation of the circle

$$x^2 + y^2 - 22x - 4y + 25 = 0 \dots\dots(1)$$

$$\text{Centre C (11, 2) } = (-g, -f)$$

Given point A $(3, -4) = (x_1, y_1)$

The equation of the normal is

$$(x - x_1)(y_1 + f) - (y - y_1)(x_1 + g) = 0$$

$$\Rightarrow (x - 3)(-4 - 2) - (y + 4)(3 - 11) = 0$$

$$\Rightarrow 3x - 4y - 25 = 0.$$

Area of the triangle formed by the normal with the

$$\begin{aligned} \text{coordinate axes} &= \frac{1}{2} \left| \frac{c^2}{a.b} \right| = \frac{1}{2} \left| \frac{(-25)^2}{3.(-4)} \right| \\ &= \frac{625}{24} \text{ sq. units} \end{aligned}$$



15. Find the inverse point of $(-2, 3)$ w.r.t the circle

$$x^2 + y^2 - 4x - 6y + 9 = 0.$$

Sol: given equation of the circle

$$x^2 + y^2 - 4x - 6y + 9 = 0 \dots\dots(1)$$

$$\text{Centre C (2, 3) } = (x_1, y_1), \text{ given point P } (-2, 3) = (x_2, y_2)$$

eq'n of CP is $(y - y_1) = m(x - x_1)$

$$\Rightarrow (y - 2) = \frac{3-3}{-2-2}(x - 2)$$

$$\Rightarrow y - 2 = 0 \dots\dots(1)$$

eq'n of polar of p $(-2, 3)$ is $S_1 = 0$

$$S_1 = xx_1 + yy_1 + \frac{2g(x+x_1)}{2} + \frac{2f(y+y_1)}{2} + c = 0$$

$$\Rightarrow x(-2) + y(3) - 2(x-2) - 3(y+3) + 9 = 0$$

$$\Rightarrow -2x + 3y - 2x + 4 - 3y - 9 + 9 = 0$$

$$\Rightarrow -4x = -4 \Rightarrow x = 1 \dots\dots(2)$$

$$\text{Solving (1) \& (2)} \Rightarrow (x, y) = (1, 3)$$

\therefore The inverse point of p is $(1, 3)$



1. Show that the circles $x^2 + y^2 - 2x - 4y - 20 = 0$,
 $x^2 + y^2 + 6x + 2y - 90 = 0$ Touch each other internally. Find the point of contact and eq'n of tangent at point of contact.

Sol: eq'n s of the given circles

$$S \equiv x^2 + y^2 - 2x - 4y - 20 = 0 \dots\dots (1)$$

$$S' \equiv x^2 + y^2 + 6x + 2y - 90 = 0 \dots\dots (2)$$

centres (-g, -f): $C_1(1, 2)$, $C_2(-3, -1)$

$$\text{radius}(r) = \sqrt{g^2 + f^2 - c}$$

$$r_1 = \sqrt{1^2 + 2^2 + 20} = \sqrt{25} = 5$$

$$r_2 = \sqrt{3^2 + 1^2 + 90} = \sqrt{100} = 10$$

$$C_1C_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$= \sqrt{(-3 - 1)^2 + (-1 - 2)^2}$$

$$= \sqrt{(-4)^2 + (-3)^2} = \sqrt{25} = 5$$

$$C_1C_2 = |r_1 - r_2|$$

the circles touch each other internally.

the point of contact p divides C_1C_2 externally in the ratio

$$r_1:r_2 = 5:10 = m:n = 1:2$$

$$P = \left[\frac{mx_2 - nx_1}{m-n}, \frac{my_2 - ny_1}{m-n} \right]$$

$$= \left(\frac{1(-3) - 2(1)}{1-2}, \frac{1(-1) - 2(2)}{1-2} \right)$$

$$= \left(\frac{-3-2}{-1}, \frac{-1-4}{-1} \right)$$

$$= (5, 5)$$



2. Show that the circles $x^2 + y^2 - 8x - 2y + 8 = 0$,
 $x^2 + y^2 - 2x + 6y + 6 = 0$ Touch each other. Find the point of contact.

Sol: eq'n s of the given circles

$$S \equiv x^2 + y^2 - 8x - 2y + 8 = 0 \dots\dots (1)$$

$$S' \equiv x^2 + y^2 - 2x + 6y + 6 = 0 \dots\dots (2)$$

centres (-g, -f): $C_1(4, 1)$, $C_2(1, -3)$

$$\text{radius}(r) = \sqrt{g^2 + f^2 - c}$$

$$r_1 = \sqrt{4^2 + 1^2 - 8} = \sqrt{17 - 8} = \sqrt{9} = 3$$

$$r_2 = \sqrt{1^2 + 3^2 - 6} = \sqrt{10 - 6} = \sqrt{4} = 2$$

$$C_1C_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$= \sqrt{(1 - 4)^2 + (-3 - 1)^2}$$

$$= \sqrt{(-3)^2 + (-4)^2} = \sqrt{25} = 5$$

$$C_1C_2 = r_1 + r_2$$

the circles touch each other externally.

the point of contact p divides C_1C_2 Internally in the ratio

$$r_1:r_2 = 3:2 = m:n$$

$$P = \left[\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n} \right]$$

$$= \left(\frac{3(1) + 2(4)}{3+2}, \frac{3(-3) + 2(1)}{3+2} \right)$$

$$= \left(\frac{3+8}{5}, \frac{-9+2}{5} \right)$$

$$= \left(\frac{11}{5}, \frac{-7}{5} \right)$$





3. Show that the circles $x^2 + y^2 + 2ax + c = 0$ and $x^2 + y^2 + 2by + c = 0$ touch each other if $\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{c}$

Sol: Given eqns of the circles

$$x^2 + y^2 + 2ax + c = 0 \dots (1)$$

$$\Rightarrow C_1(-a, 0) \text{ and } r_1 = \sqrt{a^2 - c}$$

$$x^2 + y^2 + 2by + c = 0 \dots (2)$$

$$\Rightarrow C_1(0, -b) \text{ and } r_1 = \sqrt{b^2 - c}$$

$$C_1C_2 = \sqrt{(0+a)^2 + (-b-0)^2} = \sqrt{a^2 + b^2}$$



Given circles touch each other $|r_1 + r_2| = C_1C_2$

$$\Rightarrow \sqrt{a^2 - c} + \sqrt{b^2 - c} = \sqrt{a^2 + b^2}$$

S.O.B

$$[\sqrt{a^2 - c} + \sqrt{b^2 - c}]^2 = a^2 + b^2$$

$$\Rightarrow a^2 - c + b^2 - c + 2\sqrt{a^2 - c}\sqrt{b^2 - c} = a^2 + b^2$$

$$\Rightarrow 2\sqrt{a^2 - c}\sqrt{b^2 - c} = 2c$$

$$\Rightarrow \sqrt{a^2 - c}\sqrt{b^2 - c} = c \quad \text{S.O.B} \Rightarrow (a^2 - c)(b^2 - c) = c^2$$

$$\Rightarrow a^2b^2 - a^2c - b^2c + c^2 = c^2$$

$$\Rightarrow a^2b^2 - a^2c - b^2c = 0$$

$$\Rightarrow a^2b^2 = c(a^2 + b^2)$$

$$\Rightarrow \frac{1}{c} = \frac{a^2 + b^2}{a^2b^2} \Rightarrow \frac{1}{c} = \frac{a^2}{a^2b^2} + \frac{b^2}{a^2b^2}$$

$$\therefore \frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{c}$$



4. If $x + y = 3$ is the equation of the chord of the circle $x^2 + y^2 - 2x + 4y - 8 = 0$, find the equation of the circle having AB as diameter.

Sol:

Given eqn of the circle

$$S \equiv x^2 + y^2 - 2x + 4y - 8 = 0 \dots (1)$$

$$line L \equiv x + y - 3 = 0 \dots (2)$$

Eq'n of the circle passing through the point of intersection of S=0 and L=0 is $S + \lambda L = 0$

$$\Rightarrow (x^2 + y^2 - 2x + 4y - 8) + \lambda(x + y - 3) = 0 \dots (3)$$

$$\Rightarrow (x^2 + y^2 + x(\lambda - 2) + (\lambda + 4)y - (3\lambda + 8)) = 0.$$

$$centre \left[\frac{(2-\lambda)}{2}, -\frac{(4+\lambda)}{2} \right]$$

if AB is diameter of circle (1), then centre lies on $x + y - 3 = 0$

$$\Rightarrow \left(\frac{2-\lambda}{2} \right) - \frac{(4+\lambda)}{2} = 3$$

$$\Rightarrow -2 - 2\lambda = 6$$

$$\Rightarrow -2\lambda = 8 \Rightarrow \lambda = -4$$

The required eq'n of the circle is from (3)

$$(x^2 + y^2 + x(-4 - 2) + (-4 + 4)y - (3(-4) + 8)) = 0$$

$$\therefore x^2 + y^2 - 6x + 4 = 0$$

5. If the straight line $2x + 3y = 1$ intersects the circle

$x^2 + y^2 = 4$ at the points A and B, find the equation of the circle having AB as diameter.

Sol:

Given eqn of the circle

$$S \equiv x^2 + y^2 = 4 \dots (1)$$

$$line L \equiv 2x + 3y - 1 = 0 \dots (2)$$

Eq'n of the circle passing through the point of intersection of S=0 and L=0 is $S + \lambda L = 0$



$$\Rightarrow (x^2 + y^2 - 4) + \lambda(2x + 3y - 1) = 0 \dots (3)$$

$$\Rightarrow (x^2 + y^2 + (2\lambda)x + (3\lambda)y - (\lambda + 4)) = 0.$$

$$\text{centre } \left[-\lambda, -\frac{3\lambda}{2} \right]$$

if AB a is diameter of circle (1), then centre lies on $2x + 3y - 1 = 0$

$$\Rightarrow 2(-\lambda) + 3(3\lambda/2) = 1$$

$$\Rightarrow -4\lambda - 9\lambda = 2$$

$$\Rightarrow -13\lambda = 2 \Rightarrow \lambda = -\frac{2}{13}$$

The required eq'n of the circle is from (3)

$$(x^2 + y^2 - 4) - \frac{2}{13}(2x + 3y - 1) = 0$$

$$13(x^2 + y^2) - 52 - 4x - 6y + 2 = 0$$

$$\therefore 13(x^2 + y^2) - 4x - 6y - 50 = 0$$



6. If the two circles $x^2 + y^2 + 2gx + 2fy = 0$ and

$x^2 + y^2 + 2g'x + 2f'y = 0$ touch each other then show that $f'g = fg'$.

Sol: Given eq'n of the circles

$$S \equiv x^2 + y^2 + 2gx + 2fy = 0 \dots (1)$$

centre $C_1(-g, -f)$

$$S' \equiv x^2 + y^2 + 2g'x + 2f'y = 0 \dots (2)$$

centre $C_2(-g', -f')$

above circles passes through (0, 0)

If the circles touch each other, then

OC_1C_2 are collinear \Rightarrow Area of $OC_1C_2 = 0$

$O(0, 0)$, $C_1(-g, -f)$ and $C_2(-g', -f')$

$$\text{Area} = \frac{1}{2}|x_1y_2 - x_2y_1| = 0$$

$$\Rightarrow |(-g)(-f') - (-g')(-f)| = 0$$

$$\Rightarrow gf' - g'f = 0 \quad \therefore f'g = fg'.$$



7. If the angle between the circles

$$x^2 + y^2 - 12x - 6y + 41 = 0, x^2 + y^2 + kx + 6y - 59 =$$

0, find k.

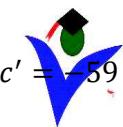
Sol: Given circles $S = x^2 + y^2 - 12x - 6y + 41 = 0 \dots (1)$

$$S' = x^2 + y^2 + kx + 6y - 59 = 0 \dots \dots \dots (2)$$

$$\theta = 45^\circ \Rightarrow \cos 45^\circ = \frac{1}{\sqrt{2}}$$

$$g = -6; f = -3; c = 41$$

$$g' = \frac{k}{2}; f' = 3; c' = -59$$



$$r_1 = \sqrt{g^2 + f^2 - c} = \sqrt{6^2 + 3^2 - 41} = 2$$

$$r_2 = \sqrt{\left(\frac{k}{2}\right)^2 + (3)^2 + 59} = \sqrt{\frac{k^2+36+236}{4}} = \frac{\sqrt{k^2+272}}{2}$$

$$\cos \theta = \frac{|c+c'-2gg'-2ff'|}{2r_1r_2}$$

$$\Rightarrow \cos 45^\circ = \frac{|41-59-(6)\left(\frac{k}{2}\right)-2(-3)(3)|}{2.2.\frac{\sqrt{k^2+272}}{2}}$$

$$\Rightarrow \frac{1}{\sqrt{2}} = \frac{|-18+6k+18|}{2\sqrt{k^2+272}} \Rightarrow \frac{1}{\sqrt{2}} = \frac{|6k|}{2\sqrt{k^2+272}}$$

$$\Rightarrow \frac{1}{\sqrt{2}} = \frac{|3k|}{\sqrt{k^2+272}} \quad \text{S.O.B} \Rightarrow \frac{1}{2} = \frac{9k^2}{k^2+272}$$

$$\Rightarrow k^2 + 272 = 18k^2$$

$$\Rightarrow 17k^2 = 272 \Rightarrow k^2 = \frac{272}{17}$$

$$\Rightarrow k^2 = 16 \quad \therefore k = \pm 4$$



8. Find the equation of the circle passing through the point of intersection of the circles $x^2 + y^2 - 8x - 6y + 21 = 0$ and $x^2 + y^2 - 2x - 15 = 0$ and (1, 2).

