# 2ND YEAR MATHEMATICS - IIA 70\% SYLLABUS WITH SOLUTIONS LONG ANSWERS TYPE QUESTIONS DE MOIVRES THEOREM Q.NO 21 

1 If $\cos \alpha+\cos \beta+\cos \gamma=0=\sin \alpha+\sin \beta+\sin \gamma$, then show that
i) $\cos 3 \alpha+\cos 3 \beta+\cos 3 \gamma=3 \cos (\alpha+\beta+\gamma)$
ii) $\sin 3 \alpha+\sin 3 \beta+\sin 3 \gamma=3 \sin (\alpha+\beta+\gamma)$

A: Given: $\cos \alpha+\cos \beta+\cos \gamma=0=\sin \alpha+\sin \beta+\sin \gamma$
Let $\mathrm{a}=\cos \alpha+\mathrm{i} \sin \alpha, \mathrm{b}=\cos \beta+\mathrm{i} \sin \beta$ and $\mathrm{c}=\cos \gamma+\mathrm{i} \sin \gamma$
$\therefore \mathrm{a}+\mathrm{b}+\mathrm{c}=(\cos \alpha+\cos \beta+\cos \gamma)+\mathrm{i}(\sin \alpha+\sin \beta+\sin \gamma)$

$$
=0+i(0)
$$

$a+b+c=0$
$a^{3}+b^{3}+c^{3}=3 a b c$
$(\cos \alpha+i \sin \alpha)^{3}+(\cos \beta+i \sin \beta)^{3}+(\cos \gamma+i \sin \gamma)^{3}=3(\operatorname{cis} \alpha)(\operatorname{cis} \beta)(\operatorname{cis} \gamma)$
By applying DeMoivre's Theorem we get
$\cos 3 \alpha+i \sin 3 \alpha+\cos 3 \beta+i \sin 3 \beta+\cos 3 \gamma+i \sin 3 \gamma=3 \operatorname{cis}(\alpha+\beta+\gamma)$
$(\cos 3 \alpha+\cos 3 \beta+\cos 3 \gamma)+i(\sin 3 \alpha+\sin 3 \beta+\sin 3 \gamma)=3[\cos (\alpha+\beta+\gamma)+i \sin (\alpha+\beta+\gamma)]$
Equating of the real and imaginary parts on bothsides, we get
(i) $\cos 3 \alpha+\cos 3 \beta+\cos 3 \gamma=3 \cos (\alpha+\beta+\gamma)$
\&
(ii) $\sin 3 \alpha+\sin 3 \beta+\sin 3 \gamma=3 \sin (\alpha+\beta+\gamma)$.
2. If $\cos \alpha+\cos \beta+\cos \gamma=0=\sin \alpha+\sin \beta+\sin \gamma$, show that $\cos ^{2} \alpha+\cos ^{2} \beta+\cos ^{2} \gamma=\frac{3}{2}=\sin ^{2} \alpha+\sin ^{2} \beta+\sin ^{2} \gamma$.

A: Let $a=\cos \alpha+i \sin \alpha, b=\cos \beta+i \sin \beta, c=\cos \gamma+i \sin \gamma$
$a+b+c=(\cos \alpha+\cos \beta+\cos \gamma)+i(\sin \alpha+\sin \beta+\sin \gamma)$

$$
\begin{aligned}
& =0+i .0 \\
& =0 .
\end{aligned}
$$

On squaring, we get $(a+b+c)^{2}=0$
$a^{2}+b^{2}+c^{2}+2(a b+b c+c a)=0$
$a^{2}+b^{2}+c^{2}=-2(a b+b c+c a)$
$\operatorname{cis} 2 \alpha+\operatorname{cis} 2 \beta+\operatorname{cis} 2 \gamma=-2 a b c\left(\frac{1}{c}+\frac{1}{a}+\frac{1}{b}\right)$
$=-2 a b c[\cos \gamma-i \sin \gamma+\cos \alpha-i \sin \alpha+\cos \beta-i \sin \beta]$
$=-2 a b c[\cos \alpha+\cos \beta+\cos \gamma)-\mathrm{i}(\sin \alpha+\sin \beta+\sin \gamma)]$
$=-2 a b c[0-i(0)]$
$=0$.
$\cos 2 \alpha+i \sin 2 \alpha+\cos 2 \beta+i \sin 2 \beta+\cos 2 \gamma+i \sin 2 \gamma=0$
$(\cos 2 \alpha+\cos 2 \beta+\cos 2 \gamma)+i(\sin 2 \alpha+\sin 2 \beta+\sin 2 \gamma)=0+i(0)$
Equating the real parts, we get
$\cos 2 \alpha+\cos 2 \beta+\cos 2 \gamma=0$
$2 \cos ^{2} \alpha-1+2 \cos ^{2} \beta-1+2 \cos ^{2} \gamma-1=0$
$2\left(\cos ^{2} \alpha+\cos ^{2} \beta+\cos ^{2} \gamma\right)=3$
$\therefore \cos ^{2} \alpha+\cos ^{2} \beta+\cos ^{2} \beta=\frac{3}{2}$
Also from (1) $\cos 2 \alpha+\cos 2 \beta+\cos 2 \gamma=0$

$$
\begin{aligned}
& 1-2 \sin ^{2} \alpha+1-2 \sin ^{2} \beta+1-2 \sin ^{2} \gamma=0 \\
& 3=2\left(\sin ^{2} \alpha+\sin ^{2} \beta+\sin ^{2} \gamma\right) \\
& \therefore \sin ^{2} \alpha+\sin ^{2} \beta+\sin ^{2} \gamma=\frac{3}{2} .
\end{aligned}
$$

3. If $n$ is an integer, Show that $(1+i)^{2 n}+(1-i)^{2 n}=2^{n+1} \cos \frac{n \pi}{2}$.

A: $1+i=\sqrt{2}\left(\frac{1}{\sqrt{2}}+i \cdot \frac{1}{\sqrt{2}}\right)=\sqrt{2}\left(\cos \frac{\pi}{4}+i \cdot \sin \frac{\pi}{4}\right)$
$1-\mathrm{i}=\sqrt{2}\left(\cos \frac{\pi}{4}-\mathrm{i} \cdot \sin \frac{\pi}{4}\right)$
Now $(1+i)^{2 n}+(1+i)^{2 n}=\left[\sqrt{2}\left(\cos \frac{\pi}{4}+i \cdot \sin \frac{\pi}{4}\right)\right]^{2 n}+\left[\sqrt{2}\left(\cos \frac{\pi}{4}-i \cdot \sin \frac{\pi}{4}\right)\right]^{2 n}$

$$
=(\sqrt{2})^{2 n}\left(\cos \frac{\pi}{4}+i \cdot \sin \frac{\pi}{4}\right)^{2 n}+(\sqrt{2})^{2 n}\left(\cos \frac{\pi}{4}-i \cdot \sin \frac{\pi}{4}\right)^{2 n}
$$

Using the De Moivre's Theorem

$$
\begin{aligned}
& =2^{n}\left(\cos 2 n \frac{\pi}{4}+i \cdot \sin 2 n \frac{\pi}{4}\right)+2^{n}\left(\cos 2 n \frac{\pi}{4}-i \cdot \sin 2 n \frac{\pi}{4}\right) \\
& =2^{n}\left(\cos \frac{n \pi}{2}+i \cdot \sin \frac{n \pi}{2}+\cos \frac{n \pi}{2}-i . \sin \frac{n \pi}{2}\right) \\
& =2^{n} \cdot 2 \cos \frac{n \pi}{2} \\
& =2^{n+1} \cos \frac{n \pi}{2} .
\end{aligned}
$$

4. If n is a positive integer, show that $(1+\mathrm{i})^{\mathrm{n}}+(1-\mathrm{i})^{\mathrm{n}}=2^{\frac{n+2}{2}} \cos \left(\frac{n \pi}{4}\right)$.

A: $1+\mathrm{i}=\sqrt{2}\left(\frac{1}{\sqrt{2}}+\mathrm{i} \cdot \frac{1}{\sqrt{2}}\right)=\sqrt{2}\left(\cos \frac{\pi}{4}+\mathrm{i} \sin \frac{\pi}{4}\right)$
$1-i=\sqrt{2}\left(\cos \frac{\pi}{4}-\mathrm{i} \sin \frac{\pi}{4}\right)$
$(1+i)^{n}+(1-i)^{n}=\left[\sqrt{2}\left(\cos \frac{\pi}{4}+i \sin \frac{\pi}{4}\right)\right]^{n}+\left[\sqrt{2}\left(\cos \frac{\pi}{4}-i \sin \frac{\pi}{4}\right)\right]^{n}$

$$
=\sqrt{2}^{n}\left(\cos \frac{n \pi}{4}+i \sin \frac{n \pi}{4}\right)+\sqrt{2}^{n}\left(\cos \frac{n \pi}{4}-i \sin \frac{n \pi}{4}\right)
$$

By applying DeMoivre's

$$
\begin{aligned}
& =2^{\frac{n}{2}}\left[\cos \frac{n \pi}{4}+i \sin \frac{n \pi \tau}{4}+\cos \frac{n \pi}{4}-i \sin \frac{n \pi \pi}{4}\right] \\
& =2^{\frac{n}{2}} \cdot 2 \cos \left(\frac{n \pi}{4}\right) \\
& =2^{\frac{n}{2}+1} \cdot \cos \left(\frac{n \pi}{4}\right) \\
& =2^{\frac{n+2}{2}} \cos \left(\frac{n \pi}{4}\right)
\end{aligned}
$$

5. If $\alpha, \beta$ are the roots of the equation $x^{2}-2 x+4=0$. Show that $\alpha^{n}+\beta^{n}=2^{n+1} \cos \left(\frac{n \pi}{3}\right)$.

A: Given equation is $x^{2}-2 x+4=0$

$$
\begin{aligned}
& x=\frac{-b \pm \sqrt{b^{2}-4 a c}}{2 a}=\frac{2 \pm \sqrt{4-16}}{2}=\frac{2 \pm \sqrt{12 i^{2}}}{2}=\frac{2 \pm 2 \sqrt{3} i}{2}=1 \pm \sqrt{3} i \\
& \text { Let } \alpha=1+\sqrt{3} i, \beta=1-\sqrt{3} i \\
& \text { Now } \alpha^{n}+\beta^{n}=(1+\sqrt{3} i)^{n}+(1-\sqrt{3} i)^{n} \\
& =\left[2\left(\frac{1}{2}+i \cdot \frac{\sqrt{3}}{2}\right)\right]^{n}+\left[2\left(\frac{1}{2}-i \cdot \frac{\sqrt{3}}{2}\right)\right]^{n} \\
& =\left[2\left(\cos \frac{\pi}{3}+i \cdot \sin \frac{\pi}{3}\right)\right]^{n}+\left[2\left(\cos \frac{\pi}{3}-i \cdot \sin \frac{\pi}{3}\right)\right]^{n} \\
& =2^{n}\left(\cos \frac{n \pi}{3}+i \cdot \sin \frac{n \pi}{3}\right)+2^{n}\left(\cos \frac{n \pi}{3}-i \cdot \sin \frac{n \pi}{3}\right) \\
& =2^{n}\left[\cos \frac{n \pi}{3}+i \cdot \sin \frac{n \pi}{3}+\cos \frac{n \pi}{3}-i \cdot \sin \frac{n \pi}{3}\right] \\
& =2^{n} \cdot 2 \cos \frac{n \pi}{3} \\
& =2^{n+1} \cos \frac{n \pi}{3} .
\end{aligned}
$$

6. Prove that $(1+\cos \theta+i \sin \theta)^{n}+(1+\cos \theta-i \sin \theta)^{n}=2^{n+1} \cos ^{n}\left(\frac{\theta}{2}\right) \cos \left(\frac{n \theta}{2}\right)$.

A: Now $(1+\cos \theta+i \sin \theta)^{n}+(1+\cos \theta-i \sin \theta)^{n}$

$$
\begin{aligned}
& =\left(2 \cos ^{2} \frac{\theta}{2}+i 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}\right)^{n}+\left(2 \cos ^{2} \frac{\theta}{2}-i 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}\right)^{n} \\
& =\left[2 \cos \frac{\theta}{2}\left(\cos \frac{\theta}{2}+i \sin \frac{\theta}{2}\right)\right]^{n}+\left[2 \cos \frac{\theta}{2}\left(\cos \frac{\theta}{2}-i \sin \frac{\theta}{2}\right)\right]^{n}
\end{aligned}
$$

By applying DeMoivre's theorem

$$
\begin{aligned}
& =2^{n} \cos ^{n}\left(\frac{\theta}{2}\right)\left[\cos \frac{n \theta}{2}+i \sin \frac{n \theta}{2}\right]+2^{n} \cos ^{n}\left(\frac{\theta}{2}\right)\left[\cos \frac{n \theta}{2}-i \sin \frac{n \theta}{2}\right] \\
& =2^{n} \cos ^{n}\left(\frac{\theta}{2}\right)\left[\cos \frac{n \theta}{2}+i \sin \frac{n \theta}{2}+\cos \frac{n \theta}{2}-i \sin \frac{n \theta}{2}\right] \\
& =2^{n} \cos ^{n}\left(\frac{\theta}{2}\right) \cdot 2 \cos \left(\frac{n \theta}{2}\right) \\
& =2^{n+1} \cos ^{n}\left(\frac{\theta}{2}\right) \cos \left(\frac{n \theta}{2}\right) .
\end{aligned}
$$

7. If $n$ is an integer and $z=\operatorname{cis} \theta$, then show that $\frac{Z^{2 n}-1}{Z^{2 n}+1}=i \tan n \theta$.

A: Given: $Z=\operatorname{cis} \theta=\cos \theta+i \sin \theta$.

$$
\text { Now } \begin{aligned}
\frac{z^{2 n}-1}{z^{2 n}+1} & =\frac{(\cos \theta+i \sin \theta)^{2 n}-1}{(\cos \theta+i \sin \theta)^{2 n}+1} \\
& =\frac{\cos 2 n \theta+i \sin 2 n \theta-1}{\cos 2 n \theta+i \sin 2 n \theta+1}
\end{aligned}
$$

By applying DeMoivre's Theorem

$$
\begin{aligned}
& =\frac{i \sin 2 n \theta-(1-\cos 2 n \theta)}{i \sin 2 n \theta+(1+\cos 2 n \theta)} \\
& =\frac{i 2 \sin n \theta \cos n \theta-2 \sin ^{2} n \theta}{i 2 \sin n \theta \cos n \theta+2 \cos ^{2} n \theta} \\
& =\frac{i 2 \sin n \theta \cos n \theta+i^{2} 2 \sin ^{2} n \theta}{i 2 \sin n \theta \cos n \theta+2 \cos ^{2} n \theta} \\
& =\frac{2 i \sin n \theta[\cos n \theta+i \sin n \theta]}{2 \cos n \theta[\cos n \theta+i \sin n \theta]} \\
& =i \tan n \theta .
\end{aligned}
$$

8. State and prove De Moivre's Theorem for an integral index.

A: De Moivre's Theorem for an integral index:
For any real number $\theta$ and any integer $n$,
$(\cos \theta+i \sin \theta)^{n}=\cos n \theta+i \sin n \theta$.
Part1: Let n be a positive integer. We prove the theorem by using the principle of mathematical induction.
Let $P(n)$ be the statement: $(\cos \theta+i \sin \theta)^{n}=\cos n \theta+i \sin n \theta$

$$
\text { If } \begin{aligned}
\mathrm{n}=1, \mathrm{LHS} & =(\cos \theta+i \sin \theta)^{1} \\
& =\cos \theta+i \sin \theta \\
\mathrm{RHS} & =\cos 1 \theta+i \sin 1 \theta \\
& =\cos \theta+i \sin \theta
\end{aligned}
$$

$\therefore$ LHS $=$ RHS
Thus $P(1)$ is TRUE.
Assume that $P(k)$ is true.
$(\cos \theta+i \sin \theta)^{\mathrm{k}}=\operatorname{cosk} \theta+\mathrm{i} \sin k \theta$
Multiplying bothsides by $\cos \theta+i \sin \theta$, we get

$$
\begin{aligned}
(\cos \theta+i \sin \theta)^{k+1} & =(\cos k \theta+i \sin k \theta)(\cos \theta+i \sin \theta) \\
& =\cos k \theta \cos \theta+i \sin k \theta \cos \theta+i \cos k \theta \sin \theta+i^{2} \sin k \theta \sin \theta \\
& =\cos (k \theta+\theta)+i \sin (k \theta+\theta) \\
& =\cos (k+1) \theta+i \sin (k+1) \theta
\end{aligned}
$$

$\therefore P(k+1)$ is TRUE
By induction, $P(n)$ is true for all positive integers $n$. i.e. $(\cos \theta+i \sin \theta)^{n}=\operatorname{cosn} \theta+i \sin n \theta$ for all $n \in z^{+}$.
Part 2: If $n=0, L H S=(\cos \theta+i \sin \theta)^{0}=1$

$$
\begin{aligned}
\mathrm{RHS} & =\cos 0 \theta+\mathrm{i} \sin 0 \theta=1 \\
\therefore \quad & \text { LHS }
\end{aligned}=\mathrm{RHS}
$$

If $\mathrm{n}=0$, the statement is TRUE.
Part 3: Let $n$ be a negative integer and $n=-m$, where $m \in z^{+}$
So for $m$, part 1 is applicable.
Now $(\cos \theta+i \sin \theta)^{\mathrm{n}}=(\cos \theta+i \sin \theta)^{-\mathrm{m}}$

$$
\begin{aligned}
& =\frac{1}{(\cos \theta+i \sin \theta)^{m}} \\
& =\frac{1}{\cos m \theta+i \sin m \theta} \text { from Part } 1 \\
& =\cos m \theta-i \sin m \theta \\
& =\cos (-m) \theta+i \sin (-m) \theta \\
& =\cos n \theta+i \sin n \theta
\end{aligned}
$$

## QUADRATIC EXPRESSIONS Q.NO 22

1. If the roots of $a x^{2}+b x+c=0$ are imaginary, show that for all $x \in R$, ' $a x^{2}+b x+c$ ' and ' $a$ ' have the same sign.

A: Given that the roots of $a x^{2}+b x+c=0$ are imaginary.
$\Rightarrow b^{2}-4 \mathrm{ac}<0$
$4 a c-b^{2}>0$ $\qquad$
Consider $\frac{a x^{2}+b x+c}{a}=x^{2}+\frac{b}{a} x+\frac{c}{a}$

$$
=x^{2}+2 x \cdot \frac{b}{2 a}+\left(\frac{b}{2 a}\right)^{2}+\frac{c}{a}-\left(\frac{b}{2 a}\right)^{2}
$$

$$
=\left(x+\frac{b}{2 a}\right)^{2}+\frac{4 a c-b^{2}}{4 a^{2}} \quad \text { from (1) }
$$

$$
\geq 0 \quad>0
$$

$\therefore$ For all $\mathrm{x} \in \mathrm{R}$, 'ax${ }^{2}+\mathrm{bx}+\mathrm{c}$ ' and ' $a$ ' have the same sign.
2. Let $\alpha, \beta$ be the real roots of $\mathbf{a x}+\mathbf{b x}+\mathbf{c}=0$ where $\alpha<\beta$, then prove the following.
i) for $\alpha<x<\beta$; ' $a x^{2}+b x+c$ ' and 'a' have opposite signs.
ii) for $x<\alpha$ or $x>\beta$; ' $a x^{2}+b x+c$ ' and ' $a$ ' have the same sign.

A: Given that $\alpha, \beta$ are the real roots of $\mathrm{ax}^{2}+\mathrm{bx}+\mathrm{c}=0$ with $\alpha<\beta$.

$$
\begin{align*}
& a x^{2}+b x+c=a(x-\alpha)(x-\beta) \quad \div a \\
& \frac{a x^{2}+b x+c}{a}=(x-\alpha)(x-\beta)----(1)
\end{align*}
$$

i) Suppose $x \in R$ and $\alpha<x<\beta$

Now $x-\alpha>0$ and $x-\beta<0$

$$
\begin{align*}
& (x-\alpha)(x-\beta)<0 \\
& \frac{a x^{2}+b x+c}{a}<0 \quad \text { from } \tag{1}
\end{align*}
$$

Thus for $x \in R$ and $\alpha<x<\beta$, then ' $a x^{2}+b x+c$ ' and ' $a$ ' have opposite signs.
ii) Suppose $x \in R$ and $x<\alpha$

$$
\text { Now } x-\alpha<0 \text { and } x-\beta<0
$$

$$
\begin{align*}
& (x-\alpha)(x-\beta)>0 \\
& \frac{a x^{2}+b x+c}{a}>0 \quad \text { from } \tag{1}
\end{align*}
$$

Suppose $x \in R$ and $x>\beta$
Now $x-\alpha>0$ and $x-\beta>0$

$$
\begin{array}{r}
(x-\alpha)(x-\beta)>0 \\
\frac{a x^{2}+b x+c}{a}>0
\end{array}
$$

Thus for $x \in R, x<\alpha$ or $x>\beta$, then ' $a x^{2}+b x+c$ ' and ' $a$ ' have the same sign.
3. If the expression $\frac{x-p}{x^{2}-3 x+2}$ takes all real values for $x \in R$, then find the bounds for $p$.

A: Let $\frac{x-p}{x^{2}-3 x+2}=y$

$$
x-p=y x^{2}-3 y x+2 y
$$

$$
y x^{2}-(3 y+1) x+(2 y+p)=0
$$

$$
b^{2}-4 a c \geq 0
$$

$\{-(3 y+1)\}^{2}-4(y)(2 y+p) \geq 0$
$9 y^{2}+6 y+1-8 y^{2}-4 y p \geq 0$
$y^{2}-2(2 p-3) y+1 \geq 0$
So, the roots of $y^{2}-2(2 p-3) y+1=0$ are imaginary or real and equal $\Rightarrow b^{2}-4 a c \leq 0$
$\{-2(2 p-3)\}^{2}-4(1)(1) \leq 0$
$\div 4$
$4 p^{2}-12 p+9-1 \leq 0$
$4 p^{2}-12 p+8 \leq 0 \quad \div 4$

$$
p^{2}-3 p+2 \leq 0
$$

$$
(p-1)(p-2) \leq 0
$$

$\therefore p \in(1,2)$.
4. Solve $4^{\mathrm{x}-1}-3 \cdot 2^{\mathrm{x}-1}+2=0$.
A. Given $4^{x-1}-3 \cdot 2^{x-1}+2=0$

$$
\Rightarrow \frac{4^{x}}{4}-\frac{3.2^{x}}{2}+2=0
$$

$$
\text { Let } 2^{\mathrm{x}}=\mathrm{t}
$$

$\Rightarrow 4^{\mathrm{x}}=2^{2 \mathrm{x}}=\mathrm{t}^{2}$
The above equation becomes
$\Rightarrow \frac{t^{2}}{4}-\frac{3 . t}{2}+2=0$
$\Rightarrow t^{2}-6 t+8=0$
$\Rightarrow \mathrm{t}^{2}-4 \mathrm{t}-2 \mathrm{t}+8=0$
$\Rightarrow t(t-4)-2(t-4)=0$
$(t-2)(t-4)=0$

$$
\mathrm{t}=2 \text { (or) } 4
$$

Case (i): If $t=2$
$2^{x}=2$
Case - (ii): If $t=4$
$\therefore x=1$
$2^{x}=2^{2}$
$\therefore \mathrm{x}=2$
$\therefore \mathrm{x}=1,2$
$(H / W)$ Solve the equation $3^{1+x}+3^{1-x}=10$. (ii)Solve $7^{1+x}+7^{1-x}=50$ for real $x$.
5.Solve $2 x^{4}+x^{3}-11 x^{2}+x+2=0$
A. Given equation $2 x^{4}+x^{3}-11 x^{2}+x+2=0 \div$ by $x^{2}$

$$
\begin{align*}
& \Rightarrow 2 x^{2}+x-11+\frac{1}{x}+\frac{2}{x^{2}}=0 \\
& \Rightarrow 2\left(x^{2}+\frac{1}{x^{2}}\right)+\left(x+\frac{1}{x}\right)-11=0-\cdots(1) \quad \text { Let } x+\frac{1}{x}=z \Rightarrow x^{2}+\frac{1}{x^{2}}=z^{2}-2 \tag{1}
\end{align*}
$$

Sub in (1)

$$
\begin{aligned}
\Rightarrow 2\left(z^{2}-2\right)+z-11 & =0 \\
2 z^{2}-4+z-11 & =0 \\
\Rightarrow 2 z^{2}+6 z-5 z-15 & =0 \\
2 z(z+3)-5(z+3) & =0 \\
(z+3)(2 z-5) & =0 \quad z=-3,5 / 2
\end{aligned}
$$

$$
\text { Case - i: If } z=-3
$$

$$
\text { Case - ii: If } z=5 / 2
$$

$$
x+\frac{1}{x}=-3
$$

$$
x+\frac{1}{x}=\frac{5}{2}
$$

$x^{2}+3 x+1=0$

$$
2 x^{2}-5 x+2=0
$$

$x=\frac{-3 \pm \sqrt{9-4}}{2}$

$$
2 x^{2}-4 x-x+2=0
$$

$$
x=\frac{-3 \pm \sqrt{5}}{2}
$$

$$
\begin{aligned}
& 2 x(x-2)-1(x-2)=0 \\
& (x-2)(2 x-1)=0
\end{aligned}
$$

$$
x=2,1 / 2 \quad \therefore \text { The roots are }\left\{\frac{-3 \pm \sqrt{5}}{2}, 2, \frac{1}{2}\right\}
$$

6. Solve $\sqrt{\frac{x}{x-3}}+\sqrt{\frac{x-3}{x}}=\frac{5}{2}$ when $x \neq 0, x \neq 3$.
A. Let $\sqrt{\frac{x}{x-3}}=z$

The above equation becomes $\quad z+\frac{1}{z}=\frac{5}{2}$

$$
\begin{array}{rlrl}
\frac{z^{2}+1}{z} & =\frac{5}{2} & \\
2 z^{2}-5 z+2 & =0 & \\
2 z^{2}-4 z-z+2 & =0 \\
2 z(z-2)-1(z-2)=0 \\
(z-2)(2 z-1) & =0 & z=2,1 / 2 \\
\text { Case -i: If } z & =2 & \text { Case }-\mathrm{ii}: \text { If } z & =1 / 2 \\
\sqrt{\frac{x}{x-3}} & =2 & \sqrt{\frac{x}{x-3}} & =\frac{1}{2} \\
\frac{x}{x-3} & =4 & \frac{x}{x-3} & =\frac{1}{4} \\
x & =4 x-12 & 4 x & =x-3 \\
12 & =3 x & 3 x & =-3 \\
x & =4 & x & =-1
\end{array}
$$

$\therefore$ The roots are $\{-1,4\}$.
$(H / W)$ Solve $\sqrt{\frac{3 x}{x+1}}+\sqrt{\frac{x+1}{3 x}}=2$ when $x \neq 0, x \neq-1$.
7. Suppose that $a, b, c \in R, a \neq 0$ and $f(x)=a x^{2}+b x+c$
i) If $a>0$, then show that $f$ has minimum at $x=\frac{-b}{2 a}$ and the minimum value of $f$ is $\frac{4 a c-b^{2}}{4 a}$.
ii) If $a<0$, then show that $f$ has maximum at $x=\frac{-b}{2 a}$ and the maximum value of $f$ is $\frac{4 a c-b^{2}}{4 a}$.

