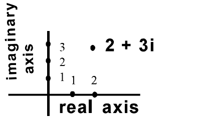
(VSAQ)

1. Represent the complex number 2 + 3i in Argand diagram.



2 Find the real and imaginary parts of the complex number $\frac{a-ib}{a+ib}$.

A: $\frac{a-ib}{a+ib}=\left(\frac{a-ib}{a+ib}\right)\left(\frac{a-ib}{a-ib}\right)$

$=\frac{\left(a^2-b^2\right)+(-2ab)i}{a^2+b^2}$

Real part $=\frac{a^2-b^2}{a^2+b^2}$, imaginary part $=\frac{-2ab}{a^2+b^2}$.

3. If $(a + ib)^2 = x + iy$, find $x^2 + y^2$.

A: Given that $(a + ib)^2 = x + iy$

Now, $|a + ib|^2 = |x + iy|$

$\Rightarrow\left(\sqrt{a^2+b^2}\right)^2=\sqrt{x^2+y^2}\Rightarrow a^2+b^2=\sqrt{x^2+y^2}$

$\Rightarrow \boxed{x^2+y^2=\left(a^2+b^2\right)^2}$

4. Find the square roots of - 5 + 12i.

A: We know that

$\sqrt{a+ib}=\pm\left[\sqrt{\frac{\sqrt{a^2+b^2}+a}{2}}+i\sqrt{\frac{\sqrt{a^2+b^2}-a}{2}}\right]$

Here a = -5, b = 12

$\sqrt{-5+12i}=\pm\left[\sqrt{\frac{\sqrt{25+144}+(-5)}{2}}+i\sqrt{\frac{\sqrt{25+144}-(-5)}{2}}\right]$

$=\pm\left[\sqrt{\frac{13-5}{2}}+i\sqrt{\frac{13+5}{2}}\right]$

$= \pm (2 + 3i)$.

5. Find the square roots of 7 + 24i.

A: We know that

$\sqrt{a+ib}=\pm\left[\sqrt{\frac{\sqrt{a^2+b^2}+a}{2}}+i\sqrt{\frac{\sqrt{a^2+b^2}-a}{2}}\right]$ $[\because b>0]$

$\therefore \sqrt{7+24i}=\pm\left[\sqrt{\frac{\sqrt{7^2+24^2}+7}{2}}+i\sqrt{\frac{\sqrt{7^2+24^2}-7}{2}}\right]$

$=\pm\left[\sqrt{\frac{\sqrt{625}+7}{2}}+i\sqrt{\frac{\sqrt{625}-7}{2}}\right]$

$=\pm\left[\sqrt{\frac{25+7}{2}}+i\sqrt{\frac{25-7}{2}}\right]$

$=\pm\left[\sqrt{\frac{32}{2}}+i\sqrt{\frac{18}{2}}\right]$

$=\pm\left[\sqrt{16}+i\sqrt{9}\right]=\pm(4+3i)$

6. Find the complex conjugate of (2 + 5i) (- 4 + 6i).

A: $(2 + 5i) (- 4 + 6i) = - 8 + 12i - 20i + 30i^2$

$= - 7 - 8i - 30 = - 38 - 8i$

Hence, its complex conjugate is - 38 + 8i.

7. Show that $z_1 = \frac{2+11i}{25}$, $z_2 = \frac{-2+i}{(1-2i)^2}$ are conjugate to each other.

A: $z_2=\frac{-2+i}{(1-2i)^2}$

$=\frac{-2+i}{1-4-4i}$

$=\frac{-2+i}{-3-4i}$

$=\frac{2-i}{3+4i}\times\frac{3-4i}{3-4i}$

$=\frac{6-8i-3i-4}{9+16}$ $\quad\because (a + ib)(a - ib) = a^2 + b^2$

$=\frac{2-11i}{25}$

$\therefore z_1$ and z_2 are conjugate to each other.

8. Find the additive inverse of $\left(\sqrt{3}, 5\right)$.

A: $\left(\sqrt{3},5\right)=\sqrt{3}+5i$

Its additive inverse $=-\left(\sqrt{3},5\right)=-\sqrt{3}-5i$

$=\left(-\sqrt{3},-5\right)$.

9. Write the multiplicative inverse of (7, 24).

A: (7, 24) = 7 + 24i.

Multiplicative inverse of 7 + 24i = $\frac{1}{7+24i}$.

$=\frac{1}{7+24i}\times\frac{7-24i}{7-24i}=\frac{7-24i}{7^2-24^2i^2}=\frac{7-24i}{49+576}$.

$=\frac{7-24i}{625}=\frac{7}{625}-i\frac{24}{625}=\left(\frac{7}{625},-\frac{24}{625}\right)$.

10. If $x+iy=\frac{1}{1+\cos\theta+i\sin\theta}$, show that $4x^2 - 1 = 0$.

A: Now $x+iy=\frac{1}{1+\cos\theta+i\sin\theta}$

$=\frac{1}{2\cos^2\frac{\theta}{2}+i\,2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}$

$=\frac{1}{2\cos\frac{\theta}{2}\left[\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right]}\times\frac{\cos\frac{\theta}{2}-i\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}-i\sin\frac{\theta}{2}}$

$=\frac{\cos\frac{\theta}{2}-i\sin\frac{\theta}{2}}{2\cos\frac{\theta}{2}\left[\cos^2\frac{\theta}{2}+\sin^2\frac{\theta}{2}\right]}$

$=\frac{\cos\frac{\theta}{2}-i\sin\frac{\theta}{2}}{2\cos\frac{\theta}{2}}$

$=\frac{1}{2}-i\frac{1}{2}\tan\frac{\theta}{2}$

Equating the real parts on both sides,

$x=\frac{1}{2}$

$\Rightarrow 2x = 1$

$\Rightarrow 4x^2 = 1$

$\Rightarrow 4x^2 - 1 = 0$.

11. If z = 2 - 3i, show tht $z^2 - 4z + 13 = 0$.

A: Given that z = 2 - 3i

$\Rightarrow z - 2 = -3i$

Squaring on both sides,

$(z - 2)^2 = (-3i)^2$

$\Rightarrow z^2 - 4z + 4 = -9$

$\Rightarrow z^2 - 4z + 13 = 0$.

12. Find the least positive integer n, satisfying $\left(\frac{1+i}{1-i}\right)^n=1$.

A: Given that $\left(\frac{1+i}{1-i}\right)^n=1\Rightarrow\left(\frac{1+i}{1-i}\times\frac{1+i}{1+i}\right)^n=1$

$\Rightarrow\left[\frac{(1+i)^2}{1^2-i^2}\right]^n=1\Rightarrow\left(\frac{1+i^2+2i}{1+1}\right)^n=1$

$\left(\frac{\not{1}-\not{1}+2i}{1+1}\right)^n=1\Rightarrow\left[\frac{\not{2}i}{\not{2}}\right]^n=1\Rightarrow i^n=1$

n = {4, 8, 12,.............∞}

∴ Required least positive integer is 4.

13. If z = (cos θ, sin θ) then find $z-\frac{1}{z}$

A: Given that z = (cos θ, sin θ) = cos θ + i sin θ

then $\frac{1}{z}=\frac{1}{\cos\theta+i\sin\theta}\times\frac{\cos\theta-i\sin\theta}{\cos\theta-i\sin\theta}$

$=\frac{\cos\theta-i\sin\theta}{(\cos\theta)^2-(i\sin\theta)^2}=\frac{\cos\theta-i\sin\theta}{\cos^2\theta+\sin^2\theta}$

$= \cos\theta - i\sin\theta$

$\therefore z-\frac{1}{z}=\cancel{\cos\theta}+i\sin\theta-\left(\cancel{\cos\theta}-i\sin\theta\right)$

$\boxed{\Rightarrow z-\frac{1}{z}=2i\sin\theta}$

14. If $z_1 = -1$, $z_2 = -i$ then find Arg ($z_1 z_2$)

A: Given that $z_1 = -1$, $z_2 = -i$

then $z_1 = \text{cis}\pi$, $z_2 = \text{cis}\left(-\frac{\pi}{2}\right)$

$\therefore$ Arg $(z_1 z_2)$ = Arg z_1 + Arg z_2.

$= \pi + \left(-\frac{\pi}{2}\right) = \frac{\pi}{2}$.

15. If $z_1 = -1$, $z_2 = i$ then find Arg$\left(\frac{z_1}{z_2}\right)$.

A: Given that $z_1 = -1$, $z_2 = i$

then $z_1 = \text{cis}\pi$, $z_2 = \text{cis}\frac{\pi}{2}$

$\therefore \text{Arg}\left(\frac{z_1}{z_2}\right)$ = Arg z_1 - Arg z_2 .

$= \pi - \frac{\pi}{2} = \frac{\pi}{2}$.

16. If Arg $\bar{z}_1$ and Arg z_2 are $\frac{\pi}{5}$ and $\frac{\pi}{3}$ respectively, find Arg z_1 + Arg z_2.

A: Given Arg $\bar{z}_1 = \frac{\pi}{5}$ and $z_2 = \frac{\pi}{3}$

$\Rightarrow$ Arg $z_1 = \frac{-\pi}{5}$ and Arg $z_2 = \frac{\pi}{3}$

$\therefore$ Arg z_1 + Arg z_2 = $\frac{-\pi}{5} + \frac{\pi}{3} = \frac{= 3\pi + 5\pi}{15} = \frac{2\pi}{5}$.

$\therefore \boxed{\text{Arg } z_1 + \text{Arg } z_2 = \frac{2\pi}{5}}$

17. Find the modulus and amplitude form of the complex number $1+\sqrt{3}i$.

A: Let x + iy = $1+\sqrt{3}i$

Here x = 1, y = $\sqrt{3}$

Now, $r = \sqrt{x^2+y^2} = \sqrt{1+3} = \sqrt{4} = 2$

$\cos\theta = \frac{x}{r} \Rightarrow \cos\theta = \frac{1}{2}$

Hence, $\sin\theta = \frac{y}{r} \Rightarrow \sin\theta = \frac{\sqrt{3}}{2}$

$\therefore$ 'θ' lies in I quadrant and $\theta = \frac{\pi}{3} \in (-\pi, \pi]$

$\therefore$ Modulus amplitude form of $1+\sqrt{3}i$

= r (cosθ + isinθ) = 2[cos $\frac{\pi}{3}$ + isin $\frac{\pi}{3}$].

18. Express $-1-i\sqrt{3}$ in polar form.

A: Let x + iy = $-1 - i\sqrt{3}$

Here x = -1, y = $-\sqrt{3}$

Now, $r = \sqrt{x^2+y^2} = \sqrt{1+3} = \sqrt{4} = 2$

$$\left.\begin{array}{l}\cos\theta = \frac{x}{r} \Rightarrow \cos\theta = \frac{-1}{2} \\ \sin\theta = \frac{y}{r} \Rightarrow \sin\theta = \frac{-\sqrt{3}}{2}\end{array}\right\}$$

$\therefore$ 'θ' lies in III quadrant and $q = -\left(\pi - \frac{\pi}{3}\right) = -\frac{2\pi}{3}$.

$\therefore$ Polar form of $-1 - i\sqrt{3}$ = r (cosθ + isinθ)

$= 2\{\cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right)\}$.

19. If $\sqrt{3}+i = r(\cos\theta + i\sin\theta)$. Find the value of θ.

A: $a+ib = \sqrt{a^2+b^2}\left(\frac{a^2}{\sqrt{a^2+b^2}} + i\frac{b}{\sqrt{a^2+b^2}}\right)$

$\sqrt{3}+i = 2\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right)$

$= 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

$\therefore \theta = \frac{\pi}{6}$.

20. If $(\cos 2\alpha + i\sin 2\alpha)(\cos 2\beta + i\sin 2\beta)$ = cos θ + i sin θ find the value of θ.

A: Given that $(\cos 2\alpha + i\sin 2\alpha)(\cos 2\beta + i\sin 2\beta)$

= cos θ + i sin θ

$\Rightarrow$ (cis 2α) (cis 2β) = cos θ + i sin θ.

$\Rightarrow$ cis (2α + 2β) = cis θ

$\therefore \boxed{\theta = 2\alpha + 2\beta}$

21. If z = x + iy = cisα.cisβ, then find the value of $x^2 + y^2$.

A: Given that x + iy

= (cosα + i sinα) (cosβ + i sinβ)

= (cosαcosβ + sinαsinβ) + i (sinαcosβ + cosαsinβ)

i.e., x + iy = cos(α + β) + i sin(α + β)

$\Rightarrow |x+iy| = \sqrt{\cos^2(\alpha+\beta) + \sin^2(\alpha+\beta)}$

$\Rightarrow \sqrt{x^2+y^2} = 1 \Rightarrow \boxed{x^2+y^2 = 1}$

22. If $\left(\sqrt{3}+i\right)^{100} = 2^{99}(a+ib)$, show that $a^2+b^2=4$.

A: Given that $\left(\sqrt{3}+i\right)^{100} = 2^{99}(a+ib)$

$\Rightarrow \left|\sqrt{3}+i\right|^{100} = \left|2^{99}(a+ib)\right|$

$\Rightarrow \left(\sqrt{\left(\sqrt{3}\right)^2+(1)^2}\right)^{100} = 2^{99}\sqrt{a^2+b^2}$

$\Rightarrow 2^{100} = 2^{99}\sqrt{a^2+b^2}$

$\Rightarrow 2 = \sqrt{a^2+b^2} \Rightarrow \boxed{a^2+b^2 = 4}$

23. If z = x + iy and |z| = 2, find the locus of z.

A: Given that z = x + iy and

|z| = 2

$\Rightarrow$ |x + iy| = 2

$\Rightarrow \sqrt{x^2+y^2} = 2$

$\Rightarrow x^2 + y^2 = 4$

so, the locus of z is $x^2 + y^2 = 4$.