Sol:

Given eq'n of the circle

$$S \equiv x^2 + y^2 - 8x - 6y + 21 = 0 \dots (1)$$

$$S' \equiv x^2 + y^2 - 2x - 15 = 0 \dots (2)$$

Eq'n of the circle passing through the point of intersection of $S=0$ and $S'=0$ is $S + \lambda S' = 0$

$$\Rightarrow (x^2 + y^2 - 8x - 6y + 21) + \lambda(x^2 + y^2 - 2x - 15) = 0 \dots (3)$$

Above eq'n passing through (1, 2)

$$\Rightarrow (1^2 + 2^2 - 8(1) - 6(2) + 21) + \lambda(1^2 + 2^2 - 2(1) - 15) = 0$$

$$\Rightarrow (1 + 4 - 8 - 12 + 21) + \lambda(1 + 4 - 2 - 15) = 0$$

$$\Rightarrow (6) + \lambda(-12) = 0$$

$$\Rightarrow 6 = 12\lambda$$

$$\Rightarrow \lambda = \frac{1}{2} \text{ Sub in (3)}$$

$$\Rightarrow (x^2 + y^2 - 8x - 6y + 21) + \frac{1}{2}(x^2 + y^2 - 2x - 15) = 0$$

$$\Rightarrow (2x^2 + 2y^2 - 16x - 12y + 42) + (x^2 + y^2 - 2x - 15) = 0$$

$$\Rightarrow (3x^2 + 3y^2 - 18x - 12y + 27) = 0 \quad (\div \text{ by } 3)$$

$$x^2 + y^2 - 6x - 4y + 9 = 0$$



9. Find the radical centre of the circles

$$x^2 + y^2 + 4x - 7 = 0,$$

$$2x^2 + 2y^2 + 3x + 5y - 9 = 0$$

$$\text{and } x^2 + y^2 + y = 0.$$

Sol:

Given circles

$$S \equiv x^2 + y^2 + 4x - 7 = 0 \dots (1)$$

$$S' \equiv 2x^2 + 2y^2 + 3x + 5y - 9 = 0 \dots (2)$$

$$S'' \equiv x^2 + y^2 + y = 0 \dots (3)$$

radical axis of (1)& (2)

$$2x^2 + 2y^2 + 8x - 14 = 0$$

$$\underline{2x^2 + 2y^2 + 3x + 5y - 9 = 0}$$

$$\underline{\underline{5x - 5y - 5 = 0}} \dots (4)$$

radical axis of (2)& (3)

$$2x^2 + 2y^2 + 3x + 5y - 9 = 0$$

$$\underline{2x^2 + 2y^2 + 0 + 2y = 0}$$

$$\underline{\underline{3x + 3y - 9 = 0}} \dots (5)$$

Solving (4) & (5) 5 -5 -5 5

 3 3 -9 3

$$(x, y) = \left[\frac{45+15}{15+15}, \frac{-15+45}{15+15} \right]$$

$$= \left[\frac{60}{30}, \frac{30}{30} \right]$$

$$= (2, 1)$$



10. Find the equation of the circle which cuts the circles

$$x^2 + y^2 + 2x + 4y + 1 = 0$$

$$2x^2 + 2y^2 + 6x + 8y - 3 = 0 \text{ and}$$

$$x^2 + y^2 - 2x + 6y - 3 = 0 \text{ orthogonally.}$$

Sol:

Given circles

$$S \equiv x^2 + y^2 + 2x + 4y + 1 = 0 \dots \dots (1)$$

$$S' \equiv 2x^2 + 2y^2 + 6x + 8y - 3 = 0 \dots \dots (2)$$

$$S'' \equiv x^2 + y^2 - 2x + 6y - 3 = 0 \dots \dots (3)$$



radical axis of (1)& (2)

$$2x^2 + 2y^2 + 4x + 8y + 2 = 0$$

$$\underline{2x^2 + 2y^2 + 6x + 8y - 3 = 0}$$

$$\underline{\underline{-2x + 0 + 5 = 0}} \dots \dots (4)$$

radical axis of (2)& (3)

$$2x^2 + 2y^2 + 6x + 8y - 3 = 0$$

$$\underline{2x^2 + 2y^2 - 4x + 12y - 6 = 0}$$

$$\underline{\underline{10x - 4y + 3 = 0}} \dots \dots (5)$$

Solving (4) & (5) -2 0 5 -2

$$\begin{array}{cccc} 10 & -4 & 3 & 10 \end{array}$$

$$(x, y) = \left[\frac{0+20}{8-0}, \frac{50+6}{8-0} \right]$$

$$= \left[\frac{20}{8}, \frac{56}{8} \right] = \left(\frac{5}{2}, 7 \right)$$

$$\text{radius} = \sqrt{S_{11}} \text{ from } \left(\frac{5}{2}, 7 \right) \text{ to } S=0$$



$$r^2 = \sqrt{S_{11}}^2 = \left(\left(\frac{5}{2} \right)^2 + 7^2 + 2 \left(\frac{5}{2} \right) + 4(7) + 2 \right)$$

$$r^2 = \left(\frac{25}{4} + 83 \right) = \frac{357}{4}$$

eq'n of the circle with centre

$$(a, b) = \left(\frac{5}{2}, 7 \right) \text{ and radius } r \text{ is}$$

$$\left[x - \frac{5}{2} \right]^2 + [y - 7]^2 = \frac{357}{4}$$

$$\Rightarrow x^2 - 5x + \frac{25}{4} + y^2 + 49 - 14y = \frac{357}{4}$$

$$\Rightarrow 4x^2 - 20x + 25 + 4y^2 + 196 - 56y - 357 = 0$$

$$\Rightarrow 4(x^2 + y^2 - 5x - 14y - 34) = 0$$

$$\therefore x^2 + y^2 - 5x - 14y - 34 = 0$$



11. Find the eq'n of the circle which passes through the point

(0, -3) and intersects the circles

$$x^2 + y^2 - 6x + 3y + 5 = 0,$$

$$x^2 + y^2 - x - 7y = 0 \text{ Orthogonally.}$$

Sol: let the required circle

$$S \equiv x^2 + y^2 + 2gx + 2fy + c = 0 \dots \dots (*)$$

$$S' \equiv x^2 + y^2 - 6x + 3y + 5 = 0,$$

$$S'' \equiv x^2 + y^2 - x - 7y = 0$$

$$(0, -3) \text{ lies on } (*) \Rightarrow 0 + 9 - 6f + c = 0$$

$$\Rightarrow 0 - 6f + c + 9 = 0 \dots \dots (1)$$

Given $S = 0$ and $S' = 0$ are orthogonal

$$2gg' + 2ff' = c + c' \text{ condition for orthogonal}$$

$$\Rightarrow 2g(-3) + 2f\left(\frac{3}{2}\right) = c + 5$$

$$\Rightarrow -6g + 3f - c - 5 = 0 \dots \dots (2)$$

Given $S = 0$ and $S'' = 0$ are orthogonal

$$2gg' + 2ff' = c + c' \text{ condition for orthogonal}$$

$$\Rightarrow 2g\left(-\frac{1}{2}\right) + 2f\left(-\frac{7}{2}\right) = c + 0$$

$$\Rightarrow -g - 7f - c = 0 \dots \dots (3)$$



$$\begin{aligned} Eq''n (1) + (2) \\ 0 - 6f + c + 9 = 0 \\ \underline{-6g + 3f - c - 5 = 0} \\ \underline{\underline{-6g - 3f + 4 = 0}} \dots \dots \dots (4) \end{aligned}$$

$$\begin{aligned} Eq''n (1) + (3) \\ 0 - 6f + c + 9 = 0 \\ \underline{-g - 7f - c + 0 = 0} \\ \underline{\underline{-g - 13f + 9 = 0}} \dots \dots \dots (5) \end{aligned}$$

Solving (4)&(5)

6	3	-4	6
1	13	-9	1



$$\begin{aligned} (x, y) &= \left[\frac{-27+54}{78-3}, \frac{-4+54}{78-3} \right] = \left[\frac{25}{75}, \frac{50}{75} \right] \\ \left(\frac{1}{3}, \frac{2}{3} \right) \text{ sub in (1)} &\Rightarrow -6 \left(\frac{2}{3} \right) + 9 + c = 0 \end{aligned}$$

$$\Rightarrow c = -9 + 4 = -5$$

$$\begin{aligned} \therefore x^2 + y^2 + 2 \left(\frac{1}{3} \right) x + 2 \left(\frac{2}{3} \right) y - 5 = 0 \\ 3x^2 + 3y^2 + 2x + 3y - 15 = 0 \end{aligned}$$

12. Find the eq'n of the circle which cuts the circles

$$x^2 + y^2 - 4x - 6y + 11 = 0,$$

$x^2 + y^2 - 10x - 4y + 21 = 0$ Orthogonally, and has the diameter along the st line $2x + 3y = 7$.

Sol: let the required circle

$$S \equiv x^2 + y^2 + 2gx + 2fy + c = 0 \dots (*)$$

$$S \equiv x^2 + y^2 - 4x - 6y + 11 = 0,$$

$$S'' \equiv x^2 + y^2 - 10x - 4y + 21 = 0$$



$$\begin{aligned} \text{given line } L &\equiv 2x + 3y - 7 = 0 \\ (-g, -f) \text{ lies on } (L) &\Rightarrow -2g - 3f - 7 = 0 \end{aligned}$$

$$\Rightarrow 2g + 3f + 7 = 0 \dots \dots (1)$$

Given $S = 0$ and $S' = 0$ are orthogonal

$$2gg' + 2ff' = c + c' \text{ condition for orthogonal}$$

$$\Rightarrow 2g(-2) + 2f(-3) = c + 11$$

$$\Rightarrow -4g - 6f - c - 11 = 0 \dots \dots (2)$$

Given $S = 0$ and $S'' = 0$ are orthogonal

$$2gg' + 2ff' = c + c' \text{ condition for orthogonal}$$

$$\Rightarrow 2g(-5) + 2f(-2) = c + 21$$

$$\Rightarrow -10g - 4f - c - 21 = 0 \dots \dots (3)$$



$$Eq''n (2) - (3)$$

$$-4g - 6f - c - 11 = 0$$

$$\underline{-10g - 4f - c - 21 = 0}$$

$$\underline{\underline{6g - 2f + 10 = 0}} \dots \dots \dots (4)$$

Solving (1) & (4)

2	3	7	2
6	-2	10	6

$$(x, y) = \left[\frac{30+14}{-4-18}, \frac{42-20}{-4-18} \right] = \left[\frac{44}{-22}, \frac{22}{-22} \right]$$

$$(-2, -1) \text{ sub in (2)} \Rightarrow -4(-2) - 6(-1) = c + 11$$

$$\Rightarrow c = 14 - 11 = 3$$

$$\therefore x^2 + y^2 + 2(-2)x + 2(-1)y + 3 = 0$$

$$x^2 + y^2 - 4x - 2y + 3 = 0$$



13. Find the eq'n of the circle which passes through the point $(2, 0)(0, 2)$ and intersects the circles

$$2x^2 + 2y^2 + 5x - 6y + 4 = 0,$$

Orthogonally.

Sol: let the required circle

$$S \equiv x^2 + y^2 + 2gx + 2fy + c = 0 \dots\dots (*)$$

$$S' \equiv x^2 + y^2 + \frac{5}{2}x - 3y + 2 = 0,$$

$$(2, 0) \text{ lies on } (*) \Rightarrow 4 + 0 + 4g + 0 + c = 0$$

$$\Rightarrow 4g + 0 + c + 4 = 0 \dots\dots (1)$$

$$(0, 2) \text{ lies on } (*) \Rightarrow 0 + 4 + 0 + 4f + c = 0$$

$$\Rightarrow 0 + 4f + c + 4 = 0 \dots\dots (2)$$

Given $S = 0$ and $S' = 0$ are orthogonal

$2gg' + 2ff' = c + c'$ condition for orthogonal

$$\Rightarrow 2g\left(\frac{5}{4}\right) + 2f\left(-\frac{3}{2}\right) = c + 2$$

$$\Rightarrow \frac{5}{2}g - 3f - c - 2 = 0 \dots\dots (3)$$

Eq'n (1) + (3)

$$4g + 0 + c + 4 = 0$$

$$\frac{5}{2}g - 3f - c - 2 = 0$$

$$\underline{\underline{\frac{13}{2}g - 3f + 2 = 0}} \Rightarrow 13g - 6f + 4 = 0 \dots\dots (4)$$

Eq'n (2) + (3)

$$0 + 4f + c + 4 = 0$$

$$\frac{5}{2}g - 3f - c - 2 = 0$$

$$\underline{\underline{\frac{5}{2}g + f + 2 = 0}} \Rightarrow 5g + 2f + 4 = 0 \dots\dots (4)$$



14. Find the eq'n and length of the common chord of the two circles $x^2 + y^2 + 3x + 5y + 4 = 0$,

$$x^2 + y^2 + 5x + 3y + 4 = 0.$$

Sol: Given circles

$$S \equiv x^2 + y^2 + 3x + 5y + 4 = 0,$$

$$S' \equiv x^2 + y^2 + 5x + 3y + 4 = 0.$$

Equation of common chord is $S - S' = 0$

$$\Rightarrow -2x + 2y = 0 \Rightarrow x - y = 0 \dots\dots (1)$$

Centre $\left(-\frac{3}{2}, -\frac{5}{2}\right)$ and

$$\text{radius}(r) = \sqrt{\left(\frac{3}{2}\right)^2 + \left(\frac{5}{2}\right)^2 - 4} = \sqrt{\frac{9+25-16}{4}}$$

$$\sqrt{\frac{18}{4}} = \sqrt{\frac{9}{2}} \text{ of } S = 0$$

d is \perp^{lar} distance from $\left(-\frac{3}{2}, -\frac{5}{2}\right)$ to (1)

$$d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

$$d = \frac{\left|\frac{-3}{2} + \frac{5}{2}\right|}{\sqrt{1^2 + 1^2}} = \frac{\left|\frac{2}{2}\right|}{\sqrt{2}} = \frac{1}{\sqrt{2}}$$

Length of chord is $= 2\sqrt{r^2 - d^2}$

$$= 2\sqrt{\frac{9}{4} - \frac{1}{2}}$$

$$= 2\sqrt{\frac{8}{2}}$$

$$= 2\sqrt{4}$$

$$= 2.2$$

$$= 4$$



15. Prove that the radical axis of the circles $x^2 + y^2 + 2gx + 2fy + c = 0$ and $x^2 + y^2 + 2g'x + 2f'y + c' = 0$ is the diameter of the latter circle if $2g'(g - g') + 2f'(f - f') = c - c'$.