A: Given quadratic function is $f(x)=a x^{2}+b x+c$.
Differentiating w.r.t. $x$ successively for two times,
$f^{\prime}(x)=2 a x+b$
$f^{\prime \prime}(x)=2 a$
For $f(x)$ to be maximum or minimum, $f^{\prime}(x)=0 \Rightarrow 2 a x+b=0 \Rightarrow x=\frac{-b}{2 a}$
If $a>0$, then $f^{\prime \prime}(x)>0$ and hence $f$ has minimum at $x=\frac{-b}{2 a}$ and the minimum value,

$$
f=a\left(\frac{-b}{2 a}\right)^{2}+b\left(\frac{-b}{2 a}\right)+c=\frac{b^{2}-2 b^{2}+4 a c}{4 a}=\frac{4 a c-b^{2}}{4 a}
$$

If $a<0$, then $f^{\prime \prime}(x)<0$ and hence $f$ has maximum at $x=\frac{-b}{2 a}$ and the maximum value of $f=a\left(\frac{-b}{2 a}\right)^{2}+b\left(\frac{-b}{2 a}\right)+c=\frac{4 a c-b^{2}}{4 a}$.

## THEORY OF EQUATIONS Q.NO 23

1. Solve the equation $x^{4}-2 x^{3}+4 x^{2}+6 x-21=0$, the sum of two roots being zero.

A: Sum of two roots of $x^{4}-2 x^{3}+4 x^{2}+6 x-21=0$ is zero. Let the roots $\alpha,-\alpha, \gamma, \delta$

$$
\begin{aligned}
& \mathrm{s}_{1}=\alpha-\alpha+\gamma+\delta=2 \\
& \gamma+\delta=2 \\
& \mathrm{~s}_{3}=\alpha \beta \gamma+\alpha \beta \delta+\alpha \gamma \delta+\beta \gamma \delta=\frac{-d}{a} \\
&-\alpha^{2} \gamma-\alpha^{2} \delta+\alpha \gamma \delta-\alpha \gamma \delta=-6 \\
& \alpha^{2}(\gamma+\delta)=6 \\
& \alpha^{2}=3 \\
& \alpha= \pm \sqrt{3} .
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{s}_{4}=(\alpha)(-\alpha)(\gamma)(\delta) & =\frac{\mathrm{e}}{\mathrm{a}} \\
-3 \gamma \delta & =-21 \\
\gamma \delta & =7 .
\end{aligned}
$$

The quadratic equation whose roots are $\gamma, \delta$ is $\mathrm{x}^{2}-(\gamma+\delta) \mathrm{x}+\gamma \delta=0$

$$
\begin{aligned}
x^{2}-2 x+7 & =0 \\
x & =\frac{-b \pm \sqrt{b^{2}-4 a c}}{2 a} \\
& =\frac{2 \pm \sqrt{(-2)^{2}-4(1)(7)}}{2(1)} \\
& =\frac{2 \pm \sqrt{-24}}{2} \\
& =\frac{2 \pm 2 \sqrt{6} i}{2} \\
& =1 \pm \sqrt{6} i
\end{aligned}
$$

Hence, the required roots of the given equation are $\sqrt{3},-\sqrt{3}, 1+\sqrt{6} i, 1-\sqrt{6} i$.
(H/W) Solve $8 x^{4}-2 x^{3}-27 x^{2}+6 x+9=0$ give that two roots have same absolute value, but are opposite in signs.
2. Solve the equation $x^{4}-5 x^{3}+5 x^{2}+5 x-6=0$, the product of two roots being 3 .

A: Product of two roots of $x^{4}-5 x^{3}+5 x^{2}+5 x-6=0$ is 3 .
Let $\alpha, \beta, \gamma, \delta$ be the roots with $\alpha \beta=3$.

$$
\begin{align*}
& s_{4}=\alpha \beta \gamma \delta=\frac{e}{a} \\
& 3 \gamma \delta=-6 \text { then } \gamma \delta=-2-\cdots---(1  \tag{1}\\
& s_{1}=\alpha+\beta+\gamma+\delta=\frac{-b}{a}=5-\cdots--(2) \\
& s_{3}=\alpha \beta \gamma+\alpha \beta \delta+\alpha \gamma \delta+\beta \gamma \delta=\frac{-d}{a} \\
& \alpha \beta(\gamma+\delta)+\gamma \delta(\alpha+\beta)=-5 \\
& 3(\gamma+\delta)-2(\alpha+\beta)=-5 \\
& 3(\alpha+\beta+\gamma+\delta)-5(\alpha+\beta)=-5 \\
& 3(5)-5(\alpha+\beta)=-5 \\
& 15+5=5(\alpha+\beta)
\end{align*}
$$

since $\alpha+\beta=4$ then $\gamma+\delta=5-4=1$.

$$
\alpha+\beta=4, \alpha \beta=3 \quad \gamma+\delta=1, \gamma \delta=-2
$$

$x^{2}-(\alpha+\beta) x+\alpha \beta=0 \quad x^{2}-(\gamma+\delta) x+\gamma \delta=0$
$x^{2}-4 x+3=0$.
$x^{2}-x-2=0$
$(x-1)(x-3)=0$
$x^{2}-2 x+x-2=0$
$\alpha=1, \beta=3$
$x(x-2)+1(x-2)=0$
$(x+1)(x-2)=0$
$\gamma=-1, \delta=2$ Hence the required roots of the given biquadratic equation are 1, 3, -1, 2 .
(H/W) Solve the equation $x^{4}+x^{3}-16 x^{2}-4 x+48=0$, given that the product of two roots is 6 .
3. Solve the equation $x^{4}+4 x^{3}-2 x^{2}-12 x+9=0$, if it has two pairs of equal roots.

A: Given that $x^{4}+4 x^{3}-2 x^{2}-12 x+9=0$ has two pairs of equal roots.
Let the roots be $\alpha, \alpha, \beta, \beta$.
$\mathrm{s}_{1}=\alpha+\alpha+\beta+\beta=-4$

$$
\begin{array}{r}
2(\alpha+\beta)=-4 \\
\alpha+\beta=-2
\end{array}
$$

$$
\begin{aligned}
s_{3}=\alpha^{2} \beta+\alpha^{2} \beta+\alpha \beta^{2}+\alpha \beta^{2} & =12 \\
2 \alpha \beta(\alpha+\beta) & =12 \\
\alpha \beta(\alpha+\beta) & =6 \\
\alpha \beta(-2) & =6 \\
\alpha \beta & =-3
\end{aligned}
$$

The quadratic equation whose roots are $\alpha, \beta$ is $x^{2}-(\alpha+\beta) x+\alpha \beta=0$

$$
\begin{array}{r}
x^{2}+2 x-3=0 \\
x^{2}+3 x-x-3=0 \\
x(x+3)-1(x+3)=0 \\
(x+3)(x-1)=0
\end{array}
$$

Hence the required roots of the given biqudratic equation are $-3,-3,1,1$.
4. Find the roots of $x^{4}-16 x^{3}+86 x^{2}-176 x+105=0$.

A: Given equation is $f(x)=x^{4}-16 x^{3}+86 x^{2}-176 x+105=0$
Now $f(1)=1-16+86-176+105=192-192=0$.
So, 1 is a root of $f(x)=0$.
By Synthetic division,

|  | 1 | -16 | 86 | -176 | 105 |
| ---: | ---: | ---: | ---: | ---: | ---: |
| 1 | 0 | 1 | -15 | 71 | -105 |
|  | 1 | -15 | 71 | -105 | 0 |

Now, $(x-1)\left(x^{3}-15 x^{2}+71 x-105\right)=0$
Let $g(x)=x^{3}-15 x^{2}+71 x-105$

$$
g(1) \neq 0, g(2) \neq 0, g(3)=0 .
$$

So, 3 is a root of $g(x)=0$.

3 | 1 | -15 | 71 | -105 |  |
| ---: | ---: | ---: | ---: | ---: |
| 3 | 0 | 3 | -36 | 105 |
|  | 1 | -12 | 35 | 0 |
|  |  |  |  |  |

Now $f(x)=0$ can be written as

$$
\begin{aligned}
(x-1)(x-3)\left(x^{2}-12 x+35\right) & =0 \\
x^{2}-12 x+35 & =0 \\
(x-5)(x-7) & =0 \\
x & =5,7
\end{aligned}
$$

Hence, the required roots of the given biquadratic equation are 1, 3, 5, 7 .
5. Find the multiple roots of the equation $x^{5}-3 x^{4}-5 x^{3}+27 x^{2}-32 x+12=0$.

A: Given equation is $f(x)=x^{5}-3 x^{4}-5 x^{3}+27 x^{2}-32 x+12=0$
Differentiating $f(x)$ w.r.t. $x$,
$f^{\prime}(x)=5 x^{4}-12 x^{3}-15 x^{2}+54 x-32$
$f^{\prime}(1)=5-12-15+54-32=59-59=0$.
$f(1)=1-3-5+27-32+12=40-40=0$.
Since $f^{\prime}(1)=0, f(1)=0$, so 1 is a multiple root for the given equation.
By Synthetic division

|  | 1 | -3 | -5 | 27 | -32 | 12 |
| :--- | ---: | ---: | ---: | ---: | ---: | ---: |
| 1 | 0 | 1 | -2 | -7 | 20 | -12 |
|  | 1 | -2 | -7 | 20 | -12 | 0 |
| 1 | 0 | 1 | -1 | -8 | 12 |  |
|  | 1 | -1 | -8 | 12 | 0 |  |

Now the given equation can be written as

$$
(x-1)(x-1)\left(x^{3}-x^{2}-8 x+12\right)=0
$$

Let $g(x)=x^{3}-x^{2}-8 x+12$
differentiating w.r.t. $x$,

$$
g^{\prime}(x)=3 x^{2}-2 x-8
$$

Now $g^{\prime}(1) \neq 0$

$$
\mathrm{g}^{\prime}(2)=12-4-8=0
$$

Also $g(2)=8-4-16+12=0$.
Since $g^{\prime}(2)=0, g(2)=0$, so 2 is a multiple root of cubic equation $x^{3}-x^{2}-8 x+12=0$

$$
\begin{gathered}
s_{1}=2+2+\gamma=\frac{-b}{a}=1 \\
\gamma=1-4=-3
\end{gathered}
$$

Hence the required roots of the given $5^{\text {th }}$ degree equation are 1, 1, 2, 2, -3 .
$(H / W)$ Find the multiple roots of the equation $8 x^{3}-20 x^{2}+6 x+9=0$.
6. Show that $x^{5}-5 x^{3}+5 x^{2}-1=0$ has three equal roots and find that root.
A. Let $f(x)=x^{5}-5 x^{3}+5 x^{2}-1$

Differentiating with respect to $x$ for two times we get
$f^{\prime}(x)=5 x^{4}-15 x^{2}+10 x$
$f^{\prime \prime}(x)=20 x^{3}-30 x+10$
$f^{\prime \prime}(1)=20-30+10=0$
$f(1)=1-5+5-1=0$
Since $f^{\prime \prime}(1)=0$ and $f(1)=0$, thus $f(x)=0$ has three equal roots and that root is 1 .
7.If $\boldsymbol{\alpha}, \boldsymbol{\beta}, \gamma$ are the roots of the $\mathbf{x}^{3}-\mathbf{6} \mathbf{x}^{2}+11 \mathrm{x}-\mathbf{6}=\mathbf{0}$ then find the equation whose roots are $\boldsymbol{\alpha}^{2}+\boldsymbol{\beta}^{2}, \boldsymbol{\beta}^{2}+\gamma^{2}, \gamma^{2}+\boldsymbol{\alpha}^{2}$.
$A$ : Let $f(x)=x^{3}-6 x^{2}+11 x-6=0$
If $x=1$ then $f(1)=1-6+11-6=12-12=0$
$(x-1)$ is a factor of $f(x)$ then

$$
\begin{array}{r}
x^{2}-5 x-6=0 \\
x^{2}-3 x-2 x-6=0 \\
(x-2)(x-3)=0 \\
(x-2)=0 ;(x-3)=0
\end{array}
$$

therefore $x^{3}-6 x^{2}+11 x-6=0$ has factors as $(x-1)(x-2)(x-3)=0 \Rightarrow x=1,2,3$
Since $\alpha, \beta, \gamma$ are the roots of the equation $f(x)=0$ then $\alpha=1, \beta=2, \gamma=3$
Now, $\alpha^{2}+\beta^{2}=1^{2}+2^{2}=5$

$$
\begin{aligned}
& \beta^{2}+\gamma^{2}=2^{2}+3^{2}=13 \\
& \gamma^{2}+\alpha^{2}=3^{2}+1^{2}=10
\end{aligned}
$$

Then equation having roots $5,13,10$ is $(x-5)(x-13)(x-10)=0$

$$
\begin{array}{r}
\left(x^{2}-18 x+65\right)(x-10)=0 \\
x^{3}-28 x^{2}+245 x-650=0
\end{array}
$$

## THEORY OF EQUATIONS Q.NO 24

## 1. Solve $x^{4}-10 x^{3}+26 x^{2}-10 x+1=0$.

A : Given equation $x^{4}-10 x^{3}+26 x^{2}-10 x+1=0 \operatorname{div}$ by $x^{2}$

$$
\begin{gathered}
x^{2}-10 x+26-\frac{10}{x}+\frac{1}{x^{2}}=0 \\
\left(x^{2}+\frac{1}{x^{2}}\right)-10\left(x+\frac{1}{x}\right)+26=0
\end{gathered}
$$

If $x+\frac{1}{x}=z \Rightarrow x^{2}+\frac{1}{x^{2}}=z^{2}-2$ then

$$
\begin{array}{r}
\left(z^{2}-2\right)-10 z+26=0 \\
z^{2}-10 z+24=0 \\
z^{2}-6 z-4 z+24=0 \\
(z-4)(z-6)=0
\end{array}
$$

If $z=4$ then $x+\frac{1}{x}=4 \Rightarrow x^{2}-4 x+1=0$

$$
x=\frac{-(-4) \pm \sqrt{16-4(1)}}{2(1)}=\frac{4 \pm \sqrt{12}}{2}=2 \pm \sqrt{3}
$$

If $z=6$ then $x+\frac{1}{x}=6 \Rightarrow x^{2}-6 x+1=0$

$$
x=\frac{-(-6) \pm \sqrt{36-4(1)(1)}}{2(1)}=\frac{6 \pm \sqrt{32}}{2}=3 \pm 2 \sqrt{2}
$$

Hence the roots are $2 \pm \sqrt{3}$ and $3 \pm 2 \sqrt{2}$
$(H / W)$ Solve $6 x^{4}-35 x^{3}+62 x^{2}-35 x+6=0$.
2. Solve $2 x^{5}+x^{4}-12 x^{3}-12 x^{2}+x+2=0$.

A: Given equation is $2 x^{5}+x^{4}-12 x^{3}-12 x^{2}+x+2=0$.
It is a reciprocal equation of first class and odd degree. So -1 is a root of it.
By Synthetic division,

|  | 2 | 1 | -12 | -12 | 1 | 2 |
| ---: | ---: | ---: | ---: | ---: | ---: | ---: |
| -1 | 0 | -2 | 1 | 11 | 1 | -2 |
| 2 | -1 | -11 | -1 | 2 | 0 |  |

Now the given equation can be written as
$(x+1)\left(2 x^{4}-x^{3}-11 x^{2}-x+2\right)=0$.
Consider $2 x^{4}-x^{3}-11 x^{2}-x+2=0 . \quad \div x^{2}$

$$
2 x^{2}-x-11-\frac{1}{x}+\frac{2}{x^{2}}=0
$$

$2\left(x^{2}+\frac{1}{x^{2}}\right)-\left(x+\frac{1}{x}\right)-11=0$.
Put $x+\frac{1}{x}=z \Rightarrow x^{2}+\frac{1}{x^{2}}=z^{2}-2$.
Now the above equation becomes,

$$
\begin{array}{cc}
2\left(z^{2}-2\right)-z-11=0 \\
2 z^{2}-z-15=0 \\
2 z^{2}-6 z+5 z-15=0 . \\
2 z(z-3)+5(z-3)=0 . \\
z-3=0 & 2 z+5=0 \\
x+\frac{1}{x}-3=0 . & 2\left(x+\frac{1}{x}\right)+5=0 \\
x^{2}-3 x+1=0 . & 2 x^{2}+5 x+2=0 \\
x=\frac{3 \pm \sqrt{9-4}}{2} & 2 x^{2}+4 x+x+2=0 \\
=\frac{3 \pm \sqrt{5}}{2} & 2 x(x+2)+1(x+2)=0 \\
& (x+2)(2 x+1)=0 \\
& x=\frac{-1}{2},-2
\end{array}
$$

Hence the roots of the given $5^{\text {th }}$ degree equation are $-1, \frac{3 \pm \sqrt{5}}{2}, \frac{-1}{2},-2$.
3. Solve the equation $x^{5}-5 x^{4}+9 x^{3}-9 x^{2}+5 x-1=0$.

A: Given equation $x^{5}-5 x^{4}+9 x^{3}-9 x^{2}+5 x-1=0$. This is a reciprocal equation of $2^{\text {nd }}$ class and odd degree.
So, 1 is a root of this equation.
Therefore ' $x-1$ ' is a factor of it
By synthetic division.

| 1 | 1 | -5 | 9 | -9 | 5 | -1 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | 0 | 1 | -4 | 5 | -4 | 1 |
|  | 1 | -4 | 5 | -4 | 1 | $\boxed{0}$ |

Now the given equation can be written as $(x-1)\left(x^{4}-4 x^{3}+5 x^{2}-4 x+1\right)=0$
Consider $x^{4}-4 x^{3}+5 x^{2}-4 x+1=0$ div by $\mathrm{x}^{2}$

$$
\begin{aligned}
x^{2}-4 x+5-\frac{4}{x}+\frac{1}{x^{2}} & =0 \\
\left(x^{2}+\frac{1}{x^{2}}\right)-4\left(x+\frac{1}{x}\right)+5 & =0
\end{aligned}
$$

Put $x+\frac{1}{x}=z \Rightarrow x^{2}+\frac{1}{x^{2}}=z^{2}-2$
Now the above equation becomes

$$
\begin{array}{r}
z^{2}-2-4 z+5=0 \\
z^{2}-4 z+3=0 \\
(z-3)(z-1)=0
\end{array}
$$

$$
z=1 \text { or } 3
$$

Case: 1. If $z=1$

$$
\text { Case: } 2 \text { If } z=3
$$

$$
\begin{array}{cc}
x+\frac{1}{x}=1 & x+\frac{1}{x}=3 \\
\mathrm{x}^{2}-\mathrm{x}+1=0 & x^{2}-3 x+1=0 \\
x=\frac{1 \pm \sqrt{1-4}}{2} & x=\frac{3 \pm \sqrt{9-4}}{2} \\
= & =\frac{1 \pm \sqrt{-3}}{2} \\
& =\frac{1 \pm i \sqrt{3}}{2}
\end{array}
$$

Hence the roots of the given equation are $1, \frac{1 \pm i \sqrt{3}}{2}, \frac{3 \pm \sqrt{5}}{2}$.
4. Solve the equation $6 x^{6}-25 x^{5}+31 x^{4}-31 x^{2}+25 x-6=0$.

A: This is a reciprocal equation of $2^{\text {nd }}$ class and even degree. So $1,-1$ are the roots of it.
By Synthetic division,

|  | 6 | -25 | 31 | 0 | -31 | 25 | -6 |
| ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: |
| 1 | 0 | 6 | -19 | 12 | 12 | -19 | 6 |
|  | 6 | -19 | 12 | 12 | -19 | 6 | 0 |
| -1 | 0 | -6 | 25 | -37 | 25 | -6 |  |
|  | 6 | -25 | 37 | -25 | 6 | 0 |  |

Now the given equation can be written as
$(x-1)(x+1)\left(6 x^{4}-25 x^{3}+37 x^{2}-25 x+6\right)=0$.
Consider $6 x^{4}-25 x^{3}+37 x^{2}-25 x+6=0 \div x^{2}$

$$
\begin{aligned}
\Rightarrow & 6 x^{2}-25 x+37-\frac{25}{x}+\frac{6}{x^{2}}=0 \\
& 6\left(x^{2}+\frac{1}{x^{2}}\right)-25\left(x+\frac{1}{x}\right)+37=0
\end{aligned}
$$

Put $x+\frac{1}{x}=z \Rightarrow x^{2}+\frac{1}{x^{2}}=z^{2}-2$.
Now the above equation becomes, $6\left(z^{2}-2\right)-25 z+37=0$.
$\Rightarrow 6 z^{2}-25 z+25=0$.
$\Rightarrow 6 z^{2}-15 z-10 z+25=0$.
$3 z(2 z-5)-5(2 z-5)=0$.

$$
(3 z-5)(2 z-5)=0
$$

$$
3 z-5=0
$$

$$
2 z-5=0 .
$$

$3\left(x+\frac{1}{x}\right)-5=0 . \quad 2\left(x+\frac{1}{x}\right)-5=0$.
$3 x^{2}-5 x+3=0 . \quad 2 x^{2}-5 x+2=0$.
$x=\frac{5 \pm \sqrt{25-36}}{2(3)}$. $2 x^{2}-4 x-x+2=0$.
$=\frac{5 \pm \sqrt{11} i}{6}$.

$$
\begin{aligned}
2 x(x-2)-1(x-2) & =0 \\
(x-2)(2 x-1) & =0 \\
x & =2, \frac{1}{2}
\end{aligned}
$$

Hence the required roots of the given $6^{\text {th }}$ degree reciprocal equation are $-1,1, \frac{5 \pm \sqrt{11} \mathrm{i}}{6}, 2, \frac{1}{2}$.
5. Solve the equation $x^{4}+2 x^{3}-5 x^{2}+6 x+2=0$, given that one root of it is $1+i$.