24. If the amplitude of z - 1 is $\frac{\pi}{2}$, find the locus of z.

A: Let z = x + iy

z - 1 = x + iy - 1

= (x - 1) + iy

Given that amplitude of z - 1 is $\frac{\pi}{2}$

$\Rightarrow \text{Tan}^{-1}\left(\frac{y}{x-1}\right) = \frac{\pi}{2}$

$\frac{y}{x-1} = \tan\frac{\pi}{2} = \frac{1}{0}$

$\Rightarrow$ x - 1 = 0

$\therefore$ Locus of z is x = 1.

25. Write $z = -\sqrt{7}+\sqrt{21}i$ in the polar form.

A: Let z = x + iy

$x + iy = -\sqrt{7}+\sqrt{21}i$

$= \sqrt{28}\left[\frac{-\sqrt{7}}{\sqrt{28}} + \frac{\sqrt{21}}{\sqrt{28}}i\right]$

$= 2\sqrt{7}\left[\frac{-1}{2} + \frac{\sqrt{3}}{2}i\right]$

$= 2\sqrt{7}\left[\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right]$.

26. If a = cos α + i sin α and b = cos β + i sin β, then find $\frac{1}{2}\left(ab + \frac{1}{ab}\right)$.

A: Now ab = (cis α) (cis β)

= cis(α + β)

= cos (α + β) + i sin (α + β)

$\Rightarrow \frac{1}{ab} = \frac{1}{\cos(\alpha+\beta)+i\sin(\alpha+\beta)} \times \frac{\cos(\alpha+\beta)-i\sin(\alpha+\beta)}{\cos(\alpha+\beta)+i\sin(\alpha+\beta)}$

$= \frac{\cos(\alpha+\beta)-i\sin(\alpha+\beta)}{1}$

= cos (α + β) - i sin (α + β)

$ab + \frac{1}{ab}$ = cos(α + β) + i sin (α + β)

+ cos (α + β) - i sin (α + β)

= 2 cos (α + β)

$\Rightarrow \frac{1}{2}\left(ab + \frac{1}{ab}\right)$ = cos (α + β).

27. Find the equation of the perpendicular bisector of the line segment joining the points 7 + 7i, 7 - 7i in the Argand diagram.

A: A(7, 7), B(7, -7) represent given two complex numbers in the Argand diagram.

Mid point on AB = $\left(\frac{7+7}{2}, \frac{7-7}{2}\right)$ = (7, 0)

Slope of $\overline{AB}$ = $\frac{-7-7}{7-7} = \frac{-14}{0} = -\infty$

Slope of $\overline{PQ}$ = 0 ($\because \overline{AB} \perp \overline{PQ}$)

$\therefore$ Equation of PQ is

y - 0 = 0 (x - 7)

$\Rightarrow$ y = 0.

28. Show that the complex numbers z satisfying $z^2 + \bar{z}^2 = 2$ constitute a hyperbola.

A: Let z = x + iy.

Now $z^2 + \bar{z}^2 = 2$

$\Rightarrow (x+iy)^2 + (x-iy)^2 = 2$

$\Rightarrow x^2 + \cancel{2ixy} + i^2y^2 + x^2 - \cancel{2ixy} + i^2y^2 = 2$

$\Rightarrow 2(x^2 - y^2 = 2)$.

$\Rightarrow x^2 - y^2 = 1$ which is a hyperbola.

29. If (1- i) (2-i) (3-i).......(1-ni) = x - iy, prove that 2.5.10......(1+n²) = x² + y².

A: Given (1- i) (2-i) (3-i).......(1-ni) = x - iy.

Taking modulus on both sides.

|1- i| |2-i| |3-i|.......|1-ni| = |x - iy|

$\Rightarrow \sqrt{1+1}\sqrt{4+1}\sqrt{9+1}.......\sqrt{1+n^2} = \sqrt{x^2+y^2}$

squaring on both sides, we get

2.5.10......$(1 + n^2) = x^2 + y^2$.

(VSAQ)

1. Find the value of $(1 + i)^{16}$.

A: $(1+i)^{16} = \left[\sqrt{2}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)\right]^{16}$

$= \left(\sqrt{2}\right)^{16}\left[\cos45^0 + i\sin45^0\right]^{16}$

$= 2^8 [\cos16(45^0) - i\sin16(45^0)]$

$= 256 [\cos720^0 + i\sin720^0] = 256 [1 - i.0] = 256$

2. Find the value of $\left(1+i\sqrt{3}\right)^3$.

A: $\left(1+i\sqrt{3}\right)^3 = \left[2\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)\right]^3 = 8\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^3$

$= 8(\cos60^0 + i\sin60^0)^3$

By applying De Moivre's theorem for an integral index.

$= 8[\cos3(60^0) + i\sin3(60^0)]$

$= 8(\cos180^0 + i\sin180^0) = 8 [-1 + i(0)] = - 8$.

3. Find the value of $(1 - i)^8$.

A: $(1-i)^8 = \left[\sqrt{2}\left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right)\right]^8$

$= \left(\sqrt{2}\right)^8 (\cos 45^0 - i \sin 45^0)^8$

By applying De Moivre's theorem for an integral index.

$= 2^4 [\cos8(45^0) - i\sin8(45^0)]$

$= 2^4 [\cos360^0 - i\sin360^0] = 16 [1 - i(0)] = 16$

4. Find the value of $\left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right)^5 - \left(\frac{\sqrt{3}}{2}-\frac{i}{2}\right)^5$.

A: $\left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right)^5 - \left(\frac{\sqrt{3}}{2}-\frac{i}{2}\right)^5$

$= (\cos 30^0 + i\sin 30^0)^5 - (\cos 30^0 - i\sin 30^0)^5$

$= \cos5(30^0) + i\sin5(30^0) - [\cos5(30^0) - i\sin5(30^0)]$

=

$\cancel{\cos150^0} + i\sin150^0 - \left[\cancel{\cos150^0} - i\sin150^0\right]$

$= 2i \sin150^0 = 2i\left(\frac{1}{2}\right) = \mathbf{i}$.

5. If A, B, C are the angles of a triangle such that x = cis A, y = cis B, z = cis C, then find xyz.

A: Given that x = cis A, y = cis B, z = cis C

Now xyz = cis A.cis B.cis C = cis(A+B+C)

= cos (A + B + C) + i sin (A + B + C)

$= \cos 180^0 + i \sin 180^0 = - 1 + i(0) = - 1$.

6. If x = cis θ, then find the value of $\left(x^6 + \frac{1}{x^6}\right)$.

A: Given that $x = \cos\theta + i\sin\theta$.

$\Rightarrow x^6 = (\cos\theta + i\sin\theta) = \cos6\theta + i\sin6\theta$.

Now, $\frac{1}{x^6} = \frac{1}{\cos6\theta + i\sin6\theta} = \cos6\theta - i\sin6\theta$.

Hence, $x^6 + \frac{1}{x^6} =$

$= \cos6\theta + \cancel{i\sin6\theta} + \cos6\theta - \cancel{i\sin6\theta}$

$= 2\cos 6\theta$.

7. Find the cube roots of 8.

A: Let $x = \sqrt[3]{8} \Rightarrow x^3 = 8$

$x^3 = 2^3 = (2.1)^3$

$\Rightarrow x = 2(1^{1/3})$

$= 2(1, \omega, \omega^2)$

$= 2, 2\omega, 2\omega^2$.

8. If α, β are the roots of the equation $x^2 + x + 1 = 0$, then prove that $\alpha^4 + \beta^4 + \alpha^{-1}\beta^{-1} = 0$.

$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{3}i}{2} = \omega, \omega^2$

A: $\alpha^4 + \beta^4 + \alpha^{-1}\beta^{-1} = \omega^4 + (\omega^2)^4 + \frac{1}{\omega}.\frac{1}{\omega^2} = \omega + \omega^2 + 1 = 0$

AIMS

9. Simplify $\dfrac{(\cos\alpha + i\sin\alpha)^4}{(\sin\beta + i\cos\beta)^8}$.

A: $= \dfrac{(\cos\alpha + i\sin\alpha)^4}{(-i^2\sin\beta + i\cos\beta)^8} = \dfrac{(\cos\alpha + i\sin\alpha)^4}{[i(\cos\beta - i\sin\beta)]^8}$

$= \dfrac{(\cos\alpha + i\sin\alpha)^4}{i^8(\cos\beta - i\sin\beta)^8} = \dfrac{\cos4\alpha + i\sin4\alpha}{\cos8\beta - i\sin8\beta}$

$= (\cos4\alpha + i\sin4\alpha)(\cos8\beta + i\sin8\beta)$

$= \cos(4\alpha + 8\beta) + i\sin(4\alpha + 8\beta) = cis(4\alpha + 8\beta)$

10. Solve $x^4 - 1 = 0$.

A: $x^4 - 1 = 0 \Rightarrow (x^2 + 1)(x^2 - 1) = 0$

$\Rightarrow x^2 + 1 = 0$ or $x^2 - 1 = 0$

$\Rightarrow x^2 = -1$ or $x^2 = 1$

$\Rightarrow x = \sqrt{-1}$ or $x = \sqrt{1}$

$\Rightarrow x = \pm i$ or $x = \pm 1$.

11. If the cube roots of unity are 1, ω, ω², then find the roots of the equation $(x - 1)^3 + 8 = 0$.

$(x - 1)^3 = -8 = (-2)^3 = -2(1)^{1/3} = -2(1, \omega, \omega^2)$

$\Rightarrow x - 1 = -2, -2\omega, -2\omega^2$

$\therefore x = 1 - 2, 1 - 2\omega, 1 - 2\omega^2$

$= -1, 1 - 2\omega, 1 - 2\omega^2$

12. If 1, ω, ω² are the cube roots of unity, then prove that $\dfrac{1}{2+\omega} + \dfrac{1}{1+2\omega} = \dfrac{1}{1+\omega}$.

$\dfrac{1}{2+\omega} + \dfrac{1}{1+2\omega} = \dfrac{1+2\omega+2+\omega}{2+4\omega+\omega+2\omega^2}$

$= \dfrac{3(1+\omega)}{2(1+\omega+\omega^2)+3\omega} = \dfrac{3(1+\omega)}{3\omega} = \dfrac{(1+\omega)^2}{\omega(1+\omega)} = \dfrac{(1+\omega+\omega^2)+\omega}{\omega(1+\omega)}$

$= \dfrac{\omega}{\omega(1+\omega)} = \dfrac{1}{1+\omega}$.

13. Prove that $-\omega$ and $-\omega^2$ are the roots of $z^2 - z + 1 = 0$, where ω and ω² are the complex cube roots of unity.

A: $z^2 - z + 1 = 0$.

$z = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \dfrac{1 \pm \sqrt{(-1)^2 - 4(1)(1)}}{2(1)}$

$= \dfrac{1 \pm \sqrt{3i^2}}{2}$

$= \dfrac{1-\sqrt{3}i}{2}, \dfrac{1+\sqrt{3}i}{2}$

$= -\left(\dfrac{-1+\sqrt{3}i}{2}\right), -\left(\dfrac{-1-\sqrt{3}i}{2}\right)$

$= -\omega, -\omega^2$.

14. If 1, ω, ω² are the cube roots of unity, find the value of $(1 - \omega + \omega^2)^3$.

$(1 - \omega + \omega^2)^3 = \left[(1+\omega^2) - \omega\right]^3 \quad \because 1 + \omega + \omega^2 = 0$

$= (-\omega - \omega)^3$

$= (-2\omega)^3$

$= -8\omega^3$

$= -8$.

15. If 1, ω, ω² are the cube roots of unity, find the value of $(1 - \omega)(1 - \omega^2)(1 - \omega^4)(1 - \omega^8)$.

A: $(1 - \omega)(1 - \omega^2)(1 - \omega^4)(1 - \omega^8)$

$= (1 - \omega)(1 - \omega^2)(1 - \omega)(1 - \omega^2)$

$= [(1 - \omega)(1 - \omega^2)]^2$

$= [1 - \omega - \omega^2 + \omega^3]^2$

$= [1 - (\omega + \omega^2) + 1]^2$

$= [2 - (-1)]^2$

$= 3^2$

$= 9$.