Sol: Given eq'n of the circles

$$S \equiv x^2 + y^2 + 2gx + 2fy + c = 0 \dots (1)$$

$$S' \equiv x^2 + y^2 + 2g'x + 2f'y + c' = 0 \dots (2)$$



Radical eq'n of the given circles is $S - S' = 0$

$$\Rightarrow 2x(g - g') + 2y(f - f') + (c - c') = 0 \dots (3)$$

Eq'n (3) becomes diameter of eq'n (2),
if centre $(-g', -f')$ lies on (3)

$$\Rightarrow 2(-g')(g - g') + 2(-f')(f - f') + (c - c') = 0$$

$$\therefore 2g'(g - g') + 2f'(f - f') = c - c'.$$



1. Find the eq'n of the ellipse with focus

$$(1, -1), e = \frac{2}{3} \text{ and directrix}$$

$$\text{as } x + y + 2 = 0.$$

Sol:

$$\text{Given } S(1, -1), e = \frac{2}{3} \text{ & } l \equiv x + y + 2 = 0$$

Let P(x, y) be any point on the locus

$$\text{W.K.T } \frac{SP}{PM} = e$$

$$\Rightarrow SP = ePM$$

$$\sqrt{(x-1)^2 + (y+1)^2} = \frac{2}{3} \frac{|x+y+2|}{\sqrt{1^2+1^2}}$$

S.O.B

$$\Rightarrow 9[(x-1)^2 + (y+1)^2] = \frac{4}{2}(x+y+2)^2$$

$$\begin{aligned} \Rightarrow 9[x^2 + 1 - 2x + y^2 + 1 + 2y] \\ = 2[x^2 + y^2 + 4 + 2xy + 4y + 4x] \end{aligned}$$

$$\begin{aligned} \Rightarrow 9x^2 + 9 - 18x + 9y^2 + 9 + 18y \\ = 2x^2 + 2y^2 + 8 + 4xy + 8y + 8x \end{aligned}$$

$$\therefore 7x^2 - 4xy + 7y^2 - 26x + 10y + 10 = 0$$



2. Find the eccentricity, foci, equations of directrices, length of latus rectum of the ellipse $9x^2 + 16y^2 = 144$.

Sol: Given eq'n of the hyperbola

$$9x^2 + 16y^2 = 144 \dots \dots (1)$$

(÷ by 144)

$$\Rightarrow \frac{x^2}{16} + \frac{y^2}{9} = 1 \quad \left[\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \right]$$

Compare with standard form

$$a^2 = 16 \mid\mid b^2 = 9$$

$$\Rightarrow a = 4 \mid\mid b = 3$$

(a>b)

centre (0, 0)

$$e = \sqrt{\frac{a^2 - b^2}{a^2}} = \frac{\sqrt{7}}{4}$$

$$\begin{aligned} \text{foci } (\pm ae, 0) &= \left(\pm 4 \frac{\sqrt{7}}{4}, 0 \right) \\ &= (\pm \sqrt{7}, 0) \end{aligned}$$

$$\text{L.L.R} = \frac{2b^2}{a} = \frac{2 \cdot 9}{4} = \frac{9}{2}$$

$$\text{L.of Ma Axis} = 2a = 2(4) = 8$$

$$\text{L.of Mi Axis} = 2b = 2(3) = 6$$

$$\text{Eq'nof directrices } x = \pm \frac{a}{e}$$

$$\Rightarrow x = \pm \frac{4}{\frac{\sqrt{7}}{4}} \Rightarrow \sqrt{7}x \pm 16 = 0.$$



3. Find the eccentricity, foci, equations of directrices, length of latus rectum of the ellipse

$$(i) 9x^2 + 16y^2 - 36x + 32y - 92 = 0$$

$$(ii) 3x^2 + y^2 - 6x - 2y - 5 = 0$$

Sol: Given eq'n of the hyperbola

$$9x^2 + 16y^2 - 36x + 32y - 92 = 0 \dots \dots (1)$$

$$\Rightarrow 9x^2 - 36x + 16y^2 + 32y - 92 = 0$$

$$\Rightarrow 9[x^2 - 4x] + 16[y^2 + 2y] - 92 = 0$$

$$\Rightarrow 9[x^2 - 4x + 4 - 4] + 16[y^2 + 2y + 1 - 1] = 92$$

$$\Rightarrow 9[(x-2)^2] - 36 + 16[(y+1)^2] - 16 = 92$$

$$\Rightarrow 9[(x-2)^2] + 16[(y+1)^2] = 144 \quad (\div \text{ by } 144)$$

$$\Rightarrow \frac{(x-2)^2}{16} + \frac{(y+1)^2}{9} = 1$$

$$\left[\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1 \right]$$

Compare with standard form

$$a^2 = 16 \parallel b^2 = 9$$

$$\Rightarrow a = 4 \parallel b = 3$$

(a>b)

centre (h, k) = (2, -1)

$$e = \sqrt{\frac{a^2 - b^2}{a^2}} = \frac{\sqrt{7}}{4}$$

$$\text{foci } (h \pm ae, k) = \left(2 \pm 4 \frac{\sqrt{7}}{4}, -1 \right) = (2 \pm \sqrt{7}, -1)$$

$$\text{L. L. R} = \frac{2b^2}{a} = \frac{2 \cdot 9}{4} = \frac{9}{2}$$

$$\text{Eq'n of directrices } x = h \pm \frac{a}{e} \Rightarrow x = -2 \pm \frac{\frac{4}{\sqrt{7}}}{4}$$

$$\Rightarrow \sqrt{7}x + 2\sqrt{7} \pm 16 = 0.$$



4. Find the eq'n of tangent and normal to the ellipse

$9x^2 + 16y^2 = 144$ at the end of the latus rectum in the first quadrant.

Sol: Given eq'n of ellipse

$$9x^2 + 16y^2 = 144 \dots \dots (1)$$

$$\Rightarrow \frac{9x^2}{144} + \frac{16y^2}{144} = 1 \Rightarrow \frac{x^2}{16} + \frac{y^2}{9} = 1$$

$$\therefore a^2 = 16, b^2 = 9$$

$$e = \sqrt{\frac{a^2 - b^2}{a^2}} = \frac{\sqrt{7}}{4}$$

End of the latus rectum in the first quadrant p $\left(ae, \frac{b^2}{a} \right)$

$$= \left(4 \frac{\sqrt{7}}{4}, \frac{9}{4} \right) = \left(\sqrt{7}, \frac{9}{4} \right)$$

eq'n of tangent at p is $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$

$$\Rightarrow \frac{x(\sqrt{7})}{16} + \frac{y(\frac{9}{4})}{9} = 1$$

$$\Rightarrow \sqrt{7}x + 4y = 16$$

eq'n of normal at p is $\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2$

$$\Rightarrow \frac{16x}{\sqrt{7}} - \frac{4y}{\frac{9}{4}} = 16 - 9$$

$$\Rightarrow 16x - 4\sqrt{7} = 7\sqrt{7}$$



5. find the eq'n of tangents to the ellipse $2x^2 + y^2 = 8$ which are (i) parallel $x - 2y - 4 = 0$ (ii) Perpendicular to the line $x + y + 2 = 0$.

Sol: Given eq'n of ellipse

$$2x^2 + y^2 = 8 \dots (1)$$

$$\Rightarrow \frac{2x^2}{8} + \frac{y^2}{8} = 1 \Rightarrow \frac{x^2}{4} + \frac{y^2}{8} = 1 \quad \therefore a^2 = 4, b^2 = 8$$

(i) Given line $x - 2y - 4 = 0 \dots (2)$

$$\Rightarrow 2y = x - 4$$

$$\Rightarrow y = \frac{x}{2} - \frac{4}{2} \quad \{y = mx + c\} \quad m = \frac{1}{2}$$

eq'n of tangent parallel to (2) is

$$y = mx \pm \sqrt{a^2m^2 - b^2}$$

$$\Rightarrow y = \frac{1}{2}x \pm \sqrt{4\left(\frac{1}{4}\right) + 8}$$

$$\Rightarrow y = \frac{x}{2} \pm \sqrt{9} \quad \text{or } x - 2y \pm 6 = 0$$

(ii) Given line $x + y + 2 = 0$

$$\Rightarrow y = -x - 2 \quad \{y = mx + c\} \quad m = -1$$

$$\text{perpendicular slope } -\frac{1}{m} = -\frac{1}{-1} = 1$$

eq'n of tangent perpendicular to (3)

$$\text{is } y = mx \pm \sqrt{a^2m^2 + b^2}$$

$$\Rightarrow y = 1x \pm \sqrt{4(1) + 8}$$

$$\Rightarrow y = x \pm \sqrt{12} \quad \text{or } x - y \pm 2\sqrt{3} = 0.$$



6. Find the eq'n of tangents to the ellipse $9x^2 + 16y^2 = 144$ which makes equal intercepts on the coordinate axes.

Sol: Given eq'n of ellipse

$$9x^2 + 16y^2 = 144 \dots (1)$$

$$\Rightarrow \frac{9x^2}{144} + \frac{16y^2}{144} = 1 \Rightarrow \frac{x^2}{16} + \frac{y^2}{9} = 1$$

$$\therefore a^2 = 16, b^2 = 9$$

Since eq'n of tangents makes equal intercepts on the coordinate axes, so $m = \pm 1$

Eq'n of tangent to the ellipse are

$$y = mx \pm \sqrt{a^2m^2 + b^2}$$

$$\Rightarrow y = \pm 1x \pm \sqrt{16(-1)^2 + 9}$$

$$\Rightarrow y = \pm x \pm 5$$

$$\Rightarrow x \pm y \pm 5 = 0$$



7. Find the eq'n of ellipse in standard form whose distance b/w foci is 8 and distance b/w the directrices is 32.

Sol: Given distance b/w

$$S(ae, 0) \text{ and } S'(-ae, 0)$$

$$\Rightarrow 2ae = 8 \Rightarrow ae = 4 \dots \dots (1)$$

And distance b/w the directrices is 32

$$\frac{2a}{e} = 32 \Rightarrow \frac{a}{e} = 16 \dots (2)$$

Multiplying (1) & (2)

$$(ae) \left(\frac{a}{e} \right) = 4 \cdot 16 = 64$$

$$a^2 = 64 \Rightarrow a = 8$$

$$\text{and } b^2 = a^2(1 - e^2)$$

$$\Rightarrow a^2 - (ae)^2$$

$$= 64 - 16 = 48$$

$$\therefore a^2 = 64, b^2 = 48$$

The required eq'n of the ellipse is

$$\frac{x^2}{64} + \frac{y^2}{48} = 1$$



8. Find the eq'n of ellipse in standard form whose distance b/w foci is 2 and the length of latus rectum is $15/2$.

Sol: Given distance b/w

$$S(ae, 0) \text{ and } S'(-ae, 0)$$

$$\Rightarrow 2ae = 2 \Rightarrow ae = 1 \dots \dots (1)$$

$$\text{And L. L. R} = \frac{2b^2}{a} = \frac{15}{2}$$

$$\Rightarrow b^2 = \frac{15a}{4} \dots \dots (2)$$

$$b^2 = a^2(1 - e^2)$$

$$\Rightarrow a^2(1 - e^2) = \frac{15a}{4}$$

$$\Rightarrow a^2 - (ae)^2 = \frac{15a}{4}$$

$$\Rightarrow a^2 - 1 = \frac{15a}{4}$$

$$\Rightarrow 4a^2 - 4 = 15a$$

$$\Rightarrow 4a^2 - 15a - 4 = 0$$

$$\Rightarrow 4a^2 - 16a + a - 4 = 0$$

$$\Rightarrow 4a(a - 4) + 1(a - 4) = 0$$

$$\Rightarrow (a - 4)(4a + 1) = 0$$

$$(a - 4) = 0, (4a + 1) = 0$$

$$a = 4 \text{ or } a = -\frac{1}{4} \times$$

$$\text{From (2)} \Rightarrow b^2 = \frac{15(4)}{4} = 15$$

$$\therefore a^2 = 16, b^2 = 15$$

The required eq'n of the ellipse is

$$\frac{x^2}{16} + \frac{y^2}{15} = 1$$



9. Show that the point of intersection of the perpendicular tangents to an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ lies on a circle.

Sol: let $y = mx \pm \sqrt{a^2m^2 + b^2}$

be the tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Let $P(x_1, y_1)$ be point of intersection of tangents

P lies on tangents

$$\therefore y_1 = mx_1 \pm \sqrt{a^2m^2 + b^2}$$

$$\Rightarrow y_1 - mx_1 = \pm \sqrt{a^2m^2 + b^2}$$

S.O.B

$$\Rightarrow (y_1 - mx_1)^2 = (\sqrt{a^2m^2 + b^2})^2$$

$$y_1^2 + m^2x_1^2 - 2mx_1y_1 - a^2m^2 - b^2 = 0$$

$$\Rightarrow m^2(x_1^2 - a^2) - 2mx_1y_1 + (y_1^2 - b^2) = 0$$

\therefore is a Q.E in "m" $\{ax^2 + bx + c = 0\}$

Let m_1, m_2 be the roots

$$\Rightarrow m_1 \cdot m_2 = \frac{c}{a} \quad \{m_1 \cdot m_2 = -1 \text{ lar tangt}\}$$

$$\Rightarrow \frac{y_1^2 - b^2}{x_1^2 - a^2} = -1 \Rightarrow y_1^2 - b^2 = -x_1^2 + a^2$$

$x_1^2 + y_1^2 = a^2 + b^2$ is an eq'n of a circle

$\therefore P$ lies on a circle.



10. If $P(x, y)$ any point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ Where S & S' are foci, then P.T
 $SP + S'P = 2a$.

Sol: The eq'n of the ellipse is given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \dots\dots (1)$$

Let S, S' be the foci

& ZM, ZM' be the directrices.

Join SP and S'P. Draw PL perpendicular to X-axis and MP perpendicular to the two directrices.

By the definition of ellipse

$$SP = ePM = e(LZ)$$

$$\Rightarrow SP = e(CZ - CL) = e\left(\frac{a}{e} - x\right)$$

$$\therefore SP = a - xe$$

$$S'P = ePM' = e(LZ')$$

$$\Rightarrow S'P = e(CZ + CL) = e\left(\frac{a}{e} + x\right)$$

$$\therefore S'P = a + xe$$

$$SP + S'P = a - xe + a + xe$$

$$\therefore SP + S'P = 2a \text{ (constant)}$$



11. If the normal at one end of a latus rectum of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ passes through one end of the minor axis, then S.T $e^4 + e^2 = 1$.