A: For the equation $x^{4}+2 x^{3}-5 x^{2}+6 x+2=0$, one root is $1+i$.
So, the other root is $1-\mathrm{i}$.
The quadratic equation whose roots are
$1+i, 1-i$ is $x^{2}-(1+i+1-i) x+(1+i)(1-i)=0$

$$
x^{2}-2 x+2=0
$$

By synthetic division,

|  | 1 | 2 | -5 | 6 | 2 |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 2 | 0 | 2 | 8 | 2 | 0 |
| -2 | 0 | 0 | -2 | -8 | -2 |
|  | 1 | 4 | 1 | 0 | 0 |

then $x^{2}+4 x+1$.
Now the given equation can be written as $\left(x^{2}-2 x+2\right)\left(x^{2}+4 x+1\right)=0$

$$
\begin{aligned}
x^{2}+4 x+1 & =0 \\
x & =\frac{-4 \pm \sqrt{16-4}}{2}=\frac{-4 \pm 2 \sqrt{3}}{2}=-2 \pm \sqrt{3}
\end{aligned}
$$

Hence the required roots of the given biquadratic equation are $1+i, 1-i,-2+\sqrt{3},-2-\sqrt{3}$.
(H/W) Solve the equation $x^{4}-9 x^{3}+27 x^{2}-29 x+6=0$, given that one root of it is $2-\sqrt{3}$.
(ii) Given that $-2+\sqrt{-7}$ is a root of the equation $x^{4}+2 x^{2}-16 x+77=0$, solve it completely.
6.Find the algebraic equation of degree 5 whose roots are the translates of roots of $x^{5}+4 x^{3}-x^{2}+11=0$ by -3.
A: Given equation is $f(x)=x^{5}+4 x^{3}-x^{2}+11=0$.
Here the roots are translated by ' -3 ', so the transformed equation is $f(x+3)=0$.
By Horner's method,

| 3 | 1 | 0 | 4 | -1 | 0 | 11 |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: |
| 3 | 0 | 3 | 9 | 39 | 114 | 342 |
| 3 | 1 | 3 | 13 | 38 | 114 | $353=\mathrm{A}_{5}$ |
| 3 | 0 | 3 | 18 | 93 | 393 |  |
| 3 | 1 | 6 | 31 | 131 | $507=\mathrm{A}_{4}$ |  |
| 3 | 0 | 3 | 27 | 174 |  |  |
| 3 | 1 | 9 | 58 | $305=\mathrm{A}_{3}$ |  |  |
| 3 | 0 | 3 | 36 |  |  |  |
|  | 1 | 12 | $94=\mathrm{A}_{2}$ |  |  |  |
|  | $1=\mathrm{A}_{0}$ | $15=\mathrm{A}_{1}$ |  |  |  |  |

$\therefore$ Required transformed equation is $x^{5}+15 x^{4}+94 x^{3}+305 x^{2}+507 x+353=0$.
7. Find the polynomial equation whose roots are the translates of those of the equation $x^{4}-5 x^{3}+7 x^{2}-17 x+11=0$ by -2.
A: Given equation is $f(x)=x^{4}-5 x^{3}+7 x^{2}-17 x+11=0$.
Here the roots are translated by -2 , so the transformed equation is $f(x+2)=0$.
By Horner's method
$f(x+2)=A_{0} x^{4}+A_{1} x^{3}+A_{2} x^{2}+A_{3} x+A_{4}=0$

$\therefore$ Required transformed equation is $\mathrm{x}^{4}+3 \mathrm{x}^{3}+\mathrm{x}^{2}-17 \mathrm{x}-19=0$
8. Transform $x^{4}+4 x^{3}+2 x^{2}-4 x-2=0$ into another equation in which the coefficient of second highest power of $x$ is zero and find the transformed equation.

A: Given equation is $f(x)=x^{4}+4 x^{3}+2 x^{2}-4 x-2=0$.
Comparing this with $p_{0} x^{4}+p_{1} x^{3}+p_{2} x^{2}+p_{3} x+p_{4}=0$. then here $p_{0}=1, p_{1}=4, n=4$ (degree)
To eliminate the second term, $f(x)=0$ is transformed to $f(x+h)=0$ where $h$ is given by $h=\frac{-p_{1}}{n \cdot p_{0}}=\frac{-4}{4(1)}=-1$
By Horner's method ; $f(x-1)=A_{0} x^{4}+A_{1} x^{3}+A_{2} x^{2}+A_{3} x+A_{4}$

|  | 1 | 4 | 2 | -4 | -2 |
| :---: | :---: | :---: | :---: | :---: | :---: |
| -1 | 0 | -1 | -3 | 1 | 3 |
|  | 1 | 3 | -1 | -1 | $1=\mathrm{A}_{4}$ |
| -1 | 0 | -1 | -2 | 3 |  |
|  | 1 | 2 | -3 | $0=\mathrm{A}_{3}$ |  |
| -1 | 0 | -1 | -1 |  |  |
|  | 1 | 1 | $-4=\mathrm{A}_{2}$ |  |  |
| -1 | 0 | -1 |  |  |  |
|  | $1=\mathrm{A}_{0}$ | $0=\mathrm{A}_{1}$ |  |  |  |

$\therefore$ Required transformed equation is $\mathrm{x}^{4}-4 \mathrm{x}^{2}+1=0$.

1. Transform the given equation $x^{3}+2 x^{2}+x+1=0$ into one in which the coefficient of the third highest power of $\mathbf{x}$ is zero.
A. Let $f(x)=x^{3}+2 x^{2}+x+1=0$.

To remove the $3^{\text {rd }}$ term, diminish the roots by $h$ such that $\frac{n(n-1)}{2} p_{0} h^{2}+(n-1) p_{1} h+p_{2}=0$
$\Rightarrow \frac{3 \cdot 2}{2} 1 \cdot h^{2}+2 \cdot 2 \cdot h+1=0$
$\Rightarrow 3 \mathrm{~h}^{2}+4 \mathrm{~h}+1=0$
$\Rightarrow 3 h^{2}+3 h+h+1=0$
$\Rightarrow(3 \mathrm{~h}+1)(\mathrm{h}+1)=0$
$\Rightarrow h=\frac{-1}{3}, \quad \mathrm{~h}=-1$
Case-1: If $h=-1$

By Horner's method, transformed equation is $x^{3}-x^{2}+1=0$
Case-2: If $h=-1 / 3$

By Horner's method, transformed equation is $x^{3}+x^{2}+\frac{23}{27}=0$.

## PERMUTATION AND COMBINATION Q.NO 25

1. Find the number of ways of arranging 6 boys and 6 girls in a row so that (i) all the girls sit together (ii) no two girls sit together (iii) boys and girls sit alternately.
A: No. of boys $=6$ and No.of girls $=6$.
i) Given condition : All the girls sit together in a row. Treat the 6 girls as one unit. So the number of units $=1+6=7$. These 7 units can be arranged in a row in 7 ! ways. Now 6 girls can be arranged among themselves in 6 ! ways. Hence, by the counting principle, the number of arrangements in which all 6 girls are together is $7!\times 6!=36,28,800$.
ii) Given condition : No two girls sit together. First of all, we shall arrange 6 boys in a row in 6 ! ways. The girls can be arranged in the 7 gaps in ${ }^{7} \mathrm{P}_{6}$ ways.
```
\checkmark B \checkmarkB\checkmarkB}\checkmarkB|\checkmarkB\checkmark
```

The number of arrangements in which no two girls sit together is $6!\times{ }^{7} P_{6}=36,28,800$.
iii) Given condition : Boys and girls sit alternatively.

$$
\begin{array}{lllllllllll}
\checkmark & B & \checkmark & B & B & \checkmark & B & \checkmark & B & \checkmark & B \\
B & \checkmark & B & \checkmark & B & \checkmark & B & \checkmark & B & \checkmark & B \\
\end{array}
$$

First of all six boys can be arranged in a row in 6! ways. Then by considering only one end gap, in 6 continuous spaces, 6 girls can be arranged in 6 ! as shown in the above figure. Hence total number of arrangements in which boys and girls set alternately $=6!6!+6!6!$

$$
\begin{aligned}
& =2(6!)(6!) \\
& =2(720)(720) \\
& =10,36,800 .
\end{aligned}
$$

2. Find the number of 4 letter words that can be formed using the letters of the word MIRACLE. How many of them i) begin with an vowe
ii) begin and end with vowels
iii) End with a consonant?

A: Given word is MIRACLE
i) Begin with an vowel

We can fil the first place with one of the 3 vowels
$\{I, A, E\}$ in ${ }^{3} P_{1}=3$ ways. Now, the remaining 3 places can be filled using the remaining 6 letters in ${ }^{6} P_{3}=120$ ways. Total number of arrangements $3 \times 120=360$ ways.
ii) Begin and end with vowels

We can fill the first and last places with $\{I, A, E\}$ in ${ }^{3} P_{2}=6$ ways. Now, the remaining 2 places can be filled using the remaining 5 letters in ${ }^{5} \mathrm{P}_{2}=20$ ways.
Total number of arrangements $6 \times 20=120$ ways
iii) End with a consonant

We can fill the last place with one of the 3 consonants $\{M, R, C, L\}$ in ${ }^{4} P_{1}=4$ ways. Now, the remaining 3 places can be fillled using the remaining 6 letters in ${ }^{6} \mathrm{P}_{3}=120$ ways.
Total number of arrangements $4 \times 120=480$ ways.
3. Find the number of four digit numbers that can be formed using the digits $1,2,5,6,7$. How many of them are divisible by (i) 2 (ii) 3 (iii) 4 (iv) 25 .
A: Number of four digit numbers formed using 1, 2, 5, 6, $7={ }^{5} \mathrm{P}_{4}=120$.
i) A number is divisible by 2 when its units place must be filled with an even digit from among the given digits.

This can be done in 2ways.
Now, the remaining 3 places can be filled with remaining 4 digits in ${ }^{4} P_{3}$ ways.
$\therefore$ The number of 4 digited numbers divisible by $2=2 \times 24=48$.
ii) A number is divisible by 3 when the sum of the digits in that number is a multiple of 3 .

The possible cases are $1256,1257,1267,1567,2567$.
The 4 digits such that their sum is a multiple of 3 from the given digits are 1, 2, 5, 7 .
They can be arranged in 4 ! ways.
$\therefore$ The number of 4 digited numbers divisible by $3=24$.
iii) A number is divisible by 4 only when the last two places (tens and unit's places) of it is a multiple of 4 .

Here, the last two places can be filled with one of the following: 12, 16, 52, 56, 72, 76. Thus the last two places can be filled in 6 ways. The remaining two places can be filled by the left over 3 digits in ${ }^{3} \mathrm{P}_{2}$ ways.
$\therefore$ The number of 4 digited numbers divisible by $4=6 \times 6=36$.
iv) A number is divisible by 25 when its last two places are filled with either 25 or 75 .

The last two places can be filled in 2 ways. The remaining 2 places from the remaining 3 digits can be filled in ${ }^{3} \mathrm{P}_{2}$ ways.
$\therefore$ The number of 4 digits numbers divisible by $25=2 \times 6=12$.
4. If ${ }^{n} C_{r}={ }^{n} C_{s}$, then prove that $r=s$ or $n=r+s$.

A: Now ${ }^{n} C_{r}={ }^{n} C_{s}$ then $r=s$ or $r \neq s$
Suppose that $r>s$ then $n-r<n-s$
Now ${ }^{n} C_{r}={ }^{n} C_{s}$

$$
\begin{aligned}
\frac{n!}{(n-r)!r!} & =\frac{n!}{(n-s)!s!} \\
(n-r)!r! & =(n-s)!s! \\
(n-r)!s!(s & +1)(s+2) \ldots \ldots . r \\
\quad(s+1)(n+2) \ldots \ldots r & =(n-r+1)(n-r+2) \ldots(n-s)
\end{aligned}
$$

Since each side of the above relation is a product of $r-s$ consecutive positive integers, we get $r=n-s$ then $n=r+s$
Similarly if $r<s$, then also we can prove that $n=r+s$.

1. Find the sum of the infinite series $1+\frac{1}{3}+\frac{1.3}{3.6}+\frac{1.3 .5}{3.6 .9}+$ $\qquad$ $\infty$.
A. $1+\frac{1}{3}+\frac{1 \cdot 3}{3 \cdot 6}+\frac{1 \cdot 3.5}{3 \cdot 6.9}+$ $\qquad$ $\infty=1+\frac{1}{1!}\left(\frac{1}{3}\right)+\frac{1.3}{2!}\left(\frac{1}{3}\right)^{2}+\frac{1.3 \cdot 5}{3!}\left(\frac{1}{3}\right)^{3}+$. $\qquad$ .$\infty$

## Comparing this with

$$
(1-x)^{-p / q}=1+\frac{p}{1!}\left(\frac{x}{q}\right)+\frac{p(p+q)}{2!}\left(\frac{x}{q}\right)^{2}+
$$

$\qquad$ ..$\infty$

Here $\mathrm{p}=1 ; \mathrm{p}+\mathrm{q}=3 \Rightarrow \mathrm{q}=2 ; \frac{\mathrm{x}}{\mathrm{q}}=\frac{1}{3} \Rightarrow \mathrm{x}=\frac{2}{3}$
$\therefore$ sum of the given infinite series $=(1-\mathrm{x})^{-\mathrm{plq} q}=\left(1-\frac{2}{3}\right)^{-1 / 2}=\left(\frac{1}{3}\right)^{-1 / 2}=\sqrt{3}$.
2. If $\mathrm{t}=\frac{4}{5}+\frac{4.6}{5.10}+\frac{4.6 .8}{5.10 .15}+$ $\qquad$ .$\infty$, then prove that $9 t=16$.

A: Given that $t=\frac{4}{5}+\frac{4.6}{5.10}+\frac{4.6 .8}{5 \cdot 10.15}+$ $\qquad$ .$\infty$

Now $\frac{4}{1!}\left(\frac{1}{5}\right)+\frac{4.6}{2!}\left(\frac{1}{5}\right)^{2}+\frac{4.6 .8}{3!}\left(\frac{1}{5}\right)^{3}+$ $\qquad$
Add 1 on both sides
$t+1=1+\frac{4}{1!}\left(\frac{1}{5}\right)+\frac{4.6}{2!}\left(\frac{1}{5}\right)^{2}+\frac{4.6 .8}{3!}\left(\frac{1}{5}\right)^{3}+$ $\qquad$

Comparing the infinite series with $(1-x)^{-p / q}=1+\frac{p}{1!}\left(\frac{x}{q}\right)+\frac{p(p+q)}{2!}\left(\frac{x}{q}\right)^{2}+$ $\qquad$ $\ldots$

Here $\mathrm{p}=4 ; \mathrm{p}+\mathrm{q}=6 \Rightarrow \mathrm{q}=2 ; \frac{\mathrm{x}}{\mathrm{q}}=\frac{1}{5} \Rightarrow \mathrm{x}=\frac{2}{5}$
$\therefore$ Sum of the given infinite series $\mathrm{t}+1=(1-\mathrm{x})^{-\mathrm{p} / q}$

$$
t+1=\left(1-\frac{1}{5}\right)^{-4 / 2}=\left(\frac{3}{5}\right)^{-2}=\left[\frac{5}{3}\right]^{2}=\frac{25}{9}
$$

$9 t+9=25 \Rightarrow 9 t=16$.
3. Find the sum of infinite series $\frac{3}{4}+\frac{3.5}{4.8}+\frac{3.5 .7}{4.8 .12}+\ldots \ldots \infty$.

A: Now let $X=\frac{3}{4}+\frac{3.5}{4.8}+\frac{3.5 .7}{4.8 .12}+$ $\qquad$
Add 1 on both sides
$X+1=1+\frac{3}{1!}\left(\frac{1}{4}\right)+\frac{3.5}{2!}\left(\frac{1}{4}\right)^{2}+\frac{3.5 \cdot 7}{3!}\left(\frac{1}{4}\right)^{3}+$ $\qquad$ Comparing the infinite series with
$(1-x)^{-p / q}=1+\frac{p}{1!}\left(\frac{x}{q}\right)+\frac{p(p+q)}{2!}\left(\frac{x}{q}\right)^{2}+$ $\qquad$ .. $\infty$

Here $\mathrm{p}=3 ; \mathrm{p}+\mathrm{q}=5 \Rightarrow \mathrm{q}=2 ; \quad \frac{\mathrm{x}}{\mathrm{q}}=\frac{1}{4} \Rightarrow \mathrm{x}=\frac{2}{4}=\frac{1}{2}$
$\therefore$ Sum of the given infinite series $X+1=(1-\mathrm{x})^{-\mathrm{p} / \mathrm{q}}-1$

$$
X=\left(1-\frac{1}{2}\right)^{-3 / 2}-1=\left(\frac{1}{2}\right)^{-3 / 2}=2^{3 / 2}-1=2 \sqrt{2}-1
$$

4. If $x=\frac{1}{5}+\frac{1.3}{5.10}+\frac{1 \cdot 3.5}{5 \cdot 10.15}+\ldots \ldots \infty$, then find the value of $3 x^{2}+6 x$.

A: Given $x=\frac{1}{5}+\frac{1.3}{5.10}+\frac{1.3 .5}{5 \cdot 10.15}+$ $\qquad$

$$
x=\frac{1}{1!}\left(\frac{1}{5}\right)+\frac{1.3}{2!}\left(\frac{1}{5}\right)^{2}+\frac{1 \cdot 3 \cdot 5}{3!}\left(\frac{1}{5}\right)^{3}+
$$

$\qquad$ $\infty$

Add 1 on both sides

$$
x+1=1+\frac{1}{1!}\left(\frac{1}{5}\right)+\frac{1.3}{2!}\left(\frac{1}{5}\right)^{2}+
$$

$\qquad$

Comparing this with $(1-x)^{-p / q}=1+\frac{p}{1!}\left(\frac{x}{q}\right)+\frac{p(p+q)}{2!}\left(\frac{x}{q}\right)^{2}+$ $\qquad$

Here $\mathrm{p}=1 ; \mathrm{p}+\mathrm{q}=3 \Rightarrow \mathrm{q}=2 ; \frac{x}{\mathrm{q}}=\frac{1}{5} \Rightarrow \mathrm{x}=\frac{2}{5}$

$$
\begin{aligned}
& \therefore x+1=(1-x)^{- \text {-/q }}=\left(1-\frac{2}{5}\right)^{-1 / 2}=\left(\frac{3}{5}\right)^{-1 / 2} \\
& x+1=\sqrt{\frac{5}{3}} \quad \text { Squaring on both sides, } \\
& x^{2}+2 x+1=\frac{5}{3} \\
& 3 x^{2}+6 x+3=5 \\
& 3 x^{2}+6 x=5-3=2
\end{aligned}
$$

5. If $x=\frac{1.3}{3.6}+\frac{1.3 .5}{3.6 .9}+\frac{1 \cdot 3.5 .7}{3 \cdot 6 \cdot 9.12}+\ldots \ldots \infty$, then prove that $9 x^{2}+24 x=11$.

A: Given $x=\frac{1.3}{3.6}+\frac{1 \cdot 3.5}{3 \cdot 6.9}+\frac{1 \cdot 3 \cdot 5 \cdot 7}{3 \cdot 6 \cdot 9 \cdot 12}+$ $\qquad$

$$
x=\frac{1.3}{2!}+\left(\frac{1}{3}\right)^{2}+\frac{1.3 .5}{3!}+\left(\frac{1}{3}\right)^{3}+\ldots \ldots \infty \quad \text { Add } 1+\frac{1}{1!}\left(\frac{1}{3}\right) \text { on both sides }
$$

$$
x+1+\frac{1}{1!}\left(\frac{1}{3}\right)=1+\frac{1}{1!}\left(\frac{1}{3}\right) \frac{1.3}{2!}+\left(\frac{1}{3}\right)^{2}+\frac{1.3 .5}{3!}+\left(\frac{1}{3}\right)^{3}+
$$

$\qquad$

Comparing with $(1-x)^{-p / q}=1+\frac{p}{1!}\left(\frac{x}{q}\right)+\frac{p(p+q)}{2!}\left(\frac{x}{q}\right)^{2}+$ $\qquad$ .$\infty$

$$
\text { Here } p=1 ; \quad p+q=3 \Rightarrow q=2 ; \quad \frac{x}{q}=\frac{1}{3} \Rightarrow x=\frac{2}{3}
$$

$\therefore x+\frac{4}{3}=(1-x)^{-p / q}=\left(1-\frac{2}{3}\right)^{-1 / 2}=\left(\frac{1}{3}\right)^{-1 / 2}$

$$
\frac{3 x+4}{3}=\sqrt{3} \quad 3 x+4=3 \sqrt{3} \quad \text { Squaring on both sides, }
$$

$$
9 x^{2}+24 x+16=27
$$

$$
\therefore 9 x^{2}+24 x=11
$$

6. Find the sum of the infinite series $\frac{3.5}{5.10}+\frac{3.5 .7}{5.10 .15}+\frac{3 \cdot 5.7 .9}{5 \cdot 10.15 .20}+\ldots . . \infty$.

A: Now $X=\frac{3.5}{5 \cdot 10}+\frac{3 \cdot 5 \cdot 7}{5 \cdot 10 \cdot 15}+\frac{3 \cdot 5 \cdot 7.9}{5 \cdot 10 \cdot 15 \cdot 20}+$ $\qquad$

$$
X=\frac{3.5}{2!}\left(\frac{1}{5}\right)^{2}+\frac{3 \cdot 5 \cdot 7}{3!} \cdot\left(\frac{1}{5}\right)^{2}+
$$

$\qquad$

$$
\text { Add } 1+\frac{3}{1!}\left(\frac{1}{5}\right) \text { on both sides }
$$

$$
X+1+\frac{3}{1!}\left(\frac{1}{5}\right)=1+\frac{3}{1!}\left(\frac{1}{5}\right)+\frac{3 \cdot 5}{2!} \cdot\left(\frac{1}{5}\right)^{2}+
$$

$\qquad$
Comparing this with $(1-x)^{-p / q}=1+\frac{p}{1!}\left(\frac{x}{q}\right)+\frac{p(p+q)}{2!}\left(\frac{x}{q}\right)^{2}+$ $\qquad$

$$
\text { Here } p=3 ; p+q=5 \Rightarrow q=2 ; \quad \frac{x}{q}=\frac{1}{5} \Rightarrow x=\frac{2}{5}
$$

Sum of the given infinite series $X+1+\frac{3}{1!}\left(\frac{1}{5}\right)=(1-x)^{-p / q}$

$$
X=\left(1-\frac{2}{5}\right)^{-3 / 2}-\frac{8}{5}=\left(\frac{3}{5}\right)^{-3 / 2}-\frac{8}{5}
$$

$$
x=\frac{5 \sqrt{5}}{3 \sqrt{3}}-\frac{8}{5} .
$$

7. If $x=\frac{5}{2!3}+\frac{5.7}{3!3^{2}}+\frac{5.7 .9}{4!3^{3}}+\ldots \ldots \infty$, then find the value of $x^{2}+4 x$.

A: Given $x=\frac{5}{2!3}+\frac{5.7}{3!3^{2}}+\frac{5.7 .9}{4!3^{3}}+\ldots \ldots \infty$
multiply and divide by 3 on both sides for proper order in numerator and dinomenator
$3 x / 3=\frac{3 \cdot 5}{2!}\left(\frac{1}{3}\right)^{2}+\frac{3 \cdot 5 \cdot 7}{3!} \cdot\left(\frac{1}{3}\right)^{3}+$ $\qquad$

Add $1+\frac{3}{1!}\left(\frac{1}{3}\right)$ on both sides

$$
\begin{aligned}
& X+1+\frac{3}{1!}\left(\frac{1}{3}\right)=1+\frac{3}{1!}\left(\frac{1}{3}\right)+\frac{3.5}{2!}\left(\frac{1}{3}\right)^{2}+\frac{3.5 .7}{3!}\left(\frac{1}{3}\right)^{3}+ \\
& x+2=1+\frac{3}{1!}\left(\frac{1}{3}\right)+\frac{3.5}{2!}\left(\frac{1}{3}\right)^{2}+\frac{3.5 .7}{3!}\left(\frac{1}{3}\right)^{3}+\ldots \ldots \infty
\end{aligned}
$$

Comparing RHS with $(1-x)^{-p / q}=1+\frac{p}{1!}\left(\frac{x}{q}\right)+\frac{p(p+q)}{2!}\left(\frac{x}{q}\right)^{2}+$ $\qquad$ ..$\infty$

Here $\mathrm{p}=3 ; \quad \mathrm{p}+\mathrm{q}=5 \Rightarrow \mathrm{q}=2 ; \quad \frac{\mathrm{x}}{\mathrm{q}}=\frac{1}{3} \Rightarrow \mathrm{y}=\frac{2}{3}$
$\therefore x+2=(1-x)^{-p / q}=\left(1-\frac{2}{3}\right)^{-3 / 2}=\left(\frac{1}{3}\right)^{-3 / 2}$
$x+2=3^{3 / 2}$
Squaring on both sides
$x^{2}+4 x+4=27$
$\therefore \mathrm{x}^{2}+4 \mathrm{x}=23$.
8. Find the sum to infinite series $\frac{7}{5}\left[1+\frac{1}{10^{2}}+\frac{1.3}{1.2}+\left(\frac{1}{10^{4}}\right)+\frac{1 \cdot 3 \cdot 5}{1.2 \cdot 3}+\left(\frac{1}{10^{6}}\right)+\ldots \ldots \infty\right]$.

A: Comparing the infinite series with $(1-x)^{-p / q}=1+\frac{p}{1!}\left(\frac{x}{q}\right)+\frac{p(p+q)}{2!}\left(\frac{x}{q}\right)^{2}+$ $\qquad$

$$
\frac{7}{5}\left[1+\frac{1}{1!}+\left(\frac{1}{100}\right)+\frac{1.3}{2!}+\left(\frac{1}{100}\right)^{2}+\frac{1.3 .5}{3!}+\left(\frac{1}{100}\right)^{3}+\ldots \ldots . . \infty\right]
$$

Here $p=1 ; \quad p+q=3 ; q=3-1=2 ; \quad \frac{x}{q}=\frac{1}{100} \quad, \quad x=\frac{2}{100}=\frac{1}{50}$

Sum of the given infinite series $=\frac{7}{5}[1-x]^{-p / q}$

$$
=\frac{7}{5}\left(1-\frac{1}{50}\right)^{-1 / 2}=\frac{7}{5}\left(\frac{49}{50}\right)^{-1 / 2}=\frac{7}{5} \sqrt{\frac{50}{49}}=\frac{7}{5}\left(\frac{5 \sqrt{2}}{7}\right)=\sqrt{2} .
$$

9. Find the sum of the infinite series $1+\frac{2}{3} \cdot \frac{1}{2}+\frac{2}{3} \cdot \frac{5}{6}\left(\frac{1}{2}\right)^{2}+\frac{2}{3} \cdot \frac{5}{6} \cdot \frac{8}{9}\left(\frac{1}{2}\right)^{3}+\ldots \ldots \infty$.