DEFINITIONS, CONCEPTS AND FORMULAE

1. If a, b, c are complex numbers and $a \neq 0$, then $ax^2 + bx + c = 0$ is called a quadratic equation.
2. The roots of the quadratic equation $ax^2 + bx + c = 0$ are $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.
3. If α, β are roots of $ax^2 + bx + c = 0$, then $\alpha + \beta = \dfrac{-b}{a}$, $\alpha\beta = \dfrac{c}{a}$.
4. $\Delta = b^2 - 4ac$ is called discriminant of $ax^2 + bx + c = 0$.
5. If a, b, c are real, then the nature of the roots of $ax^2 + bx + c = 0$ is as follows :
   i) If $b^2 - 4ac < 0$, then the roots are imaginary and they are conjugate complex numbers.
   ii) If $b^2 - 4ac = 0$, then the roots are real and equal.
   iii) If $b^2 - 4ac > 0$, then the roots are real and distinct.
6. If a, b, c are rational, then the nature of the roots of $ax^2 + bx + c = 0$ is as follows :
   i) If $b^2 - 4ac < 0$, then the roots are imaginary and they are conjugate complex numbers.
   ii) If $b^2 - 4ac = 0$, then the roots are rational and equal.
   iii) If $b^2 - 4ac > 0$ and $b^2 - 4ac$ is a perfect square, then the roots are rational and distinct.
   iv) If $b^2 - 4ac > 0$ and $b^2 - 4ac$ is not a perfect square, then the roots are irrational and distinct. They are conjugate surds.
7. The quadratic equation whose roots are α, β is $x^2 - (\alpha + \beta)x + \alpha\beta = 0$.
8. If α, β are the roots of $ax^2+bx+c=0$, then $ax^2+bx+c = a(x - \alpha)(x - \beta)$.
9. A necessary and sufficient condition for the quadratic equations $a_1x^2 + b_1x + c_1 = 0$ and $a_2x^2 + b_2x + c_2 = 0$ to have a common root is $(c_1a_2 - c_2a_1)^2 = (a_1b_2 - a_2b_1)(b_1c_2 - b_2c_1)$.
   Here the common root is $\dfrac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}$.
10. If the roots of $ax^2 + bx + c = 0$ are imaginary (complex roots) then for $x \in R$, '$ax^2 + bx + c$' and 'a' have the same sign.
11. If the roots of $ax^2 + bx + c = 0$ are real and equal to $\alpha = \dfrac{-b}{2a}$, then for $\alpha \neq x \in R$, '$ax^2 + bx + c$' and 'a' have the same sign.
12. Let α, β be the real roots of $ax^2 + bx + c = 0$ and $\alpha < \beta$, then for
    i) $x \in R$, $\alpha < x < \beta \Rightarrow$ '$ax^2 + bx + c$' and 'a' have opposite signs.
    ii) $x \in R$, $x < \alpha$ or $x > \beta \Rightarrow$ 'ax^2+bx+c' and 'a' have the same sign.
13. Let $f(x) = ax^2+bx+c$ be a quadratic function.
    i) If $a > 0$, then f (x) has minimum at $x = \dfrac{-b}{2a}$ and the minimum value $= \dfrac{4ac - b^2}{4a}$.
    ii) If $a < 0$, then f (x) has maximum at $x = \dfrac{-b}{2a}$ and the maximum value $= \dfrac{4ac - b^2}{4a}$.
14. Let α, β be the roots of $ax^2+bx+c = 0$, then the equation whose roots are
    i) $\dfrac{1}{\alpha}$ and $\dfrac{1}{\beta}$ is $f\left(\dfrac{1}{x}\right) = 0$
    ii) $\alpha + k$ and $\beta + k$ is $f(x - k) = 0$
    iii) $\alpha - k$ and $\beta - k$ is $f(x + k) = 0$
    iv) $-\alpha$ and $-\beta$ is $f(-x) = 0$
    v) $k\alpha$ and $k\beta$ is $f\left(\dfrac{x}{k}\right) = 0$
15. If $ax^2 + bx + c$ is a quadratic expression, then $ax^2 + bx + c > 0$ or $ax^2 + bx + c \geq 0$ or $ax^2 + bx + c < 0$ or $ax^2 + bx + c \leq 0$ are called a 'quadratic inequations'.

(VSAQ)

1. Form a quadratic equation whose roots are -3 ± 5i.

A: The quadratic equation whose roots are

-3 + 5i and -3 - 5i is $x^2 - (\alpha + \beta)x + \alpha\beta = 0$

$\Rightarrow x^2 - (-3 + 5i - 3 - 5i)x + (-3 + 5i)(-3 - 5i) = 0$

$\because (a + ib)(a - ib) = a^2 + b^2$

$\Rightarrow x^2 + 6x + 34 = 0.$

2. Obtain a quadratic equation whose roots are $\frac{p-q}{p+q}, \frac{-(p+q)}{p-q}$.

A: The quadratic equation whose roots are $\frac{p-q}{p+q}, \frac{-(p+q)}{p-q}$ is

$x^2 - \left[\frac{p-q}{p+q} - \frac{(p+q)}{p-q}\right]x + \left(\frac{p-q}{p+q}\right)\left\{\frac{-(p+q)}{p-q}\right\} = 0$

$\Rightarrow x^2 - \left\{\frac{(p-q)^2 - (p+q)^2}{p^2 - q^2}\right\}x - 1 = 0$

$\Rightarrow (p^2 - q^2)x^2 + 4pqx - (p^2 - q^2) = 0$

3. Find the quadratic equation, the sum of whose roots is 7 and the sum of the squares of the roots is 25.

A: Let α, β be the roots of the required quadratic equation.

Given that $\alpha + \beta = 7$, $\alpha^2 + \beta^2 = 25$.

$(\alpha + \beta)^2 = 7^2$

$\Rightarrow \alpha^2 + \beta^2 + 2\alpha\beta = 49$

$\Rightarrow 25 + 2\alpha\beta = 49$

$\Rightarrow 2\alpha\beta = 49 - 25 = 24$

$\Rightarrow \alpha\beta = 12$

$\therefore$ Required quadratic equation is

$x^2 - (\alpha + \beta)x + \alpha\beta = 0$

$\Rightarrow x^2 - 7x + 12 = 0.$

4. If α, β are the roots of the equation $ax^2 + bx + c = 0$. Find the value of $\frac{1}{\alpha} + \frac{1}{\beta}$.

A: α, β are the roots of $ax^2 + bx + c = 0$

$\Rightarrow \alpha + \beta = \frac{-b}{a}, \alpha\beta = \frac{c}{a}$

Now $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\beta + \alpha}{\alpha\beta} = \frac{-b/a}{c/a} = \frac{-b}{c}$.

5. If α and β are the roots of the equation $ax^2 + bx + c = 0$, then find the value of $\frac{1}{\alpha^2} + \frac{1}{\beta^2}$.

A: α, β are the roots of $ax^2 + bx + c = 0$.

$\Rightarrow \alpha + \beta = \frac{-b}{a}, \alpha\beta = \frac{c}{a}$

Now $\frac{1}{\alpha^2} + \frac{1}{\beta^2} = \frac{\beta^2 + \alpha^2}{(\alpha\beta)^2}$

$= \frac{(\alpha + \beta)^2 - 2\alpha\beta}{(\alpha\beta)^2}$

$= \frac{(-b/a)^2 - 2c/a}{(c/a)^2}$

$= \frac{b^2 - 2ac}{a^2} \cdot \frac{a^2}{c^2}$

$= \frac{b^2 - 2ac}{c^2}$.

6. If α and β are the roots of the equation $x^2 + x + 1 = 0$, find the value of $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$.

A: α and β are the roots of $x^2 + x + 1 = 0$.

$\Rightarrow \alpha + \beta = -b/a = -1$; $\alpha\beta = c/a = 1$

Now $\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha^2 + \beta^2}{\alpha\beta}$

$= \frac{(\alpha + \beta)^2 - 2\alpha\beta}{\alpha\beta}$

$= \frac{(-1)^2 - 2(1)}{1}$

$= -1.$

7. If α and β are the roots of the equation $2x^2 + 3x + 6 = 0$, find the quadratic equation whose roots are α^3 and β^3.

A: α, β are the roots of $2x^2 + 3x + 6 = 0$

$\Rightarrow \alpha + \beta = \frac{-b}{a} = \frac{-3}{2}$; $\alpha\beta = \frac{c}{a} = \frac{6}{2} = 3$

$\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$

$= \frac{-27}{8} + \frac{27}{2}$

$= \frac{-27 + 108}{8}$

$= \frac{81}{8}$

$\alpha^3\beta^3 = 3^3 = 27$

Required quadratic equation is

$x^2 - (\alpha^3 + \beta^3)x + \alpha^3\beta^3 = 0$

$x^2 - \frac{81}{8}x + 27 = 0$

$8x^2 - 81x + 216 = 0.$

8. If the equation $x^2 - 15 - m(2x - 8) = 0$ has equal roots, then find the values of m.

A: Given equation is $x^2 - 2mx + (8m - 15) = 0$

since it has equal roots $b^2 - 4ac = 0$

$\Rightarrow (-2m)^2 - 4(1)(8m - 15) = 0$

$\Rightarrow 4m^2 - 4(8m - 15) = 0$ $\quad \div 4$

$\Rightarrow m^2 - 8m + 15 = 0$

$\Rightarrow (m - 3)(m - 5) = 0$

$\therefore m = 3$ or 5.

9. If $(m + 1)x^2 + 2(m + 3)x + m + 8 = 0$ has equal roots, find m.

A: Given equation is $(m + 1)x^2 + 2(m + 3)x + (m + 8) = 0$

Since it has equal roots $b^2 - 4ac = 0$

$\Rightarrow \{2(m+3)\}^2 - 4(m + 1)(m + 8) = 0$ $\quad \div 4$

$\Rightarrow m^2 + 6m + 9 - (m^2 + 9m + 8) = 0$

$\Rightarrow -3m + 1 = 0$

$\Rightarrow m = 1/3.$

10. Prove that the roots of $(x - a)(x - b) = h^2$ are always real.

A: Given equation is $(x - a)(x - b) = h^2$

$\Rightarrow x^2 - (a + b)x + (ab - h^2) = 0$

Its discriminant

$= \{-(a + b)\}^2 - 4(1)(ab - h^2)$

$= a^2 + b^2 + 2ab - 4ab + 4h^2$

$= (a - b)^2 + (2h)^2$

≥ 0

Hence the roots of the given equation are always real.

11. If $x^2 - 6x + 5 = 0$ and $x^2 - 12x + p = 0$ have a common root, then find p.

A: $x^2 - 6x + 5 = 0$

$\Rightarrow (x - 1)(x - 5) = 0$

$\Rightarrow x = 1, 5$

$\Rightarrow$ If $x = 1$, $1 - 12 + p = 0 \Rightarrow p = 11$

$\Rightarrow$ If $x = 5$, $25 - 60 + p = 0 \Rightarrow p = 35$

$\therefore p = 11$ or 35

12. If the quadratic equations $ax^2 + 2bx + c = 0$ and $ax^2 + 2cx + b = 0$, $(b \neq c)$ have a common root, then show that $a + 4b + 4c = 0$.

A: Let α be the common root of given two equations.

$a\alpha^2 + 2b\alpha + c = 0$

$a\alpha^2 + 2c\alpha + b = 0$

on subtraction $2(b - c)\alpha - (b - c) = 0$

$2\alpha - 1 = 0$ $\quad \because b - c \neq 0$

$\alpha = 1/2$

$\Rightarrow a(1/2)^2 + 2b(1/2) + c = 0$

$\Rightarrow a + 4b + 4c = 0.$

13. For what values of x, the expression $3x^2 + 4x + 4$ is positive.

A: Given expression is $3x^2 + 4x + 4$

Consider $3x^2 + 4x + 4 = 0$

Roots are $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \frac{-4 \pm \sqrt{16 - 4(3)(4)}}{2(3)}$

$= \frac{-4 \pm \sqrt{-32}}{6}$

$= \frac{-4 \pm 4\sqrt{2}\, i}{6}$

which are complex numbers.

Thus, $\forall x \in R$, $3x^2 + 4x + 4$ is positive.

14. For what values of x, the expression $15 + 4x - 3x^2$ is negative.

A: Given exprssion is $15 + 4x - 3x^2$.

Here $a = -3 < 0$.

Consider $15 + 4x - 3x^2 = 0$

$\Rightarrow 3x^2 - 4x - 15 = 0$

$\Rightarrow 3x^2 - 9x + 5x - 15 = 0$

$\Rightarrow 3x(x - 3) + 5(x - 3) = 0$

$\Rightarrow (3x + 5)(x - 3) = 0$

$\Rightarrow \alpha = -5/3, \beta = 3 \quad \because \alpha < \beta$

Thus for $x \in R$ and $x < -5/3$ or $x > 3$, then $15 + 4x - 3x^2$ is negative.

15. Find the maximum value of $2x - 7 - 5x^2$ for $x \in R$.

A: Comparing $2x - 7 - 5x^2$ with $ax^2 + bx + c$, we get $a = -5$, $b = 2$, $c = -7$.
Maximum value of $2x - 7 - 5x^2$

$= \frac{4ac - b^2}{4a}$

$= \frac{4(-5)(-7) - 2^2}{4(-5)}$

$= \frac{140 - 4}{-20}$

$= \frac{136}{-20}$

$= -\frac{34}{5}$.

16. Find the nature of the roots of $3x^2 + 7x + 2 = 0$.

A: Given equation is $3x^2 + 7x + 2 = 0$.

Now, $\Delta = b^2 - 4ac = (7)^2 - 4(3)\ (2)$

$= 49 - 24 = 25 = 5^2 > 0$.

$\therefore$ Roots are rational and not equal.