Sol: eq'n of the normal to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ is } \frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2$$

Let L($ae, \frac{b^2}{a}$) one end of the latus rectum

The eq'n of the normal at L is

$$\frac{a^2x}{ae} - \frac{b^2y}{\frac{b^2}{a}} = a^2 - b^2$$

$$\frac{ax}{e} - \frac{ay}{1} = a^2 - b^2$$

Since it passes through one end of minor axis (0, -b)

$$\frac{a(0)}{e} - \frac{a(-b)}{1} = a^2 - b^2$$

$$ab = a^2 - b^2$$

$$\Rightarrow ab = a^2 e^2$$

$$\Rightarrow b = ae^2 \text{ S.O.B}$$

$$\Rightarrow b^2 = a^2 e^4$$

$$a^2(1 - e^2) = a^2 e^4$$

$$(1 - e^2) = e^4 \therefore e^4 + e^2 = 1.$$

$$\begin{aligned} e^2 &= \frac{a^2 - b^2}{a^2} \\ a^2 e^2 &= a^2 - b^2 \\ b^2 &= a^2(1 - e^2) \end{aligned}$$



12. Show that the condition for the line $y = mx + c$ to be a tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $c^2 = a^2 m^2 + b^2$

Sol: Given eq'n of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \dots\dots (1)$$

Eq'n of tangent to (1) at P(θ)

$$\frac{x}{a} \cos \theta + \frac{y}{b} \sin \theta - 1 = 0 \dots\dots (2)$$

Given eq'n of tangent

$$mx - y + c = 0 \dots\dots (3)$$

Eq'n (2) & (3) represents same line

Comparing coefficients

$$\Rightarrow \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} \Rightarrow \frac{\cos \theta}{a.m} = \frac{\sin \theta}{b.(-1)} = \frac{-1}{c}$$

$$\Rightarrow \cos \theta = -\frac{am}{c} \quad \& \quad \sin \theta = \frac{b}{c}$$

$$\cos^2 \theta + \sin^2 \theta = 1 \Rightarrow \left[-\frac{am}{c} \right]^2 + \left[\frac{b}{c} \right]^2 = 1$$

$$\Rightarrow \frac{a^2 m^2}{c^2} + \frac{b^2}{c^2} = 1 \therefore a^2 m^2 + b^2 = c^2.$$



13. Show that the condition for the line

$lx + my + n = 0$ to be a tangent to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ is } a^2l^2 + b^2m^2 = n^2.$$

Sol: Given eq'n of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \dots \dots (1)$$

Eq'n of tangent to (1) at $P(\theta)$

$$\frac{x}{a} \cos\theta + \frac{y}{b} \sin\theta - 1 = 0 \dots \dots (2)$$

Given eq'n of tangent

$$lx + my + n = 0 \dots \dots (3)$$

Eq'n (2) & (3) represents same line

Comparing coefficients

$$\Rightarrow \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} \Rightarrow \frac{\cos\theta}{a.l} = \frac{\sin}{b.m} = \frac{-1}{n}$$

$$\Rightarrow \cos\theta = -\frac{al}{n} \quad \& \quad \sin\theta = -\frac{bm}{n}$$

$$\cos^2\theta + \sin^2\theta = 1 \Rightarrow \left[-\frac{al}{n}\right]^2 + \left[-\frac{bm}{n}\right]^2 = 1$$

$$\Rightarrow \frac{a^2l^2}{n^2} + \frac{b^2m^2}{n^2} = 1 \therefore a^2l^2 + b^2m^2 = n^2.$$



14. Find the condition for the line

$x\cos\alpha + y\sin\alpha = p$ to be a tangent to the ellipse

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

Sol: Given eq'n of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \dots \dots (1)$$

the condition for the line

$lx + my + n = 0$ to be a tangent to the ellipse

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ is } a^2l^2 + b^2m^2 = n^2$$

Given eq'n of tangent

$$x\cos\alpha + y\sin\alpha = p \dots \dots (3)$$

here $l = \cos\alpha, m = \sin\alpha \text{ & } n = -p$

hence required condition is

$$a^2l^2 + b^2m^2 = n^2$$

$$\Rightarrow a^2\cos^2\alpha + b^2\sin^2\alpha = p^2$$



15. Find the value of k if $4x + y + k = 0$ is a tangent to the ellipse $x^2 + 3y^2 = 3$.

Sol: Sol: Given eq'n of ellipse

$$x^2 + 3y^2 = 3 \dots (1)$$

$$\Rightarrow \frac{x^2}{3} + \frac{3y^2}{3} = 1 \Rightarrow \frac{x^2}{3} + \frac{y^2}{1} = 1$$

$$\therefore a^2 = 3, b^2 = 1$$

$$\text{Given line } 4x + y + k = 0 \dots (2)$$

$$\Rightarrow y = -4x + k \quad \{y = mx + c\}$$

$$m = -4, c = -k$$

$$\text{condition for tangency } c^2 = a^2m^2 + b^2$$

$$\Rightarrow k^2 = (3)(16) + 1$$

$$\Rightarrow k^2 = 49$$

$$\therefore k = \pm 7$$



16. The distance of a point on the ellipse $x^2 + 3y^2 = 6$ from its centre is equal to 2. Find the eccentric angles.

Sol: given eq'n of the ellipse $x^2 + 3y^2 = 6 \dots (1)$

$$\Rightarrow \frac{x^2}{6} + \frac{y^2}{2} = 1 \quad \left[\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \right]$$

Compare with standard form

$$a^2 = 6 \parallel b^2 = 2$$

$$\Rightarrow a = \sqrt{6} \parallel b = \sqrt{2}$$

Any point on the ellipse

$$P(\sqrt{6}\cos\theta, \sqrt{2}\sin\theta)$$

$$\text{Centre } C(0, 0)$$

$$CP=2$$

$$\sqrt{(\sqrt{6}\cos\theta - 0)^2 + (\sqrt{2}\sin\theta - 0)^2} = 2$$

$$\Rightarrow 6\cos^2\theta + 2\sin^2\theta = 4$$

$$\Rightarrow 6\cos^2\theta + 2(1 - \cos^2\theta) = 4$$

$$\Rightarrow 6\cos^2\theta + 2 - 2\cos^2\theta = 4$$

$$\Rightarrow 4\cos^2\theta = 2$$

$$\Rightarrow 4\cos^2\theta = \frac{1}{2} \quad \therefore \cos\theta = \frac{1}{\sqrt{2}}$$



1. One focus of a hyperbola is $(1, -3)$ and the corresponding directrix is $y = 2$. find the eq'n of the hyperbola if its eccentricity is $3/2$.

Sol: Given S $(1, -3)$,

$$e = \frac{3}{2} \text{ &}$$

$$l \equiv y = 2$$

Let P (x, y) be any point on hyperbola

$$\text{W.K.T } \frac{SP}{PM} = e$$

$$\Rightarrow SP = ePM$$

$$\sqrt{(x-1)^2 + (y+3)^2} = \frac{3}{2} \frac{|y-2|}{\sqrt{0^2+1^2}}$$

S.O.B

$$\Rightarrow 4[(x-1)^2 + (y+3)^2] = 9(y-2)^2$$

$$\Rightarrow 4[x^2 + 1 - 2x + y^2 + 9 + 6y] = 9[y^2 + 4 - 4y]$$

$$\Rightarrow 4x^2 + 4 - 8x + 4y^2 + 36 + 24y = 9y^2 + 36 - 36y$$

$$\therefore 4x^2 - 5y^2 - 8x + 60y + 4 = 0$$



2. Find the eccentricity, foci, equations of directrices, length of latus rectum of the hyperbola $x^2 - 4y^2 = 4$.

Sol: Given eq'n of the hyperbola

$$x^2 - 4y^2 = 4 \dots \dots (1) \quad (\div 4)$$

$$\Rightarrow \frac{x^2}{4} - \frac{y^2}{1} = 1 \quad \left[\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \right]$$

Compare with standard form

$$a^2 = 4 \parallel b^2 = 1$$

$$\Rightarrow a = 2 \parallel b = 1$$

(1). centre $(0, 0)$

$$(2). e = \sqrt{\frac{a^2+b^2}{a^2}} = \frac{\sqrt{5}}{2}$$

$$(3). foci \left(\pm ae, 0 \right) = \left(\pm 2 \frac{\sqrt{5}}{2}, 0 \right)$$

$$= (\pm \sqrt{5}, 0)$$

$$(4). L.L.R = \frac{2b^2}{a} = \frac{2 \cdot 1}{2} = 1$$

$$(7). Eq'n of directrices x = \pm \frac{a}{e}$$

$$\Rightarrow x = \pm \frac{2}{\frac{\sqrt{5}}{2}} \Rightarrow \sqrt{5}x \pm 4 = 0.$$



3. find the eccentricity, foci & eqns of directrices of the ellipse

$$16y^2 - 9x^2 = 144$$

Sol: Given eqn of the hyperbola

$$16y^2 - 9x^2 = 144 \dots \dots (1) \quad (\div \text{ by } 4)$$

$$\Rightarrow \frac{x^2}{16} - \frac{y^2}{9} = -1 \quad \left[\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1 \right]$$

Compare with standard form

$$a^2 = 16 \parallel b^2 = 9$$

$$\Rightarrow a = 4 \parallel b = 3$$

(1). centre (0, 0)

$$(2). e = \sqrt{\frac{a^2+b^2}{b^2}} = \sqrt{\frac{9+16}{9}} = \frac{5}{3}$$

$$(3). \text{foci } (0, \pm be) = \left(0, \pm 3 \cdot \frac{5}{3}\right) = (0, \pm 5)$$

$$(4). L.L.R = \frac{2a^2}{b} = \frac{2 \cdot 16}{3} = \frac{32}{3}$$

(5). L. of transverse Axis = $2a = 8$

(6). L. of conjugate Axis = $2b = 6$

$$(7). \text{Eq'n of directrices } y = \pm \frac{b}{e}$$

$$\Rightarrow x = \pm \frac{3}{\frac{5}{3}} \Rightarrow 5x \pm 9 = 0.$$



4. Show that the condition for the line

$$lx + my + n = 0 \text{ to be a tangent to the hyperbola}$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ is}$$

$$a^2l^2 - b^2m^2 = n^2.$$

Sol: Given eqn of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \dots \dots (1)$$

Eqn of tangent to (1) at P(θ) is

$$\frac{x}{a} \sec \theta + \frac{y}{b} \tan \theta - 1 = 0 \dots \dots (2)$$

Given eqn of tangent

$$lx + my + n = 0 \dots \dots (3)$$

Eqn (2) & (3) represents same line
Comparing coefficients

$$\Rightarrow \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

$$\Rightarrow \frac{\sec \theta}{a.l} = \frac{\tan \theta}{b.m} = \frac{-1}{n}$$

$$\Rightarrow \sec \theta = -\frac{al}{n} \quad \& \quad \tan \theta = -\frac{bm}{n}$$

$$\sec^2 \theta - \tan^2 \theta = 1$$

$$\Rightarrow \left[-\frac{al}{n}\right]^2 - \left[-\frac{bm}{n}\right]^2 = 1$$

$$\Rightarrow \frac{a^2l^2}{n^2} - \frac{b^2m^2}{n^2} = 1$$

$$\therefore a^2l^2 - b^2m^2 = n^2.$$



5. find the eq'n of tangents to the hyperbola $3x^2 - 4y^2 = 12$
 which are (i) parallel
 (ii) Perpendicular to the line $y = x - 7$.

Sol: Given hyperbola

$$3x^2 - 4y^2 = 12 \dots (1)$$

$$\Rightarrow \frac{3x^2}{12} - \frac{4y^2}{12} = 1$$

$$\Rightarrow \frac{x^2}{4} - \frac{y^2}{3} = 1 \quad \therefore a^2 = 4, b^2 = 3$$

- (i) eq'n of tangent parallel to $y = x - 7$

$$\{y = mx + c\} \quad m = 1$$

$$\text{is } y = mx \pm \sqrt{a^2m^2 - b^2}$$

$$\Rightarrow y = 1x \pm \sqrt{4(1) - 3}$$

$$\Rightarrow y = x \pm 1 \quad \text{or } x - y \pm 1 = 0$$

- (ii) eq'n of tangent perpendicular to $y = x - 7$

$$\{y = mx + c\} \quad m = 1$$

$$\text{perpendicular slope } -\frac{1}{m} = -\frac{1}{1} = -1$$

$$\text{is } y = mx \pm \sqrt{a^2m^2 - b^2}$$

$$\Rightarrow y = -1x \pm \sqrt{4(1) - 3}$$

$$\Rightarrow y = -x \pm 1 \quad \text{or } x + y \pm 1 = 0.$$



6. find the eq'n of tangents to the hyperbola $x^2 - 4y^2 = 4$
 which are (i) parallel
 (ii) Perpendicular to the line $x + 2y = 0$.

Sol: Given hyperbola

$$x^2 - 4y^2 = 4 \dots (1)$$

$$\Rightarrow \frac{x^2}{4} - \frac{4y^2}{4} = 1$$

$$\Rightarrow \frac{x^2}{4} - \frac{y^2}{1} = 1 \quad \therefore a^2 = 4, b^2 = 1$$

- (i) eq'n of tangent parallel to $x + 2y = 0$

$$\{m = -\frac{a}{b}\} \quad m = -\frac{1}{2}$$

$$\text{is } y = mx \pm \sqrt{a^2m^2 - b^2}$$

$$\Rightarrow y = \frac{1}{-2}x \pm \sqrt{4(-\frac{1}{2})^2 - 1}$$

$$\Rightarrow 2y = -x \pm 0 \quad \text{or } x + 2y = 0$$

- (ii) eq'n of tangent perpendicular to $x + 2y = 0$

$$\{m = -\frac{a}{b}n\} \quad m = -\frac{1}{2}$$

$$\text{perpendicular slope } -\frac{1}{m} = -\frac{1}{-\frac{1}{2}} = 2$$

$$\text{is } y = 2x \pm \sqrt{a^2m^2 - b^2}$$

$$\Rightarrow y = 2x \pm \sqrt{4(4) - 1}$$

$$\Rightarrow y = 2x \pm \sqrt{15} \quad \text{or } 2x - y \pm \sqrt{15} = 0$$



7. Find the eq'n of the hyperbola whose foci are (4, 2), (8, 2) and eccentricity is 2.

Sol: given foci are S (4, 2), S' (8, 2)
and $e = 2$.

Centre C = midpoint of SS'

$$(h, k) = \left(\frac{4+8}{2}, \frac{2+2}{2} \right) = (6, 2)$$

distance between foci = 8

$$\Rightarrow 2ae = SS' = \sqrt{(8-4)^2 + (2-2)^2} = 4$$

$$\Rightarrow 2ae = 4$$

$$\Rightarrow ae = 2$$

$$\Rightarrow a = 2/e \Rightarrow a = \frac{2}{2} = 1$$

$$W.K.T b^2 = a^2(e^2 - 1)$$

$$\Rightarrow b^2 = 1(4-1) = 3$$

$$Eq. of hyperbola \frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

$$\Rightarrow \frac{(x-6)^2}{1} - \frac{(y-2)^2}{3} = 1$$



8. Find the eccentricity, foci, equations of directrices, length of latus rectum of the ellipse

$$5x^2 - 4y^2 + 20x + 8y = 4$$

Sol: Given eq'n of the hyperbola

$$5x^2 - 4y^2 + 20x + 8y = 4 \dots \dots (1)$$

$$\Rightarrow 5x^2 + 20x - 4y^2 + 8y = 4$$

$$\Rightarrow 5[x^2 + 4x] - 4[y^2 - 2y] = 4$$

$$\Rightarrow 5[x^2 + 4x + 4 - 4] - 4[y^2 - 2y + 1 - 1] = 98$$

$$\Rightarrow 5[(x+2)^2] - 20 - 4[(y-1)^2] + 4 = 4$$

$$\Rightarrow 5[(x+2)^2] - 4[(y-1)^2] = 20$$

(\div by 20)

$$\Rightarrow \frac{(x+2)^2}{4} - \frac{(y-1)^2}{5} = 1$$

$$\left[\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 \right]$$

Compare with standard form

$$a^2 = 4 \parallel b^2 = 5$$

$$\Rightarrow a = 2 \parallel b = \sqrt{5}$$

(1). centre $(h, k) = (-2, 1)$

$$(2). e = \sqrt{\frac{a^2+b^2}{a^2}} = \sqrt{\frac{4+5}{4}} = \frac{\sqrt{9}}{2} = \frac{3}{2}$$

$$(3). foci (h \pm ae, k) = \left(-2 \pm 2 \cdot \frac{3}{2}, 1 \right)$$

$$= (-2 \pm 3, 1) = (1, 1) \text{ and } (-5, 1)$$

$$(4). L.L.R = \frac{2b^2}{a} = \frac{2 \cdot 5}{2} = 5$$

$$(5). Eq'n of directrices x - h = \pm \frac{a}{e}$$

$$\Rightarrow x + 2 = \pm \frac{2}{\frac{3}{2}} \Rightarrow x + 2 = \pm \frac{4}{3}$$

$$\Rightarrow 3x + 2 = 0 \text{ and } 3x + 10 = 0$$

9. If e, e_1 are the eccentricities of a hyperbola and its conjugate hyperbola, prove that $\frac{1}{e^2} + \frac{1}{e_1^2} = 1$.