A: Now $1+\frac{2}{3} \cdot \frac{1}{2}+\frac{2}{3} \cdot \frac{5}{6}\left(\frac{1}{2}\right)^{2}+\frac{2}{3} \cdot \frac{5}{6} \cdot \frac{8}{9}\left(\frac{1}{2}\right)^{3}+\ldots \ldots \infty=1+2\left(\frac{1}{6}\right)+\frac{2 \cdot 5}{2!}\left(\frac{1}{6}\right)^{2}+\frac{2 \cdot 5 \cdot 8}{3!}\left(\frac{1}{6}\right)^{3}+$ $\qquad$
Comparing this with
$(1-x)^{-p / q}=1+\frac{p}{1!}\left(\frac{x}{q}\right)+\frac{p(p+q)}{2!}\left(\frac{x}{q}\right)^{2}+$ $\qquad$
Here $p=2 ; p+q=5 \Rightarrow q=3 ; \quad \frac{x}{q}=\frac{1}{6} \Rightarrow x=\frac{3}{6}=\frac{1}{2}$
$\therefore$ Sum of the given infinite series $=(1-x)^{- \text {-// }}=\left(1-\frac{1}{2}\right)^{-2 / 3}=\left(\frac{1}{2}\right)^{-2 / 3}=2^{2 / 3}=\sqrt[3]{4}$.
10. Show that for any non - zero rational number $x$,

$$
1+\frac{x}{2}+\frac{x(x-1)}{2.4}+\frac{x(x-1)(x-2)}{2.4 .6}+\ldots \ldots \infty=1+\frac{x}{3}+\frac{x(x+1)}{3.6}+\frac{x(x+1)(x+2)}{3.6 .9}+\ldots \ldots \infty .
$$

A: LHS $=1+\frac{x}{2}+\frac{x(x-1)}{2.4}+\frac{x(x-1)(x-2)}{2.4 .6}+\ldots \ldots \infty$

$$
R H S=1+\frac{x}{3}+\frac{x(x+1)}{3.6}+\frac{x(x+1)(x+2)}{3.6 .9}+\ldots \ldots \ldots
$$

$1+\frac{x}{1!}\left(\frac{1}{2}\right)+\frac{x(x-1)}{2!}\left(\frac{1}{2}\right)^{2}+\frac{x(x-1)(x-2)}{3!}\left(\frac{1}{2}\right)^{3}+\ldots \ldots$ $\ldots \quad 1+\frac{x}{1!}\left(\frac{1}{3}\right)+\frac{x(x+1)}{2!}\left(\frac{1}{3}\right)^{2}+\frac{x(x+1)(x+2)}{3!}\left(\frac{1}{3}\right)^{3}+$

$$
\begin{align*}
\because(1-x)^{-n} & =1+\frac{n}{1!} x+\frac{n(n+1)}{2!} x^{2}+\ldots . . \infty=\left(1+\frac{1}{2}\right)^{-x} & \because(1-x)^{-n}=1+\frac{n}{1!} x+\frac{n(n+1)}{2!} x^{2}+\ldots . \infty=\left(1-\frac{1}{3}\right)^{x} \\
& =\left(\frac{3}{2}\right)^{-x}=\left(\frac{2}{3}\right)^{x}-\cdots(1) & =\left(\frac{2}{3}\right)^{x}-\cdots(2)
\end{align*}
$$

From (1) \& (2), $1+\frac{x}{2}+\frac{x(x-1)}{2.4}+\frac{x(x-1)(x-2)}{2.4 .6}+\ldots \ldots \infty=1+\frac{x}{3}+\frac{x(x+1)}{3.6}+\frac{x(x+1)(x+2)}{3.6 .9}+\ldots \ldots \infty$. 11. Find the sum of the infinite series $\frac{3}{4.8}-\frac{3.5}{4.8 .12}+\frac{3.5 .7}{4.8 .12 .16}=\ldots . . \infty$.

A: Given $\frac{3}{4.8}-\frac{3.5}{4.8 .12}+\frac{3 \cdot 5.7}{4 \cdot 8 \cdot 12.16}-\ldots \ldots \infty$
$X=\frac{1.3}{4.8}-\frac{1 \cdot 3 \cdot 5}{4 \cdot 8 \cdot 12}+\frac{1 \cdot 3 \cdot 5 \cdot 7}{4 \cdot 8 \cdot 12 \cdot 16}-\ldots \ldots$ .$\infty$
$X=\frac{1.3}{2!}\left(\frac{1}{4}\right)^{2}-\frac{1.3 .5}{3!}\left(\frac{1}{4}\right)^{2}+$ $\qquad$ $\infty$

Add $1-\frac{1}{1!}\left(\frac{1}{4}\right)$ on both sides
$X+1-\frac{1}{1!}\left(\frac{1}{4}\right)=1-\frac{1}{1!}\left(\frac{1}{4}\right)+\frac{1.3}{2!}\left(\frac{1}{4}\right)^{2}-\frac{1.3 .5}{3!}\left(\frac{1}{4}\right)^{3}$ $\qquad$ $\infty$

Comparing the infinite series with $(1+x)^{-p / q}=1-\frac{p}{1!}\left(\frac{x}{q}\right)+\frac{p(p+q)}{2!}\left(\frac{x}{q}\right)^{2}-$ $\qquad$

Here $\mathrm{p}=1 ; \mathrm{p}+\mathrm{q}=3 \Rightarrow \mathrm{q}=2 ; \quad \frac{\mathrm{x}}{\mathrm{q}}=\frac{1}{4} \Rightarrow \mathrm{x}=\frac{2}{4}=\frac{1}{2}$
Sum of the given infinite series $X+1-\frac{1}{1!}\left(\frac{1}{4}\right)=(1+x)^{- \text {-// }}$

$$
X=(1+x)^{-p / q}-\frac{3}{4}=\left(1+\frac{1}{2}\right)^{-1 / 2}-\frac{3}{4}=\left(\frac{3}{2}\right)^{-1 / 2}-\frac{3}{4}=\sqrt{\frac{2}{3}}-\frac{3}{4} .
$$

12. If the coefficient of $x^{10}$ in the expansion of $\left(a x^{2}+\frac{1}{b x}\right)^{11}$ is equal to the coefficient of $x^{-10}$ in the expansion of $\left(a x-\frac{1}{b x^{2}}\right)^{11}$, find the relation between and $a$ and $b$, where $a$ and $b$ are real numbers.
$A: \ln \left(a x^{2}+\frac{1}{b x^{2}}\right)^{11}$,
$T_{r+1}={ }^{11} C_{r} \cdot\left(a x^{2}\right)^{11-r}\left(\frac{1}{b x}\right)^{r}={ }^{11} C_{r} \cdot \frac{a^{11-r}}{b^{r}} \cdot x^{22-3 r}$
To get the coefficient of $x^{10}, 22-3 r=10$

$$
\begin{aligned}
12 & =3 r \\
r & =4
\end{aligned}
$$

$\therefore$ Coefficient of $x^{10}$ in $\left(a x^{2}+\frac{1}{b x^{2}}\right)^{11}={ }^{11} C_{4} \cdot \frac{a^{7}}{b^{4}}$
$\ln \left(a x^{2}-\frac{1}{b x^{2}}\right)^{11}, T_{r+1}={ }^{11} C_{r} \cdot(a x)^{11-r}\left(\frac{-1}{b x^{2}}\right)^{r}={ }^{11} C_{r} \cdot(-1)^{r} \cdot \frac{a^{11-r}}{b^{r}} \cdot x^{11-3 r}$
To get the coefficient of $x^{-10}, 11-3 r=-10$

$$
\begin{aligned}
21 & =3 r \\
r & =7 .
\end{aligned}
$$

Thus, coefficient of $x^{-10}$ in $\left(a x-\frac{1}{b x^{2}}\right)^{11}$ is $=(-1)^{7{ }^{11}} C_{7} \cdot \frac{a^{4}}{b^{7}}$

$$
\begin{aligned}
\text { But }{ }^{11} C_{4} \cdot \frac{a^{7}}{b^{4}} & =-{ }^{11} C_{7} \cdot \frac{a^{4}}{b^{7}} \quad \because{ }^{n} C_{r}={ }^{n} C_{n-r} \\
a^{3} & =\frac{-1}{b^{3}} \\
(a b)^{3} & =-1 \\
\therefore a b & =-1 . \quad(\because a, b \in R)
\end{aligned}
$$

13. Find the numerically greatest terms in the expansion of $(4 a-6 b)^{13}$ when $a=3, b=5$.
A.Given $(4 a-6 b)^{13}=(4 a)^{13}\left(1-\frac{6 b}{4 a}\right)^{13}$

$$
=(4 a)^{13}(1+x)^{n}
$$

where $x=\frac{-6 b}{4 a}, n=13$
$|x|=\left|\frac{-6 b}{4 a}\right|=\left|\frac{-6.5}{4.3}\right|=\frac{5}{2}$
Now, $\frac{(n+1)|x|}{|x|+1}=\frac{(13+1) \frac{5}{2}}{\frac{5}{2}+1}=\frac{14.5}{5+2}=\frac{70}{7}=10$
$\therefore\left|\mathrm{T}_{10}\right|$ and $\left|\mathrm{T}_{11}\right|$ are numerically greatest.
$T_{10}: r+1=10 \Rightarrow r=9$
The general term in the expansion of $(1-x)^{n}$ is $T_{r-}(1-x)^{n}$ is $T_{r-1}={ }^{n} C_{r} x^{n-r} a^{r}$
$\mathrm{T}_{10}$ in this expansion is
$T_{9+1}={ }^{13} C_{9}(4 a)^{13-9}(-6 b)^{9}$
$\mathrm{T}_{10}={ }^{13} \mathrm{C}_{9}(4 \mathrm{a})^{4}(-6 \mathrm{~b})^{9}={ }^{13} \mathrm{C}_{9}\left(4^{4} \cdot 3^{4}\right)(-6)^{9}\left(5^{9}\right)={ }^{-13} \mathrm{C}_{9} 12^{4} .30^{9}$
$\left|T_{10}\right|={ }^{13} \mathrm{C}_{9} 12^{4} .30^{9}$
$T_{11}: r+1=11 \Rightarrow r=10$.
The general term in the expansion of
$(1+x)^{n}$ is $T_{r+1}={ }^{n} C_{r} x^{n-r} \cdot a^{r}$
$T_{11}$ in this expansion is
$\mathrm{T}_{10+1}={ }^{13} \mathrm{C}_{10}(4 \mathrm{a})^{13-10}(-6 \mathrm{~b})^{10}={ }^{13} \mathrm{C}_{10}(4 a)^{3}(6 \mathrm{~b})^{10}$
$\mathrm{T}_{11}={ }^{13} \mathrm{C}_{10}(4.3)^{3}(6.5)^{10}={ }^{13} \mathrm{C}_{0} 12^{3} \cdot 30^{10}$
$\left|T_{11}\right|={ }^{13} \mathrm{C}_{10} 12^{3} .30^{10}$
$\therefore\left|T_{10}\right|=\left|T_{11}\right|$
$(H / W)$ Find the numerically greatest term in the expansion of $(4+3 x)^{15}$ where $x=\frac{7}{2}$.

## PROBABILITY Q.NO 27

1. State and explain the axioms that define 'Probability function'. Prove addition theorem on probability.

A: Probability function: Let $S$ be the sample space of a random experiment.
Then a function $P: P(S) \rightarrow R$ satisfying the following axioms is called a probability function.

1) $P(A) \geq 0 \forall A \in P(S)$. This is called axiom of non-negativity.
2) $P(S)=1$. This called axiom of certainity.
3) If $A, B \in S$ and $A \cap B=\phi$, then $P(A \cup B)=P(A)+P(B)$. This is known as axiom of union.

Addition theorem : If $E_{1}, E_{2}$ are any two events in a sample space $S$, then $P\left(E_{1} \cup E_{2}\right)=P\left(E_{1}\right)+P\left(E_{2}\right)-P\left(E_{1} \cap E_{2}\right)$
Proof: Case 1 : Suppose that $E_{1} \cap E_{2}=\phi$.

$$
P\left(E_{1} \cup E_{2}\right)=P\left(E_{1}\right)+P\left(E_{2}\right)
$$

By the axiom of union

$$
\begin{aligned}
& P\left(E_{1} \cup E_{2}\right)=P\left(E_{1}\right)+P\left(E_{2}\right)-0 \\
& P\left(E_{1} \cup E_{2}\right)=P\left(E_{1}\right)+P\left(E_{2}\right)-P\left(E_{1} \cap E_{2}\right)
\end{aligned}
$$

Case 2 : Suppose that $E_{1} \cap E_{2} \neq \phi$.

$$
\begin{aligned}
& P\left(E_{1} \cup E_{2}\right)=P\left[E_{1} \cup\left(E_{2}-E_{1}\right)\right] \\
& P\left(E_{1} \cup E_{2}\right)=P\left(E_{1}\right)+P\left(E_{2}-E_{1}\right) \\
& \because E_{2}-E_{1}=E_{2}-\left(E_{1} \cap E_{2}\right) \\
& P\left(E_{1} \cup E_{2}\right)=P\left(E_{1}+\left(E_{2}-\left(E_{1} \cap E_{2}\right)\right)\right. \\
& P\left(E_{1} \cup E_{2}\right)=P\left(E_{1}\right)+P\left(E_{2}\right)-P\left(E_{1} \cap E_{2}\right)
\end{aligned}
$$

$\therefore$ From (1), (2) Addition theorem is proved.
2. $A, B, C$ are three horses running in a race. The probability of $A$ to win the race is twice that of $B$, and probability of $B$ is twice that of $C$. What are the probabilities of $A, B, C$ to win the race. Also find the probability that horse A loses in the race.
$A$ : Let $A, B, C$ be the events of winning in the race by the horses $A, B, C$ respectively.
Given that $P(A)=2 P(B)$ and $P(B)=2 P(C)$

$$
\mathrm{P}(\mathrm{~A})=2 \mathrm{P}(\mathrm{~B})=2(2 \mathrm{P}(\mathrm{C}))=4 \mathrm{P}(\mathrm{C})
$$

Clearly the events $A, B, C$ are mutually exclusive and exhanstive.

$$
\begin{gathered}
A \cup B \cup C=S \\
P(A \cup B \cup C)=P(S) \\
P(A)+P(B)+P(C)=1 \\
4 P(C)+2 P(C)+P(C)=1 \\
7 P(C)=1 \\
P(C)=\frac{1}{7} \\
P(B)=2 P(C)=2\left(\frac{1}{7}\right)=\frac{2}{7} \\
P(A)=4 P(C)=4\left(\frac{1}{7}\right)=\frac{4}{7} \\
\therefore P(A)=\frac{4}{7}, P(B)=\frac{2}{7}, P(C)=\frac{1}{7}
\end{gathered}
$$

Probability that horse $A$ loses in the race $P(\bar{A})=1-P(A)=1-\frac{4}{7}=\frac{3}{7}$.
3. $A, B, C$ are 3 newspapers from a city. $20 \%$ of the population read $A, 16 \%$ read $B, 14 \%$ read $C, 8 \%$ read both $A$ and $B, 5 \%$ read both $A$ and $C, 4 \%$ read both $B$ and $C$ and $2 \%$ read all the three. Find the percentage of population who read atleast one newspaper and also find the percentage of population who read the newspaper A only.
$A$ : Let $A, B, C$ be the events that a person selected from the city reads newspapers $A, B, C$ respectively
Given that $P(A)=\frac{20}{100}, P(B)=\frac{16}{100}, P(C)=\frac{14}{100}$
$P(A \cap B)=\frac{8}{100}, P(A \cap C)=\frac{5}{100}, P(B \cap C)=\frac{4}{100}$
$P(A \cap B \cap C)=\frac{2}{100}, P(A \cup B \cup C)=$ ?
Probability that a person selected from the city reads atleast one newspaper $P(A \cup B \cup C)=P(A)+P(B)+P(C)-P(A \cap B)-P(B \cap C)-P(C \cap A)+P(A \cap B \cap C)$

$$
\begin{aligned}
& =\frac{20}{100}+\frac{16}{100}+\frac{14}{100}-\frac{8}{100}-\frac{4}{100}-\frac{5}{100}+\frac{2}{100} \\
& =\frac{20+16+14-8-4-5+2}{100}=\frac{52-17}{100}=\frac{35}{100}=35 \%
\end{aligned}
$$

$\therefore$ Required percentage of population who read atleast one newspaper $=P(A \cup B \cup C) \times 100=\frac{35}{100} \times 100=35 \%$
Probability that the selected person read the newspaper A only $=$
$P(A)-P(A \cap B)-P(A \cap C)+P(A \cap B \cap C)=\frac{20}{100}-\frac{8}{100}-\frac{5}{100}+\frac{2}{100}=\frac{9}{100}$
$\therefore$ Percentage of population who read the newspaper A only $=\frac{9}{100} \times 100=9 \%$.
4.The probabilities of three events $A, B, C$ are such that $P(A)=0.3, P(B)=0.4, P(C)=0.8, P(A \cap B)=0.08$, $P(A \cap C)=0.28, P(A \cap B \cap C)=0.09$ and $P(A \cup B \cup C) \geq 0.75$. Show that $P(B \cap C)$ lies in the interval [0.23, 0.48].
A: Given that $P(A)=0.3, P(B)=0.4, P(C)=0.8$

$$
\mathrm{P}(\mathrm{~A} \cap \mathrm{~B})=0.08, \mathrm{P}(\mathrm{~A} \cap \mathrm{C})=0.28, \mathrm{P}(\mathrm{~A} \cap \mathrm{~B} \cap \mathrm{C})=0.09
$$

$P(A \cup B \cup C) \geq 0.75$
$P(A \cup B \cup C)=P(A)+P(B)+P(C)-P(A \cap B)-P(B \cap C)-P(C \cap A)+P(A \cap B \cap C)$

$$
\begin{aligned}
& =0.3+0.4+0.8-0.08-P(B \cap C)-0.28+0.09 \\
& =1.23-P(B \cap C)
\end{aligned}
$$

It is given that $P(A \cup B \cup C) \geq 0.75$

$$
\begin{aligned}
1.23-P(B \cap C) & \geq 0.75 \\
1.23-0.75 & \geq P(B \cap C) \\
0.48 & \geq P(B \cap C) \\
P(B \cap C) & \leq 0.48----
\end{aligned}
$$

We know that $A \cup B \cup C \subseteq S$

$$
P(A \cup B \cup C) \leq P(S)
$$

$1.23-P(B \cap C) \leq 1 \quad$ by the axiom of certainity 1.23-1 $\leq P(B \cap C)$

$$
0.23 \leq P(B \cap C)----(2)
$$

Combining (1) \& (2), we get $P(B \cap C) \in[0,23,0.48]$.
5. The probabilities of three mutually exclusive events are respectively given as $\frac{1+3 p}{3}, \frac{1-p}{4}, \frac{1-2 p}{2}$. Prove that $\frac{1}{3} \leq p \leq \frac{1}{2}$.
A. Let $\mathrm{A}, \mathrm{B}, \mathrm{C}$ be the given three mutually exclusive events.

$$
\begin{align*}
& P(A)=\frac{1+3 p}{3}, \\
& P(B)=\frac{1-p}{4}, \\
& P(C)=\frac{1-2 p}{2} \\
& 0 \leq P(A) \leq 1 \\
& 0 \leq P(B) \leq 1 \\
& 0 \leq P(C) \leq 1 \\
& 0 \leq \frac{1+3 p}{3} \leq 1 \\
& 0 \leq \frac{1-p}{4} \leq 1 \\
& 0 \leq \frac{1-2 p}{2} \leq 1 \\
& 0 \leq 1+3 p \leq 3 \\
& 0 \leq 1-p \leq 4 \\
& 0 \leq 1-2 p \leq 2 \\
& -1 \leq 3 p \leq 2 \\
& -1 \leq-p \leq 3 \\
& -1 \leq-2 p \leq 1 \\
& -\frac{1}{3} \leq p \leq \frac{2}{3}  \tag{1}\\
& -3 \leq p \leq 1  \tag{3}\\
& -\frac{1}{2} \leq p \leq \frac{1}{2}  \tag{2}\\
& \text { Also } 0 \leq P(A \cup B \cup C) \leq 1 \\
& \Rightarrow 0 \leq P(A)+P(B)+P(C) \leq 1 \\
& \Rightarrow 0 \leq \frac{1+3 p}{3}+\frac{1-p}{4}+\frac{1-2 p}{2} \leq 1 \\
& \Rightarrow 0 \leq 4+12 p+3-3 p+6-12 p \leq 12 \\
& \Rightarrow 0 \leq 13-3 p \leq 12 \\
& -13 \leq-3 p \leq-1 \\
& \Rightarrow 1 \leq 3 p \leq 13 \\
& \Rightarrow \frac{1}{3} \leq \mathrm{p} \leq \frac{13}{3} \rightarrow(4)
\end{align*}
$$

Combining (1),(2), (3), (4) we get $\frac{1}{3} \leq p \leq \frac{1}{2}$.
6. If two numbers are selected randomly from 20 consecutive natural numbers, find the probability that the sum of two numbers is (i) an even number (ii) an odd number.
A: Given set is $\{1,2,3,4 \ldots, 19,20\}$
Consider the sets $\{2,4,6, \ldots .20\},\{1,3,5, \ldots . .19\}$.
Let $A$ be event that the sum of two numbers is even and $B$ be the event that sum of two numbers is odd when two numbers are selected from $\{1,2,3, \ldots .20\}$.
Sum of two numbers is even if both of them are even or both are odd.
$\mathrm{P}(\mathrm{B})=\frac{\mathrm{n}(\mathrm{A})}{\mathrm{n}(\mathrm{S})}=\frac{{ }^{10} \mathrm{C}_{2}+{ }^{10} \mathrm{C}_{2}}{{ }^{20} \mathrm{C}_{2}}=\frac{45+45}{\frac{20 \times 19}{2}}=\frac{90}{190}=\frac{9}{19}$
Sum of two numbers is odd if one number is even, one is odd.
$P(B)=\frac{\mathrm{n}(\mathrm{A})}{\mathrm{n}(\mathrm{S})}=\frac{{ }^{10} \mathrm{C}_{1} \cdot{ }^{10} \mathrm{C}_{1}}{{ }^{20} \mathrm{C}_{2}}=\frac{10.10}{190}=\frac{100}{190}=\frac{10}{19}$.
7. A bag contains 12 two rupee coins, 7 one rupee coins and 4 half a rupee coins. If three of them are selected at random, then find the probability that (i) the sum of three coins is maximum (ii) the sum of three coins is minimum (iii) each coin is of different value.
A: Number of two rupee coins $=12$
Number of one rupee coins $=7$
Number of half a rupee coins $=4$
Total number of coins $=23$
(i) To have sum of three coins as maximum, we shall select all the three coins are from two rupee coins.
$\therefore$ Probability that the sum of three coins is maximum $=\frac{{ }^{12} \mathrm{C}_{3}}{{ }^{23} \mathrm{C}_{3}}=\frac{12.11 .10}{6} \times \frac{6}{23.22 .21}=\frac{20}{161}$
(ii) To have sum of three coins as minimum, we shall select all the three coins are from half a rupee coins.
$\therefore$ Probability that the sum of three coins is minimum $=\frac{{ }^{4} \mathrm{C}_{3}}{{ }^{23} \mathrm{C}_{3}}=\frac{4.3 .2}{6} \times \frac{6}{23.22 .21}=\frac{4}{1771}$
(iii) Probability that each coin is of different value when three coins are selected

$$
=\frac{{ }^{12} \mathrm{C}_{1} \cdot{ }^{7} \mathrm{C}_{1} \cdot{ }^{4} \mathrm{C}_{1}}{{ }^{23} \mathrm{C}_{3}}=\frac{12.7 .4}{1} \times \frac{6}{23.22 .21}=\frac{48}{253}
$$

8. In a box containing 15 bulbs, 5 are defective. If 5 bulbs are selected at random from the box, find the probability of the event, that (i) None of them is defective (ii) Atleast one of them is defective
(iii) Only one of them is defective

A: Total number of bulbs $=15$
Number of defective bulbs $=5$
$\therefore$ Number of good bulbs $=10$
(i) Probability that none of them is defective, when 5 bulbs are selected. $=\frac{{ }^{10} \mathrm{C}_{5}}{{ }^{15} \mathrm{C}_{5}}=\frac{10.9 .8 .7 .6}{5!} \times \frac{5!}{15.14 .13 .12 .11}=\frac{12}{143}$
(ii) Probability that atleast one bulb is defective $=1$ - probability that none of them is defective $=1-\frac{12}{143}=\frac{143-12}{143}=\frac{131}{143}$
(iii) Probability that only one of them is defective $=\frac{{ }^{5} \mathrm{C}_{1} \cdot{ }^{10} \mathrm{C}_{4}}{{ }^{15} \mathrm{C}_{5}}=\frac{5 \cdot 10.9 .8 \cdot 7}{24} \times \frac{120}{15 \cdot 14.13 .12 .11}=\frac{50}{143}$.