17. If α, β are the roots of the equation $ax^2 + bx + c = 0$, then find the value of $\frac{\alpha^2 + \beta^2}{\alpha^{-2} + \beta^{-2}}$.

A: Clearly, $\alpha + \beta = \frac{-b}{a}, \alpha\beta = \frac{c}{a}$.

Now $\frac{\alpha^2 + \beta^2}{\alpha^{-2} + \beta^{-2}} = \frac{\alpha^2 + \beta^2}{\frac{1}{\alpha^2} + \frac{1}{\beta^2}}$

$= \frac{\alpha^2 + \beta^2}{\frac{\beta^2 + \alpha^2}{\alpha^2\beta^2}} = (\alpha\beta)^2 = \left(\frac{c}{a}\right)^2 = \frac{c^2}{a^2}$.

18. If α, β are the roots of $ax^2 + bx + c = 0$, find the values of $\alpha^2 + \beta^2$ and $\alpha^3 + \beta^3$.

A: If α, β are the roots of $ax^2 + bx + c = 0$ then

$\alpha + \beta = \frac{-b}{a}, \alpha\beta = \frac{c}{a}$

(i) $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

$= \left(\frac{-b}{a}\right)^2 - 2\left(\frac{c}{a}\right) = \frac{b^2 - 2ac}{a^2}$.

(ii) $\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$

$= \left(\frac{-b}{a}\right)^3 - 3\left(\frac{c}{a}\right)\left(\frac{-b}{a}\right)$

$= \frac{b^3}{a^3} + \frac{3bc}{a^2} = \frac{3abc - b^3}{a^3}$.

19. Find a quadratic equation, the sum of whose roots is 1 and sum of the squares of roots is 13.

A: Let a, b be the roots of required equation then
$\alpha + \beta = 1, \alpha^2 + \beta^2 = 13$
We have, $(\alpha + \beta)^2 = \alpha^2 + \beta^2 + 2\alpha\beta$
$\Rightarrow (1)^2 = 13 + 2\alpha\beta$
$\Rightarrow 1 - 13 = 2\alpha\beta \Rightarrow 2\alpha\beta = -12$

$\Rightarrow \boxed{\alpha\beta = -6}$

$\therefore$ Required equation : $x^2 - (\alpha + \beta)x + \alpha\beta = 0$

$\Rightarrow x^2 - (1)x - 6 = 0 \Rightarrow \boxed{x^2 - x - 6 = 0}$.

20. If $x^2 + bx + c = 0$ and $x^2 + cx + b = 0$ ($b \neq c$) have a common root then show that $1 + b + c = 0$.

A: Let α be the common root of the given equations
then $\alpha^2 + b\alpha + c = 0$ ....(1)
and $\alpha^2 + c\alpha + b = 0$ ....(2)
Solving (1) & (2)

$\cancel{\alpha^2} + b\alpha + c = 0 - \left(\cancel{\alpha^2} + c\alpha + b\right) = 0$

$\Rightarrow (b - c)\alpha + (c - b) = 0$

$\Rightarrow (b - c)\alpha = (b - c) \Rightarrow \boxed{\alpha = 1}$

Substitute in (1) $\Rightarrow (1)^2 + b(1) + c = 0$

$\Rightarrow \boxed{1 + b + c = 0}$.

21. If the equations $x^2 + ax + b = 0$ and $x^2 + cx + d = 0$ have a common root and the first equation has equal roots then prove that $2(b + d) = ac$.

A: Let α be the common root.

Then $\alpha^2 + c\alpha + d = 0 \rightarrow (1)$

Also, $x^2 + ax + b = 0$ has equal roots.

$\Rightarrow \alpha + \alpha = -a, \alpha\alpha = b \Rightarrow \alpha = -a/2, \alpha^2 = b$.

$(1) \Rightarrow b + c(-a/2) + d = 0$.

$\Rightarrow b + d = ac/2$

$\Rightarrow \boxed{2(b + d) = ac}$.

22. Determine the sign of the expression $x^2 - 5x + 6$.

A: (i) Take $x^2 - 5x + 6 > 0 \Rightarrow (x - 2)(x - 3) > 0$.

$\Rightarrow$ for $x < 2, x > 3$ the expression is positive.

(ii) Take $x^2 - 5x + 6 < 0 \Rightarrow (x - 2)(x - 3) < 0$.

$\Rightarrow$ for $2 < x < 3$ the expression is negative.

23. Find the maximum or minimum of the expression $ax^2 + bx + a$ ($a, b \in R$ and $a \neq 0$).

A: Case: (i) Suppose $a > 0$

$\Rightarrow$ the expression has absolute minimum at

$x = \frac{-b}{2a}$

That minimum value is

$= \frac{4a(a) - b^2}{4a} = \frac{4a^2 - b^2}{4a}$

Case: (ii) Suppose $a < 0$

$\Rightarrow$ the expression has absolute maximum at

$x = \frac{-b}{2a}$

That maximum value is

$= \frac{4a(a) - b^2}{4a} = \frac{4a^2 - b^2}{4a}$.

24. Find the maximum or minimum of the expression $3x^2 + 2x + 11$.

A: Given expression is $3x^2 + 2x + 11$
compare with $ax^2 + bx + c = 0$
then we get $a = 3 > 0, b = 2, c = 11$
Since $a > 0$

$\Rightarrow$ the expression has absolute minimum at

$x = \frac{-b}{2a} = \frac{-2}{2(3)} = \frac{-1}{3}$

That minimum value $= \frac{4ac - b^2}{4a} = \frac{4(3)(11) - 2^2}{4(3)}$

$= \frac{132 - 4}{12} = \frac{128}{12} = \frac{32}{3}$.

(VSAQ)

1. Form a polynomial equation of lowest degree, whose roots are 1, -1, 3.

A: Polynomial equation of lowest degree whose roots are 1, -1, 3 is $(x - 1)(x + 1)(x - 3) = 0$

$\Rightarrow (x^2 - 1)(x + 3) = 0$

$\Rightarrow x^3 - 3x^2 - x + 3 = 0.$

2. Form a polynomial equation with rational coefficients and whose roots are $2 \pm \sqrt{3}, 1 \pm 2i$.

A: Given roots are $2 \pm \sqrt{3}, 1 \pm 2i$.

Required biquadratic equation is

$[x^2 - (2+\sqrt{3} + 2-\sqrt{3})x + (2+\sqrt{3})(2-\sqrt{3})]$

$[x^2 - (1 + 2i + 1 - 2i)x + (1 + 2i)(1 - 2i)] = 0$

$\Rightarrow (x^2 - 4x + 1)(x^2 - 2x + 5) = 0$

$\Rightarrow x^4 - 2x^3 + 5x^2 - 4x^3 + 8x^2 - 20x + x^2 - 2x + 5 = 0$

$\Rightarrow x^4 - 6x^3 + 14x^2 - 22x + 5 = 0.$

3. If -1, 2, α are the roots of $2x^3 + x^2 - 7x - 6 = 0$, then find α.

A: Given that -1, 2, α are the roots of $2x^3 + x^2 - 7x - 6 = 0$

$\Rightarrow s_1 = \alpha + \beta + \gamma = -b/a$

$\Rightarrow -1 + 2 + \alpha = -1/2$

$\Rightarrow \alpha = -3/2.$

4. If the product of the roots of the equation $4x^3 + 16x^2 - 9x - a = 0$ is 9, then find a.

A: Product of the roots of $4x^3 + 16x^2 - 9x - a = 0$ is 9.

$\Rightarrow s_3 = \alpha\beta\gamma = -d/a = 9$

$\Rightarrow -(-a)/4 = 9$

$\Rightarrow a = 36.$

5. If α, β and 1 are the roots of $x^3 - 2x^2 - 5x + 6 = 0$, then find α and β.

A: Given α, β, 1 are the roots of $x^3 - 2x^2 - 5x + 6 = 0$

$\Rightarrow s_1 = \alpha + \beta + 1 = 2$

$\Rightarrow \alpha + \beta = 1.$

Also $s_3 = \alpha\beta\gamma = \alpha\beta(1) = -d/a = -6/1 = -6$

$\Rightarrow \alpha\beta = -6$

By observation $\alpha = 3, \beta = -2.$

6. If 1, -2, 3 are the roots of $x^3 - 2x^2 + ax + 6 = 0$, then find a.

A: Given that 1 is a root of $x^3 - 2x^2 + ax + 6 = 0$

$\Rightarrow 1 - 2 + a + 6 = 0$

$\Rightarrow a = -5.$

7. Solve the equation $x^3 - 3x^2 - 6x + 8 = 0$, given that the roots are in A.P.

A: Let the roots be a- d , a, a + d.

$s_1 = a - d + a + a + d = 3$

$\Rightarrow 3a = 3 \Rightarrow a = 1$

$s_3 = (a - d)(a)(a + d) = -8$

$\Rightarrow (1- d)(1)(1 + d) = -8$

$\Rightarrow 1 - d^2 = -8$

$\Rightarrow d^2 = 9$

$\therefore d = \pm 3$

If d = 3, the roots are 1 - 3, 1, 1 + 3

i.e - 2, 1, 4.

8. Find s_1, s_2, s_3 and s_4 for the equation $8x^4 - 2x^3 - 27x^2 + 6x + 9 = 0$.

A: Given equation is $8x^4 - 2x^3 - 27x^2 + 6x + 9 = 0$

Comparing this with $ax^4 + bx^3 + cx^2 + dx + e = 0$.

$s_1 = -\frac{b}{a} = \frac{2}{8} = \frac{1}{4}$

$s_2 = \frac{c}{a} = \frac{-27}{8}$

$s_3 = -\frac{d}{a} = \frac{-6}{8} = \frac{-3}{4}$

$s_4 = \frac{e}{a} = \frac{9}{8}.$

9. Solve the equation $x^3 - 3x^2 - 16x + 48 = 0$, one root being 3.

A: Given equation is $x^3 - 3x^2 - 16x + 48 = 0$

$s_1 = 3 + \beta + \gamma = 3$

$\Rightarrow \beta + \gamma = 0$

$s_3 = 3(\beta)(\gamma) = -48$

$\Rightarrow \beta\gamma = -16$

The quadratic equation whose roots are β, γ is $x^2 - (\beta + \gamma)x + \beta\gamma = 0$

$\Rightarrow x^2 - (0)x - 16 = 0$

$x = \pm 4.$

$\therefore$ The other two roots are 4, -4.

10. If 1, 2, 3, 4 are the roots of $x^4 + ax^3 + bx^2 + cx + d = 0$, then find the values of a, b, c, d.

A: Given that 1, 2, 3, 4 are the roots of

$x^4 + ax^3 + bx^2 + cx + d = 0.$

$\Rightarrow a = -s_1 = -(\alpha + \beta + \gamma + \delta) = -(1 + 2 + 3 + 4) = -10$

$b = s_2 = \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta$

$= 1(2) + 1(3) + 1(4) + 2(3) + 2(4) + 3(4) = 35$

$c = -s_3 = -(\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta)$

$= -(1.2.3 + 1.2.4 + 1.3.4 + 2.3.4) = -50$

$d = s_4 = \alpha\beta\gamma\delta = 1.2.3.4 = 24$

$\therefore a = -10, b = 35, c = -50, d = 24$

11. If α, β, γ are the roots of $x^3 + px^2 + qx + r = 0$, then find $\alpha^2 + \beta^2 + \gamma^2$.

A: Given that α, β, γ are the roots of $x^3 + px^2 + qx + r = 0$.

Now $\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$

$= (-p)^2 - 2(q)$

$= p^2 - 2q.$

12. If α, β, γ are the roots of $x^3 + px^2 + qx + r = 0$, then find the value of $\alpha^3 + \beta^3 + \gamma^3$.

A: We know that

$\alpha^3 + \beta^3 + \gamma^3 - 3\alpha\beta\gamma = (\alpha+\beta+\gamma)[\alpha^2 + \beta^2 + \gamma^2 - \alpha\beta - \beta\gamma - \gamma\alpha]$

$\Rightarrow \alpha^3+\beta^3+\gamma^3 = (\alpha+\beta+\gamma)[(\alpha+\beta+\gamma)^2 - 3(\alpha\beta + \beta\gamma + \gamma\alpha)] + 3\alpha\beta\gamma$

$= (-p)[(-p)^2 - 3(q)] + 3(-r)$

$= 3pq - p^3 - 3r.$

13. If α, β, γ are the roots of $x^3 - 2x^2 + 3x - 4 = 0$, then find $\Sigma\alpha^2\beta^2$

A: Given that α, β, γ are the roots of $x^3 - 2x^2 + 3x - 4 = 0$

$\Sigma\alpha^2\beta^2 = \alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2$

$= (\alpha\beta + \beta\gamma + \gamma\alpha)^2 - 2(\alpha\beta.\beta\gamma + \beta\gamma.\gamma\alpha + \gamma\alpha.\alpha\beta)$

$= (\alpha\beta + \beta\gamma + \gamma\alpha)^2 - 2\alpha\beta\gamma(\beta + \gamma + \alpha)$

$= 3^2 - 2(4)(2)$

$= 9 - 16$

$= -7.$

14. Find the quotient and remainder, when $2x^5 - 3x^4 + 5x^3 - 3x^2 + 7x - 9$ divided by $x^2 - x - 3$.

A: By synthetic division, dividing

$2x^5 - 3x^4 + 5x^3 - 3x^2 + 7x - 9$ is by $x^2 - x - 3$

	2	-3	5	-3	7	-9
1	0	2	-1	10	4	0
3	0	0	6	-3	30	12
	2	-1	10	4	41	3

Required quotient is $2x^3 - x^2 + 10x + 4$ and the remainder is $41x + 3$.