Sol:

$$\text{Eq'n of the hyperbola } \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \dots (1)$$

$$\Rightarrow \text{its eccentricity } e = \sqrt{\frac{a^2+b^2}{a^2}}$$

Eq'n of conjugate hyperbola to (1) is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1 \dots (2)$$

$$\Rightarrow \text{its eccentricity } e_1 = \sqrt{\frac{a^2+b^2}{b^2}}$$

$$L.H.S = \frac{1}{e^2} + \frac{1}{e_1^2}$$

$$= \frac{1}{\frac{a^2+b^2}{a^2}} + \frac{1}{\frac{a^2+b^2}{b^2}}$$

$$= \frac{a^2}{a^2+b^2} + \frac{b^2}{a^2+b^2}$$

$$= \frac{a^2+b^2}{a^2+b^2} = 1 R.H.S$$

10. If the eccentricity of a hyperbola is $5/4$, then find the eccentricity of its conjugate hyperbola.

Sol:

$$\text{Given } e = \frac{5}{4}, e_1 = ?$$

W.K.T

$$\frac{1}{e^2} + \frac{1}{e_1^2} = 1.$$

$$\Rightarrow \frac{1}{\left(\frac{5}{4}\right)^2} + \frac{1}{e_1^2} = 1$$

$$\Rightarrow \frac{16}{25} + \frac{1}{e_1^2} = 1$$

$$\Rightarrow \frac{1}{e_1^2} = 1 - \frac{16}{25}$$

$$\Rightarrow \frac{1}{e_1^2} = \frac{25-16}{25}$$

$$\Rightarrow \frac{1}{e_1^2} = \frac{9}{25}$$

$$\Rightarrow e_1^2 = \frac{25}{9}$$

$$e_1 = \frac{5}{3}$$



11. Show the angle b/w the two asymptotes of a

$$\text{hyperbola } \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ is } 2 \tan^{-1}\left(\frac{b}{a}\right) \text{ or } 2 \sec^{-1}(e).$$

Sol: eqns of asymptotes of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are

$$\frac{x}{a} - \frac{y}{b} = 0 \text{ and } \frac{x}{a} + \frac{y}{b} = 0$$

let 2θ be the angle b/w the asymptotes.

Slope of the asymptote $\frac{x}{a} - \frac{y}{b} = 0$ is

$$\tan\theta(m) = \frac{-1/a}{-1/b} \Rightarrow \tan\theta = \frac{b}{a}$$

$$\Rightarrow \theta = \tan^{-1}\left(\frac{b}{a}\right)$$

$$\Rightarrow 2\theta = 2 \tan^{-1}\left(\frac{b}{a}\right)$$

and $\sec^2\theta = 1 + \tan^2\theta$

$$= 1 + \frac{b^2}{a^2}$$

$$\sec^2\theta = e^2$$

$$\sec\theta = e \Rightarrow \theta = \sec^{-1}e$$

$$\therefore 2\theta = \sec^{-1}e$$

$$\text{hence } 2\theta = 2 \tan^{-1}\left(\frac{b}{a}\right) \text{ or } 2 \sec^{-1}(e)$$



12. Show that eq'n of normal at P (θ) to the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ is } \frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2 + b^2.$$

Sol: Eq'n of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \dots (1)$

$p(a \sec\theta, b \tan\theta)$ be any point on (1)

Equation of the tangent at P (θ) is

$$\frac{x \sec\theta}{a} - \frac{y \tan\theta}{b} = 1$$

$$\Rightarrow -\frac{y \tan\theta}{b} = -\frac{x \sec\theta}{a} + 1$$

$$\Rightarrow -y = -\frac{x b \sec\theta}{a \tan\theta} + 1$$

$$\text{Slope of the tangent} = \frac{b \sec\theta}{a \tan\theta}$$

$$\text{Slope of the tangent} = -\frac{a \tan\theta}{b \sec\theta}$$

Now eq'n of normal at P (θ)

$$(y - y_1) = m(x - x_1)$$

$$\Rightarrow (y - b \tan\theta) = -\frac{a \tan\theta}{b \sec\theta} (x - a \sec\theta)$$

$$\Rightarrow b \sec\theta (y - b \tan\theta) = -a \tan\theta (x - a \sec\theta)$$

$$\Rightarrow by \sec\theta - b^2 \sec\theta \tan\theta$$

$$= -a x \tan\theta + a^2 \tan\theta \sec\theta$$

$$\Rightarrow ax \tan\theta + by \sec\theta = (a^2 + b^2) \tan\theta \sec\theta$$

$$\Rightarrow ax \frac{\tan\theta}{\tan\theta \sec\theta} + by \frac{\sec\theta}{\tan\theta \sec\theta} = (a^2 + b^2)$$

$$\therefore \frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = (a^2 + b^2)$$



1.Evaluate $\lim_{n \rightarrow \infty} \sum_{i=0}^n \frac{i}{i^4 + n^4}$.

$$\text{Sol: } \lim_{n \rightarrow \infty} \sum_{i=0}^n \frac{\frac{i^3}{n^4}}{\frac{i^4}{n^4} + \frac{n^4}{n^4}} \quad [\div \text{ by } n^4]$$

$$= \lim_{n \rightarrow \infty} \sum_{i=0}^n \frac{\frac{i^3}{n^4}}{\frac{i^4}{n^4} + \frac{n^4}{n^4}}$$

$$= \lim_{n \rightarrow \infty} \sum_{i=0}^n \frac{\frac{1}{n} \left(\frac{i}{n}\right)^3}{\left(\frac{i}{n}\right)^4 + 1}$$

$$= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^n \frac{\left(\frac{i}{n}\right)^3}{1 + \left(\frac{i}{n}\right)^4}$$

$$\therefore \int_0^1 f(x) dx = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^n f\left(\frac{i}{n}\right)$$

$$= \int_0^1 \frac{x^3}{1+x^4} dx$$

$$= \frac{1}{4} \int_0^1 \frac{4x^3}{1+x^4} dx$$

$$= \frac{1}{4} [\log(1+x^4)]_0^1$$

$$= \frac{1}{4} [\log 2 - \log 1]$$

$$= \frac{1}{4} \log 2$$



2. $\lim_{n \rightarrow \infty} \frac{\sqrt{n+1} + \sqrt{n+2} + \dots + \sqrt{n+n}}{n\sqrt{n}}$.

$$\text{sol: } \lim_{n \rightarrow \infty} \frac{\sqrt{n+1} + \sqrt{n+2} + \dots + \sqrt{n+n}}{n\sqrt{n}}$$

$$\lim_{n \rightarrow \infty} \left(\frac{1}{n}\right) \sum_{i=0}^n \frac{\sqrt{n+i}}{\sqrt{n}}$$

$$\lim_{n \rightarrow \infty} \left(\frac{1}{n}\right) \sum_{i=0}^n \sqrt{\frac{n+i}{n}}$$

$$\lim_{n \rightarrow \infty} \left(\frac{1}{n}\right) \sum_{i=0}^n \sqrt{1 + \frac{i}{n}}$$

$$\therefore \int_0^1 f(x) dx = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^n f\left(\frac{i}{n}\right)$$

$$= \int_0^1 \sqrt{1+x} dx$$

$$= \int_0^1 (1+x)^{1/2} dx$$

$$= \left[\frac{(1+x)^{\frac{3}{2}}}{\frac{3}{2}} \right]_0^1$$

$$= \frac{2}{3} [2^{3/2} - 1^{3/2}]$$

$$= \frac{2}{3} [2\sqrt{2} - 1]$$



$$3. \lim_{n \rightarrow \infty} \left[\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{6n} \right].$$

$$\text{Sol: } \lim_{n \rightarrow \infty} \left[\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{6n} \right]$$

$$= \lim_{n \rightarrow \infty} \left[\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+5n} \right]$$

$$= \lim_{n \rightarrow \infty} \left(\frac{1}{n} \right) \left[\frac{1}{1+\frac{1}{n}} + \frac{1}{1+\frac{2}{n}} + \cdots + \frac{1}{1+\frac{5n}{n}} \right]$$

$$= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^n \left[\frac{1}{1+\frac{i}{n}} \right]$$

$$\therefore \int_0^p f(x) dx = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{pn} f\left(\frac{i}{n}\right)$$

$$= \int_0^5 \left[\frac{1}{1+x} \right] dx$$

$$= [\log(1+x)]_0^5$$

$$= [\log(1+5) - \log(1+0)]$$

$$= [\log 6 - \log 1] = \log 6$$



$$4. \int_0^{\pi/2} \frac{a\sin x + b\cos x}{\cos x + \sin x} dx$$

$$sol: I = \int_0^{\pi/2} \frac{a\sin x + b\cos x}{\cos x + \sin x} dx \dots (1)$$

$$\therefore \int_0^a f(x) dx = \int_0^a f(a-x) dx$$

$$I = \int_0^{\pi/2} \frac{a\sin\left(\frac{\pi}{2}-x\right) + b\cos\left(\frac{\pi}{2}-x\right)}{\cos\left(\frac{\pi}{2}-x\right) + \sin\left(\frac{\pi}{2}-x\right)} dx$$

$$I = \int_0^{\pi/2} \frac{a\cos x + b\sin x}{\cos x + \sin x} dx \dots (2)$$

adding (1)&(2)

$$= \int_0^{\pi/2} \frac{a\sin x + b\cos x}{\cos x + \sin x} dx + \int_0^{\pi/2} \frac{a\cos x + b\sin x}{\cos x + \sin x} dx$$

$$I + I = \int_0^{\pi/2} \frac{a\sin x + b\cos x + a\cos x + b\sin x}{\cos x + \sin x} dx$$

$$2I = \int_0^{\pi/2} \frac{\sin x(a+b) + \cos x(a+b)}{\cos x + \sin x} dx$$

$$2I = \int_0^{\pi/2} \frac{(a+b)(\sin x + \cos x)}{\cos x + \sin x} dx = (a+b) \int_0^{\pi/2} 1 dx$$

$$2I = (a+b)[x]_0^{\pi/2}$$

$$2I = (a+b) \left[\frac{\pi}{2} - 0 \right] = (a+b) \cdot \frac{\pi}{4}$$



$$5. \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$$

$$sol: I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx \dots (1)$$

$$\therefore \int_0^a f(x)dx = \int_0^a f(a+b-x)dx$$

$$I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin(\frac{\pi}{2}-x)}}{\sqrt{\sin(\frac{\pi}{2}-x)} + \sqrt{\cos(\frac{\pi}{2}-x)}} dx$$

$$I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}} dx \dots (2)$$

adding (1)& (2)

$$I + I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin x} + \sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$$

$$2I = \int_{\pi/6}^{\pi/3} 1 dx$$

$$2I = [x]$$

$$2I = \left[\frac{\pi}{3} - \frac{\pi}{6} \right]$$

$$2I = \frac{\pi}{6}$$

$$I = \frac{\pi}{12}$$



$$6. \int_0^{2\pi} \sin^4 x \cos^6 x dx$$

Sol:

$$I = \int_0^{2\pi} \sin^4 x \cos^6 x dx$$

$$\therefore \int_0^{2a} f(x)dx = 2 \int_0^a f(x)dx$$

$$I = 2 \int_0^{\pi} \sin^4 x \cos^6 x dx$$

$$\therefore \int_0^{2a} f(x)dx = 2 \int_0^a f(x)dx$$

$$I = 2.2 \int_0^{\pi/2} \sin^4 x \cos^6 x dx$$

$$\int_0^{\pi/2} \cos^n x \sin^m x dx =$$

$$\frac{(n-1)(n-3)(n-5)\dots(m-1)(m-3)\dots}{(m+n)(m+n-2)(m+n-4)\dots} \left(\frac{\pi}{2}\right)$$

$$I = 4 \cdot \frac{3.1.5.3.1}{10.8.6.4.2} \frac{\pi}{2} = \frac{3\pi}{128}$$

$$7. \int_{-\pi/2}^{\pi/2} \sin^2 x \cos^4 x dx$$

Sol:

$$I = \int_{-\pi/2}^{\pi/2} \sin^2 x \cos^4 x dx$$

$$\therefore \int_{-a}^a f(x)dx = 2 \int_0^a f(x)dx$$

$$I = 2 \int_0^{\frac{\pi}{2}} \sin^2 x \cos^4 x dx$$

$$\int_0^{\pi/2} \cos^n x \sin^m x dx =$$

$$\frac{(n-1)(n-3)(n-5)\dots(m-1)(m-3)\dots}{(m+n)(m+n-2)(m+n-4)\dots} \left(\frac{\pi}{2}\right)$$

$$I = 2 \cdot \frac{1.3.1}{6.4.2} \frac{\pi}{2} = \frac{\pi}{16}$$



8. $\int_0^{\pi/2} \frac{1}{4+5\cos x} dx$
sol:

L.L: $x = 0 \Rightarrow t = 0$

U.L: $x = \frac{\pi}{2} \Rightarrow t = 1$

$$I = \int_0^{\pi/2} \frac{1}{4+5\cos x} dx$$

$$= \int_0^1 \frac{1}{4+5[\frac{1-t^2}{1+t^2}]} \left(\frac{2dt}{1+t^2} \right)$$

$$= \int_0^1 \frac{1}{\frac{4(1+t^2)+5(1-t^2)}{1+t^2}} \left(\frac{2d}{1+t^2} \right)$$

$$= 2 \int_0^1 \frac{1}{4+4t^2+5-5t^2} dt$$

$$= 2 \int_0^1 \frac{1}{9-t^2} dt$$

$$= 2 \int_0^1 \frac{1}{3^2-t^2} dt$$

$$= \frac{2}{2.(3)} \left(\log \left| \frac{3+t}{3-t} \right| \right)_0^1$$

$$= \frac{1}{3} \left(\log \left| \frac{3+1}{3-1} \right| - \log \left| \frac{3+0}{3-0} \right| \right)$$

$$= \frac{1}{3} (\log 2 - \log 1)$$

$$= \frac{1}{3} (\log 2) .$$

Let $t = \tan \left(\frac{x}{2} \right);$

$$dx = \frac{2dt}{1+t^2};$$

$$\cos x = \frac{1-t^2}{1+t^2}$$



9. Obtain a reduction formula for $\int_0^{\pi/2} \sin^n x dx.$

Sol: Sol: $I_n = \int_0^{\pi/2} \sin^n x dx = \int \sin^{n-1} x \cdot \sin x dx$

Here $U = \sin^{n-1} x \Rightarrow U' = (n-1) \sin^{n-2} x (\cos x)$

$$V = \sin x \Rightarrow \int \sin x dx = -\cos x + c$$

By using integration by parts

$$\int (UV) dx = U \int V dx - \int [U' \int V dx] dx$$

$$I_n = [\sin^{n-1} x \cdot (-\cos x)]_0^{\pi/2}$$

$$- \int_0^{\pi/2} (n-1) \sin^{n-2} x (\cos x) (-\cos x) dx$$

$$I_n = [0 - 0] + (n-1) \int_0^{\pi/2} \sin^{n-2} x \cos^2 x dx$$

$$= (n+1) \int_0^{\pi/2} \sin^{n-2} x (1 - \sin^2 x) dx$$

$$= (n+1) \int_0^{\pi/2} (\sin^{n-2} x - \sin^{n-2} x \sin^2 x) dx$$

$$= (n+1) \int_0^{\pi/2} \sin^{n-2} x dx - (n+1) \int_0^{\pi/2} \sin^n x dx$$

$$I_n = (n-1) I_{n-2} - (n-1) I_n$$

$$I_n + (n-1) I_n = (n-1) I_{n-2}$$

$$I_n (1 + n - 1) = (n-1) I_{n-2}$$

$$I_n (n) = (n-1) I_{n-2}$$

$$I_n = \frac{(n-1)}{n} \cdot \frac{(n-3)}{(n-2)} \cdot \frac{(n-5)}{(n-4)} \cdots \frac{\pi}{2} \quad \text{if } n \text{ is even}$$