## PROBABILITY Q.NO 28

1. $A, B, C$ are aiming to shoot a balloon. A will succeed 4 times out of 5 attempts. The chance of $B$ to shoot the balloon is 3 out of 4 and that of $C$ is 2 out of 3 . If the three aim the balloon simultaneously, then find the probability that atleast two of them hit the balloon.
$A$ : Let $A, B, C$ be the events that the shooters $A, B, C$ succeed in shooting the balloon.
Given that $P(A)=\frac{4}{5}, P(B)=\frac{3}{4}, P(C)=\frac{2}{3}$
Clearly $A, B, C$ are indepedent events.
Probability that atleast two of them hit the balloon $=P(A \cap B \cap \bar{C})+P(A \cap \bar{B} \cap C)+P(\bar{A} \cap B \cap C)+P(A \cap B \cap C)$
$P(A) P(B) P(C)+P(A) P(\bar{B}) P(C)+P(\bar{A}) P(B) P(C)+P(A) P(B) P(C)$

$$
\begin{aligned}
& =\frac{4}{5} \cdot \frac{3}{4}\left(1-\frac{2}{3}\right)+\frac{4}{5}\left(1-\frac{3}{4}\right)\left(\frac{2}{3}\right)+\left(1-\frac{4}{5}\right)\left(\frac{3}{4}\right)\left(\frac{2}{3}\right)+\left(\frac{4}{5}\right)\left(\frac{3}{4}\right)\left(\frac{2}{3}\right) \\
& =\frac{4}{5} \cdot \frac{3}{4} \cdot \frac{1}{3}+\frac{4}{5} \cdot \frac{1}{4} \cdot \frac{2}{3}+\frac{1}{5} \cdot \frac{3}{4} \cdot \frac{2}{3}+\frac{4}{5} \cdot \frac{3}{4} \cdot \frac{2}{3}=\frac{12+8+6+24}{60}=\frac{50}{60}=\frac{5}{6}
\end{aligned}
$$

2. In a shooting test the probabilities of $A, B, C$ hitting the targets are $\frac{1}{2}, \frac{2}{3}$ and $\frac{3}{4}$ respectively. If all of them fire at the same time, find the probability that i) Only one of them hits the target
ii) Atleast one of them hits the target
$A$ : Let $A, B, C$ be the events that the three persons $A, B, C$ respectively hitting the target.
Given that $P(A)=\frac{1}{2}, P(B)=\frac{2}{3}, P(C)=\frac{3}{4}$
Clearly the events are independent.
i) Probability that only one of them hits the target.

$$
\begin{aligned}
P(A \cap \bar{B} & \cap \bar{C})+P(\bar{A} \cap B \cap \bar{C})+P(\bar{A} \cap \bar{B} \cap C)=P(A) P(\bar{B}) P(\bar{C})+P(\bar{A}) P(B) P(\bar{C})+P(\bar{A}) P(\bar{B}) P(C) \\
& =\frac{1}{2}\left(1-\frac{2}{3}\right)\left(1-\frac{3}{4}\right)+\left(1-\frac{1}{2}\right)\left(\frac{2}{3}\right)\left(1-\frac{3}{4}\right)+\left(1-\frac{1}{2}\right)\left(1-\frac{2}{3}\right)+\left(\frac{3}{4}\right) \\
& =\frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{4}+\frac{1}{2} \cdot \frac{2}{3} \cdot \frac{1}{4}+\frac{1}{2} \cdot \frac{1}{3} \cdot \frac{3}{4} \\
& =\frac{1+2+3}{24}=\frac{6}{24}=\frac{1}{4} .
\end{aligned}
$$

ii) Probability that atleast one of them hits the target $=1$ - probability that all the three fail to hit the target

$$
\begin{aligned}
1-\mathrm{P}(\overline{\mathrm{~A}} \cap \overline{\mathrm{~B}} \cap \overline{\mathrm{C}}) & =1-\mathrm{P}(\overline{\mathrm{~A}}) \mathrm{P}(\overline{\mathrm{~B}}) \mathrm{P}(\overline{\mathrm{C}}) \\
& =1-\left(1-\frac{1}{2}\right)\left(1-\frac{2}{3}\right)\left(1-\frac{3}{4}\right) \\
& =1-\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)\left(\frac{1}{4}\right) \\
& =1-\frac{1}{24}=\frac{23}{24} .
\end{aligned}
$$

3. If $A, B, C$ are three independent events of an experiment such that $P(A \cap \overline{\mathbf{B}} \cap \overline{\mathbf{C}})=\frac{\mathbf{1}}{4}, \mathbf{P}(\overline{\mathbf{A}} \cap \mathbf{B} \cap \overline{\mathbf{C}})=\frac{\mathbf{1}}{\mathbf{8}}$, $\mathbf{P}(\overline{\mathbf{A}} \cap \overline{\mathbf{B}} \cap \overline{\mathbf{C}})=\frac{1}{4}$ then find $\mathrm{P}(\mathrm{A}), \mathrm{P}(\mathrm{B})$ and $\mathrm{P}(\mathrm{C})$.
A : Given that $\mathrm{A}, \mathrm{B}, \mathrm{C}$ are independent events and
$P(A \cap \bar{B} \cap \bar{C})=\frac{1}{4} \Rightarrow P(A) P(\bar{B}) P(\bar{C})=\frac{1}{4}----(1)$
$P(\bar{A} \cap B \cap \bar{C})=\frac{1}{8} \Rightarrow P(\bar{A}) P(B) P(\bar{C})=\frac{1}{8}$
$\mathrm{P}(\overline{\mathrm{A}} \cap \overline{\mathrm{B}} \cap \overline{\mathrm{C}})=\frac{1}{4} \Rightarrow \mathrm{P}(\overline{\mathrm{A}}) \mathrm{P}(\overline{\mathrm{B}}) \mathrm{P}(\overline{\mathrm{C}})=\frac{1}{4}$
$\frac{(1)}{(3)} \Rightarrow \frac{P(\bar{A}) P(\bar{B}) P(\bar{C})}{P(\bar{A}) P(\bar{B}) P(\bar{C})}=\frac{\left(\frac{1}{4}\right)}{\left(\frac{1}{4}\right)}$

$$
\begin{aligned}
\frac{P(A)}{1-P(A)} & =1 \\
P(A) & =1-P(A) \\
2 P(A) & =1 \\
P(A) & =\frac{1}{2}
\end{aligned}
$$

$\frac{(2)}{(3)} \Rightarrow \frac{\mathrm{P}(\overline{\mathrm{A}}) \mathrm{P}(\mathrm{B}) \mathrm{P}(\overline{\mathrm{C}})}{\mathrm{P}(\overline{\mathrm{A}}) \mathrm{P}(\overline{\mathrm{B}}) \mathrm{P}(\overline{\mathrm{C}})}=\frac{\left(\frac{1}{8}\right)}{\left(\frac{1}{4}\right)}$

$$
\begin{aligned}
\frac{P(B)}{1-P(B)} & =\frac{1}{2} \\
2 P(B) & =1-P(B) \\
3 P(B) & =1 \\
P(B) & =\frac{1}{3}
\end{aligned}
$$

From (1), $P(A) P(\bar{B}) P(\bar{C})=\frac{1}{4}$

$$
\begin{gathered}
\frac{1}{2}\left(1-\frac{1}{3}\right) P(\bar{C})=\frac{1}{4} \\
\frac{1}{2}\left(\frac{2}{3}\right) P(\bar{C})=\frac{1}{4} \\
P(\bar{C})=\frac{3}{4} \\
P(C)=1-P(\bar{C})=1-\frac{3}{4}=\frac{1}{4} \\
\therefore P(A)=\frac{1}{2}, P(B)=\frac{1}{3}, P(C)=\frac{1}{4} .
\end{gathered}
$$

4. Define conditional probability. There are 3 black and 4 white balls in first bag; 4 black and 3 white balls in the second bag. A die is rolled and the first bag is selected if it is 1 or 3 , and the second bag for the rest. Find the probability of drawing a black ball from the selected bag.

A: Conditional probability : If $A, B$ are two events in a sample space $S$ and $P(A) \neq 0$, then the probability of $B$ after The event $A$ has occured is called conditional probability of $B$ given $A$. It is denoted by $P\left(\frac{B}{A}\right)$.

$$
P\left(\frac{B}{A}\right)=\frac{n(A \cap B)}{n(A)}=\frac{P(A \cap B)^{\prime \prime}}{P(A)}
$$

Let $A_{1}, A_{2}$ be the events of selecting first and second bags respectively
Let $E$ be the event of drawing a black ball from the selected bag.
Now $P\left(A_{1}\right)=\frac{2}{6}=\frac{1}{3}$ and $P\left(A_{2}\right)=\frac{4}{6}=\frac{2}{3}$
Probability of drawing a black ball from the first bag $P\left(\frac{E}{A_{1}}\right)=\frac{{ }^{3} C_{1}}{{ }^{7} C_{1}}=\frac{3}{7}$
Probability of drawing a black ball from the second bag $P\left(\frac{E}{A_{2}}\right)=\frac{{ }^{4} C_{1}}{{ }^{7} C_{1}}=\frac{4}{7}$
By total probability theorem, $P(E)=P\left(A_{1}\right) P\left(\frac{E}{A_{1}}\right)+P\left(A_{2}\right) P\left(\frac{E}{A_{2}}\right)=\frac{1}{3}\left(\frac{3}{7}\right)+\frac{2}{3}\left(\frac{4}{7}\right)=\frac{3+8}{21}=\frac{11}{21}$.

5 The probability that Australia wins a match against India in a cricket game is given to be $\frac{1}{3}$. If India and Austrlia play 3 matches, what is the probability that (i) Australia will loose all the three matches?
ii) Australia will win atleast one match?

A: Let $A_{1}, A_{2}, A_{3}$ be the events that Australia wins a match against India in the $1^{\text {st }}, 2^{\text {nd }}, 3^{\text {rd }}$ games respectively.
Given that $P\left(A_{1}\right)=P\left(A_{2}\right)=P\left(A_{3}\right)=\frac{1}{3}$
i) Probability that Australia will loose all the three matches $P\left(\overline{A_{1}} \cap \overline{A_{2}} \cap \overline{A_{3}}\right)=P\left(\overline{A_{1}}\right) P\left(\overline{A_{2}}\right) P\left(\overline{A_{3}}\right)$

$$
=\left(1-\frac{1}{3}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{3}\right)=\left(\frac{2}{3}\right)\left(\frac{2}{3}\right)\left(\frac{2}{3}\right)=\frac{8}{27} .
$$

ii) Probability that Australia will win atleast one match = 1 - probability that Australia will loose all the three matches

$$
=1-\frac{8}{27}=\frac{27-8}{27}=\frac{19}{27} .
$$

6. Three screws are drawn at random from a lot of 50 screws, 5 of which are defective. Find the probability of the event that all 3 screws are non - defective, assuming that the drawing is i) with replacement ii) without replacement.

A: Number of defective screws $=5$
Number of good screws $=45$
Total number of screws $=50$.
i) Consider the drawing of 3 screws with replacement.

Let $A, B, C$ be the events of drawing $1^{\text {st }}, 2^{\text {nd }}, 3^{\text {rd }}$ screws as non - defective, when three screws are drawn with replacement. Now $P(A \cap B \cap C)=P(A) P\left(\frac{B}{A}\right) P\left(\frac{C}{A \cap B}\right)=\frac{{ }^{45} C_{1}}{{ }^{50} C_{1}} \cdot \frac{{ }^{45} C_{1}}{{ }^{50} C_{1}} \cdot \frac{{ }^{45} C_{1}}{{ }^{50} C_{1}}=\frac{9^{3}}{10^{3}}=\frac{729}{1000}$.
ii) Consider the drawing of 3 screws without replacement.

Let $A, B, C$ be the events of drawing $1^{\text {st }}, 2^{\text {nd }}, 3^{\text {rd }}$ screws as non defective, when three screws are drawn without replacement.

Required probability, $P(A \cap B \cap C)=P(A) P\left(\frac{B}{A}\right) P\left(\frac{C}{A \cap B}\right)=\frac{{ }^{45} C_{1}}{{ }^{50} C_{1}} \cdot \frac{{ }^{44} C_{1}}{{ }^{49} C_{1}} \cdot \frac{{ }^{43} C_{1}}{{ }^{48} C_{1}}=\frac{45}{50} \cdot \frac{44}{49} \cdot \frac{43}{48}=\frac{1419}{1960}$.

## RANDOM VARIABLE \& DISTRIBUTIONS Q.NO 29

1. The Probability distribution of a random variable $X$ is given below.

| $X=x_{i}$ | 1 | 2 | 3 | 4 | 5 |
| :--- | :--- | :--- | :--- | :--- | :--- |
| $P\left(X=x_{i}\right)$ | $k$ | $2 k$ | $3 k$ | $4 k$ | $5 k$ |

Find the value of $k$ and the mean and variance of $X$.
$A$ : Given that $X$ is a random variable
Sum of all probabilities $\sum_{i=1}^{5} P\left(X=x_{i}\right)=1$

$$
\begin{aligned}
k+2 k+3 k+4 k+5 k & =1 \\
15 k & =1 \\
k & =\frac{1}{15}
\end{aligned}
$$

Let $\mu$ be the mean and $\sigma^{2}$ be the variance of $X$
is the probability distribution of a random variable $X$. Find the value of $k$ and the variance of $X$.
A: Given that $X$ is a random variable.
Let $\mu$ be the mean and $\sigma^{2}$ be the variance of $X$.

$$
\begin{aligned}
& \text { Sum of all probabilities } \begin{aligned}
\sum_{i=-2}^{3} P\left(X=x_{i}\right) & =1 \\
0.1+k+0.2+2 k+0.3+k & =1 \\
4 k+0.6 & =1 \\
4 k & =1-0.6=0.4 \\
k & =0.1 .
\end{aligned}
\end{aligned}
$$

$$
\text { Mean } \mu=\sum_{i=-2}^{3} x_{i} \cdot P\left(X=x_{i}\right)
$$

$$
=(-2)(0.1)-1(0.1)+0(0.2)+1(0.2)+2(0.3)+3(0.1)
$$

$$
=0.8
$$

We know that
Variance, $\sigma^{2}=\sum_{i=-2}^{3} x_{i}^{2} \cdot P\left(X=x_{i}\right)-\mu^{2}$

$$
\begin{aligned}
\sigma^{2}+0.8^{2} & =(-2)^{2}(0.1)+(-1)^{2}(0.1)+0^{2}(0.2)+\left(1^{2}\right)(0.2)+2^{2}(0.3)+3^{2}(0.1) \\
\sigma^{2} & =2.80-0.64 \\
\sigma^{2} & =2.16
\end{aligned}
$$

$$
\begin{aligned}
& \text { Mean, } \mu=\sum_{i=1}^{5} x_{i} \cdot P\left(X=x_{i}\right) \\
& =1(k)+2(2 k)+3(3 k)+4(4 k)+5(5 k) \\
& =\mathrm{k}+4 \mathrm{k}+9 \mathrm{k}+16 \mathrm{k}+25 \mathrm{k} \\
& =55 \mathrm{k}=\frac{55}{15}=\frac{11}{3} \text {. } \\
& \text { Variance, } \sigma^{2}=\sum_{i=1}^{5} x_{i}^{2} P\left(X=x_{i}\right)-\mu^{2} \\
& =1(k)+4(2 k)+9(3 k)+16(4 k)+25(5 k)-\left(\frac{11}{3}\right)^{2} \\
& =225 \mathrm{k}-\frac{121}{9} \\
& =225\left(\frac{1}{15}\right)-\frac{121}{9} \\
& =15-\frac{121}{9}=\frac{135-121}{9}=\frac{14}{9} \text {. } \\
& \text { 2. } X=x_{i} \quad-2 \quad-1 \quad 0 \quad 1 \quad 2 \quad 3 \\
& \begin{array}{lllllll}
\mathrm{P}\left(\mathrm{X}=\mathrm{x}_{\mathrm{i}}\right) & 0.1 & \mathrm{k} & 0.2 & 2 k & 0.3 & \mathrm{k}
\end{array}
\end{aligned}
$$

3. A random variable $X$ has the following probability distribution.

| X | $=\mathrm{x}_{\mathrm{i}}$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| ---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\mathrm{P}(\mathrm{X}$ | $\left.=\mathrm{x}_{\mathrm{i}}\right)$ | 0 | k | 2 k | 2 k | 3 k | $\mathrm{k}^{2}$ | $2 \mathrm{k}^{2}$ |

Find (i) $k$ (ii) the mean (iii) $P(0<X<5)$
A: Given that $X$ is a random variable

$$
\begin{aligned}
\text { Sum of all probabilities } \sum_{i=0} P\left(X=x_{i}\right) & =1 \\
0+k+2 k+2 k+3 k+k^{2}+2 k^{2}+7 k^{2}+k & =1 \\
10 k^{2}+9 k-1 & =0 \\
10 k^{2}+10 k-k-1 & =0 \\
10 k(k+1)-1(k+1) & =0 \\
(k+1)(10 k-1) & =0 \\
k=-1 \text { is false or } k & =\frac{1}{10} .
\end{aligned}
$$

Let $\mu$ be the mean of $X$.

$$
\text { Mean, } \begin{aligned}
\mu & =\sum_{i=0}^{7} x_{i} P\left(X=x_{i}\right) \\
& =0(0)+1(0.1)+2(0.2)+3(0.2)+4(0.3)+5(0.01)+6(0.02)+7(0.17) \\
& =0.1+0.4+0.6+1.2+0.05+0.12+1.19 \\
& =3.66
\end{aligned}
$$

$$
P(0<X<5)=P(X=1)+P(X=2)+P(X=3)+P(X=4)
$$

$$
=\mathrm{k}+2 \mathrm{k}+2 \mathrm{k}+3 \mathrm{k}
$$

$$
=8 \mathrm{k}
$$

$$
=8(0.1)
$$

$$
=0.8 .
$$

4. A cubical die is thrown. Find the mean and variance of $X$, giving the number on the face that shows up.

A: Let $X$ be the number on the face that shows up when a die is thrown. Here $X$ is a random variable. Probability distribution of $X$ is shown below.

| $\mathrm{X}=\mathrm{x}_{\mathrm{i}}$ | 1 | 2 | 3 | 4 | 5 | 6 |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: |
| $\mathrm{P}\left(\mathrm{X}=\mathrm{x}_{\mathrm{i}}\right)$ | $\frac{\mathbf{1}}{\mathbf{6}}$ | $\frac{\mathbf{1}}{\mathbf{6}}$ | $\frac{\mathbf{1}}{\mathbf{6}}$ | $\frac{\mathbf{1}}{\mathbf{6}}$ | $\frac{\mathbf{1}}{\mathbf{6}}$ | $\frac{\mathbf{1}}{\mathbf{6}}$ |

Let $\mu$ be the mean and $\sigma^{2}$ be the variance of $X$

$$
\begin{aligned}
\mu & =\sum_{i=1}^{6} x_{i} P\left(X=x_{i}\right) \\
& =1\left(\frac{1}{6}\right)+2\left(\frac{1}{6}\right)+3\left(\frac{1}{6}\right)+4\left(\frac{1}{6}\right)+5\left(\frac{1}{6}\right)+6\left(\frac{1}{6}\right)=\frac{21}{6}=\frac{7}{2} . \\
\sigma^{2} & =\sum_{i=1}^{6} x_{i}^{2} P\left(X=x_{i}\right)-\mu^{2} \\
& =1^{2}\left(\frac{1}{6}\right)+2^{2}\left(\frac{1}{6}\right)+3^{2}\left(\frac{1}{6}\right)+4^{2}\left(\frac{1}{6}\right)+5^{2}\left(\frac{1}{6}\right)+6^{2}\left(\frac{1}{6}\right)-\left(\frac{7}{2}\right)^{2}=\frac{91}{6}-\frac{49}{4}=\frac{182-147}{12}=\frac{35}{12} .
\end{aligned}
$$

5. The range of random variable $X$ is $\{0,1,2\}$, Given that $P(X=0)=3 C^{3}, P(X=1)=4 C-10 C^{2}, P(X=2)=5 C-1$.

Find (i) the value of $C$
(ii) $P(X<1)$
(iii) $\mathrm{P}(1<\mathrm{X} \leq 2)$
(iv) $\mathrm{P}(0<\mathrm{X} \leq 3)$

A: Given that range of random variable $X$ is $\{0,1,2\}$.
$P(X=0)=3 C^{3}, P(X=1)=4 C-10 C^{2}, P(X=2)=5 C-1$
Since $X$ is a random variable,
Sum of all probabilities, $\sum_{i=0}^{2} P\left(X=x_{i}\right)=1$

$$
\begin{array}{r}
P(X=0)+P(X=1)+P(X=2)=1 \\
3 C^{3}+4 C-10 C^{2}+5 C-1=1 \\
3 C^{3}-10 C^{2}+9 C-2=0
\end{array}
$$

Clearly C = 1 satisfies this equation By Synthetic division,

$$
\begin{array}{l|cccc} 
& 3 & -10 & 9 & -2 \\
1 & 0 & 3 & -7 & 2 \\
\hline & 3 & -7 & 2 & \boxed{0}
\end{array}
$$

Now the above equation becomes, $(C-1)\left(3 C^{2}-7 C+2\right)=0$
$C=1,3 C^{2}-6 C-C+2=0$

$$
3 C(C-2)-1(C-2)=0
$$

$$
(C-2)(3 C-1)=0 \quad \therefore C=1,2, \frac{1}{3} .
$$

$C=1,2$ are not possible. So, the value of $C=\frac{1}{3}$.
ii) $P(X<1)=P(X=0)=3 C^{3}=3\left(\frac{1}{3}\right)^{3}=\frac{1}{9}$.
iii) $P(1<x \leq 2)=P(X=2)=5 C-1=\frac{5}{3}-1=\frac{2}{3}$.
iv) $\mathrm{P}(0<\mathrm{X} \leq 3)=\mathrm{P}(\mathrm{X}=1)+\mathrm{P}(\mathrm{X}=2)=\frac{2}{9}+\frac{2}{3}=\frac{8}{9}$.
6. The range of a random variable $X$ is $\{1,2,3, \ldots\}$ and $P(X=k)=\frac{C^{k}}{k!}, k=1,2,3, \ldots \ldots$. Find the value of $C$ and $P(0<X<3)$.
A: Given that $X$ is a random variable $\Rightarrow \sum_{k=1}^{\infty} P(X=k)=1 \Rightarrow \sum_{k=1}^{\infty} \frac{C^{k}}{k!}=1$

$$
\Rightarrow \frac{C}{1!}+\frac{C^{2}}{2!}+\frac{C^{3}}{3!}+\ldots \infty=1
$$

Adding 1 on both sides $\Rightarrow 1+\frac{C}{1!}+\frac{C^{2}}{2!}+\frac{C^{3}}{3!}+\ldots \infty=1+1$

$$
\begin{aligned}
& e^{C}=2 \\
& C=\log _{\mathrm{e}} 2
\end{aligned}
$$

ii) $\mathrm{P}(0<\mathrm{X}<3) \Rightarrow \mathrm{P}(\mathrm{X}=1)+\mathrm{P}(\mathrm{X}=2)=\frac{C}{1!}+\frac{C^{2}}{2!}$

$$
=\log _{\mathrm{e}} 2+\frac{1}{2}\left(\log _{e} 2\right)^{2}
$$

7. If $X$ is a random variable with the probablity distribution $P(X=k)=\frac{(k+1) C}{2^{k}}(k=0,1,2, \ldots$.$) then find C$ ? A: Sum of the infinite A.G.P $(|r|<1)$.

$$
a(1)+(a+d) r+(a+2 d) r^{2}+\ldots \ldots \infty=\frac{a}{1-r}+\frac{d r}{(1-r)^{2}}
$$

Given that $X$ is a random variable with
$P(X=k)=\frac{(k+1) C}{2^{k}}, k=0,1,2 \ldots$
$\sum_{k=0}^{\infty} P(X=k)=1$
$\sum_{k=0}^{\infty} \frac{(k+1) C}{2^{k}}=1$.
$C \sum_{k=0}^{\infty}(k+1) \cdot \frac{1}{2^{k}}=1$
$C\left[1.1+2 \cdot \frac{1}{2}+3 \cdot \frac{1}{2^{2}}+4 \cdot \frac{1}{2^{3}}+\ldots \ldots \ldots . . \infty\right]=1$; Here $a=1, d=1, r=\frac{1}{2}$
$C\left[\frac{a}{1-r}+\frac{d r}{(1-r)^{2}}\right]=1$.
c $\left[\frac{1}{1-\frac{1}{2}}+\frac{1 \cdot \frac{1}{2}}{\left(1-\frac{1}{2}\right)^{2}}\right]=1$
$C\left[\frac{1}{\frac{1}{2}}+\frac{\frac{1}{2}}{\frac{1}{4}}\right]=1 \quad \Rightarrow C[2+2]=1 \quad \Rightarrow 4 C=1 \quad \Rightarrow C=\frac{1}{4}$.