15. Find the polynomial equation of degree 4 whose roots are negatives of the roots of $x^4 - 6x^3 + 7x^2 - 2x + 1 = 0$

A: Required transformed equation is $f(-x) = 0$

$\Rightarrow (-x)^4 - 6(-x)^3 + 7(-x)^2 - 2(-x) + 1 = 0$

$\Rightarrow x^4 + 6x^3 + 7x^2 + 2x + 1 = 0.$

16. Find the algebraic equation whose roots are 3 times the roots of $x^3 + 2x^2 - 4x + 1 = 0$.

A: Given equation is $f(x) = x^3 + 2x^2 - 4x + 1 = 0.$

Required transformed equation is $f(x/3) = 0$

$\Rightarrow \frac{x^3}{27} + \frac{2x^2}{9} - \frac{4x}{3} + 1 = 0$

$\Rightarrow x^3 + 6x^2 - 36x + 27 = 0.$

17. If α, β, γ are the roots of the equation $x^3 + 2x^2 - 4x - 3 = 0$, find the equation whose roots are $\frac{\alpha}{3}, \frac{\beta}{3}, \frac{\gamma}{3}$.

A: Given equation is $f(x) = x^3 + 2x^2 - 4x - 3 = 0$

Required transformed equation is $f(3x) = 0$

$\Rightarrow 27x^3 + 2(9x^2) - 4(3x) - 3 = 0$

$\Rightarrow 9x^3 + 6x^2 - 4x - 1 = 0.$

18. Find the equation whose roots are squares of the roots of $x^3 + 3x^2 - 7x + 6 = 0$.

A: Given equation is $f(x) = x^3 + 3x^2 - 7x + 6 = 0$.

Required transformed equation is $f(\sqrt{x}) = 0$

$\Rightarrow (\sqrt{x})^3 + 3(\sqrt{x})^2 - 7\sqrt{x} + 6 = 0$

$\Rightarrow x\sqrt{x} + 3x - 7\sqrt{x} + 6 = 0$

$\Rightarrow \sqrt{x}\,(x - 7) = -(3x + 6)$

Squaring on both sides,

$\Rightarrow x(x^2 - 14x + 49) = 9x^2 + 36x + 36$

$\Rightarrow x^3 - 14x^2 + 49x - 9x^2 - 36x - 36 = 0$

$\Rightarrow x^3 - 23x^2 + 13x - 36 = 0.$

19. Form the monic polynomial equation of degree 3 whose roots are 2, 3 and 6.

A: The polynomial equation whose roots 2, 3, 6 is $(x - 2)(x-3)(x - 6) = 0$.

$\Rightarrow (x^2 - 5x + 6)(x - 6) = 0$

$\Rightarrow x^3 - 6x^2 - 5x^2 + 30x + 6x - 36 = 0$

$\Rightarrow x^3 - 11^2 + 36x - 36 = 0.$

20. If 1, 1, α are the roots of $x^3 - 6x^2 + 9x - 4 = 0$ then find 'α'.

A: Given equation is $x^3 - 6x^2 + 9x - 4 = 0$

Given that 1, 1, α are the roots of given equation

then $s_1 = 1 + 1 + \alpha = -p_1$

$\Rightarrow 2 + \alpha = -(-6)$

$\Rightarrow \alpha = 6 - 2 \quad \Rightarrow \boxed{\alpha = 4}$.

21. If a, b, c are the roots of $x^3 - px^2 + qx - r = 0$ then find $\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}$.

A: Given equation is $x^3 - px^2 + qx - r = 0$.........(1).

Given that 'a, b, c' are the roots of (1), then

$a + b + c = p$, $ab + bc + ca = q$, $abc = r$.

$\therefore \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{b^2c^2 + a^2c^2 + a^2b^2}{a^2b^2c^2}$

$= \frac{(bc + ac + ab)^2 - 2abc(a + b + c)}{(abc)^2} = \frac{q^2 - 2rp}{r^2}$.

22. If α, β, γ are the roots of $x^3 - ax^2 + bx + c = 0$, then find $\Sigma\alpha^2\beta + \Sigma\alpha\beta^2$.

A: Given that α, β, γ are the roots of $x^3 + ax^2 + bx + c = 0$ then

$s_1 \Rightarrow \alpha + \beta + \gamma = -a$

$s_2 \Rightarrow \alpha\beta + \beta\gamma + \gamma\alpha = b$

$s_3 \Rightarrow \alpha\beta\gamma = -c$

$\Sigma\alpha^2\beta + \Sigma\alpha\beta^2 = \alpha^2\beta + \beta^2\gamma + \gamma^2\alpha + \alpha\beta^2 + \beta\gamma^2 + \gamma\alpha^2.$

$= (\alpha + \beta + \gamma)(\alpha\beta + \beta\gamma + \gamma\alpha) - 3\alpha\beta\gamma$

$= (-a)(b) - 3(-c)$

$= s_1s_2 - 3s_3 = 3c - ab.$

23. Find the equation whose roots are the cubes of the roots of $x^3 + 3x^2 + 2 = 0$.

A: Let $f(x) = x^3 + 3x^2 + 2$.

Formula : The equation whose roots are the cubes of the roots of $f(x) = 0$ is $f\left(\sqrt[3]{x}\right) = 0$.

$\Rightarrow \left(\sqrt[3]{x}\right)^3 + 3\left(\sqrt[3]{x}\right)^2 + 2 = 0 \Rightarrow x + 3\left(x^{2/3}\right) + 2 = 0$.

$\Rightarrow (x + 2) = -3x^{2/3}$ [cubing on both sides]

$\Rightarrow (x + 2)^3 = \left(-3x^{2/3}\right)^3$

$= x^3 + 2^3 + 3(x^2)(2) + 3x(2^2) = -27x^2$

$= x^3 + 8 + 6x^2 + 12x + 27x^2 = 0$

$\Rightarrow \boxed{x^3 + 33x^2 + 12x + 8 = 0}$

(VSAQ)

1. If $^nP_4 = 1680$, then find n.

A: $^nP_4 = 1680$

$\Rightarrow n(n-1)(n - 2)(n - 3) = 168 \times 10$

$= 8 \times 21 \times 10$

$= 8 \times 7 \times 3 \times 5 \times 2$

$= 8 \times 7 \times 6 \times 5$

$\therefore$ n = 8.

2. If $^nP_7 = 42 \cdot {}^nP_5$, then find n.

A: Given that $^nP_7 = 42 \cdot {}^nP_5$

$\Rightarrow n(n - 1)(n - 2)(n - 3)(n - 4)(n - 5)(n - 6)$

$= 42 \cdot n(n - 1)(n - 2)(n - 3)(n - 4)$

$\Rightarrow (n - 5)(n - 6) = 42$

$\Rightarrow (n - 5)(n - 6) = (7)(6)$

$\Rightarrow n - 5 = 7$

$\therefore$ n = 12.

3. If $^{n=1}P_5 : {}^nP_6 = 2 : 7$, find n

A: $\frac{^{n+1}P_5}{^nP_6} = \frac{2}{7}$

$\Rightarrow 7 \cdot {}^{n+1}P_5 = 2 \cdot {}^nP_6$

$\Rightarrow 7 \cdot (n+1)n(n-1)(n - 2)(n - 3)$

$= 2 \cdot n(n-1)(n - 2)(n - 3)(n - 4)(n - 5)$

$\Rightarrow 7(n + 1) = 2(n - 4)(n - 5)$

$\Rightarrow 7n + 7 = 2(n^2 - 9n + 20)$

$\Rightarrow 2n^2 - 25n + 33 = 0$

$\Rightarrow 2n^2 - 3n - 22n + 33 = 0$

$\Rightarrow n(2n - 3) - 11(2n - 3) = 0$

$\Rightarrow (n - 11)(2n - 3) = 0$

$\Rightarrow$ n = 11 or n = 3/2 is not possible

$\therefore$ n = 11.

4. If $^{18}P_{r-1} : {}^{17}P_{r-1} = 9 : 7$, find r

A: $\frac{^{18}P_{r-1}}{^{17}P_{r-1}} = \frac{9}{7}$

$\Rightarrow 7 \cdot {}^{18}P_{r-1} = 9 \cdot {}^{17}P_{r-1}$

$\Rightarrow 7 \cdot \frac{18!}{[18 - (r - 1)]!} = 9 \cdot \frac{17!}{[17 - (r - 1)]!}$

$\Rightarrow \frac{7 \cdot (18)(17!)}{(19 - r)(18 - r)!} = \frac{9(17!)}{(18 - r)!}$

$\Rightarrow 14 = 19 - r$

$\therefore$ r = 5.

5. If $^{12}P_r = 1320$, find r.

A: $^{12}P_r = 1320$

$= 132 \times 10$

$= 12 \times 11 \times 10$

$= {}^{12}P_3$

$\therefore$ r = 3.

6. If $^{12}P_5 + 5 \cdot {}^{12}P_4 = {}^{13}P_r$, find r.

A: Given that $^{12}P_5 + 5 \cdot {}^{12}P_4 = {}^{13}P_r$

Comparing this with $^{n-1}P_r + r \cdot {}^{n-1}P_{r-1} = {}^nP_r$

Here r = 5.

7. Find the number of 5 letter words that can be formed using the letters of the word "NATURE" that begin with 'N" when repetition is allowed.

A: Number of 5 letter words that can be formed using the letters of the word 'NATURE' that begin with 'N' when repetition is allowed = $1 \times 6^4 = 1296$.

8. Find the number of 4 letter words that can be formed using the letters of the word 'PISTON' in which atleast one letter is repeated.

A: Number of 4 letter words that can be formed using the letters of the word 'PISTON' in which atleast one letter is repeated = $n^r - {}^nP_r$

$= 6^4 - {}^6P_4$

$= 1296 - 360$

$= 936.$

9. Find the number of ways of arranging 7 persons around a circle.

A: Number of ways of arranging 7 persons around a circle = (n - 1)!

= (7 - 1)!

= 6!

= 720!

10. Find the number of chains that can be prepared using 7 different coloured beads.

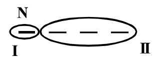
A: Number of chains that can be prepared using 7 differnt coloured beads

= ½ (n - 1)!

= ½ (7 -1)!

= ½ x 720

= 360.



11. Find the number of ways of arranging the letters of the word 'MATHEMATICS'.

A: Number of ways of arranging the letters of the word 'MATHEMATICS'

$= \frac{11!}{2!2!2!}$ since it contains M's - 2, A's - 2, T's - 2.

12. Find the number of ways of arranging letters of the word 'ASSOCIATIONS'.

A: Given word 'ASSOCIATIONS' contains

A's - 2, S's - 3, O's - 2, I's - 2.

Number of ways of arranging the letters of the given word $= \frac{12!}{2!3!2!2!}$

13. Find the number of ways of forming a committee of 5 members, out of 6 Indians and 5 Americans so that always Indians will be in majority in the committee.

A: No.of Indians = 6

No.of Americans = 5

The committee should contian 5 members with majority for Indians

6 Indians	5 Americans
3	2
4	1
5	0

Total number of ways of forming committee

$= {}^6C_3 \cdot {}^5C_2 + {}^6C_4 \cdot {}^5C_1 + {}^6C_5 \cdot {}^5C_0$

$= 20(10 + 15(5) + 6(1)$

$= 200 + 75 + 6$

$= 281.$

14. Find the number of ways of selecting a Cricket team of 11 players from 7 bats men and 6 bowlers such that there will be atleast 5 bowlers in the team.

A: Number of batsmen = 7

Number of bowlers = 6

We shall form a team of 11 players wiht atleast 5 bowlers

7 Batsmen	6 Bowlers
6	5
5	6

Total number of teams formed

$= {}^7C_6 \cdot {}^6C_5 + {}^7C_5 \cdot {}^6C_6$

$= 7(6) + 21(1)$

$= 42 + 21$

$= 63.$

15. For $1 \le 4 \le n$, with usual notation, if ${}^nC_{r-1} + {}^nC_r = {}^{n+1}C_{r-1}$, find r.

A: Given ${}^nC_{r-1} + {}^nC_r = {}^{n+1}C_{r-1}$

$\Rightarrow {}^{n+1}C_r = {}^{n+1}C_{r-1}$

it ${}^nC_r = {}^nC_s$ then r = s or n = r + s.

Here n + 1 = r + r - 1   r = r - 1

n + 2 = 2r   It is impossible

$r = \frac{n+2}{2}.$

COMBINATIONS

(VSAQ)

1. If ${}^nC_5 = {}^nC_6$, then find ${}^{13}C_n$.

A: Given that ${}^nC_5 = {}^nC_6$

$\Rightarrow$ r = s or n = r + s

Here $5 \neq 6$, n = 5 + 6 = 11

$\therefore {}^{13}C_n = {}^{13}C_{11} = {}^{13}C_2 = \frac{13.12}{2} = 78.$

2. If ${}^{12}C_{r+1} = {}^{12}C_{3r-5}$, find r.

A: Now ${}^{12}C_{r+1} = {}^{12}C_{3r-5}$

$\Rightarrow$ r = s or n = r + s.