10. Evaluate $\int_0^1 x \tan^{-1} x dx$.

Sol: $\int_0^1 x \tan^{-1} x dx$ by using integration by parts

$$\text{here } u = \tan^{-1} x \Rightarrow u' = \frac{1}{1+x^2}$$

$$v = x \Rightarrow \int x dx = \frac{x^2}{2} + c$$

$$\int (uv)dx = U \int V dx - \int [U' \int V dx] dx$$

$$I = \left[\tan^{-1} x \cdot \frac{x^2}{2} \right]_0^1 - \int_0^1 \frac{1}{1+x^2} \cdot \frac{x^2}{2} dx$$

$$I = \left[\tan^{-1} 1 \cdot \frac{1}{2} - \tan^{-1} 0 \cdot \frac{0}{2} \right] - \frac{1}{2} \int_0^1 \frac{x^2}{1+x^2} dx$$

$$I = \left[\frac{\pi}{4} \cdot \frac{1}{2} - 0 \right] - \frac{1}{2} \int_0^1 \frac{1+x^2-1}{1+x^2} dx$$

$$I = \left[\frac{\pi}{8} \right] - \frac{1}{2} \int_0^1 \frac{1+x^2}{1+x^2} dx + \frac{1}{2} \int_0^1 \frac{1}{1+x^2} dx$$

$$I = \left[\frac{\pi}{8} \right] - \frac{1}{2} \int_0^1 1 dx + \frac{1}{2} \int_0^1 \frac{1}{1+x^2} dx$$

$$I = \left[\frac{\pi}{8} \right] - \frac{1}{2} [x]_0^1 + \frac{1}{2} [\tan^{-1} x]_0^1$$

$$I = \left[\frac{\pi}{8} \right] - \frac{1}{2} [1 - 0] + \frac{1}{2} [\tan^{-1} 1 - \tan^{-1} 0]$$

$$I = \left[\frac{\pi}{8} \right] - \frac{1}{2} + \frac{1}{2} \left[\frac{\pi}{4} \right]$$

$$I = \left[\frac{\pi}{8} \right] - \frac{1}{2} + \left[\frac{\pi}{8} \right] = I = \left[\frac{\pi}{4} \right] - \frac{1}{2}$$



11. Evaluate $\int_0^1 \sin \left[\frac{2x}{1+x^2} \right] dx$

Sol:

$$I = \int_0^1 \sin \left[\frac{2x}{1+x^2} \right] dx$$

$$\text{let } x = \tan \theta \Rightarrow \theta = \tan^{-1} x \\ dx = \sec^2 \theta d\theta$$

U.L	L.L
$\theta = \tan^{-1} x$	$\theta = \tan^{-1} x$
$\theta = \tan^{-1} 1$	$\theta = \tan^{-1} 0$
$\theta = \frac{\pi}{4}$	$\theta = 0$

$$= \int_0^{\pi/4} \sin \left[\frac{2 \tan \theta}{1+\tan^2 \theta} \right] \sec^2 \theta d\theta \\ = \int_0^{\pi/4} \sin[2\theta] \sec^2 \theta d\theta \\ = 2 \int_0^{\pi/4} \theta \cdot \sec^2 \theta d\theta$$

by using integration by parts

here $u = \theta \Rightarrow u' = 1$

$$v = \sec^2 \theta \Rightarrow \int \sec^2 \theta d\theta = \tan \theta + c$$

$$\int (uv)dx = U \int V dx - \int [U' \int V dx] dx$$



$$I = 2[\theta \tan \theta]_0^{\pi/4} - \int_0^{\pi/4} \tan \theta \, d\theta$$

$$I = 2\left[\frac{\pi}{4} \tan \frac{\pi}{4}\right] - [\log \sec \theta]_0^{\pi/4}$$

$$I = 2\left[\frac{\pi}{4} 1 - 0\right] - \left[\log \sec \frac{\pi}{4} - \log \sec 0\right]$$

$$I = \left[\frac{\pi}{2}\right] - [\log \sqrt{2} - \log 1]$$

$$I = \frac{\pi}{2} - \log \sqrt{2}$$

12. $\int_0^4 (16 - x^2)^{5/2} dx$

Sol:

$$I = \int_0^4 (16 - x^2)^{5/2} dx$$

put $x = 4\sin\theta \Rightarrow dx = 4\cos\theta d\theta$

U.L	L.L
$4 = 4\sin\theta$	$x = 4\sin\theta$
$\theta = \sin^{-1} 1$	$\theta = \sin^{-1} 0$
$\theta = \frac{\pi}{2}$	$\theta = 0$

$$I = 4 \int_0^{\pi/4} [16 - 16\sin^2\theta]^{5/2} \cos\theta \, d\theta$$

$$I = 4 \int_0^{\pi/4} (16)^{5/2} [1 - \sin^2\theta]^{5/2} \cos\theta \, d\theta$$



$$I = 4(4)^5 \int_0^{\pi/4} [\cos^2\theta]^{5/2} \cos\theta \, d\theta$$

$$I = (4)^6 \int_0^{\pi/4} \cos^6\theta \, d\theta$$

$$I_n = \frac{(n-1)}{n} \cdot \frac{(n-3)}{(n-2)} \cdot \frac{(n-5)}{(n-4)} \cdots \frac{\pi}{2}$$

$$I = 4^6 \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = 640\pi$$

13. Find the area enclosed by the curves

$$y = 3x \text{ and } y = 6x - x^2.$$

Sol: Given eq'n

$$y = 6x - x^2 \dots (1) \quad y = 3x \dots (2)$$

solving (1)and (2)

$$6x - x^2 = 3x$$

$$\Rightarrow x^2 - 3x = 0$$

$$\Rightarrow x(x - 3) = 0$$

$$x = 0 \text{ or } x = 3$$

$$\text{Required Area} = \int_0^3 [(1) - (2)] \, dx$$

$$= \int_0^3 [6x - x^2 - 3x] \, dx$$

$$= \int_0^3 [3x - x^2] \, dx$$

$$= \left[\frac{3x^2}{2} - \frac{x^3}{3} \right]_0^3$$

$$= \frac{3}{2}[3^2 - 0^2] - \frac{1}{3}[3^3 - 0^3]$$

$$= \frac{27}{2} - 9 = \frac{27-18}{2} = \frac{9}{2} \text{ sq. units}$$



14. Find the area enclosed by the curves

$$y^2 = 4x \text{ and } x^2 = 4y.$$

Sol:

$$\text{Given eq'n } y^2 = 4x \Rightarrow y = \sqrt{4x} \dots (1)$$

$$x^2 = 4y \Rightarrow y = \frac{x^2}{4} \dots (2)$$

solving (1)and (2)

$$\sqrt{4x} = \frac{x^2}{4} \text{ S.O.B}$$

$$\Rightarrow 4x = \frac{x^4}{16}$$

$$\Rightarrow 64x = x^4$$

$$\Rightarrow 64x - x^4 = 0$$

$$\Rightarrow x(64 - x^3) = 0$$

$$x = 0 \text{ or } x^3 = 64 \Rightarrow x = 4$$

$$\text{Required Area} = \int_0^4 [(1) - (2)] dx$$

$$= \int_0^4 \left[\sqrt{4x} - \frac{x^2}{4} \right] dx$$

$$= \int_0^4 \left[2x^{1/2} - \frac{x^2}{4} \right] dx$$

$$= \left[\frac{2x^{3/2}}{3/2} - \frac{x^3}{4 \cdot 3} \right]_0^4$$

$$= \frac{4}{3} [4^{3/2} - 0^2] - \frac{1}{12} [4^3 - 0^3]$$

$$= \frac{4}{3} [8] - \frac{1}{12} [64]$$

$$= \frac{32}{3} - \frac{16}{3} = \frac{16}{3} \text{ sq.units}$$



Differential equations: An equation involving one dependent variable and its derivatives w. r. t one or more independent variables is called a differential equation.

Order of differential equation: The order of a differential equation is the order of the highest derivative occurring in it.

Degree of differential equation: Degree of a differential equation is the degree of the highest derivative occurring in it when the derivatives are made free from the radical sign.

Solutions of differential equations of the first order and first degree:

- Variables separable method.
- Homogeneous equations.
- Non-Homogeneous equations:
- Linear equation:
- Equation reducible to linear form:

Variable separable method.

To solve $\frac{dy}{dx} = XY$, where X is a function of x only and Y is a function of y only.

Bring all the terms of x and dx on one side, the terms of y and dy on the other side.

Integrate both sides and add an arbitrary constant on one side.

$$1. \text{ Solve } \frac{dy}{dx} = e^{x-y} + x^2 e^{-y}$$

$$\text{Sol: } \frac{dy}{dx} = e^{x-y} + x^2 e^{-y}$$

$$\Rightarrow \frac{dy}{dx} = \frac{e^x}{e^y} + \frac{x^2}{e^y}$$



$$\Rightarrow \frac{dy}{dx} = \frac{e^x + x^2}{e^y}$$

$$\Rightarrow e^y dy = (e^x + x^2) dx$$

$$\Rightarrow \int e^y dy = \int (e^x + x^2) dx$$

$$\Rightarrow e^y = e^x + 2x + c$$

$$\therefore e^y = e^x + 2x + c$$

$$2. \text{ Solve } (xy^2 + x) dx + (yx^2 + y) dy = 0.$$

$$\text{Sol: } (xy^2 + x) dx + (yx^2 + y) dy = 0.$$

$$\Rightarrow (y^2 + 1)x dx + (x^2 + 1)y dy = 0.$$

$$\div by (y^2 + 1)(x^2 + 1)$$

$$\Rightarrow \frac{x}{(x^2+1)} + \frac{y}{(y^2+1)} = 0$$

$$\Rightarrow \frac{1}{2} \int \frac{2x}{(x^2+1)} dx + \frac{1}{2} \int \frac{2y}{(y^2+1)} dy = 0$$

$$\Rightarrow \frac{1}{2} \log(1+x^2) + \frac{1}{2} \log(1+y^2) = \frac{1}{2} \log c$$

$$\Rightarrow \log(1+x^2)(1+y^2) = \log c$$

$$\therefore (1+x^2)(1+y^2) = c$$



3. Solve $\frac{dy}{dx} + \frac{y^2+y+1}{x^2+x+1} = 0$

Sol: $\frac{dy}{dx} = -\frac{y^2+y+1}{x^2+x+1}$

$$\Rightarrow \frac{dy}{y^2+y+1} = -\frac{dx}{x^2+x+1}$$

$$\Rightarrow \frac{dy}{y^2+y+\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2+1} = -\frac{dx}{x^2+x+\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2+1}$$

$$\Rightarrow \int \frac{1}{\left(y+\frac{1}{2}\right)^2-\left(\frac{\sqrt{3}}{2}\right)^2} dy + \int \frac{1}{\left(x+\frac{1}{2}\right)^2-\left(\frac{\sqrt{3}}{2}\right)^2} dx$$

$$\therefore \int \frac{1}{x^2+a^2} dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + c$$

$$\Rightarrow \frac{1}{\frac{\sqrt{3}}{2}} \tan^{-1}\left(\frac{y+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) + \frac{1}{\frac{\sqrt{3}}{2}} \tan^{-1}\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) = c$$

$$\therefore \tan^{-1}\left(\frac{2y+1}{\sqrt{3}}\right) + \tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) = c$$



4. Solve $(e^x + 1)ydy + (y + 1)dx = 0$

Sol: $(e^x + 1)ydy + (y + 1)dx = 0$

$\div by (e^x + 1)(y + 1)$

$$\Rightarrow \frac{(e^x+1)ydy}{(e^x+1)(y+1)} + \frac{(y+1)dx}{(e^x+1)(y+1)} = 0$$

$$\Rightarrow \int \frac{y}{(y+1)} dy + \int \frac{1}{(e^x+1)} dx = 0$$

$$\Rightarrow \int \frac{y+1-1}{(y+1)} dy + \int \frac{1}{\left(\frac{1}{e^{-x}}+1\right)} dx = 0$$

$$\Rightarrow \int \left(\frac{y+1}{(y+1)} - \frac{1}{(y+1)}\right) dy + \int \frac{e^{-x}}{(1+e^{-x})} dx = 0$$

$$\Rightarrow \int 1 dy - \int \frac{1}{(y+1)} dy - \int \frac{-e^{-x}}{(1+e^{-x})} dx = 0$$

$$\Rightarrow y - \log(y+1) - \log(1+e^{-x}) = \log c$$

$$\Rightarrow y = \log(y+1) + \log(1+e^{-x}) + \log c$$

$$\Rightarrow y = \log(y+1)(1+e^{-x})c$$

$$\Rightarrow e^y = c(y+1)(1+e^{-x})$$

or

$$e^y e^x = c(y+1)(1+e^x)$$

$$e^{x+y} = c(y+1)(1+e^x)$$



5. Solve $y - x \frac{dy}{dx} = 5(y^2 + \frac{dy}{dx})$

Sol: $y - x \frac{dy}{dx} = 5y^2 + 5 \frac{dy}{dx}$

$$\Rightarrow x \frac{dy}{dx} + 5 \frac{dy}{dx} = y - 5y^2$$

$$\Rightarrow (x+5) \frac{dy}{dx} = y(1 - 5y)$$

$$\Rightarrow \frac{dy}{y(1-5y)} = \frac{dx}{(x+5)}$$



6. Solve $\frac{dy}{dx} = \frac{x(2\log x + 1)}{\sin y + y \cos y}$

Sol: $\frac{dy}{dx} = \frac{x(2\log x + 1)}{\sin y + y \cos y}$

$$\Rightarrow (\sin y + y \cos y) dy = (2x \log x + x) dx$$

$$\Rightarrow \int \sin y dy + \int y \cos y dy$$

$$= 2 \int x \cdot \log x dx + \int x dx$$

$$\Rightarrow -\cos y + y \int \cos y dy - \int \left[\frac{dy}{dx} \int \cos y dy \right] dy$$

$$= 2 \left\{ \log x \int x dx - \int \left[\frac{d}{dx} (\log x) \int x dx \right] dx \right\}$$

$$\Rightarrow -\cos y + y \sin y + \cos y$$

$$= 2 \left\{ \frac{x^2}{2} \log x - \int \frac{1}{x} \frac{x^2}{2} \right\} + \frac{x^2}{2} + c$$

$$\Rightarrow -\cos y + y \sin y + \cos y$$

$$= x^2 \log x - \frac{x^2}{2} + \frac{x^2}{2} + c \Rightarrow y \sin y = x^2 \log x + c$$