## RANDOM VARIABLE \& DISTRIBUTIONS Q.NO 30

1. One in nine ships is likely to be wrecked when they set on sail. When 6 ships are set on sail, find the probablity for (i) atleast one will arrive safely, (ii) exactly three will arrive safely.
A: Let $X$ be the number of ships arriving safely, when six ships are set on sail. $X$ follows binomial distribution, Let $\mathrm{n}, \mathrm{p}$ be the parameters.
then $\mathrm{n}=6, \mathrm{q}=\frac{1}{9}$

$$
p=1-q=1-\frac{1}{9}=\frac{8}{9}
$$

(i) Probablity that atleast one ship will arrive safely $=P(X \geq 1)$

$$
1-P(X=0)=1-{ }^{6} C_{0}\left(\frac{1}{9}\right)^{6}\left(\frac{8}{9}\right)^{0}=1-\frac{1}{9^{6}}
$$

(ii) Probablity that exactly three ships will arrive safely $=P(X=3)$

$$
={ }^{6} C_{3}\left(\frac{1}{9}\right)^{3}\left(\frac{8}{9}\right)^{3}=20\left(\frac{8^{3}}{9^{6}}\right)
$$

2. If the mean and variance of a binomial variate $X$ are 2.4 and 1.44 respectively find $P(1<X \leq 4)$.

A: Let $\mathrm{n}, \mathrm{p}$ be the parameters of the binomial distribution.
Given that $n p=24, n p q=1.44$
Now $q=\frac{n p q}{n p}=\frac{1.44}{1.4}=\frac{144}{240}=\frac{12}{20}=\frac{3}{5}$
$\therefore \mathrm{p}=1-\mathrm{q}=1-\frac{3}{5}=\frac{2}{5}$
Also $\mathrm{np}=2.4$
$\mathrm{n}\left(\frac{2}{5}\right)=2.4=\frac{24}{10}=\frac{12}{5}$

$$
n=6
$$

Required probablity $P(1<X \leq 4)=P(X=2)+P(X=3)+P(X=4)$

$$
\begin{aligned}
& ={ }^{6} \mathrm{C}_{2}\left(\frac{3}{5}\right)^{4}\left(\frac{2}{5}\right)^{2}+{ }^{6} \mathrm{C}_{3}\left(\frac{3}{5}\right)^{3}\left(\frac{2}{5}\right)^{3}+{ }^{6} \mathrm{C}_{4}\left(\frac{3}{5}\right)^{2}\left(\frac{2}{5}\right)^{4} \\
& =\frac{15(81) 4}{5^{6}}+\frac{20(27)(8)}{5^{6}}+\frac{15(9)(16)}{5^{6}} \\
& =\frac{972+864+432}{5^{5}} \\
& =\frac{2268}{3125}
\end{aligned}
$$

3. In the experiment of tossing a coin $n$ times, if the variable $X$ denotes the number of heads and $P(X=4)$, $P(X=5), P(X=6)$ are in A.P, then find $n$.
$A$ : Let $X$ denotes the number of heads, when a coin is tossed for $n$ times.
Let $\mathrm{n}, \mathrm{p}$ be the parameters of binomial distribution.
Here $p=\frac{1}{2} ; q=1-p=1-\frac{1}{2}=\frac{1}{2}$
Given that $P(X=4), P(X=5), P(X=6)$ are in A.P
${ }^{n} C_{4}\left(\frac{1}{2}\right)^{n-4}\left(\frac{1}{2}\right)^{4},{ }^{n} C_{5}\left(\frac{1}{2}\right)^{n-5}\left(\frac{1}{2}\right)^{5},{ }^{n} C_{6}\left(\frac{1}{2}\right)^{n-6}\left(\frac{1}{2}\right)^{6}$ are in A.P.
${ }^{n} C_{4} \cdot \frac{1}{2^{n}},{ }^{n} C_{5} \cdot \frac{1}{2^{n}},{ }^{n} C_{6} \frac{1}{2^{n}}$ are in A.P
${ }^{n} C_{4},{ }^{n} C_{5},{ }^{n} C_{6}$, are in A.P
$2 b=a+c$
4. ${ }^{n} \mathrm{C}_{5}={ }^{\mathrm{n}} \mathrm{C}_{4}+{ }^{\mathrm{n}} \mathrm{C}_{6}$
$2=\frac{{ }^{n} C_{4}}{{ }^{n} C_{5}}+\frac{{ }^{n} C_{6}}{{ }^{n} C_{5}}$
$2=\frac{5}{n-4}+\frac{n-5}{6}$
$2=\frac{30+(n-4)(n-5)}{6(n-4)}$
$12(n-4)=30+n^{2}-9 n+20$
$12 n-48=30+n^{2}-9 n+20$
$n^{2}-21 n+98=0$
$n^{2}-7 n-14 n+98=0$
$\mathrm{n}(\mathrm{n}-7)-14(\mathrm{n}-7)=0$
$\mathrm{n}=7$ or 14 .
5. If the difference between the mean and the variance of a binomial distribution is $\frac{\mathbf{5}}{\mathbf{9}}$, then find the probablity for the event of 2 successes, when the experiment is conducted 5 times.
A: Let $\mathrm{n}, \mathrm{p}$ be the parameters of the binomial distribution.
Given that $\mathrm{n}=5$,

$$
\begin{aligned}
n p-n p q & =\frac{5}{9} \\
5 p(1-q) & =\frac{5}{9} \\
p^{2} & =\frac{1}{9} \\
p & =\frac{1}{3}
\end{aligned}
$$

Required probablity is $P(X=2)={ }^{5} C_{2}\left(\frac{2}{3}\right)^{3}\left(\frac{1}{3}\right)^{2}=10\left(\frac{8}{27}\right)\left(\frac{1}{9}\right)=\frac{80}{243}$.

## SHORT ANSWERS TYPE QUESTIONS <br> COMPLEX NUMBERS Q.NO 11

1. If $z=3-5 i$, then show that $z^{3}-10 z^{2}+58 z-136=0$.

A: Given $z=3-5 i$

$$
\begin{aligned}
& z-3=-5 i \\
& (z-3)^{2}=(-5 i)^{2} \\
& z^{2}-6 z+9=-25 . \\
& z^{2}-6 z+34=0 .
\end{aligned}
$$

Now $z^{3}-10 z^{2}+58 z-136=z\left(z^{2}-6 z+34\right)-4 z^{2}+24 z-136$.

$$
\begin{aligned}
& =z(0)-4\left(z^{2}-6 z+34\right) \\
& =0-4(0)=0 .
\end{aligned}
$$

$(H / W)$ If $z=2-i \sqrt{7}$, then show that $3 z^{3}-4 z^{2}+z+88=0$.
2. If $(x-i y)^{1 / 3}=a-i b$, then show that $\frac{x}{a}+\frac{y}{b}=4\left(a^{2}-b^{2}\right)$.

A: Given that $(x-i y)^{1 / 3}=a-i b$
cubing on both sides,

$$
\begin{aligned}
& x-i y=(a-i b)^{3} \\
& x-i y=a^{3}-3 a^{2} i b+3 a i^{2} b^{2}-i^{3} b^{3} \\
& x-i y=\left(a^{3}-3 a b^{2}\right)-i\left(3 a^{2} b-b^{3}\right)
\end{aligned}
$$

equating the real and imaginary parts on both sides
$x=a^{3}-3 a b^{2} ; y=3 a^{2} b-b^{3}$
Now $\frac{x}{a}+\frac{y}{b}=\frac{a^{3}-3 a b^{2}}{a}+\frac{3 a^{2} b-b^{3}}{b}=a^{2}-3 b^{2}+3 a^{2}-b^{2}=4 a^{2}-4 b^{2}=4\left(a^{2}-b^{2}\right)$.
3. Determine the locus of $z, z \neq 2 i$, such that $\operatorname{Re}\left(\frac{z-4}{z-2 i}\right)=0$.

A: Let $z=x+i y$

$$
\text { Now } \begin{aligned}
\frac{z-4}{z-2 i} & =\frac{x+i y-4}{x+i y-2 i}=\frac{(x-4)+i y}{x+i(y-2)} \cdot \frac{x-i(y-2)}{x-i(y-2)} \\
& =\frac{x^{2}-4 x+y^{2}-2 y+i(x y-x y+2 x+4 y-8)}{x^{2}+(y-2)^{2}}
\end{aligned}
$$

Real part of $\frac{z-4}{z-2 i}=\frac{x^{2}+y^{2}-4 x-2 y}{x^{2}+(y-2)^{2}}=0 \quad \therefore$ Equation of locus is $x^{2}+y^{2}-4 x-2 y=0$
4. Find the real values of $\theta$ in order that $\frac{3+2 i \sin \theta}{1-2 i \sin \theta}$ (i) real number $\quad$ (ii) purely imaginary number.

A: $\frac{3+2 i \sin \theta}{1-2 i \sin \theta}=\frac{3+2 i \sin \theta}{1-2 i \sin \theta} \times \frac{1+2 i \sin \theta}{1+2 i \sin \theta}$

$$
\begin{aligned}
& =\frac{3+6 i \sin \theta+2 i \sin \theta-4 \sin ^{2} \theta}{1+4 \sin ^{2} \theta} \\
& =\frac{3-4 \sin ^{2} \theta}{1+4 \sin ^{2} \theta}+i \frac{8 \sin \theta}{1+4 \sin ^{2} \theta}
\end{aligned}
$$

i) $\frac{3+2 i \sin \theta}{1-2 i \sin \theta}$ is a real number Its imaginary part is zero.

$$
\begin{array}{r}
\frac{8 \sin \theta}{1+4 \sin ^{2} \theta}=0 \\
\sin \theta=0
\end{array}
$$

General solution is $\theta=n \pi, n \in Z$.
ii) $\frac{3+2 i \sin \theta}{1-2 i \sin \theta}$ is a purely imaginary number Its real part is zero.

$$
\begin{aligned}
\frac{3-4 \sin ^{2} \theta}{1+4 \sin ^{2} \theta} & =0 \\
3-4 \sin ^{2} \theta & =0 \\
\sin ^{2} \theta=\frac{3}{4} & =\left(\frac{\sqrt{3}}{2}\right)^{2}=\sin ^{2} \frac{\pi}{3}
\end{aligned}
$$

General solution is $\theta=n \pi \pm \frac{\pi}{3}, n \in Z$.
5. If $x+i y=\frac{3}{2+\cos \theta+i \sin \theta}$ then, show that $x^{2}+y^{2}=4 x-3$.
A. Given that, $x+i y=\frac{3}{2+\cos \theta+i \sin \theta}$

$$
\begin{aligned}
\frac{3}{2+\cos \theta+i \sin \theta} & =\frac{3}{2+\cos \theta+i \sin \theta} \times \frac{2+\cos \theta-i \sin \theta}{2+\cos \theta-i \sin \theta} \\
& =\frac{3(2+\cos \theta-i \sin \theta)}{(2+\cos \theta)^{2}-i^{2} \sin ^{2} \theta} \\
& =\frac{3(2+\cos \theta)-3 i \sin \theta}{4+\cos ^{2} \theta+4 \cos \theta+\sin ^{2} \theta} \\
& =\frac{3(2+\cos \theta)-3 i \sin \theta}{4+\left(\sin ^{2} \theta+\cos ^{2} \theta\right)+4 \cos \theta} \\
& =\frac{3(2+\cos \theta)-3 i \sin \theta}{4+4 \cos \theta+1}
\end{aligned}
$$

$x+i y=\frac{3(2+\cos \theta)}{5+4 \cos \theta}-\frac{3 \sin \theta \cdot i}{5+4 \cos \theta}$
Comparing real and imaginary parts on both sides.
$x=\frac{3(2+\cos \theta)}{5+4 \cos \theta}, y=\frac{-3 \sin \theta}{5+4 \cos \theta}$
L.H.S.: $x^{2}+y^{2}=\left[\frac{3(2+\cos \theta)}{5+4 \cos \theta}\right]^{2}+\frac{(-3 \sin \theta)^{2}}{(5+4 \cos \theta)^{2}}=\frac{9\left(4+\cos ^{2} \theta+4 \cos \theta\right)+9 \sin ^{2} \theta}{(5+4 \cos \theta)^{2}}$

$$
=\frac{9\left(4+\cos ^{2} \theta+4 \cos \theta+\sin ^{2} \theta\right)}{(5+4 \cos \theta)^{2}}
$$

$$
=\frac{9(4+1+4 \cos \theta)}{(5+4 \cos \theta)^{2}}
$$

$$
\begin{equation*}
=\frac{9(5+4 \cos \theta)}{(5+4 \cos \theta)^{2}}=\frac{9}{5+4 \cos \theta} \tag{1}
\end{equation*}
$$

R.H.S. $: 4 x-3=\frac{4[3(2+\cos \theta)]}{5+4 \cos \theta}-3=\frac{24+12 \cos \theta-15-12 \cos \theta}{5+4 \cos \theta}=\frac{9}{5+4 \cos \theta}$

From (1) and (2)
L.H.S. $=$ R.H.S.
$x^{2}+y^{2}=4 x-3$
6. If $x+i y=\frac{1}{1+\cos \theta+i \sin \theta}$, show that $4 x^{2}-1=0$.

A: Now $x+i y=\frac{1}{1+\cos \theta+i \sin \theta}$

$$
\begin{aligned}
\frac{1}{1+\cos \theta+i \sin \theta} & =\frac{1}{2 \cos ^{2} \frac{\theta}{2}+i 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}} \\
& =\frac{1}{2 \cos \frac{\theta}{2}\left[\cos \frac{\theta}{2}+i \sin \frac{\theta}{2}\right]} \times \frac{\cos \frac{\theta}{2}-i \sin \frac{\theta}{2}}{\cos \frac{\theta}{2}-i \sin \frac{\theta}{2}}
\end{aligned}
$$

$$
\begin{aligned}
& =\frac{\cos \frac{\theta}{2}-i \sin \frac{\theta}{2}}{2 \cos \frac{\theta}{2}\left[\cos ^{2} \frac{\theta}{2}+\sin ^{2} \frac{\theta}{2}\right]} \\
& =\frac{\cos \frac{\theta}{2}-i \sin \frac{\theta}{2}}{2 \cos \frac{\theta}{2}} \\
& =\frac{1}{2}-i \frac{1}{2} \tan \frac{\theta}{2}
\end{aligned}
$$

Equating the real parts on both sides, $x=\frac{1}{2}=>2 x=1$
S.o.b.s $4 x^{2}=1 \Rightarrow 4 x^{2}-1=0$.
7. Express $\frac{4+2 i}{1-2 i}+\frac{3+4 i}{2+3 i}$ in the form $a+i b$.

Sol. Let, $z=\frac{4+2 i}{1-2 i}+\frac{3+4 i}{2+3 i}$

$$
\begin{aligned}
& =\frac{(4+2 i)(1+2 i)}{(1-2 i)(1+2 i)}+\frac{(3+4 i)(2-3 i)}{(2+3 i)(2-3 i)} \\
& =\frac{\left(4+2 i+8 i+4 i^{2}\right)}{1-4 i^{2}}+\frac{6-9 i+8 i-12 i^{2}}{4-9 i^{2}} \\
& =\frac{4+10 i-4}{1+4}+\frac{6-i+8 i-12 i^{2}}{4-9 i^{2}} \\
& =\frac{4+10 i-4}{1+4}+\frac{6-i+12}{4+9} \\
& =\frac{10 i}{5}+\frac{18-i}{13} \\
& =\frac{130 i+90-5 i}{65} \\
& =\frac{9 i+125 i}{65}=\frac{18}{13}+\frac{25}{13} i .
\end{aligned}
$$

8. If x and y are real numbers such that $\frac{(1+i) x-2 i}{3+i}+\frac{(2-3 i) y+i}{3-i}=\mathrm{i}$, then determine the values of $x$ and $y$.

Sol: Given that $\frac{(1+i) x-2 i}{3+i}+\frac{(2-3 i) y+i}{3-i}$

$$
\begin{gathered}
\frac{x+i(x-2)}{3+i}+\frac{2 y-i(3 y-1)}{3-i}=\frac{[x+i(x-2)]}{3+i} \times \frac{3-i}{3-i}+\frac{[2 y-i(3 y-1)]}{3-i} \times \frac{3+i}{3+i} \\
\frac{[x+i(x-2)](3-i)}{9+1}+\frac{[2 y-i(3 y-1)](3+i)}{9+1}=\frac{[3 x+3 i(x-2)-i(x+i(x-2))]}{10}+\frac{[6 y-3 i(3 y-1)+i(2 y-i(3 y-1))]}{10} \\
=\frac{3 x+3 i x-6 i-i x+x-2}{10}+\frac{6 y-9 i y+3 i+2 i y+3 y-1}{10}
\end{gathered}
$$

$$
=\frac{4 x+2 i x-6 i-2+9 y-7 i y+3 i-1}{10}
$$

Then $\frac{(4 x+9 y-3)+i(2 x-3-7 y)}{10}=\mathrm{i}$
Equating real and imaginary both sides

$$
\begin{align*}
& 4 x+9 y-3=0 \text { and } \frac{2 x-3-7 y}{10}=1 \\
& 4 x+9 y-3=0----(1)  \tag{1}\\
& 2 x-7 y-13=0----(2) \tag{2}
\end{align*}
$$

Solving (1) and (2) we get

$$
23 y=-23 \quad \text { then } \mathbf{y}=-\mathbf{1}
$$

Sub in (2), $2 x+7-13=0 ; 2 x-6=0$ then $x=3$.
9. If $a=\cos \alpha+i \sin \alpha$ and $b=\cos \beta+i \sin \beta$, then find $\frac{1}{2}\left(a b+\frac{1}{a b}\right)$.

A: Now ab $=($ cis $\alpha)($ cis $\beta$ )

$$
\begin{aligned}
& =\operatorname{cis}(\alpha+\beta) \\
& =\cos (\alpha+\beta)+i \sin (\alpha+\beta)
\end{aligned}
$$

$$
\frac{1}{\mathrm{ab}}=\frac{1}{\cos (\alpha+\beta)+i \sin (\alpha+\beta)} \times \frac{\cos (\alpha+\beta)-i \sin (\alpha+\beta)}{\cos (\alpha+\beta)-i \sin (\alpha+\beta)}
$$

$$
=\frac{\cos (\alpha+\beta)-i \sin (\alpha+\beta)}{1}
$$

$$
=\cos (\alpha+\beta)-i \sin (\alpha+\beta)
$$

$a b+\frac{1}{a b}=\cos (\alpha+\beta)+i \sin (\alpha+\beta)+\cos (\alpha+\beta)-i \sin (\alpha+\beta)$

$$
=2 \cos (\alpha+\beta)
$$

$\frac{1}{2}\left(a b+\frac{1}{a b}\right)=\cos (\alpha+\beta)$.

## DE MOIVRES THEOREM Q.NO 12

1. If $m, n$ are integers and $x=\cos \alpha+i \sin \alpha, y=\cos \beta+i \sin \beta$ then prove that $x^{m} y^{n}+\frac{1}{x^{m} y^{n}}=2 \cos (m \alpha+n \beta)$ and $x^{m} y^{n}-\frac{1}{x^{m} y^{n}}=2 i \sin (m \alpha+n \beta)$.
A. Given that $x=\cos \alpha+i \sin \alpha$ and $y=\cos \beta+i \sin \beta$

$$
\text { Now, } \begin{aligned}
x^{m} & =(\cos \alpha+i \sin \alpha)^{m} \\
& =\cos m \alpha+i \sin m \alpha=\operatorname{cis} m \alpha \\
y^{n} & =(\cos \beta+i \sin \beta)^{n} \\
& =\cos n \beta+i \sin \beta=\operatorname{cis} n \beta \\
x^{m} y^{n} & =\operatorname{cis} m \alpha \cdot \operatorname{cis} n \beta \\
& =\operatorname{cis}(m \alpha+n \beta) \\
& =\cos (m \alpha+n \beta)+i \sin (m \alpha+n \beta)
\end{aligned}
$$

$$
x^{m} y^{n}+\frac{1}{x^{m} y^{n}}=2 \cos (m \alpha+n \beta) \text { and }
$$

$$
x^{m} y^{n}-\frac{1}{x^{m} y^{n}}=2 i \sin (m \alpha+n \beta)
$$

2. If $(1+x)^{n}=a_{0}+a_{1} x+a_{2} x^{2}+$ $\qquad$ $a_{n} x^{n}$, then show that
(i) $a_{0}-a_{2}+a_{4}$ $\qquad$ $=2^{\frac{n}{2}} \cos \frac{n \pi}{4}$.
(ii) $a_{1}-a_{3}+a_{5} \ldots \ldots \ldots$ $=2^{\frac{n}{2}} \sin \frac{n \pi}{4}$.

A: Now $(1+x)^{n}=a_{0}+a_{1} x+a_{2} x^{2}+$ $\qquad$ .$+a_{n} x^{n}$.
Put $x=i$, then $a_{0}+a_{1} i+a_{2} i^{2}+\ldots \ldots .+a_{n} i^{n}=(1+i)^{n}=\left[\sqrt{2}\left(\cos \frac{\pi}{4}+i \sin \frac{\pi}{4}\right)\right]^{n}$.
By applying De Moivre's theorem
$a_{0}+a_{1} i-a_{2}-a_{3} i+a_{4}+\ldots \ldots \ldots=2^{\frac{n}{2}}\left(\cos \frac{n \pi}{4}+i \sin \frac{n \pi}{4}\right)$
$\left(a_{0}-a_{2}+a_{4} \ldots \ldots\right)+i\left(a_{1}-a_{3}+a_{5} \ldots \ldots.\right)=2^{\frac{n}{2}} \cos \frac{n \pi}{4}+i 2^{\frac{n}{2}} \sin \frac{n \pi}{4}$.
Equating real and imaginary parts both sides.

$$
\therefore a_{0}-a_{2}+a_{4} \ldots \ldots \ldots \ldots \ldots=2^{\frac{n}{2}} \cos \frac{n \pi}{4} \text { and } a_{1}-a_{3}+a_{5} \ldots \ldots \ldots \ldots \ldots=2^{\frac{n}{2}} \sin \frac{n \pi}{4} .
$$

3. If $1, \omega, \omega^{2}$ are the cube roots of unity, then find the value of $(a+2 b)^{2}+\left(a \omega^{2}+2 b \omega\right)^{2}+\left(a \omega+2 b \omega^{2}\right)^{2}$.

Sol: We know that $1+\omega+\omega^{2}=0$ and $\omega^{3}=1$.

$$
\begin{aligned}
& \text { Now }(a+2 b)^{2}+\left(a \omega^{2}+2 b \omega\right)^{2}+\left(a \omega+2 b \omega^{2}\right)^{2} \text {. } \\
& =a^{2}+4 b^{2}+4 a b+a^{2} \omega^{4}+4 b^{2} \omega^{2}+4 a b \omega^{3}+a^{2} \omega^{2}+4 b^{2} \omega^{4}+4 a b \omega^{3} \\
& =a^{2}+4 b^{2}+4 a b+a^{2} \omega+4 b^{2} \omega^{2}+4 a b+a^{2} \omega^{2}+4 b^{2} \omega+4 a b \\
& =a^{2}\left(1+\omega+\omega^{2}\right)+4 b^{2}\left(1+\omega+\omega^{2}\right)+12 a b \\
& =a^{2}(0)+4 b^{2}(0)+12 a b \\
& =12 a b .
\end{aligned}
$$

4. If $1, \omega, \omega^{2}$ are the cube roots of unity, then prove that $\frac{1}{2+\omega}+\frac{1}{1+2 \omega}=\frac{1}{1+\omega}$.

A: $\frac{1}{2+\omega}+\frac{1}{1+2 \omega}=\frac{1+2 \omega+2+\omega}{2+4 \omega+\omega+2 \omega^{2}}$

$$
=\frac{3(1+\omega)}{2\left(1+\omega+\omega^{2}\right)+3 \omega}=\frac{3(1+\omega)}{3 \omega}=\frac{(1+\omega)^{2}}{\omega(1+\omega)}=\frac{\left(1+\omega+\omega^{2}\right)+\omega}{\omega(1+\omega)}=\frac{\omega}{\omega(1+\omega)}=\frac{1}{1+\omega} .
$$

## QUADRATIC EXPRESSIONS Q.NO 13

1. Determine the range of the expression $\frac{x^{2}+x+1}{x^{2}-x+1}, x \in R$.

A: Let $\frac{x^{2}+x+1}{x^{2}-x+1}=y$

$$
x^{2}+x+1=y x^{2}-y x+y
$$

$$
(y-1) x^{2}-(y+1) x+(y-1)=0
$$

$$
\text { For } x \in R, b^{2}-4 a c \geq 0
$$

$$
\{-(y+1)\}^{2}-4(y-1)(y-1) \geq 0
$$

$$
-3 y^{2}+10 y-3 \geq 0
$$

$$
3 y^{2}-10 y+3 \leq 0
$$

$$
3 y^{2}-y-9 y+3 \leq 0
$$

$$
y(3 y-1)-3(3 y-1) \leq 0
$$

$$
(3 y-1)(y-3) \leq 0
$$

$$
\left(y-\frac{1}{3}\right)(y-3) \leq 0
$$

$$
y \in\left[\frac{1}{3}, 3\right]
$$

2. If $x$ is a real number, find the range $\frac{x+2}{2 x^{2}+3 x+6}$.

A: Let $\frac{x+2}{2 x^{2}+3 x+6}=y$

$$
\begin{gathered}
2 x+3 x+0 \\
x+2=2 y x^{2}+3 y x+6 y \\
2 y x^{2}+(3 y-1) x+2(3 y-1)=0 \\
\text { For } x \in R, b^{2}-4 a c \geq 0 \\
(3 y-1)^{2}-4(2 y)(2)(3 y-1) \geq 0 \\
(3 y-1)[3 y-1-16 y] \geq 0 \\
(3 y-1)(-13 y-1) \geq 0 \\
(3 y-1)(13 y+1) \leq 0 \\
{\left[y-\left(\frac{-1}{13}\right)\right]\left(y-\frac{1}{3}\right) \leq 0} \\
y \in\left[\frac{-1}{13}, \frac{1}{3}\right]
\end{gathered}
$$

3. Show that $\frac{x}{x^{2}-5 x+9}$ lies between $\frac{-1}{11}, 1$.

A: Let $\frac{x}{x^{2}-5 x+9}=y$

$$
\begin{gathered}
x=y x^{2}-5 y x+9 y \\
y x^{2}-(5 y+1) x+9 y=0 \\
\text { For } x \in R, b^{2}-4 a c \geq 0 \\
\{-(5 y+1)\}^{2}-4(y)(9 y) \geq 0 \\
25 y^{2}+10 y+1-36 y^{2} \geq 0 \\
-11 y^{2}+10 y+1 \geq 0
\end{gathered}
$$

$$
\begin{aligned}
11 y^{2}-10 y-1 & \leq 0 \\
11 y^{2}-11 y+y-1 & \leq 0 \\
11 y(y-1)+1(y-1) & \leq 0 \\
(11 y+1)(y-1) & \leq 0 \\
(11 y+1)(y-1) & \leq 0 \\
{\left[y-\left(\frac{-1}{11}\right)\right](y-1) } & \leq 0 \quad y \in\left[\frac{-1}{11}, 1\right]
\end{aligned}
$$

4. If $x$ is real, show that the values of the expression $\frac{x^{2}+34 x-71}{x^{2}+2 x-7}$ do not lie between 5 and 9 .