$\Rightarrow$ r + 1 = 3r - 5   12 = r + 1 + 3r - 5

$\Rightarrow$ 2r = 6   4r = 16

$\Rightarrow$ r = 3   r = 4

$\therefore$ r = 3 or 4

3. If $10.{}^nC_2 = 3.{}^{n+1}C_3$, find n.

A: Given that $10.{}^nC_2 = 3.{}^{n+1}C_3$

$\Rightarrow 10\frac{n(n-1)}{2} = \frac{3(n+1)n(n-1)}{6}$

$\Rightarrow$ 10 = n + 1

$\therefore$ n = 9.

4. If ${}^nC_4 = 210$, find n.

A: ${}^nC_4 = \frac{210x24}{24}$

$= \frac{21x10 \text{ x } 8 \text{ x } 3}{4!}$

$= \frac{7x3 \text{ x } 10 \text{ x } 8 \text{ x } 3}{4!}$

$= \frac{10 \text{ x } 9 \text{ x } 8 \text{ x } 7}{4!}$

$= {}^{10}C_4$

$\therefore$ n = 10.

5. If ${}^{12}C_r = 495$, find the possible values of r.

A: ${}^{12}C_r = \frac{495 \text{ x } 24}{24}$

$= \frac{5 \text{ x } 99 \text{ x } 12 \text{ x } 2}{4!}$

$= \frac{12 \text{ x } 11 \text{ x } 10 \text{ x } 9}{4!}$

$= {}^{12}C_4$ or ${}^{12}C_8$

$\therefore$ r = 4 or 8.

6. If ${}^nP_r = 5040$ and ${}^nC_r = 210$, find n and r.

A: We know that $r! = \frac{{}^nP_r}{{}^nC_r} = \frac{5040}{210} = 24 = 4!$

$\therefore$ r = 4.

Also ${}^nP_4 = 5040$

= 10 x 504

= 10 x 9 x 56

= 10 x 9 x 8 x 7

$= {}^{10}P_4$

$\therefore$ n = 10.

7. Find the value of ${}^{10}C_5 + 2.{}^{10}C_4 + {}^{10}C_3$

A: ${}^{10}C_5 + 2.{}^{10}C_4 + {}^{10}C_3$

$= \{{}^{10}C_5 + {}^{10}C_4\} + \{{}^{10}C_4 + {}^{10}C_3\} \because {}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

$= {}^{11}C_5 + {}^{11}C_4$

$= {}^{12}C_5$

8. If a set A has 12 elements, then find the number of subsets of A having 4 elements.

$n!\left[\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + ...(-1)^n \frac{1}{n!}\right]$

$\therefore$ Required number of ways $= = 4!\left[\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right]$

$= 24\left[\frac{1}{2} - \frac{1}{6} + \frac{1}{24}\right] = 24\left[\frac{12-4+1}{24}\right] = 9.$

7. Find th number of palindromes with 6 digits that can be formed using the digits (i) 0, 2, 4, 6, 8 (ii) 1, 3, 5, 7, 9

A: (i) The number of palindromes formed

$= 4 \text{ x } 5^{4/2} = 4 \text{ x } 5^2 = 100.$

(ii) The number of palindromes formed

$= 5^{6/2} = 5^3 = 125.$

8. Find the number of functions from a set A containing 5 element s into a set B containing 4 elements.

A: Required number of functions $= [n(B]^{n(A)}$

$= 4^5 = 1024.$

9. Find the number of injections from a set A containing 4 elements into a set B containing 6 elements.

A: Required number of injections (one - one functions).

$= {}^{n(B)}P_{n(A)} = {}^6P_4 = 6.5.4.3 = 360.$

10. Find the number of surjections from a set A containing 6 elements onto a set B containing 2 elements.

A: Required no. of surjections (onto functions)

$= 2^{n(A)} - 2 - 2^6 - 2 = 64 - 2 = 62.$

11. Find the number of bijections from a set A containing 7 elements into itself.

A: Required number of bijections - n(A)! = 7!..

12. Find the number of ways of arranging 7 gents, 4 ladies around a circular table if no two ladies wish to sit together.

A: First we arrange 7 gents around a circular table as (7 - 1)! = 6!.

In between there are 7 gaps, these 7 gaps can be filled with 4 ladies as ${}^7P_4$ ways.

$\therefore$ Total number of ways = 6! x ${}^7P_4$.

13. Find the number of ways of arranging the letters of the word
i) INDEPENDENCE
ii) INTERMEDIATE
iii) PERMUTATION
iv) COMBINATION

A: Given word contains 12 letters

In which there are 3N's, 2D's, 4E's

$\therefore$ The number of required arrangements $= \frac{12!}{3!2!4!}$

ii) The no. of letters in the word 'INTERMEDIATE' = 12

In this there are 2I's, 2T's, 3E's.

$\therefore$ The number of required arrangements $= \frac{12!}{2!2!3!}$.

iii) Given word 'PERMUTATION' contains 11 letters in which there are 2T's

The number of required arrangements = $\frac{(11)!}{2!}$.

iv) Given word 'COMBINATION' contains 11 letters. In which there are 2O's, 2I's, 2N's

$\therefore$ The number of required arrangements = $\frac{(11)!}{2!2!2!} = \frac{(11)!}{(2!)^3}$.

14. In a class there are 30 students. If each student plays a chess game with each of the other student, then find the total number of chess games played by them.

A: No of students in the class = 30.

$\therefore$ Total no. of chess games played by them

$$= {}^{30}C_2 = \frac{30 \text{ x } 29}{2 \text{ x } 1} = 435.$$

15. Find the number of ways of selecting 4 boys and 3 girls from a group of 8 boys and 5 girls.

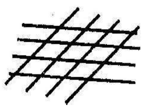
A: The no. of ways of selecting 4 boys from 8 boys is ${}^8C_4$.

The no. of ways of selecting 3 girls from 5 girls is ${}^5C_3$.

$\therefore$ Total no. of selection = ${}^8C_4$ x ${}^5C_3$

$$= \frac{8 \text{ x } 7 \text{ x } 6 \text{ x } 5}{4 \text{ x } 3 \text{ x } 2 \text{ x } 1} = \frac{5 \text{ x } 4 \text{ x } 3}{3 \text{ x } 2 \text{ x } 1} = 70 \text{ x } 10 = 700.$$

16. Find the number of parallelograms formed by the set of 'm' parallel lines intersects an other set of 'n' parallel lines.



A: A parallelogram is formed with 2 lines from the first set of 'm' lines and 2 lines from the second set of 'n' lines.

$\therefore$ The number of required parallelograms = ${}^mC_2$ x ${}^nC_2$.

17. If there are 5 alike pens, 6 alike pencils and 7 alike erasers, find the no. of ways of selecting any number of (one or more) things out of them.

A: Formula: If 'p; things are alike of one kind 'q' things are alike of second kind and 'r' things are alike of third kind then the no. of ways of selecting any number of things (one or more) is (p + 1) (q + 1) (r + 1) -1.

$\therefore$ The required number of ways

= (5 + 1) (6 + 1) (7 + 1) - 1

= 6.7.8 - 1 = 336 - 1 = 335.

18. Find the no. of zeros in 100!

A: $100! = 2^{\alpha}3^{\beta}5^{\gamma}7^{\delta}$............... $\boxed{100 = 2^2 \text{ x } 5^2}$

where

$$\alpha = \left[\frac{100}{2}\right] + \left[\frac{100}{2^2}\right] + \left[\frac{100}{2^3}\right] + \left[\frac{100}{2^4}\right] + \left[\frac{100}{2^5}\right] + \left[\frac{100}{2^6}\right]$$

= 50 + 25 + 12 + 6 + 3 + 1 = 97

and $\gamma = \left[\frac{100}{5}\right] + \left[\frac{100}{5^2}\right] = 20 + 4 = 24$

Now, the number of zero's in 100! is the power of 10(2 x 5) in 100! which is 24.

19. Find te numbr of ways in which 12 things can be (i) divided into 4 equal groups (ii) distributed to 4 persons equally.

A: i) The number of ways of dividing 12 things into 4 equal groups = $\frac{12!}{(3!)^4.4!}$.

ii) The no. of ways of distributing 12 things to 4 persons = $\frac{12!}{(3!)^4}$.

(VSAQ)

1. Find the number of terms in (2a + 3b + c)^5.

A: Number of terms in $(2a + 3b + C)^5$

$$= \frac{(n+1)(n+2)}{2!}$$

$$= \frac{(5+1)(n+2)}{2}$$

$$= \frac{6 \text{ x } 7}{2}$$

= 21.

2. Find the 3rd term from the end in the expansion of $\left(x^{\frac{-2}{3}} - \frac{3}{x^2}\right)^8$.

A: In $\left(x^{\frac{-2}{3}} - \frac{3}{x^2}\right)^8$, 3rd term from the end

$= T_7$

$= T_{6+1}$

$= {}^8C_6\,(x^{-2/3})^{8-6}\,(-3/x^2)^6$

$= {}^8C_2 \text{ x } x^{-4/3} \text{ x } 36/x^{12}$

$= {}^8C_2 \text{ x } 3^6/x^{40/3}$.

3. Find the coefficient of x^{-6} in (3x - 4/x)^{10}

A: $T_{r+1} = {}^{10}C_r\,(3x)^{10-r}\,(-4/x)^r$

$= {}^{10}C_r\,.3^{10-r}.(-4)^r.\,x^{10-r-r}$

To get the coefficient of x^{-6},

10 - 2r = - 6

$\Rightarrow$ 2r = 16

$\Rightarrow$ r = 8

Coefficient of $x^{-6} = {}^{10}C_8\,.\,3^2.\,(-4)^8$

$= {}^{10}C_2\,.\,3^2\,.\,4^8$.

4. Find the coefficient of x^{-7} in $\left(\frac{2x^2}{3} - \frac{5}{4x^5}\right)^7$.

A: In $\left(\frac{2x^2}{3} - \frac{5}{4x^5}\right)^7$,

$$T_{r+1} = {}^7C_r\,.\left(\frac{2x^2}{3}\right)^{7-r}\left(\frac{-5}{4x^5}\right)^r$$

$$= {}^7C_r\,.\left(\frac{2}{3}\right)^{7-r}\left(\frac{-5}{4}\right)^r.\,x^{14-7r}$$

To get the coefficient of x^{-7},

14 - 7r = 7

7r = 21

r = 3

$\therefore$ Coefficient of x^{-7}

$$= {}^7C_3\left(\frac{2}{3}\right)^{7-3}\left(\frac{-5}{4}\right)^3$$

$$= -35\left(\frac{2^4}{3^4}\right)\left(\frac{5^3}{4^3}\right)$$

$$= \frac{-4375}{324}.$$

5. Find the term independent of in $\left(\frac{\sqrt{x}}{3} - \frac{4}{x^2}\right)^{10}$.

A: General term $T_{r+1} = {}^{10}C_r.\left(\frac{\sqrt{x}}{3}\right)^{10-r}\left(\frac{-4}{x^2}\right)^r$

$$= {}^{10}C_r\,\frac{(-4)^r}{3^{10-r}}.x^{\frac{10-r}{2}-2r}$$

To get the term independent of x, $\frac{10-r}{2} - 2r = 0$.

10 - 5r = 0 $\Rightarrow$ r = 2.

$\therefore$ Term independent of x

$$= {}^{10}C_2\,\frac{(-4)^2}{3^8}$$

$$= \frac{45 \text{ x } 16}{3^8} = \frac{80}{729}.$$

6. Find the numerically greatest terms the expansion of (3 + 2a)^{15} when a = 5/2.

A: $(3 + 2a)^{15} = 3^{15}\left(1 + \frac{2a}{3}\right)^{15}$

$|x| = \left|\frac{2a}{3}\right| = \left|\frac{2}{3}.\frac{5}{2}\right| = \frac{5}{3}$

Now $\frac{(n+1)|x|}{|x|+1} = \frac{(15+1).5/3}{8/3} = 10$

$\therefore$ $|T_{10}|$ and $|T_{11}|$ are numerically greatest.

$|T_{10}| = {}^{15}C_9\, 3^6 \left(2 . \frac{5}{2}\right)^9 = {}^{15}C_9 . 3^6 . 5^9$

$|T_{11}| = {}^{15}C_{10}\, 3^5 \left(2 . \frac{5}{2}\right)^{10} = {}^{15}C_0 . 3^5 . 3^{10}$

and $|T_{10}| = |T_{11}|$.