7. Solve $\sqrt{1-x^2} dx + \sqrt{1-y^2} dy = 0$

Sol: $\sqrt{1-x^2} dx + \sqrt{1-y^2} dy = 0$

Integrating on both sides

$$\int \sqrt{1-x^2} dx + \int \sqrt{1-y^2} dy = 0$$

$$\left\{ \because \int \sqrt{a^2 - x^2} dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \left(\frac{x}{a} \right) \right\}$$

$$\frac{x}{2} \sqrt{1-x^2} + \frac{1}{2} \sin^{-1}(x) + \frac{y}{2} \sqrt{1-y^2} + \frac{1}{2} \sin^{-1}(y) = c$$

$$\therefore x \sqrt{1-x^2} + \sin^{-1} x + y \sqrt{1-y^2} + \sin^{-1} y = 2c$$



8. Solve $\sin^{-1}\left(\frac{dy}{dx}\right) = x + y$

$$\text{Sol: } \sin^{-1}\left(\frac{dy}{dx}\right) = x + y$$

$$\Rightarrow \frac{dy}{dx} = \sin(x + y)$$

$$\text{put } x + y = t \Rightarrow 1 + \frac{dy}{dx} = \frac{dt}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{dt}{dx} - 1$$

$$\Rightarrow \frac{dt}{dx} - 1 = \sin t$$

$$\Rightarrow \frac{dt}{dx} = 1 + \sin t$$

$$\Rightarrow \int \frac{1}{1+\sin} dt = \int 1 dx$$

$$\Rightarrow \int \frac{1-\sin}{1-\sin^2 t} dt = \int 1 dx$$

$$\Rightarrow \int \frac{1-\sin t}{\cos^2 t} dt = \int 1 dx$$

$$\Rightarrow \int \left[\frac{1}{\cos^2 t} - \frac{\sin t}{\cos t} \right] dt = \int 1 dx$$

$$\Rightarrow \int \{ \sec^2 t - \sec t \tan t \} dt = \int 1 dx$$

$$\therefore \tan t - \sec t = x + c$$



9. Solve $\frac{dy}{dx} - x \tan(x - y) = 1$

$$\text{Sol: put } y - x = t \Rightarrow \frac{dy}{dx} - 1 = \frac{dt}{dx}$$

The given eq'n becomes

$$1 + \frac{dt}{dx} - x \tan t = 1$$

$$\Rightarrow \frac{dt}{dx} = x \tan t$$

$$\Rightarrow \frac{dt}{\tan t} = x dx$$

$$\Rightarrow \int \cot t dt = \int x dx$$

$$\Rightarrow \log|\sin t| = \frac{x^2}{2} + c$$

$$\therefore \log|\sin(y - x)| = \frac{x^2}{2} + c$$

10. Solve $\frac{dy}{dx} = \tan^2(x + y)$.

Homogeneous differential equations

To solve the equation $\frac{dy}{dx} = \frac{f_1(x,y)}{f_2(x,y)}$, where $f_1(x,y), f_2(x,y)$ are homogeneous functions of the same degree in x and y.

Put $y = vx$, so that $\frac{dy}{dx} = v + x\frac{dv}{dx}$

Substitute the values of y and $\frac{dy}{dx}$ in the given equation.

Separate the variables v and x.

Integrate both sides and add an arbitrary constant on one side.

Separate back the variables $v = \frac{y}{x}$.

1. Solve $(x^2 + y^2)dy = 2xy \cdot dx$.

$$\text{Sol: } (x^2 + y^2)dy = 2xy \cdot dx$$

$$\Rightarrow \frac{dy}{dx} = \frac{2xy}{x^2 + y^2} \dots\dots\dots(1)$$

$$\text{let } y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\text{Eq'n (1)} \Rightarrow v + x\frac{dv}{dx} = \frac{2x(vx)}{x^2 + (vx)^2}$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{2x^2v}{x^2 + v^2x^2} \Rightarrow v + x\frac{dv}{dx} = \frac{x^2}{x^2} \left(\frac{2v}{1+v^2}\right)$$

$$\Rightarrow x\frac{dv}{dx} = \frac{2v}{1+v^2} - v$$

$$\Rightarrow x\frac{dv}{dx} = \frac{2v-v-v^3}{1+v^2}$$



$$\Rightarrow x \frac{dv}{dx} = \frac{v-v^3}{1+v^2}$$

$$\Rightarrow \frac{1+v^2}{v-v^3} dv = \frac{dx}{x}$$

$$\Rightarrow \frac{1+v^2}{v(1-v^2)} dv = \frac{dx}{x}$$

$$\Rightarrow \int \frac{1}{v(1-v^2)} dv + \int \frac{v^2}{v(1-v^2)} dv = \int \frac{1}{x} dx$$

$$\Rightarrow \int \frac{(1-v^2)+v^2}{v(1-v^2)} dv + \int \frac{v}{(1-v^2)} dv = \int \frac{1}{x} dx$$

$$\Rightarrow \int \frac{(1-v^2)}{v(1-v^2)} dv + \int \frac{v^2}{v(1-v^2)} dv + \int \frac{v}{(1-v^2)} dv = \int \frac{1}{x} dx$$

$$\Rightarrow \int \frac{(1-v^2)}{v(1-v^2)} dv + \int \frac{v}{(1-v^2)} dv + \int \frac{v}{(1-v^2)} dv = \int \frac{1}{x} dx$$

$$\Rightarrow \int \frac{1}{v} dv - \int \frac{-2v}{(1-v^2)} dv = \int \frac{1}{x} dx$$

$$\therefore \int \frac{f'(x)}{f(x)} dx = \log |f(x)| + c.$$

$$\Rightarrow \log|v| - \log|1-v^2| = \log x + \log c$$

$$\Rightarrow \log \left| \frac{v}{1-v^2} \right| = \log x c$$

$$\Rightarrow \frac{v}{1-v^2} = xc \text{ substituting } v=y/x$$

$$\Rightarrow \frac{\frac{y}{x}}{1-\left(\frac{y}{x}\right)^2} = xc \quad \Rightarrow \frac{y}{x} \left(\frac{x^2}{x^2-y^2} \right) = xc$$



2. Solve $(x^2 - y^2)dx - xy \cdot dy = 0$.

$$\text{Sol: } (x^2 - y^2)dx - xy \cdot dy = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{x^2 - y^2}{2xy} \dots\dots$$

this is homogenous D.E

$$\text{let } y = vx \Rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx}$$

$$\text{Eq'n 1} \Rightarrow v + x \frac{dv}{dx} = \frac{x^2 - (vx)^2}{x(vx)} = \frac{x^2 - v^2 x^2}{v x^2}$$

$$\Rightarrow v + x \frac{dv}{dx} = \frac{x^2 - v^2 x^2}{v x^2}$$

$$\Rightarrow v + x \frac{dv}{dx} = \frac{x^2}{x^2} \left(\frac{1-v^2}{v} \right)$$

$$\Rightarrow v + x \frac{dv}{dx} = \left(\frac{1-v^2}{v} \right)$$

$$\Rightarrow x \frac{dv}{dx} = \left(\frac{1-v^2}{v} \right) - v$$

$$\Rightarrow x \frac{dv}{dx} = \left(\frac{1-v^2-v^2}{v} \right) \Rightarrow x \frac{dv}{dx} = \left(\frac{1-2v^2}{v} \right)$$

$$\Rightarrow \int \frac{v}{1-2v^2} dv = \int \frac{1}{x} dx$$

$$\Rightarrow -\frac{1}{4} \int \frac{-4v}{1-2v^2} dv = \int \frac{1}{x} dx$$



$$\therefore \int \frac{f'(x)}{f(x)} dx = \log |f(x)| + c.$$

$$\Rightarrow -\frac{1}{4} \log|1 - 2v^2| = \log x + \log c$$

$$\Rightarrow \log|1 - 2v^2| = -4 \log(cx)$$

$$\Rightarrow \log|1 - 2v^2| = \log(cx)^{-4}$$

$$\Rightarrow (1 - 2v^2) = (cx)^{-4}$$

$$\Rightarrow \left(1 - 2 \left(\frac{y}{x} \right)^2 \right) = \frac{1}{x^4 c^4} \Rightarrow \frac{x^2 - 2y^2}{x^2} = \frac{1}{x^4 c^4}$$

$$\Rightarrow x^2(x^2 - 2y^2) = \frac{1}{c^4}$$



3. Solve $(x^2 - y^2) \frac{dy}{dx} = xy$.

$$\text{Sol: } (x^2 - y^2) \frac{dy}{dx} = xy$$

$$\Rightarrow \frac{dy}{dx} = \frac{xy}{x^2 - y^2} \dots$$

$$\text{let } y = vx \Rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx}$$

$$\text{Eq'n 1} \Rightarrow v + x \frac{dv}{dx} = \frac{x(vx)}{x^2 - (vx)^2}$$

$$\Rightarrow v + x \frac{dv}{dx} = \frac{x^2 v}{x^2 - v^2 x^2} \Rightarrow v + x \frac{dv}{dx} = \frac{x^2}{x^2} \left(\frac{v}{1 - v^2} \right)$$

$$\Rightarrow x \frac{dv}{dx} = \frac{v}{1 - v^2} - v$$

$$\Rightarrow x \frac{dv}{dx} = \frac{v - v + v^3}{1 - v^2}$$

$$\Rightarrow x \frac{dv}{dx} = \frac{v^3}{1 - v^2}$$

$$\Rightarrow \frac{1 - v^2}{v^3} dv = \frac{dx}{x}$$

$$\Rightarrow \int \left(\frac{1}{v^3} - \frac{v^2}{v^3} \right) dv = \int \frac{1}{x} dx$$

$$\Rightarrow -\frac{1}{2} \frac{1}{v^2} - \log v = \log x + \log c$$

substituting $v = y/x$

$$\Rightarrow -\frac{1}{2} \frac{x^2}{y^2} - \log y/x = \log xc$$

$$\Rightarrow -\frac{1}{2} \frac{x^2}{y^2} = \log y/x + \log xc$$

$$\Rightarrow -\frac{1}{2} \frac{x^2}{y^2} = \log \frac{yx}{x} c$$

$$\Rightarrow -x^2 = 2y^2 \log yc$$

$$\Rightarrow x^2 + 2y^2 \log yc = 0$$



4. solve $\frac{dy}{dx} = \frac{x-y}{x+y}$

Sol:

this is homogenous D.E

$$\text{let } y = vx \Rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx}$$

$$\text{Eq'n 1} \Rightarrow v + x \frac{dv}{dx} = \frac{x - vx}{x + vx}$$

$$\Rightarrow v + x \frac{dv}{dx} = \frac{x(1-v)}{x(1+v)}$$

$$\Rightarrow v + x \frac{dv}{dx} = \frac{(1-v)}{(1+v)}$$

$$\Rightarrow x \frac{dv}{dx} = \frac{(1-v)}{(1+v)} - v$$

$$\Rightarrow x \frac{dv}{dx} = \left(\frac{(1-v-v-v^2)}{(1+v)} \right)$$

$$\Rightarrow x \frac{dv}{dx} = \left(\frac{(1-2v-v^2)}{(1+v)} \right)$$

$$\Rightarrow \int \frac{1+v}{1-2v-v^2} dv = \int \frac{1}{x} dx$$

$$\Rightarrow -\frac{1}{2} \int \frac{-2(1+v)}{1-2v-v^2} dv = \int \frac{1}{x} dx$$

$$\Rightarrow -\frac{1}{2} \log|1 - 2v - v^2| = \log x + \log c$$

$$\Rightarrow \log|1 - 2v - v^2| = -2 \log|xc|$$



$$\Rightarrow \log|1 - 2v - v^2| = \log(cx)^{-2}$$

$$\Rightarrow 1 - 2v - v^2 = (cx)^{-2}$$

$$\Rightarrow 1 - \frac{2y}{x} - \left(\frac{y}{x}\right)^2 = \frac{1}{c^2 x^2}$$

$$\Rightarrow \frac{x^2 - 2xy - y^2}{x^2} = \frac{1}{c^2 x^2}$$

$$\therefore x^2 - 2xy - y^2 = \frac{1}{c^2}$$

5. solve $(2x-y)dy = (2y-x)dx$

sol: $(2x-y)dy = (2y-x)dx$

$$\Rightarrow \frac{dy}{dx} = \frac{2y-x}{2x-y}$$

this is homogenous D.E

$$\text{let } y = vx \Rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx}$$

$$\text{Eq'n 1} \Rightarrow v + x \frac{dv}{dx} = \frac{2vx-x}{2x-(vx)}$$

$$\Rightarrow v + x \frac{dv}{dx} = \frac{x(2v-1)}{x(2-v)}$$

$$\Rightarrow v + x \frac{dv}{dx} = \frac{(2v-1)}{(2-v)}$$



$$\Rightarrow x \frac{dv}{dx} = \frac{(2v-1)}{(2-v)} - v$$

$$\Rightarrow x \frac{dv}{dx} = \frac{(2v-1-2v+v^2)}{(2-v)}$$

$$\Rightarrow x \frac{dv}{dx} = \frac{(v^2-1)}{(2-v)}$$

$$\Rightarrow \int \frac{2-v}{v^2-1} dv = \int \frac{1}{x} dx$$

$$\Rightarrow \int \frac{2}{(v-1)(v+1)} dv = \int \frac{1}{x} dx$$

$$\Rightarrow \int \left[\frac{\frac{1}{2}}{(v-1)} + \frac{\frac{-3}{2}}{(v+1)} \right] dv = \int \frac{1}{x} dx$$

$$\Rightarrow \int \frac{1}{(v-1)} dv - 3 \int \frac{1}{(v+1)} dv = 2 \int \frac{1}{x} dx$$

$$\Rightarrow \log(v-1) - 3 \log(v+1) = 2 \log x + \log c$$

$$\Rightarrow \log(v-1) - \log(v+1)^3 = \log x^2 c$$

$$\Rightarrow \log \frac{(v-1)}{(v+1)^3} = \log x^2 c$$

$$\Rightarrow \frac{(v-1)}{(v+1)^3} = x^2 c \Rightarrow \left(\frac{y}{x} - 1\right) = x^2 \left(\frac{y}{x} + 1\right)^3 c$$

$$\Rightarrow (y-x) = \frac{x^3}{x^3} (y+x)^3 c$$

$$\therefore (y-x) = (y+x)^3 c$$



6. solve $x dy = \left(y + x \cos^2 \frac{y}{x} \right) dx$

Sol: $x dy = \left(y + x \cos^2 \frac{y}{x} \right) dx$

$$\frac{dy}{dx} = \frac{y}{x} + \frac{x \cos^2 \left(\frac{y}{x} \right)}{x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{x} + \cos^2 \left(\frac{y}{x} \right) \dots$$

this is homogenous D.E

$$\text{let } y = vx \Rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx}$$

$$\text{Eq'n 1} \Rightarrow v + x \frac{dv}{dx} = v + \cos^2(v)$$

$$\Rightarrow x \frac{dv}{dx} = \cos^2(v)$$

$$\Rightarrow \int \frac{1}{\cos^2(v)} dv = \int \frac{1}{x} dx$$

$$\Rightarrow \int \sec^2(v) dv = \int \frac{1}{x} dx$$

$$\Rightarrow \tan v = \log x + c$$

$$\therefore \tan \left(\frac{y}{x} \right) = \log x + c$$



7. Given the solution of

$$x \sin^2 \left(\frac{y}{x} \right) dx = y dx - x dy \text{ Which passes through the point } (1, \frac{\pi}{4})$$