A: Let $\frac{x^{2}+34 x-71}{x^{2}+2 x-7}=y$

$$
x^{2}+34 x-71=y x^{2}+2 y x-7 y
$$

$$
(y-1) x^{2}+2(y-17) x+(71-7 y)=0
$$

For $x \in R, b^{2}-4 a c \geq 0$

$$
\{2(y-17)\}^{2}-4(y-1)(71-7 y) \geq 0
$$

$4\left(y^{2}-34 y+289\right)+4\left(7 y^{2}-78 y+71\right) \geq 0$

$$
y^{2}-34 y+289+7 y^{2}-78 y+71 \geq 0
$$

$$
8 y^{2}-112 y+360 \geq 0
$$

$$
y^{2}-14 y+45 \geq 0
$$

$$
y^{2}-9 y-5 y+45 \geq 0
$$

$$
y(y-9)-5(y-9) \geq 0
$$

$$
y \in(-\infty, 5] \cup[9, \infty)
$$

$$
(y-5)(y-9) \geq 0
$$

5. If $x$ is real, find the maximum and minimum values of the expression $\frac{x^{2}+14 x+9}{x^{2}+2 x+3}$.

A: Let $\frac{x^{2}+14 x+9}{x^{2}+2 x+3}=y$

$$
x^{2}+14 x+9=y x^{2}+2 y x+3 y
$$

$$
(y-1) x^{2}+2(y-7) x+(3 y-9)=0
$$

For $x \in R, b^{2}-4 a c \geq 0$

$$
\{2(y-7)\}^{2}-4(y-1)(3 y-9) \geq 0
$$

$$
y^{2}-14 y+49-\left(3 y^{2}-12 y+9\right) \geq 0
$$

$$
-2 y^{2}-2 y+40 \geq 0
$$

$$
y^{2}+y-20 \leq 0
$$

$$
y^{2}+5 y-4 y-20 \leq 0
$$

$$
y(y+5)-4(y+5) \leq 0
$$

$y \in[-5,4]$.

$$
(y+5)(y-4) \leq 0
$$

6. Prove that $\frac{1}{3 x+1}+\frac{1}{x+1}-\frac{1}{(3 x+1)(x+1)}$ does not lie between 1 and 4 , if $x$ is real.

A: $\frac{1}{3 x+1}+\frac{1}{x+1}-\frac{1}{(3 x+1)(x+1)}=\frac{x+1+3 x+1-1}{(3 x+1)(x+1)}=\frac{4 x+1}{(3 x+1)(x+1)}$

$$
\text { Let } \begin{aligned}
\frac{4 x+1}{3 x^{2}+4 x+1} & =y \\
4 x+1 & =3 y x^{2}+4 y x+y
\end{aligned}
$$

$$
3 y x^{2}+4(y-1) x+(y-1)=0
$$

$$
\text { For } x \in R, b^{2}-4 a c \geq 0
$$

$$
\left\{(4(y-1)\}^{2}-4(3 y)(y-1) \geq 0\right.
$$

$$
(y-1)[4(y-1)-3 y] \geq 0
$$

$y \in(-\infty, 1] \cup[4, \infty)$.

$$
(y-1)(y-4) \geq 0
$$

7. If $c^{2} \neq a b$ and the roots of $\left(c^{2}-a b\right) x^{2}-2\left(a^{2}-b c\right) x+\left(b^{2}-a c\right)=0$ are equal then show that $a^{3}+b^{3}+c^{3}=3 a b c$ or $a=0$.
A. Given quadratic equation is $\left(c^{2}-a b\right) x^{2}-2\left(a^{2}-b c\right) x+\left(b^{2}-a c\right)=0$

Since (1) has equation of roots $b^{2}-4 a c=0$
$\left[-2\left(a^{2}-b c\right)\right]^{2}-4\left(c^{2}-a b\right)\left(b^{2}-a c\right)=0$
$4\left[a^{4}+b^{2} c^{2}-2 a^{2} b c\right]-4\left[b^{2} c^{2}-a c^{3}-a b^{3}+a^{2} b c\right]=0$
$4 a^{4}+4 b^{2} c^{2}-8 a^{2} b c-4 b^{2} c^{2}+4 a^{2} b c=0$
$4 a\left[a^{3}-3 a b c+c^{3}+b^{3}\right]=0$
$a=0$ (or) $a^{3}+b^{3}+c^{3}=3 a b c$.
8. If the equations $x^{2}+a x+b=0$ and $x^{2}+c x+d=0$ have a common root and the first equation has equal roots then prove that $2(b+d)=a c$.
A: Let $\alpha$ be the common root.
Then $\alpha^{2}+c \alpha+d=0 \rightarrow(1)$
Also, $x^{2}+a x+b=0$ has equal roots.
$\alpha+\alpha=-a, \alpha \alpha=b$
$\alpha=-\mathrm{a} / 2, \alpha^{2}=\mathrm{b}$.
From (1); $b+c(-a / 2)+d=0$.
$b+d=a c / 2$
$\Rightarrow 2(b+d)=a c$.
9. If $\alpha, \beta$ are the roots of $a x^{2}+b x+c=0$, find the values of $\alpha^{2}+\beta^{2}$ and $\alpha^{3}+\beta^{3}$.

A: If $\alpha, \beta$ are the roots of $a x^{2}+b x+c=0$ then $\alpha+\beta=\frac{-b}{a}, \alpha \beta=\frac{c}{a}$
(i) $\alpha^{2}+\beta^{2}=(\alpha+\beta)^{2}-2 \alpha \beta=\left(\frac{-b}{a}\right)^{2}-2\left(\frac{c}{a}\right)=\frac{b^{2}-2 a c}{a^{2}}$.
(ii) $\alpha^{3}+\beta^{3}=(\alpha+\beta)^{3}-3 \alpha \beta(\alpha+\beta)=\left(\frac{-b}{a}\right)^{3}-3\left(\frac{c}{a}\right)\left(\frac{-b}{a}\right)=-\frac{b^{3}}{a^{3}}+\frac{3 b c}{a^{2}}=\frac{3 a b c-b^{3}}{a^{3}}$.
10. If $x_{1}, x_{2}$ are the roots of the quadratic equation $a x^{2}+b x+c=0$ and $c \neq 0$. Find the value of $\left(a x_{1}+b\right)^{-2}+\left(a x_{2}+b\right)^{-2}$ in terms of $a, b, c$.
A. Given that $x_{1}, x_{2}$ are the roots of $a x^{2}+b x+c=0$
$x_{1}+x_{2}=-b / a, x_{1} x_{2}=c / a$
Also $a x_{1}^{2}+b x_{1}+c=0, a x_{2}^{2}+b x_{2}+c=0$

$$
\begin{aligned}
x_{1}\left(a x_{1}+b\right) & =-c, x_{2}\left(a x_{2}+b\right)=-c \\
a x_{1}+b & =\frac{-c}{x_{1}} \quad a x_{2}+b=\frac{-c}{x_{2}}
\end{aligned}
$$

Now $\left(a x_{1}+b\right)^{-2}+\left(a x_{2}+b\right)^{-2}=\left(\frac{-c}{x_{1}}\right)^{-2}+\left(\frac{-c}{x_{2}}\right)^{-2}=\frac{x_{1}^{2}}{c^{2}}+\frac{x_{2}^{2}}{c^{2}}=\frac{x_{1}^{2}+x_{2}^{2}}{c^{2}}=\frac{\left(x_{1}+x_{2}\right)^{2}-2 x_{1} x_{2}}{c^{2}}=\frac{\left(\frac{-b}{a}\right)^{2}-2 \cdot \frac{c}{a}}{c^{2}}$

$$
\left(a x_{1}+b\right)^{-2}+\left(a x_{2}+b\right)^{-2}=\frac{b^{2}-2 a c}{c^{2} a^{2}}
$$

## THEORY OF EQUATIONS Q.NO 14

1. Solve $4 x^{3}-24 x^{2}+23 x+18=0$, given that the roots are in A.P.

A: Given that the roots of $4 x^{3}-24 x^{2}+23 x+18=0$ are in A.P.
Let the roots be $a-d, a, a+d$.
Sum of the roots $a-d+a+a+d=\frac{-b}{a}$

$$
\begin{aligned}
3 a & =\frac{24}{4} \\
a & =\frac{6}{3}=2 .
\end{aligned}
$$

Product of the roots $(a-d)(a)(a+d)=\frac{-d}{a}$

$$
\begin{aligned}
(2-d)(2)(2+d) & =\frac{-18}{4} \\
4-d^{2} & =\frac{-9}{4} \\
d^{2} & =4+\frac{9}{4}=\frac{25}{4} \\
d & = \pm \frac{5}{2}
\end{aligned}
$$

Hence the roots of the given equation are $2-\frac{5}{2}, 2,2+\frac{5}{2}=\frac{-1}{2}, 2, \frac{9}{2}$.
(a). Solve $8 x^{3}-36 x^{2}-18 x+81=0$, given that the roots are in A.P.
(b). Solve $x^{3}-3 x^{2}-6 x+8=0$, given that the roots are in A.P.
2. Solve the equation $x^{3}-7 x^{2}+14 x-8=0$, given that the roots are in geometric progression.

A: Given that the roots of $x^{3}-7 x^{2}+14 x-8=0$ are in G.P.
Let the roots be $\frac{a}{r}$, $a$, ar

$$
\begin{aligned}
s_{3}=\left(\frac{a}{r}\right)(a)(a r) & =\frac{-d}{a} \\
a^{3} & =8 \\
a & =2
\end{aligned}
$$

$$
\mathrm{s}_{1}=\frac{\mathrm{a}}{\mathrm{r}}+\mathrm{a}+\mathrm{ar}=\frac{-\mathrm{b}}{\mathrm{a}}
$$

$$
2\left(r+\frac{1}{r}+1\right)=7
$$

$$
r+\frac{1}{r}+1=\frac{7}{2}
$$

$$
r+\frac{1}{r}=\frac{7}{2}-1=\frac{5}{2}
$$

$$
r+\frac{1}{r}=2+\frac{1}{2} \text { by observation If } r=2, \text { the required roots are } \frac{2}{2}, 2,2(2)=1,2,4
$$

(a). Solve the equation $3 x^{3}-26 x^{2}+52 x-24=0$, given that the roots are in geometric progression.
3. Solve the equation $15 x^{3}-23 x^{2}+9 x-1=0$, given that the roots are in H.P.

A: Given that the roots of
$f(x)=15 x^{3}-23 x^{2}+9 x-1=0$ $\qquad$ (1) are in H.P.

Roots of $f\left(\frac{1}{x}\right)=0$ are in A.P.

$$
\begin{align*}
f\left(\frac{1}{x}\right)=\frac{15}{x^{3}}-\frac{23}{x^{2}}+\frac{9}{x}-1 & =0 \\
15-23 x+9 x^{2}-x^{3} & =0 \\
x^{3}-9 x^{2}+23 x-15 & =0- \tag{2}
\end{align*}
$$

Let the roots of (2) be $a-d, a, a+d$.

$$
\begin{aligned}
s_{1}=a-d+a+a+d & =\frac{-b}{a} \\
3 a & =9 \quad \Rightarrow a=3 \\
s_{3}=(a-d)(a)(a+d) & =\frac{-d}{a} \\
3\left(9-d^{2}\right) & =15 \\
9-d^{2} & =\frac{15}{3}=5 \\
d^{2} & =4 \\
d & = \pm 2
\end{aligned}
$$

If $d=2$, the roots of (2), are $3-2,3,3+2=1,3,5$.
Hence the roots of the given equation are $1, \frac{1}{3}, \frac{1}{5}$.
(a): Solve the equation $6 x^{3}-11 x^{2}+6 x-1=0$, given that the roots are in H.P.
4. Solve the equation $18 x^{3}+81 x^{2}+121 x+60=0$, given that one root is equal to half the sum of the remaining roots.

A: Let $\alpha, \beta, \gamma$ be the roots such that $\beta=\frac{\alpha+\gamma}{2}$

$$
\alpha+\gamma=2 \beta
$$

$$
\begin{aligned}
\text { Now } s_{1}=\alpha+\beta+\gamma & =\frac{-b}{a} & s_{3}=\alpha \beta \gamma & =\frac{-d}{a} \\
2 \beta+\beta & =\frac{-81}{18} & \alpha \gamma\left(\frac{-3}{2}\right) & =\frac{-60}{18} \\
3 \beta & =\frac{-9}{2} \Rightarrow \beta=\frac{-3}{2} & \alpha \gamma & =\frac{20}{9}
\end{aligned}
$$

$\therefore \alpha+\gamma=-3$
The quadratic equation, whose roots are $\alpha, \gamma$ is $x^{2}-(\alpha+\gamma) x+\alpha \gamma=0$.

$$
\begin{aligned}
& x^{2}-(-3) x+\frac{20}{9}=0 \\
& 9 x^{2}+27 x+20=0 \\
& 9 x^{2}+12 x+15 x+20=0 \\
& 3 x(3 x+4)+5(3 x+4)=0 \\
& (3 x+4)(3 x+5)=0 \\
& x=\frac{-4}{3}, \frac{-5}{3}
\end{aligned}
$$

5. Solve $x^{3}-9 x^{2}+14 x+24=0$ given that two of the roots are in the ratio $3: 2$.

A: Given that two of the roots of $x^{3}-9 x^{2}+14 x+24=0$ are in the ratio $3: 2$.
Let the roots be $3 \alpha, 2 \alpha, \gamma$.
$\mathrm{s}_{1}=3 \alpha+2 \alpha+\gamma=9$

$$
\begin{equation*}
\gamma=9-5 \alpha \tag{1}
\end{equation*}
$$

$s_{2}=(3 \alpha)(2 \alpha)+(2 \alpha)(\gamma)+\gamma(3 \alpha)=14$

$$
\begin{array}{r}
6 \alpha^{2}+2 \alpha \gamma+3 \alpha \gamma=14 \\
6 \alpha^{2}+5 \alpha \gamma=14 \tag{2}
\end{array}
$$

$\mathrm{s}_{3}=(3 \alpha)(2 \alpha)(\gamma)=-24$

$$
\begin{align*}
6 \alpha^{2} \gamma & =-24  \tag{3}\\
\alpha^{2} \gamma & =-4
\end{align*}
$$

From (1) \& (2), $6 \alpha^{2}+5 \alpha(9-5 \alpha)=14$

$$
\begin{aligned}
6 \alpha^{2}+45 \alpha-25 \alpha^{2}-14 & =0 \\
-19 \alpha^{2}+45 \alpha-14 & =0 \\
19 \alpha^{2}-45 \alpha+14 & =0 \\
19 \alpha^{2}-38 \alpha-7 \alpha+14 & =0 \\
19 \alpha(\alpha-2)-7(\alpha-2) & =0 \\
(19 \alpha-7)(\alpha-2) & =0 \\
\alpha & =2, \frac{7}{19}
\end{aligned}
$$

If $\alpha=2$, then $\gamma=9-5(2)=-1$.
Substituting $\alpha, \gamma$ values in LHS of (3), $\alpha^{2} \gamma=2^{2}(-1)=-4=$ RHS, which is satisfied.
whereas $\alpha=\frac{7}{19}$ does not satisfy equation (3).
Hence, the required roots of given cubic equation are 3(2), 2(2), $-1=6,4,-1$.
6. If the roots of the equation $x^{3}+3 p x^{2}+3 q x+r=0$ are in A.P., then show that $2 p^{3}-3 p q+r=0$.

A: Given that the roots of $x^{3}+3 p x^{2}+3 q x+r=0---(1)$ are in A.P.
Let the roots be a-d, $a, a+d$
Sum of the roots $a-d+a+a+d=\frac{-b}{a}$

$$
\begin{aligned}
3 a & =-3 p \\
a & =-p
\end{aligned}
$$

Substituting $x=-p$ in $(1)$, we get $(-p)^{3}+3 p(-p)^{2}+3 q(-p)+r=0$

$$
\begin{aligned}
-p^{3}+3 p^{3}-3 p q+r & =0 \\
2 p^{3}-3 p q+r & =0 .
\end{aligned}
$$

7. Show that the condition that the roots of $x^{3}+3 p x^{2}+3 q x+r=0$ may be in G.P. is $p^{3} r=q^{3}$.

A: Given that the roots of $x^{3}+3 p x^{2}+3 q x+r=0---(1)$ are in G.P.
Let the roots be $\frac{\alpha}{\beta}, \alpha, \alpha \beta$.
$\left(\frac{\alpha}{\beta}\right)(\alpha)(\alpha \beta)=\frac{-d}{a}$
$\alpha^{3}=-r, \quad \alpha^{3}+r=0$
substituting $x=\alpha$ in (1), $\alpha^{3}+3 p \alpha^{2}+3 q \alpha+r=0$

$$
\begin{aligned}
\left(\alpha^{3}+r\right)+3 \alpha(p \alpha+q) & =0 \\
0+3 \alpha(p \alpha+q) & =0
\end{aligned}
$$

$$
\begin{aligned}
p \alpha+q & =0 \\
p \alpha & =-q
\end{aligned} \quad \because \alpha \neq 0
$$

Cubing on both sides, $p^{3} \alpha^{3}=-q^{3} \Rightarrow p^{3}(-r)=-q^{3}$
$p^{3} r=q^{3}$ is the required condition.
8. Given that the roots of $x^{3}+3 p x^{2}+3 q x+r=0$ are in H.P, show that $2 q^{3}=r(3 p q-r)$.

A: Let $f(x)=x^{3}+3 p x^{2}+3 q x+r=0$
Given that roots of $f(x)=0$ are in H.P.
Roots of $f\left(\frac{1}{x}\right)=0$ are in A.P.
$f\left(\frac{1}{x}\right)=0 \Rightarrow \frac{1}{x^{3}}+\frac{3 p}{x^{2}}+\frac{3 q}{x}+r=0$.

$$
\Rightarrow r x^{3}+3 q x^{2}+3 p x+1=0
$$

Let the roots of this equation be $a-d, a, a+d$.
sum $=a-d+a+a+d=-\frac{3 q}{r}$.

$$
\Rightarrow 3 a=-\frac{3 q}{r} .
$$

$\Rightarrow \mathrm{a}=\frac{-\mathrm{q}}{\mathrm{r}}$.
Since a is root, $\mathrm{ra}^{3}+3 q a^{2}+3 p a+1=0$.
$\Rightarrow r\left(\frac{-q}{r}\right)^{3}+3 q\left(\frac{-q}{r}\right)^{2}+3 p\left(\frac{-q}{r}\right)+1=0$

$$
\frac{-q^{3}}{r^{2}}+\frac{3 q^{3}}{r^{2}}-\frac{3 p q}{r}+1=0
$$

$$
=\frac{2 q^{3}-3 p q r+r^{2}}{r^{2}}=0
$$

$$
2 q^{3}-3 p q r+r^{2}=0
$$

$$
2 q^{3}=3 p q r-r^{2}
$$

$\therefore 2 q^{3}=r(3 p q-r)$.

## PERMUTATION Q.NO 15

1. Find the rank of the word 'MASTER'.

A: We shall find the rank of the word 'MASTER'. Alphabatical order of letters of the word is A, E, M, R, S, T.
No.of 6 letter words formed which begin with ' $A$ ' $=5$ !
No.of Gletter words formed which begin with ' $E$ ' $=5$ !
No.of 6 letter words formed which begin with 'MAE' $=3$ !
No.of 6letter words formed which begin with 'MAR' $=3$ !
No.of 6letter words formed which begin with 'MASE' $=2$ !
No.of 6letter words formed which begin with 'MASR' $=2$ !
Next word formed is MASTER $=1$
$\therefore$ Rank of the word 'MASTER' $=2(5!)+2(3!)+2(2!)+1$

$$
=2(120)+2(6)+2(2)+1=257 .
$$

(ii). Find the rank of the word 'REMAST'.

A: We shall find the rank of the word 'REMAST'. Alphabatical order of letters of the word is A, E, M, R, S, T.
No.of Gletter words formed which begin with ' $A$ ' $=5$ !
No.of 6letter words formed which begin with ' $E$ ' $=5$ !
No.of 6letter words formed which begin with ' $M$ ' $=5$ !
No.of 6letter words formed which begin with 'RA' $=4$ !
No.of 6letter words formed which begin with 'REA' $=3$ !
Next word formed is REMAST $=1$
$\therefore$ Rank of the word 'REMAST' $=3(5!)+4!+3!+1$

$$
=3(120)+24+6+1=391
$$

(iii). Find the rank of the word 'PRISON'.

A: We shall find the rank of the word 'PRISON'. Alphabetical order of the letters is INOPRS.
No. of 6 letter words formed which being with 'I' = 5!
No. of 6 letter words formed which being with ' $N$ ' = 5 !
No. of 6 letter words formed which being with ' O ' $=5$ !
No. of 6 letter words formed which being with 'PI' = 4!
No. of 6 letter words formed which being with 'PN' $=4$ !
No. of 6 letter words formed which being with'PO' $=4$ !
No. of 6 letter words formed which being with 'PRIN' $=2$ !
No. of 6 letter words formed which being with 'PRIO' $=2$ !
No. of 6 letter words formed which being with 'PRISN' $=1$ !
Next word formed is PRISON $=1$.
$\therefore$ Rank of 'PRISON'. $=3(5!)+3(4!)+2(2!)+1+1$.
$=3(120)+3(24)+2(2)+1+1=438$.
2. Find the number of 4 letter words that can be formed using the letters of the word 'MIXTURE' which (i) contain the letter $X$ (ii) do not contain the letter $X$.
A: We have fill up 4 blanks using 7 letters of the word 'MIXTURE'. Take 4 blanks.
i) First we put $x$ in one of the 4 blanks. This can be done in 4 ways.

Now we can fill the remaining 3 places with the remaining 6 letters in ${ }^{6} \mathrm{P}_{3}$ ways.
Thus the number of 4 letter words formed which contain the letter $X=4 .{ }^{6} P_{3}=4 \times 120=480$.
ii) Remove the letter $X$. From the remaining 6 letters, number of 4 letter words formed $={ }^{6} P_{4}$. Thus the number of 4 letter words that do not contain the letter $X={ }^{6} \mathrm{P}_{4}=360$.
3. Find the number of ways of arranging 5 different mathematics books, 4 different physics books and 3 different chemistry books such that the books of the same subject are together.
A: Number of mathematics books $=5$
Number of physics books $=4$
Number of chemistry books $=3$
Given condition: Books of same subject are together.
Treat 5 mathematics books as $1^{\text {st }}$ unit, 4 physics books as $2^{\text {nd }}$ unit, 3 chemistry books as $3^{\text {rd }}$ unit.
Now the number of units is 3 . These 3 units can be arranged in a row in 3 ! ways.
Then 5 mathematics books can be shuffled internally in 5 ! ways, 4 physics books can be shuffled internally in 4 ! ways,
3 chemistry books can be shuffled internally in 3! ways.
By the counting principle, total number of arrangements such that books of same subject are together.

$$
\begin{aligned}
& =3!\times 5!\times 4!\times 3! \\
& =6 \times 120 \times 24 \times 6 \\
& =720 \times 144 \\
& =1,03,680 .
\end{aligned}
$$

4. If the letters of the word BRING are permuted in all possible ways and the words thus formed are arranged in the dictionary order, then find the $59^{\text {th }}$ word.
A: Given word is BRING.
The alphabetical order of the letters is B, G, I, N, R
In the dictionary order, first we write all words beginning with $B$.
Clearly the number of words begining with $B$ are $4!=24$
Similarly the number of words begining with $G$ are $4!=24$
Since the words begin with $B$ and $G$ sum to 48 , the 59 word must start with I
Number of words begin with IB is $3!=6$
Hence the $59^{\text {th }}$ word must start with IG
Number of words begin with IGB $=2!=2$
Number of words begin with IGN = 2! =2
the next word is $59^{\text {th }}=$ IGRBN.