7. Find the numerically greatest term in the expansion of $(3x+ 5y)^{12}$ when x = 1/2, y = 4/3.

A: $(3x + 5y)^{12} = (3x)^{12} \left(1+\frac{5y}{3x}\right)^{12}$

$|x| = \left|\frac{5}{3}.\frac{4}{3}.\frac{2}{1}\right| = \frac{40}{9}$

Now $\frac{(n+1)|x|}{|x|+1} = \frac{13x\frac{40}{9}}{\frac{49}{9}} = \frac{520}{9} = 10.4$

$\therefore$ Numerically greatest term

$= |T_{10+1}|$

$= |{}^{12}C_{10} (3.½)^{12-10} (5.4/3)^{10}|$

$= {}^{12}C_2 . (3/2)^2 (20/3)^{10}$

8. If the coefficients of $(2r + 4)^{th}$ and $(r - 2)^{nd}$ terms in the expansion of $(1 + x)^{18}$ are equal, find r

A: In $(1 + x)^{18}$, $T_{2r+4} = T_{(2r+3)+1}$

$= {}^{18}C_{2r+3}$

$T_{r-2} = T_{(r-3)+1}$

$= {}^{18}C_{r-3}$

But ${}^{18}C_{2r+3} = {}^{18}C_{r-3}$

$\Rightarrow r = s$ $\quad$ $n = r + s$

$\Rightarrow 2r + 3 = r - 3$ $\quad$ $18 = 2r + 3 + r - 3$

$\Rightarrow r = -6$ $\quad$ $18 = 3r$

is not possible $\quad$ r = 6

$\therefore$ r = 6.

9. If ${}^{22}C_r$ is the largest binomial coefficient in the expansion of $(1 + x)^{22}$ find the value of ${}^{13}C_r$.

A: $= {}^nC_{n/2}$ if n is even

$= {}^{22}C_{11}$

r = 11.

Now ${}^{13}C_r = {}^{13}C_{11}$

$= {}^{13}C_2$

$= \frac{13 \times 12}{2}$

= 78.

10. If $(1 + 3x - 2x^2)^{10} = a_0 + a_1 x + a_2x^2 + ... a_{20} x^{20}$, then prove that (i) $a_0 + a_1 + a_2 + ... + a_{20} = 2^{10}$

(ii) $a_0 - a_1 + a_2 - ..... + a_{20} = 4^{10}$

A: $(1 + 3x - 2x^2)^{10} = a_0 + a_1x + a_2x^2 + ... + a_{20} x^{20}$

Put x = 1 in the above relation,

$a_0 + a_1.1 + a_2 - 1^2 + ... fa_{20} .1^{20} = (1 + 3 - 2)^{10}$

$\Rightarrow a^0 + a_1 + a_2 + ... = a_{20} = 2^{10}$

Put x = -1 in the given relation,

$a_0 + a_1 (-1) + a_2 (-1)^2 + ... + a_{20} (-1)^{20} = (1 - 3 - 2)^{10}$

$a_0 - a_1 + a_2 - .... + a_{20} = 4^{10}$

11. Obtain the values of x for which the binomial expansion of $(2 + 3x)^{-2/3}$ is valid.

A: $(2 + 3x)^{-2/3} + 2^{-2/3} (1 + 3x/x)^{-2/3}$

The above expansion is valid if

$|3x/x| < 1$

$\Rightarrow |x| < 2/3$

$\Rightarrow x \in \left(\frac{-2}{3}, \frac{2}{3}\right)$.

12. Find the values of x for which the binomial expansion $(7 + 3x)^{-5}$ is valid.

A: $(7 + 3x)^{-5} = 7^{-5} \left(1+\frac{3x}{7}\right)^{-5}$

The avove expansion is valid if $\left|\frac{3x}{7}\right| < 1$

$\Rightarrow |x| < \frac{7}{3}$

$\Rightarrow x \in \left(\frac{-7}{3}, \frac{7}{3}\right)$

13. Find the number of terms with non-zero coefficients in $(4x - 7y)^{49} + (4x - 7y)^{49}$.

A: The number of terms in the expansion $(x + y)^n + (x - y)^n$ when 'n'is odd is

$\frac{n+1}{2} = \frac{49+1}{2} = 25$.

14. Write down and simplify 6^{th} term in $\left(\frac{2x}{3}+\frac{3y}{2}\right)^9$.

A: 6^{th} term $= T_6 = T_{5+1}$.

$= {}^9C_5 \left(\frac{2x}{3}\right)^{9-5} \left(\frac{3y}{2}\right)^5 = {}^9C_5 \left(\frac{2x}{3}\right)^4 \left(\frac{3y}{2}\right)^5$

$= 126.\left[\frac{3}{2}\right] x^4y^5 = 189\, x^4y^5$.

15. If A and B are coefficients of x^n in the expansion of $(1 + x)^{2n}$ and $(1 + x)^{2n-1}$ respectively, then find the value of $\frac{A}{B}$.

A: A = coefficient of x^n in $(1 + x)^{2n} = {}^{2n}C_n$.

B = Coefficient of x^n in $(1 + x)^{2n-1} = {}^{2n-1}C_n$.

$\frac{A}{B} = \frac{{}^{2n}C_n}{{}^{2n-1}C_n} = \frac{(2n)!}{(2n-n)!n!} x \frac{(2n-1-n)!n!}{(2n-1)!}$

$= \frac{(2n)!}{(n!)^2} x \frac{(n-1)!n!}{(2n-1)!} = \frac{(2n)(2n-1)!}{n!n(n-1)!} x \frac{(n-1)n!}{(2n-1)!}$

$= \frac{2n}{n} = 2$.

16. Find the largest binomial coefficient(s) in the expansion of $(1 + x)^{24}$.

A: Here n = 24, an even integer.

Hence there is only one largest binomial coefficient, that is ${}^nC_{\frac{n}{2}} = {}^{24}C_{12}$.

17. Find the largest binomial coefficient(s) in the expansion of $(1 + x)^{19}$.

A: Here n = 19 (odd).

$\therefore$ The largest binomial coefficients are

${}^nC_{\frac{n-1}{2}}, {}^nC_{\frac{n+1}{2}} = {}^{19}C_9, {}^{19}C_{10}$.

$\left[\text{Note that } {}^{19}C_9 = {}^{19}C_{10}\right]$.

18. Find the middle terms in the expansion of $\left(4a+\frac{3}{2}b\right)^{11}$.

A: Given expansion is $\left(4a+\frac{3}{2}b\right)^{11}$.

Here $\boxed{n=1}$, odd

So, middle terms are $\frac{T_{11+1}}{2}, \frac{T_{11+3}}{2} = T_6, T_7$.

$T_6 = T_{5+1} = {}^{11}C_5 (4a)^{11-5} \left(\frac{3}{5}b\right)^5$

$= {}^{11}C_5 4^6.a^6.\left(\frac{3}{2}\right)^5 . b^2 = {}^{11}C_5.4^6.\left(\frac{3}{2}\right)^5 a^6 b^6$

$T_7 = T_{6+1} = {}^{11}C_6 (4a)^{11-6} \left(\frac{3}{5}b\right)^6$

$= {}^{11}C_5 4^5.a^5.\left(\frac{3}{2}\right)^6 . b^6 = {}^{11}C_6.4^5.\left(\frac{3}{2}\right)^6 a^5 b^6$.

19. Find the middle term in the expansion of $\left(\frac{3x}{7}-2y\right)^{10}$.

A: Here $\boxed{n=10}$, even

So, middle term $= T_{\frac{10}{2}+1} = T_{5+1}$.

$\therefore T_{5+1} = {}^{10}C_5.\left(\frac{3x}{7}\right)^{10-5} . (-2y)^5$

$= -{}^{10}C_5.\left(\frac{3x}{7}\right)^5.(2y)^5 == -{}^{10}C_5.\left(\frac{6}{7}\right)^5 x^5y^5$.

20. Find the coefficient of x^7 in $\left[\frac{3x^2}{7}+\frac{4}{5x^3}\right]^{11}$.

A: Given expansion is $\left[\frac{3x^2}{7}+\frac{4}{5x^3}\right]^{11}$.

General term $T_{r+1} = {}^nC_r.x^{n-4}. a^r$.

$= {}^{11}C_r.\left(\frac{3x^2}{7}\right)^{11-r} . \left(\frac{4}{5x^3}\right)^r$.

$= {}^{11}C_r.\left(\frac{3}{7}\right)^{11-r} . x^{22-2r} \left(\frac{4}{5}\right)^r .x^{-3r}$

$= {}^{11}C_r.\left(\frac{3}{7}\right)^{11-r} . \left(\frac{4}{5}\right)^r .x^{22-5r}$

take $22 - 5r = 7 \Rightarrow 5r = 15 \Rightarrow \boxed{r=3}$

Coefficient of x^7

is ${}^{11}C_3.\left(\frac{3}{7}\right)^{11-3}.\left(\frac{4}{5}\right)^3 = {}^{11}C_3.\left(\frac{3}{7}\right)^8.\left(\frac{4}{5}\right)^3$.

21.Find the term independent of x in the expansion of $\left(\frac{3}{\sqrt[3]{x}}+5\sqrt{x}\right)^{25}$.

A: General term $T_{r+1} = {}^nC_r\, x^{n-r} a^r$.

$= {}^{25}C_r\left(\frac{3}{\sqrt[3]{x}}\right)^{25-r}\left(5\sqrt{x}\right)^r = {}^{25}C_r\ 3^{25-r}\left(\frac{1}{x^{\frac{25-r}{3}}}\right)\left(5\sqrt{x}\right)^r$

$= {}^{25}C_r 3^{25-r}5^r x^{-\left(\frac{25-r}{3}\right)+\frac{r}{2}}$

take $\frac{-25+r}{3}+\frac{r}{2}=0$ then

$-50+2r+3r=0 \Rightarrow \boxed{r=10}$.

$\therefore$ The term independent of x is

$T_{11} = {}^{25}C_{10}\, 3^{25-10}\, 5^{10} = {}^{25}C_{10} 3^{15} 5^{10}$.

22.Find the term independent of x in the expansion of $\left(4x^3+\frac{7}{x^2}\right)^{14}$.

A: General term $T_{r+1} = {}^nC_r\, x^{n-r} a^r$.

$T_{r+1} = {}^{14}C_r (4x^3)^{14-r}\left(\frac{7}{x^2}\right)^r = {}^{14}C_r 4^{14-r} 7^r x^{42-5r}$.

If 42 - 5r = 0 then $r = \frac{42}{5}$.

Which is not possible.

$\therefore$ The term indepenent of x is '0'.

23.Prove that $C_0 + 2.C_1 + 2^2.c_2 + ...+2^n . C_n. = 3^n$.

A: We know that

$C_0 + C_1.x + C_2.x^2 + ...+C_n.x_n = (1+x)^n$.

Let $\boxed{x=2}$

then we get

$\boxed{C_0 + 2.C_1 + 2^2.C_2 + ...+2^n.C_n = 3^n}$.

24.Find the sum of $3.C_0 + 6.C_1 + 12.C_2 + ...3.2^n C_n$.

A: take $3.C_0 + 6.C_1 + 12.C_2 + ...3.2^n C_n$.

$= 3.C_0 + 3.2.C_1 + 3.2^2.C_2 + ... + 3.2^n C_n$

$= 3\left[C_0 + 2.C_1 + 2^2.C_2 + ... + 2^n C_n\right]$

$= 3\left[(1+2)^n\right] = 3.3^n = 3^{n+1}$.

25.Prove that

$$\frac{C_1}{C_0}+2.\frac{C_2}{C_1}+3.\frac{C_3}{C_2}+.........+n.\frac{C_n}{C_{n-1}} = \frac{n(n+1)}{2}.$$

A: L.H.S. $= \frac{C_1}{C_0}+2.\frac{C_2}{C_1}+3.\frac{C_3}{C_2}+.........+n.\frac{C_n}{C_{n-1}}$

$= \frac{n}{1}+2\frac{\frac{n(n-1)}{2}}{n}+3\frac{\frac{n(n-1)(n-2)}{3!}}{\frac{n(n-1)}{2}}+......+n\frac{1}{n}$

= n + (n - 1) + (n - 2) + ...+ 2 + 1.

= 1 + 2 + 3 ....+ n = $\frac{n(n+1)}{2}$ = R.H.S.

I (VSAQ)

1. Find the mean deviation from the mean of the following discrete data : 3, 6, 10, 4, 9, 10.

A: Mean of the data 3, 6, 10, 4, 9, 10 is

$\bar{x} = \frac{3+6+10+4+9+10}{6}$

$= \frac{42}{6}$

$= 7$

$\therefore$ Mean deviation from the mean

$= \frac{\sum_{i=1}^{6}|x_i - \bar{x}|}{n}$

$= \frac{4+1+3+3+2+3}{6}$

$= \frac{16}{6}$

$= 2.67$.

2. Compute the mean deviation about the median of the data 6, 7, 10, 12, 13, 4, 12, 16.

A:Ascending order of the given data is 6, 7, 10, 12, 12, 13, 16.

Median $M = \frac{x_4 + x_5}{2}$

$= \frac{10+12}{2}$

$= 11$.

$\therefore$ Mean deviation from the median

$= \frac{\sum_{i=1}^{8}(x_i - M)}{8}$

$= \frac{7+5+4+1+1+1+2+5}{8}$

$= \frac{26}{8}$

$= 3.25$.

3. Find the variance and standard deviation of the data 5, 12, 3, 18, 6, 8, 2, 10.

Mean $\bar{x} = \frac{5+12+3+18+6+8+2+10}{8}$

$\bar{x} = \frac{64}{8} = 8$

Variance $\sigma^2 = \frac{\sum_{i=1}^{8}(x_i - \bar{x})^2}{n}$

$= \frac{9+16+25+100+4+36+4}{8}$

$= \frac{194}{8}$

= 24.25.

Standard deviation $\sigma = \sqrt{24.25}$

= 4.95.