Sol:

$$x \sin^2 \left(\frac{y}{x} \right) dx = y dx - x dy$$

$$\Rightarrow x dy = y dx - x \sin^2 \left(\frac{y}{x} \right) dx$$

$$\Rightarrow x dy = \left[y - x \sin^2 \left(\frac{y}{x} \right) \right] dx$$

$$\Rightarrow dy = \left[\frac{y}{x} - \frac{x \sin^2 \left(\frac{y}{x} \right)}{x} \right] dx$$

$$\Rightarrow \frac{dy}{dx} = \left[\frac{y}{x} - \sin^2 \left(\frac{y}{x} \right) \right] \dots$$

$$\text{let } y = vx \Rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx}$$

$$\Rightarrow v + x \frac{dv}{dx} = v - \sin^2(v)$$

$$\Rightarrow x \frac{dv}{dx} = -\sin^2(v)$$

$$\Rightarrow \int \frac{1}{\sin^2(v)} dv = - \int \frac{1}{x} dx$$

$$\Rightarrow \int \cosec^2 v dv = - \int \frac{1}{x} dx$$

$$\Rightarrow -\cot v = -\log x + c$$

$$\Rightarrow -\cot \left(\frac{y}{x} \right) = -\log x + c$$



this is passing Through the point $(1, \frac{\pi}{4})$

$$\Rightarrow -\cot\left(\frac{\pi}{4}\right) = -\log 1 + c$$

$$\Rightarrow -1 = 0 + c \Rightarrow c = -1$$

$$\therefore -\cot\left(\frac{y}{x}\right) = -\log x - 1$$

$$8. \text{ solve } \frac{dy}{dx} = \frac{2x-y+1}{x+2y-3}.$$

$$\text{Sol: } \frac{dy}{dx} = \frac{2x-y+1}{x+2y-3}$$

$$[a' + b = 1 - 1 = 0]$$

this is non-homogeneous D.E of case(1)

Re grouping the terms properly

$$\Rightarrow (x + 2y - 3)dy = (2x - y + 1)dx$$

$$\Rightarrow (2x + 1)dx - (2y - 3)dy - ydx - xdy = 0$$

$$\Rightarrow (2x + 1)dx - (2y - 3)dy - [ydx + xdy] = 0$$

$$\Rightarrow \int (2x + 1)dx - \int (2y - 3)dy - \int d(xy) = \int 0$$

$$\Rightarrow 2\frac{x^2}{2} + x - 2\frac{y^2}{2} - 3y - xy = c$$

$$\therefore x^2 - y^2 - xy + x - 3y = c$$



$$9. \text{ solve } \frac{dy}{dx} = \frac{x-y+3}{2x-2y+5}.$$

$$\text{Sol: } \frac{dy}{dx} = \frac{x-y+3}{2x-2y+5}$$

$$\left[\frac{a}{a'} = \frac{b}{b'} \right]$$

this is non-homogeneous D.E of case(2)

$$\frac{dy}{dx} = \frac{x-y+3}{2(x-y)+5} \dots\dots$$

$$\text{let } (x - y) = v \Rightarrow 1 - \frac{dy}{dx} = \frac{dv}{dx}$$

$$\Rightarrow \frac{dy}{dx} = 1 - \frac{dv}{dx}$$

Now eq'n 1 becomes

$$\Rightarrow 1 - \frac{dv}{dx} = \frac{v+3}{2v+5}$$

$$\Rightarrow 1 - \frac{v+3}{2v+5} = \frac{dv}{dx}$$

$$\Rightarrow \frac{2v+5-v-3}{2v+5} = \frac{dv}{dx}$$

$$\Rightarrow \frac{v+2}{2v+5} = \frac{dv}{dx}$$

$$\Rightarrow \int 1dx = \int \frac{2v+5}{v+2} dv$$

$$\Rightarrow \int 1dx = \int \frac{2v+4+1}{v+2} dv$$



$$\Rightarrow \int 1 dx = \int \frac{2(v+2)+1}{v+2} dv$$

$$\Rightarrow \int 1 dx = \int \left(\frac{2(v+2)}{v+2} + \frac{1}{v+2} \right) dv$$

$$\Rightarrow \int 1 dx = \int \left(2 + \frac{1}{v+2} \right) dv$$

$$\Rightarrow x = 2v + \log(v+2) + c$$

$$\Rightarrow x = 2(x-y) + \log(x-y+2) + c$$

$$\therefore x - 2y + \log(x-y+2) = c$$

Linear differential equations

To solve $\frac{dy}{dx} + Py = Q$, where P and Q are

functions of x only.

Make the co-efficient of $\frac{dy}{dx}$ unity, if not so already.

Find $I.F = e^{\int p dx}$ and remember that $e^{\int \log f(x) dx} = f(x)$

the solution is $y(I.F) = \int Q(I.F) dx + c$



1. Solve $(1+x^2) \frac{dy}{dx} + y = e^{\tan^{-1} x}$

Sol: $(1+x^2) \frac{dy}{dx} + y = e^{\tan^{-1} x}$

$$\Rightarrow \frac{dy}{dx} + \frac{y}{1+x^2} = \frac{e^{\tan^{-1} x}}{1+x^2}$$

this is in the form of $\frac{dy}{dx} + Py = Q$

$$\text{where } P = \frac{1}{1+x^2}, Q = \frac{e^{\tan^{-1} x}}{1+x^2}$$

$$\therefore I.F = e^{\int P dx}$$

$$I.F = e^{\int \frac{1}{1+x^2} dx} = e^{\tan^{-1} x}$$

the solution is $y(I.F) = \int Q(I.F) dx + c$

$$\Rightarrow y \cdot e^{\tan^{-1} x} = \int \frac{e^{\tan^{-1} x}}{1+x^2} \cdot e^{\tan^{-1} x} dx$$

$$\text{let } e^{\tan^{-1} x} = t \Rightarrow \frac{e^{\tan^{-1} x}}{1+x^2} dx = dt$$

$$\Rightarrow y \cdot e^{\tan^{-1} x} = \int t dt$$

$$\Rightarrow y \cdot e^{\tan^{-1} x} = \frac{t^2}{2}$$

$$\therefore y \cdot e^{\tan^{-1} x} = \frac{(e^{\tan^{-1} x})^2}{2} + c$$



2. Solve $(1 + x^2) \frac{dy}{dx} + 2xy - 4x^2 = 0$

$$\text{Sol: } (1 + x^2) \frac{dy}{dx} + 2xy - 4x^2 = 0$$

$$\frac{dy}{dx} + \frac{2x}{1+x^2} = \frac{4x^2}{1+x^2}$$

this is in the form of $\frac{dy}{dx} + Py = Q$

$$\text{where } P = \frac{2x}{1+x^2}, Q = \frac{4x^2}{1+x^2}$$

$$\therefore I.F = e^{\int P dx}$$

$$I.F = e^{\int \frac{2x}{1+x^2} dx} = e^{\log(1+x^2)} = 1 + x^2$$

the solution is $y(I.F) = \int Q(I.F) dx + c$

$$\Rightarrow y(1+x^2) = \int \frac{4x^2}{1+x^2} \cdot (1+x^2) dx$$

$$\Rightarrow y(1+x^2) = \int 4x^2 dx$$

$$\Rightarrow y(1+x^2) = \frac{4x^3}{3} + c$$

$$\therefore y(1+x^2) = \frac{4x^3}{3} + c$$



3. Solve $\frac{dy}{dx} + y \tan x = \sin x$

$$\text{Sol: } \frac{dy}{dx} + y \tan x = \sin x$$

this is in the form of $\frac{dy}{dx} + Py = Q$

where $P = \tan x, Q = \sin x$

$$\therefore I.F = e^{\int P dx}$$

$$I.F = e^{\int \tan x dx} = e^{\log|\sec x|}$$

$$I.F = \sec x$$

the solution is $y(I.F) = \int Q(I.F) dx + c$

$$\Rightarrow y(\sec x) = \int \sin x \sec x dx$$

$$\Rightarrow y(\sec x) = \int \frac{\sin x}{\cos x} dx$$

$$\Rightarrow y(\sec x) = \int \tan x dx$$

$$\therefore y(\sec x) = \log|\sec x| + c$$



4. Solve $\frac{dy}{dx} - y \tan x = e^x \sec x$

Sol: $\frac{dy}{dx} - y \tan x = e^x \sec x$

this is in the form of $\frac{dy}{dx} + Py = Q$

where $P = \tan x, Q = e^x \sec x$

$$\therefore I.F = e^{\int P dx}$$

$$I.F = e^{\int -\tan x dx} = e^{-\log|\sec x|}$$

$$I.F = \frac{1}{\sec x} = \cos x$$

the solution is $y(I.F) = \int Q(I.F) dx + c$

$$\Rightarrow y(\cos x) = \int e^x \sec x \cdot \cos x dx$$

$$\Rightarrow y(\cos x) = \int e^x dx$$

$$\Rightarrow y(\cos x) = e^x + c.$$



5. Solve $(x + y + 1) \frac{dy}{dx} = 1$

Sol: $(x + y + 1) \frac{dy}{dx} = 1$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{x+y+1} \quad \Rightarrow \frac{dx}{dy} = x + y + 1$$

$$\Rightarrow \frac{dx}{dy} - x = y + 1$$

this is in the form of $\frac{dx}{dy} + Px = Q$

where $P = -1, Q = y + 1$

$$\therefore I.F = e^{\int P dy}$$

$$I.F = e^{\int -1 dy} = e^{-y}$$

$$I.F = e^{-y}$$

the solution is $x(I.F) = \int Q(I.F) dy + c$

$$\Rightarrow x(e^{-y}) = \int e^{-y}(y + 1) dy$$

$$\Rightarrow x(e^{-y}) = (y + 1) \int e^{-y} dy - \int \left[\frac{d}{dx} (y + 1) \int e^{-y} dy \right] dy$$

$$\Rightarrow x(e^{-y}) = -(y + 1)e^{-y} - \int 1 \cdot \frac{e^{-y}}{-1} dy$$

$$\Rightarrow x(e^{-y}) = -(y + 1)e^{-y} + \int e^{-y} dy$$

$$\Rightarrow x(e^{-y}) = -(y + 1)e^{-y} + \frac{e^{-y}}{-1} + c$$

$$x = -(y + 1) - 1 + ce^y$$

$x + y + 2 = ce^y$ which is the required solution.



6. Solve $(1 + y^2)dx = (\tan^{-1} y - x)dy$

Sol:

$$(1 + y^2)dx = (\tan^{-1} y - x)dy$$

$$\Rightarrow \frac{dx}{dy} = \frac{\tan^{-1} y - x}{(1+y^2)}$$

$$\Rightarrow \frac{dx}{dy} + \frac{x}{(1+y^2)} = \frac{\tan^{-1} y}{(1+y^2)}$$

this is in the form of $\frac{dx}{dy} + Px = Q$

$$\text{where } P = \frac{1}{(1+y^2)}, Q = \frac{\tan^{-1} y}{(1+y^2)}$$

$$\therefore I.F = e^{\int P dx}$$

$$I.F = e^{\int \frac{1}{(1+y^2)} dy} = e^{\tan^{-1} y}$$

$$I.F = e^{\tan^{-1} y}$$

the solution is $x(I.F) = \int Q(I.F)dy + c$

$$\Rightarrow x(e^{\tan^{-1} y}) = \int e^{\tan^{-1} y} \cdot \frac{\tan^{-1} y}{(1+y^2)} dy$$

$$\text{let } \tan^{-1} y = t \Rightarrow \frac{1}{(1+y^2)} dy = dt$$

$$\Rightarrow x(e^{\tan^{-1} y}) = \int e^t \cdot t \cdot dt$$

$$\Rightarrow x(e^{\tan^{-1} y}) = e^t(e^t - 1) + c$$

$$\Rightarrow x(e^{\tan^{-1} y}) = e^{\tan^{-1} y}(e^{\tan^{-1} y} - 1) + c$$



7. Solve $\cos x \frac{dy}{dx} + y \sin x = \sec^2 x$

Sol:

$$\cos x \frac{dy}{dx} + y \sin x = \sec^2 x$$

$$\Rightarrow \frac{dy}{dx} + y \cdot \frac{\sin x}{\cos x} = \frac{\sec^2 x}{\cos x}$$

$$\Rightarrow \frac{dy}{dx} + y \cdot \tan x = \sec^3 x$$

this is in the form of $\frac{dy}{dx} + Py = Q$

where $P = \tan x, Q = \sec^4 x$

$$\therefore I.F = e^{\int P dx}$$

$$I.F = e^{\int \tan x dx} = e^{\log|\sec x|}$$

$$I.F = \sec x$$

the solution is $y(I.F) = \int Q(I.F)dx + c$

$$\Rightarrow y(\sec x) = \int \sec^3 x \cdot \sec x dx$$

$$\Rightarrow y(\sec x) = \int \sec^4 x dx$$

$$\Rightarrow y(\sec x) = \int (1 + \tan^2 x) \sec^2 x dx$$

$$\text{let } \tan x = t \Rightarrow \sec^2 x dx = dt$$

$$\Rightarrow y(\sec x) = \int (1 + t^2) dt$$

$$\therefore y(\sec x) = t + \frac{t^3}{3} + c$$

$$\therefore y(\sec x) = \tan x + \frac{\tan^3 x}{3} + c$$



8. Solve $x \log x \frac{dy}{dx} + y = 2 \log x$