4. The coefficient of variation of two distributions are 60 and 70 and their standard deviations are 21 and 16 respectively. Find their arithmetic means.

A: Let $\bar{x}$ and $\bar{y}$ be the means of given two distributions

Coefficient of variation $C.V. = \frac{\sigma}{\bar{x}} \times 100$

$60 = \frac{21}{\bar{x}}(100)$

$\Rightarrow \bar{x} = 35$.

For the second distribution $C.V. = \frac{\sigma}{\bar{y}} \times 100$

$70 = \frac{16}{y} \times 100$

$\Rightarrow \bar{y} = 22.85$.

5. The variance of 20 observations is 5. If each of the observations is multiplied by 2, find the variance of the resulting observations.

A: We know that if each observation in a data multiplied by a constant k, then the variance of the resulting observations is k^2 times that of the variance of original observations.

Here each of the observation is multiplied by 2 .

$\therefore$ Variance of resulting observations

$= 2^2 (5)$

$= 4(5)$

$= 20.$

6. If each of the observations $x_1, x_2, ...x_n$ is increased by k, where k is a positive or negative number, then show that the variance remains unchanged.

A: For the observations $x_1, x_2, ...x_n$,

Mean $\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}$

Variance $\sigma_1^2 = \frac{\sum(x_i - \bar{x})^2}{n}$

Mean of new observations $\bar{y} = \frac{\sum_{i=1}^{n}(x_i + k)}{n}$

$= \frac{\sum_{i=1}^{n} x_i}{n} + \frac{kn}{n}$

$= \bar{x} + k$

$\therefore$ Variance of new observations $\sigma_2^2 = \frac{\sum_{i=1}^{n}(y_i - \bar{y})^2}{n}$

$= \frac{\sum_{i=1}^{n}\left[x_i + k - (\bar{x} + k)\right]^2}{n}$

$= \frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n}$

$= \sigma_1^2$.

Thus the variance of new observations is the same as that of the original observations.

7. Find the mean deviation from the mean of the following discrete data : 6, 7, 10, 12, 13, 4, 12, 16.

Mean of the data $= \frac{6+7+10+12+13+4+12+16}{8}$.

$\bar{x} = \frac{80}{8}$

= 10.

$\therefore$ Mean deviation from the mean $= \frac{\sum_{i=1}^{8}|x_i - \bar{x}|}{8}$

$= \frac{4+3+0+2+3+6+2+6}{8}$

$= \frac{26}{8} = 3.25.$

8. Find the mean deviation about the median for the data : 4, 6, 9, 3, 10, 13, 2.

A: The ascending order of the data is 2, 3, 4, 6, 9, 10, 13.

Median M = x_4 = 6.

$\therefore$ Mean deviation from the median = $\dfrac{\sum_{i=1}^{7}|x_i - M|}{7}$

$= \dfrac{4+3+2+0+3+4+7}{7}$

$\dfrac{23}{7}$

= 3.29.

9. Find the variance for the discrete data : 6, 7, 10, 12, 13, 4, 8, 12.

A: Mean = $\dfrac{6+7+10+12+13+4+8+12}{8}$

$= \dfrac{72}{8}$

= 9.

Variance $\sigma^2 = \dfrac{\sum_{i=1}^{8}(x_i - \bar{x})^2}{8}$

$= \dfrac{9+4+1+9+16+25+1+9}{8}$

$= \dfrac{74}{8}$

= 9.25.

(VSAQ)

1. The probability distribution of a random variable X is

X = x	0	1	2	3
P(X = x)	k	3k	3k	k

Find the value of k and the mean of X.

A: Given that X is a random variable.

$\Rightarrow \sum_{i=1}^{3} P(X = x_i) = 1$

$\Rightarrow$ k + 3k + 3k + k = 1

$\Rightarrow$ 8k = 1

$\Rightarrow$ k = 1/8

Let μ be the mean of X.

$\therefore \mu = \sum_{i=1}^{3} x_i P(x = x_i)$

= 0(k) +1(3k) + 2(3k) + 3(k)

= 12k

$= \dfrac{12}{8} = \dfrac{3}{2}$

$\therefore$ k = 1/8, μ = 3/2

2. A random variable X has the range {1, 2, 3 ....} If P(X = r) = $\dfrac{C^r}{r!}$ for r = 1, 2, 3, ..., then find C.

A: Given that X is a common variable

$\Rightarrow \sum_{r=1}^{\infty} P(x = r) = 1$

$\Rightarrow \sum_{r=1}^{\infty} \dfrac{C^r}{r!} = 1$

$\dfrac{C}{1!} + \dfrac{C^2}{2!} + \dfrac{C^3}{3!} + ...\infty = 1$

Adding 1 on bothsides,

$1 + \dfrac{C}{1!} + \dfrac{C^2}{2!} + ...\infty = 1 + 1$

$\Rightarrow e^c = 2$

$\Rightarrow C = \log_e 2.$

3. Find the constant C, so that f(x) = $C\left(\dfrac{2}{3}\right)^x$, x = 1, 2, 3 ...... is the probability distribution function of a discrete random variable X.

A: Given that X is a discrete random variable

$\Rightarrow \sum_{x=1}^{\infty} f(x) = 1$

$\Rightarrow \sum_{x=1}^{\infty} C\left(\dfrac{2}{3}\right)^x = 1$

$\Rightarrow C\left[\dfrac{2}{3} + \left(\dfrac{2}{3}\right)^2 + \left(\dfrac{2}{3}\right)^3 + ...\infty\right] = 1$

$\Rightarrow C\left[\dfrac{\frac{2}{3}}{1-\frac{2}{3}}\right] = 1 \qquad \because S_\infty = \dfrac{a}{1-r}$

$\Rightarrow$ 2C = 1

$\therefore C = \dfrac{1}{2}.$

4. For a binomial ditribution with mean 6 and variance 2, Find the first two terms of hte distribution..

A: Let n, p be the paremeters of the binomial distribution.

np = 6, npq = 2.

$q = \dfrac{npq}{np} = \dfrac{2}{6} = \dfrac{1}{3}.$

$\Rightarrow$ p = 1 - q = 1 - 1/3 = 2/3.

Also np = 6

$\Rightarrow$ n x 2/3 = 6

$\Rightarrow$ n = 9

First two terms of the binomial distribution are

P(X = 0), P(X = 1)

$= {}^9C_0\left(\dfrac{1}{3}\right)^9\left(\dfrac{2}{3}\right)^0, {}^9C_1\left(\dfrac{1}{3}\right)^8\left(\dfrac{2}{3}\right)^1$

$= \dfrac{1}{3^9}, \dfrac{2}{3^7}.$

5. The mean and variance of a binomial distribution are 4 and 3 respectively. Find P(X $\geq$ 1)

A: Let n, p be the parameters of the binomial distribution.
np = 4, npq = 3

$$q = \frac{npq}{np} = \frac{3}{4}$$

$$\Rightarrow p = 1 - q = 1\ 1 - \frac{3}{4} = \frac{1}{4}$$

Also np = 4
n.1/4 / 4
n = 16
Required probability P(X $\geq$ 1)

$$= \sum_{r=1}^{16} P(X = r)$$

$$= \sum_{r=1}^{16} {}^{16}C_r \left(\frac{3}{4}\right)^{16-r} \left(\frac{1}{4}\right)^r.$$

6. If the mean and variance of a binomial variate X are 2. 4 and 1. 44 respectively, then find p and n.

A: Let n, p be the parameter of the binomial distribution.
Given that np = 2.4, npq = 1.44

$$q = \frac{npq}{np} = \frac{1.44}{2.4} = \frac{144}{240} = \frac{12}{20} = \frac{3}{5}$$

$$\Rightarrow p = 1 - q = 1 - \frac{3}{5} = \frac{2}{5}$$

Also np = 2.4

$$\Rightarrow n\left(\frac{2}{5}\right) = 2.4$$

$\Rightarrow n = 6$ $\quad \because n = 6,\ p = 2/5$

7. X follows Poisson distribution such that P(X = 1) = 3P (X = 2). Find the variance of X.

A: Let X be the parameter of the poisson distribution.
Given that P(X = 1) = 3P(X = 2)

$$\Rightarrow \frac{e^{-\lambda}\lambda^1}{1!} = 3\frac{e^{-\lambda}\lambda^2}{2!}$$

$$\Rightarrow 1 = \frac{3\lambda}{2}$$

$$\Rightarrow \lambda = \frac{2}{3}$$

Variance of X = $\frac{2}{3}$.

8. A poisson variate X satisties P(X = 1) = P(X = 2). Find P(X = 5).

A: Let λ be the parameter of the poisson distribution.

Given that P(X = 1) = P(X = 2)

$$\Rightarrow \frac{e^{-\lambda}\lambda^1}{1!} = \frac{e^{-\lambda}\lambda^2}{2!}$$

$$\Rightarrow 1 = \frac{\lambda}{2}$$

$\Rightarrow \lambda = 2$

$$\text{Now } P(X = 5) = \frac{e^{-\lambda}\lambda^5}{5!}$$

$$= \frac{e^{-2}.2^5}{5!}$$

$$= \frac{32e^{-2}}{120}$$

$$= \frac{4e^{-2}}{15}.$$

9. The probability that a person choosen at random is left handed (in hand writing) is 0.1. What is the probability that in a group of 10 people, there is one who is left handed.

A: Here n = 10, p = 0.1 = $\frac{1}{10}$, q = 0.9 = $\frac{9}{10}$.

$\therefore$ The required probability that exactly one out of 10 is left handed is P(X = 1) = ${}^{10}C_1\, p^1\, q^{10-1}$.

$$= 10.\left(\frac{1}{10}\right)^1.\left(\frac{9}{10}\right)^9 = \left(\frac{9}{10}\right)^9.$$

10. It is given that 10% of the electric bulbs manufactured by a company are defective. In a sample of 20 bulbs, find the probability that more than 2 are defective.

A: Let X be number of defective bulbs in the sample of 20 bulbs.

The probability that a bulb will be defective is

$$p = \frac{10}{100} = \frac{1}{10},\ \text{Hence } q = 1 - \frac{1}{10} = \frac{9}{10}$$

Now, X follows the binomial distribution with parameters n = 20, p = $\frac{1}{10}$, q = $\frac{9}{10}$.

$\therefore$ The required probability is

$$P(X > 2) = \sum_{k=3}^{20} {}^{20}C_k \left(\frac{1}{10}\right)^k \left(\frac{9}{10}\right)^{20-k}$$

$$= \sum_{k=3}^{20} {}^{20}C_k.\frac{9^{20-k}}{10^{20}}.$$

11. On an average, rain falls on 12 days in every 30 days, find the probability that, rain will fall on just 3 days of a given week.

A: Let X be the number of days rain falls in a week.

The probability that rain will fall on a day.

$$p = \frac{12}{30} = \frac{2}{5},\ \text{Hence } q = 1 - \frac{2}{5} = \frac{3}{5}.$$

Now, X follows the binomial distribution with parameters n = 7, $p = \frac{2}{5}$, $q = \frac{3}{5}$

$\therefore$ The required probability is

$$P(X = 3) = {}^{7}C_3\left(\frac{2}{5}\right)^3\left(\frac{3}{5}\right)^4 = 35.\frac{2^3.3^4}{5^7}$$

12. In a book of 450 pages, there are 400 typographical errors. Assuming that the number of errors per page follow the Poisson law, find the probability that a random sample of 5 pages will contain no typographical error.

A: Let the average number of errors per page in the book is $\lambda = \frac{400}{450} = \frac{8}{9}$.

The probability that a page contain 'r' errors is

$$P(X = r) = \frac{e^{-\lambda}\lambda^r}{r!}$$

The probability that a page contain no errors is

$$P(X = 0) = \frac{e^{-\lambda}\lambda^r}{r!} = e^{-8/9}.$$

$\therefore$ The required probability that a random sample of 5 pages will contain no error is

$$\left[P(X = 0)\right]^5 = \left[e^{-8/9}\right]^5.$$

13. Deficiency of red cells in the blood cells is determined by examining a specimen of blood under a microscope, Suppose a small volume contains on an average 20 red cells for normal persons. Using the poisson distribution, find the probability that a specimen of blood taken from a normal person will contain less than 15 red cells.

A: Here $\lambda = 20$.

Let $P(X = r)$ denote the probability that a specimen taken from a normal person will contain 'r' red cells.

$\therefore$ The required probability is

$$P(X<15)=\sum_{r=0}^{14}P(X=r)=\sum_{r=0}^{14}\frac{e^{-20}20^{r}}{r!}\ .$$

14. In a city, 10 accidents take place in a span of 50 days. Assuming that the number of accidents follows the Poisson distribution, find the probability that there will be 3 or more accidents in a day.

A: Here $\lambda = \frac{10}{50} = 0.2$

The required probability is

$$P(X\geq 3)=1-\left[P(X=0)+P(X=1)+P(X=2)\right]$$

$$=1-\left[\frac{e^{-0.2}(0.2)^{0}}{0!}+\frac{e^{-0.2}(0.2)^{1}}{1!}+\frac{e^{-0.2}(0.2)^{2}}{2!}\right]$$

$$=1-\left[\frac{1}{e^{0.2}}+\frac{1}{5e^{0.2}}+\frac{1}{50e^{0.2}}\right]=1-\frac{61}{50e^{0.2}}\ .$$

AIMS