## COMBINATION Q.NO 16

1. Simplify ${ }^{25} C_{4}+\sum_{r=0}^{4}{ }^{29-r} C_{3}$.

A: Now ${ }^{25} \mathrm{C}_{4}+\sum_{\mathrm{r}=0}^{4}{ }^{29-r} \mathrm{C}_{3}={ }^{29} \mathrm{C}_{3}+{ }^{28} \mathrm{C}_{3}+{ }^{27} \mathrm{C}_{3}+{ }^{26} \mathrm{C}_{3}+\left\{{ }^{25} \mathrm{C}_{3}+{ }^{25} \mathrm{C}_{4}\right\}$
$\because{ }^{n} C_{r-1}+{ }^{n} C_{r}={ }^{n+1} C_{r}$
$={ }^{29} \mathrm{C}_{3}+{ }^{28} \mathrm{C}_{3}+{ }^{27} \mathrm{C}_{3}+\left\{{ }^{26} \mathrm{C}_{3}+{ }^{26} \mathrm{C}_{4}\right\}$
$={ }^{29} \mathrm{C}_{3}+{ }^{28} \mathrm{C}_{3}+\left\{{ }^{27} \mathrm{C}_{3}+{ }^{27} \mathrm{C}_{4}\right\}$
$={ }^{29} \mathrm{C}_{3}+\left\{{ }^{28} \mathrm{C}_{3}+{ }^{28} \mathrm{C}_{4}\right\}$
$={ }^{29} \mathrm{C}_{3}+{ }^{29} \mathrm{C}_{4}={ }^{30} \mathrm{C}_{4}$.
2. Simplify ${ }^{34} \mathrm{C}_{5}+\sum_{\mathrm{r}=0}^{4}{ }^{38-\mathrm{r}} \mathrm{C}_{4}$.
A. Now ${ }^{34} \mathrm{C}_{5}+\sum_{\mathrm{r}=0}^{4}{ }^{38-\mathrm{r}} \mathrm{C}_{4}={ }^{38} \mathrm{C}_{4}+{ }^{37} \mathrm{C}_{4}+{ }^{36} \mathrm{C}_{4}+{ }^{35} \mathrm{C}_{4}+\left\{{ }^{34} \mathrm{C}_{4}+{ }^{34} \mathrm{C}_{5}\right\}$
$={ }^{38} \mathrm{C}_{4}+{ }^{37} \mathrm{C}_{4}+{ }^{36} \mathrm{C}_{4}+\left\{{ }^{35} \mathrm{C}_{4}+{ }^{35} \mathrm{C}_{5}\right\}$
$={ }^{38} \mathrm{C}_{4}+{ }^{37} \mathrm{C}_{4}\left\{{ }^{36} \mathrm{C}_{4}+{ }^{36} \mathrm{C}_{5}\right\}$
$={ }^{38} \mathrm{C}_{4}+\left\{{ }^{37} \mathrm{C}_{4}+{ }^{37} \mathrm{C}_{5}\right\}$
$={ }^{38} \mathrm{C}_{4}+{ }^{38} \mathrm{C}_{5}$
$={ }^{39} \mathrm{C}_{5}$.
3. Prove that for $3 \leq r \leq n,{ }^{n-3} C_{r}+3 .{ }^{n-3} C_{r-1}+3 \cdot{ }^{n-3} C_{r-2}+{ }^{n-3} C_{r-3}={ }^{n} C_{r}$.

A: Now ${ }^{n-3} C_{r}+3 .{ }^{n-3} C_{r-1}+3 .{ }^{n-3} C_{r-2}+{ }^{n-3} C_{r-3}$

$$
\because{ }^{n} C_{r}+{ }^{n} C_{r-1}={ }^{n+1} C_{r}
$$

$$
\begin{aligned}
& =\left\{n^{n-3} C_{r}+{ }^{n-3} C_{r-1}\right\}+2\left\{{ }^{n-3} C_{r-1}+{ }^{n-3} C_{r-2}\right\}+\left\{{ }^{n-3} C_{r-2}+{ }^{n-3} C_{r-3}\right\} \\
& ={ }^{n-2} C_{r}+2 \cdot{ }^{n-2} C_{r-1}+{ }^{n-2} C_{r-2} \\
& =\left\{{ }^{n-2} C_{r}+{ }^{n-2} C_{r-1}\right\}+\left\{{ }^{n-2} C_{r-1}+{ }^{n-2} C_{r-2}\right\} \\
& ={ }^{n+1} C_{r} \\
& ={ }^{n-1} C_{r}+{ }^{n-1} C_{r-1} \\
& ={ }^{n} C_{r} .
\end{aligned}
$$

4. Prove that ${ }^{n} C_{r}+{ }^{n} C_{r-1}={ }^{n+1} C_{r}$.

A: Now ${ }^{n} C_{r}+{ }^{n} C_{r-1}=\frac{n!}{(n-r)!r!}+\frac{n!}{[n-(r-1)]!(r-1)!}$

$$
\begin{aligned}
& =\frac{n!(n+1-r)+n!r}{r!(n+1-r)!} \\
& =\frac{n!(n+1-r+r)}{r!(n+1-r)!} \\
& =\frac{(n+1)!}{(n+1-r)!r!} \\
& ={ }^{n+1} C_{r} .
\end{aligned}
$$

5. Find the number of ways of selecting 3 vowels and 2 consonants from the letters of the word 'EQUATION'.

A: Given word 'EQUATION' contains 5 vowels and 3 consonants.
Number of ways of selecting 3 vowels and 2 consonants from 5 vowels and 3 consonants $={ }^{5} C_{3} \times{ }^{3} C_{2}$

$$
\begin{aligned}
& =10 \times 3 \\
& =30 .
\end{aligned}
$$

## PARTIAL FRACTIONS Q.NO 17

1. Resolve $\frac{x+4}{\left(x^{2}-4\right)(x+1)}$ into partial fractions.

A: Let $\frac{x+4}{(x-2)(x+2)(x+1)}=\frac{A}{x-2}+\frac{B}{x+2}+\frac{C}{x+1}$

$$
\begin{align*}
& \frac{x+4}{(x-2)(x+2)(x+1)}=\frac{A(x+2)(x+1)+B(x-2)(x+1)+C(x-2)(x+2)}{(x-2)(x+2)(x+1)} \\
& x+4=A(x+2)(x+1)+B(x-2)(x+1)+C(x-2)(x+2)------(1) \tag{1}
\end{align*}
$$

Put $x=2$ in (1), we get
$6=A(4)(3)$
$\therefore A=\frac{1}{2}$
Put $x=-2$ in (1), we get

$$
2=B(-4)(-1)
$$

$$
4 B=2
$$

$\therefore B=\frac{1}{2}$
Put $x=-1$ in (1), we get
$3=C(-3)(1)$
$\therefore C=-1$

$$
\therefore \frac{x+4}{\left(x^{2}-4\right)(x+1)}=\frac{1}{2(x-2)}+\frac{1}{2(x+2)}-\frac{1}{x+1}
$$

(H/W) $\frac{3 x+7}{x^{2}+3 x+2}$ Ans $\frac{-10}{x-1}+\frac{13}{x-2}$
2. Resolve $\frac{x^{2}+13 x+15}{(2 x+3)(x+3)^{2}}$ into sum of partial fractions.

A: Let $\frac{x^{2}+13 x+15}{(2 x+3)(x+3)^{2}}=\frac{A}{2 x+3}+\frac{B}{x+3}+\frac{C}{(x+3)^{2}}$

$$
\frac{x^{2}+13 x+15}{(2 x+3)(x+3)^{2}}=\frac{A(x+3)^{2}+B(2 x+3)(x+3)+C(2 x+3)}{(2 x+3)(x+3)^{2}}
$$

$$
\begin{equation*}
x^{2}+13 x+15=A(x+3)^{2}+B(2 x+3)(x+3)+C(2 x+3) \tag{1}
\end{equation*}
$$

Put $x=-3$ in (1), we get

$$
\begin{aligned}
9-39+15 & =C(-3) \\
-3 C & =-15 \\
\therefore C & =5
\end{aligned}
$$

Put $x=\frac{-3}{2}$ in (1), we get

$$
\frac{9}{4}-\frac{39}{2}+15=\mathrm{A}\left(\frac{9}{4}\right)
$$

$$
\frac{9 \mathrm{~A}}{4}=\frac{-9}{4}
$$

$\therefore \mathrm{A}=-1$

Equating the coefficient of $x^{2}$, we get $A+2 B=1$

$$
\begin{aligned}
-1+2 B & =1 \\
B & =1
\end{aligned}
$$

$\therefore \frac{\mathrm{x}^{2}+13 \mathrm{x}+15}{(2 \mathrm{x}+3)(\mathrm{x}+3)^{2}}=\frac{-1}{2 \mathrm{x}+3}+\frac{1}{\mathrm{x}+3}+\frac{5}{(\mathrm{x}+3)^{2}}$.
(ii) $\frac{3 x-18}{x^{3}(x+3)}$.

A: Let $\frac{3 x-18}{x^{3}(x+3)}=\frac{A}{x}+\frac{B}{x^{2}}+\frac{C}{x^{3}}+\frac{D}{x+3}$

$$
\begin{align*}
\frac{3 x-18}{x^{3}(x+3)} & =\frac{A x^{2}(x+3)+B x(x+3)+C(x+3)+D x^{3}}{x^{3}(x+3)} \\
3 x-18 & =A x^{2}(x+3)+B x(x+3)+C(x+3)+D x^{3}---(1) \\
3 x-18 & =A\left(x^{3}+3 x^{2}\right)+B\left(x^{2}+3 x\right)+C(x+3)+D x^{3}---( \tag{2}
\end{align*}
$$

Put $x=0$ in (1), we get

$$
\begin{aligned}
& -18=C(3) \\
& \therefore C=-6
\end{aligned}
$$

Put $x=-3$ in (1), we get
$-9-18=D(-27)$
$\therefore \mathrm{D}=1$
Equating the coefficient of $x^{3}$ on both sides in (2), $0=A+D$

$$
\begin{array}{r}
A+1=0 \\
\therefore A=-1
\end{array}
$$

Equating the coefficient of $x^{2}$ on both sides in (2), $0=3 A+B$

$$
B=-3(-1)
$$

$$
\therefore \frac{3 x-18}{x^{3}(x+3)}=\frac{-1}{x}+\frac{3}{x^{2}}-\frac{6}{x^{3}}+\frac{1}{x+3}
$$

(iii) $\frac{2 x^{2}+2 x+1}{x^{3}+x^{2}}$ into partial fractions.

A: Let $\frac{2 x^{2}+2 x+1}{x^{2}(x+1)}=\frac{A}{x}+\frac{B}{x^{2}}+\frac{C}{x+1}$

$$
\begin{align*}
& \frac{2 x^{2}+2 x+1}{x^{2}(x+1)}=\frac{A x(x+1)+B(x+1)+C x^{2}}{x^{2}(x+1)} \\
& 2 x^{2}+2 x+1=A x(x+1)+B(x+1)+C x^{2} \tag{1}
\end{align*}
$$

Put $x=0$ in (1), we get

$$
1=B
$$

$$
\therefore B=1
$$

Put $x=-1$ in (1), we get

$$
\begin{gathered}
2-2+1=C(1) \\
\therefore C=1
\end{gathered}
$$

Put $x=1$ in (1), we get

$$
\begin{aligned}
2+2+1 & =2 A+2 B+C \\
5 & =2 A+2+1 \\
2 A & =2 \\
\therefore A & =1 . \\
\therefore \frac{2 x^{2}+2 x+1}{x^{3}+x^{2}} & =\frac{1}{x}+\frac{1}{x^{2}}+\frac{1}{x+1} .
\end{aligned}
$$

3. Resolve $\frac{3 x^{3}-8 x^{2}+10}{(x-1)^{4}}$ into partial fractions.
A. Given $\frac{3 x^{3}-8 x^{2}+10}{(x-1)^{4}}$
put $\mathrm{x}-1=\mathrm{y} \Rightarrow \mathrm{x}=\mathrm{y}+1$

$$
\text { Now } \begin{aligned}
\frac{3 x^{3}-8 x^{2}+10}{(x-1)^{4}}=\frac{3(y+1)^{3}-8(y+1)^{2}+10}{y^{4}} & =\frac{3\left(y^{3}+3 y^{2}+3 y+1\right)-8\left(y^{2}+1+2 y\right)+10}{y^{4}} \\
& =\frac{3 y^{3}+9 y^{2}+9 y+3-8 y^{2}-8-16 y+10}{y^{4}} \\
& =\frac{3 y^{3}+y^{2}-7 y+5}{y^{4}} \\
& =\frac{3}{y}+\frac{1}{y^{2}}-\frac{7}{y^{3}}+\frac{5}{y^{4}} \\
& =\frac{3}{x-1}+\frac{1}{(x-1)^{2}}-\frac{7}{(x-1)^{3}}+\frac{5}{(x-1)^{4}} .
\end{aligned}
$$

(ii) $\frac{x^{2}+5 x+7}{(x-3)^{3}}$

A: Put $x-3=y$

$$
x=y+3
$$

Now $\frac{x^{2}+5 x+7}{(x-3)^{3}}=\frac{(y+3)^{2}+5(y+3)+7}{y^{3}}$

$$
\begin{aligned}
& =\frac{y^{2}+6 y+9+5 y+15+7}{y^{3}} \\
& =\frac{y^{2}+11 y+31}{y^{3}} \\
& =\frac{1}{y}+\frac{11}{y^{2}}+\frac{31}{y^{3}} \\
& =\frac{1}{x-3}+\frac{11}{(x-3)^{2}}+\frac{31}{(x-3)^{3}} .
\end{aligned}
$$

## PARTIAL FRACTIONS Q.NO 18

1. Resolve $\frac{x^{2}-3}{(x+2)\left(x^{2}+1\right)}$ into partial fractions.

A: Let $\frac{x^{2}-3}{(x+2)\left(x^{2}+1\right)}=\frac{A}{x+2}+\frac{B x+C}{x^{2}+1}$

$$
\frac{x^{2}-3}{(x+2)\left(x^{2}+1\right)}=\frac{A\left(x^{2}+1\right)+(B x+C)(x+2)}{(x+2)\left(x^{2}+1\right)}
$$

$$
x^{2}-3=A\left(x^{2}+1\right)+(B x+C)(x+2)-\cdots---(1)
$$

$$
x^{2}-3=A\left(x^{2}+1\right)+B\left(x^{2}+2 x\right)+C(x+2)-\cdots-\cdots(2)
$$

Put $x=-2$ in (1), we get
$4-3=A(4+1)$

$$
5 A=1 \quad \therefore A=\frac{1}{5}
$$

Equating the coefficient of $x^{2}$ on both sides in (2), then

$$
1=\mathrm{A}+\mathrm{B} \quad \therefore \mathrm{~B}=\frac{4}{5}
$$

Equating the coefficient of $x$ on both sides in (2),

$$
0=2 B+C
$$

$$
C=-2\left(\frac{4}{5}\right) \quad \therefore C=\frac{-8}{5}
$$

$$
\therefore \frac{x^{2}-3}{(x+2)\left(x^{2}+1\right)}=\frac{1}{5(x+2)}+\frac{4 x-8}{5\left(x^{2}+1\right)} .
$$

(ii) $\frac{2 x^{2}+3 x+4}{(x-1)\left(x^{2}+2\right)}$.

A: Let $\frac{2 x^{2}+3 x+4}{(x-1)\left(x^{2}+2\right)}=\frac{A}{x-1}+\frac{B x+C}{x^{2}+2}$

$$
\begin{align*}
& \frac{2 x^{2}+3 x+4}{(x-1)\left(x^{2}+2\right)}=\frac{A\left(x^{2}+2\right)+(B x+C)(x-1)}{(x-1)\left(x^{2}+2\right)} \\
& 2 x^{2}+3 x+4=A\left(x^{2}+2\right)+(B x+C)(x-1)---  \tag{1}\\
& 2 x^{2}+3 x+4=A\left(x^{2}+2\right)+B\left(x^{2}-x\right)+C(x-1) \tag{2}
\end{align*}
$$

Put $x=1$ in equation (1), we get $2+3+4=A(1+2)$

$$
3 A=9 \quad \therefore A=3
$$

Equating the coefficient of $x^{2}$ on both sides in (2), we get $2=A+B \Rightarrow B=-1$
Equating the coefficient of $x$ on both sides in (2), we get $3=-B+C$

$$
\begin{aligned}
& \quad \begin{array}{l}
C=3-1 \\
\therefore C=2 \\
\therefore \frac{2 x^{2}+3 x+4}{(x-1)\left(x^{2}+2\right)}
\end{array}=\frac{3}{x-1}+\frac{(-x+2)}{x^{2}+2} \\
& =\frac{3}{x-1}-\frac{(x-2)}{x^{2}+2} .
\end{aligned}
$$

(iii) $\frac{3 x-1}{\left(1-x+x^{2}\right)(x+2)}$ into partial fractions.

A: $\frac{3 x-1}{\left(1-x+x^{2}\right)(x+2)}=\frac{A x+B}{1-x+x^{2}}+\frac{C}{x+2}$.

$$
\begin{aligned}
\Rightarrow \frac{3 x-1}{\left(1-x+x^{2}\right)(x+2)} & =\frac{(A x+B)(x+2)+C\left(1-x+x^{2}\right)}{\left(1-x+x^{2}\right)(x+2)} \\
\Rightarrow 3 x-1 & =(A x+B)(x+2)+C\left(1-x+x^{2}\right) \rightarrow \text { (1) } \\
3 x-1 & =A\left(x^{2}+2 x\right)+B(x+2)+C\left(1-x+x^{2}\right) \rightarrow \text { (2) }
\end{aligned}
$$

put $x=-2$ in (1), we get
$3(-2)-1=0+C(1+2+4)$

$$
7 C=-7 \text { then } C=-1
$$

Equating the coefficient of $x^{2}$ in (2)

$$
\begin{aligned}
0 & =A+C . \\
\therefore A & =1 .
\end{aligned}
$$

Equating the coefficient of $x$ in (2)

$$
3=2 A+B-C
$$

$$
3=2+B+1 \text { then } B=0
$$

$\therefore \frac{3 \mathrm{x}-1}{\left(1-\mathrm{x}+\mathrm{x}^{2}\right)(\mathrm{x}+2)}=\frac{1 \cdot \mathrm{x}+0}{1-\mathrm{x}+\mathrm{x}^{2}}-\frac{1}{\mathrm{x}+2}$.

$$
=\frac{x}{1-x+x^{2}}-\frac{1}{x+2}
$$

(iv) Resolve into partial fractions $\frac{2 x^{2}+1}{x^{3}-1}$.

A: $\frac{2 x^{2}+1}{x^{3}-1}=\frac{2 x^{2}+1}{(x-1)\left(x^{2}+x+1\right)}$

$$
\begin{aligned}
\frac{2 x^{2}+1}{(x-1)\left(x^{2}+x+1\right)} & =\frac{A}{(x-1)}+\frac{B x+c}{\left(x^{2}+x+1\right)} \\
& =\frac{A\left(x^{2}+x+1\right)+(B x+c)(x-1)}{\left(x^{2}+x+1\right)}
\end{aligned}
$$

$$
2 x^{2}+1=A\left(x^{2}+x+1\right)+(B x+c)(x-1)
$$

Comparing the corresponding coefficients we have

$$
A+B=2, A-B+C=0, A-C=1
$$

By solving above equtaions we get

$$
A=1, B=1, C=0 \text { then }
$$

$$
\frac{2 x^{2}+1}{\left(x^{3}-1\right)}=\frac{1}{x-1}+\frac{x}{x^{2}+x+1}
$$

2. Resolve $\frac{x^{3}}{(x-a)(x-b)(x-c)}$ into partial fractions.

A: Let $\frac{x^{3}}{(x-a)(x-b)(x-c)}=\frac{1}{1}+\frac{A}{x-a}+\frac{B}{x-b}+\frac{C}{x-c}$

$$
\frac{x^{3}}{(x-a)(x-b)(x-c)}=\frac{(x-a)(x-b)(x-c)+A(x-b)(x-c)+B(x-a)(x-c)+C(x-a)(x-b)}{(x-a)(x-b)(x-c)}
$$

$$
\begin{equation*}
x^{3}=(x-a)(x-b)(x-c)+A(x-b)(x-c)+B(x-a)(x-c)+C(x-a)(x-b) \tag{1}
\end{equation*}
$$

Put $x=a$ in (1), we get

$$
a^{3}=A(a-b)(a-c)
$$

$$
\therefore A=\frac{a^{3}}{(a-b)(a-c)}
$$

Put $x=b$ in (1), we get

$$
b^{3}=B(b-a)(b-c)
$$

$\therefore B=\frac{b^{3}}{(b-a)(b-c)}$
Put $x=c$ in (1), we get $c^{3}=C(c-a)(c-b)$

$$
\therefore C=\frac{c^{3}}{(c-a)(c-b)}
$$

$$
\therefore \frac{x^{3}}{(x-a)(x-b)(x-c)}=1+\frac{a^{3}}{(a-b)(a-c)(x-a)}+\frac{b^{3}}{(b-a)(b-c)(x-b)}+\frac{c^{3}}{(c-a)(c-b)(x-c)} .
$$

(ii) $\frac{x^{3}}{(2 x-1)(x+2)(x-3)}$.

A: $\frac{x^{3}}{(2 x-1)(x+2)(x-3)}$ is an improper fraction.

$$
\begin{aligned}
& \text { Let } \frac{x^{3}}{(2 x-1)(x+2)(x-3)}=\frac{(1 / 2)}{1}+\frac{A}{2 x-1}+\frac{B}{x+2}+\frac{C}{x-3} \\
& \frac{x^{3}}{(2 x-1)(x+2)(x-3)}=\frac{\frac{1}{2}(2 x-1)(x+2)(x-3)+A(x+2)(x-3)+B(2 x-1)(x-3)+C(2 x-1)(x+2)}{(2 x-1)(x+2)(x-3)}
\end{aligned}
$$

$$
\begin{equation*}
x^{3}=\frac{1}{2}(2 x-1)(x+2)(x-3)+A(x+2)(x-3)+B(2 x-1)(x-3)+C(2 x-1)(x+2) \tag{1}
\end{equation*}
$$

Put $x=\frac{1}{2}$ in (1), we get

$$
\begin{aligned}
& \frac{1}{8}=A\left(\frac{1}{2}+2\right)\left(\frac{1}{2}-3\right) \\
& \frac{1}{8}=A\left(\frac{5}{2}\right)\left(\frac{-5}{2}\right)
\end{aligned}
$$

$$
\therefore \mathrm{A}=\frac{-1}{50}
$$

$$
\text { Put } x=-2 \text { in (1), we get }
$$

$$
\begin{aligned}
& -8=\mathrm{B}(-4-1)(-5) \\
& \therefore \mathrm{B}=\frac{-8}{25}
\end{aligned}
$$

$$
\text { Put } x=3 \text { in (1), we get }
$$

$$
27=\mathrm{C}(5)(5)
$$

$$
\therefore C=\frac{27}{25} \text {. }
$$

$$
\therefore \frac{x^{3}}{(2 x-1)(x+2)(x-3)}=\frac{1}{2}-\frac{1}{50(2 x-1)}-\frac{8}{25(x+2)}+\frac{27}{25(x-3)} .
$$

$$
\text { (iii). } \frac{x^{4}}{(x-1)(x-2)}
$$

A:

$$
\begin{gathered}
\left.x^{2}-3 x+2\right) x^{4} \quad\left(x^{2}+3 x+7\right. \\
\frac{x^{4}-3 x^{3}+2 x^{2}}{3 x^{3}-2 x^{2}} \\
\frac{3 x^{3}-9 x^{2}}{7 x^{2}}+6 x \\
\frac{7 x^{2}}{}-\frac{+21 x+14}{15 x-14}
\end{gathered}
$$

$$
\therefore \frac{\mathrm{x}^{4}}{(\mathrm{x}-1)(\mathrm{x}-2)}=\mathrm{x}^{2}+3 \mathrm{x}+7+\frac{15 \mathrm{x}-14}{(\mathrm{x}-1)(\mathrm{x}-2)}
$$

$$
\text { Let } \frac{15 x-14}{(x-1)(x-2)}=\frac{A}{x-1}+\frac{B}{x-2}
$$

$$
\frac{15 x-14}{(x-1)(x-2)}=\frac{A(x-2)+B(x-1)}{(x-1)(x-2)}
$$

$$
\begin{equation*}
15 x-14=A(x-2)+B(x-1) \tag{1}
\end{equation*}
$$

Put $x=1$ in (1), we get

$$
15-14=A(-1)
$$

$$
\therefore \mathrm{A}=-1 \text {. }
$$

Put $x=2$ in (1), we get

$$
30-14=B(2-1)
$$

$$
B=16
$$

$\therefore \frac{x^{4}}{(x-1)(x-2)}=x^{2}+3 x+7-\frac{1}{x-1}+\frac{16}{x-2}$.

## PROBABILITY Q.NO 19

1. In a committee of 25 members, each member is proficient either in mathematics or in statistics or in both. If 19 of these are proficient in mathematics, 16 in statistics, find the probability that a person selected from the committee is proficient in both.
$A$ : Let $A$ be the event that the selected member is proficient in mathematics and $B$ be the event that the selected member is proficient in statistics.
Given that out of 25 members, each member is proficient either in mathematics or in satistics or in both.
So $A, B$ are exhanstive events then $A \cup B=S$

$$
P(A \cup B)=P(S)
$$

$$
P(A)+P(B)-P(A \cap B)=1
$$

$$
P(A \cap B)=P(A)+P(B)-1
$$

$$
=\frac{19}{25}+\frac{16}{25}-1
$$

$$
=\frac{19+16-25}{25}=\frac{10}{25}=\frac{2}{5} .
$$

2. Find the probability of drawing an ace or spade from a well shuffled pack of 52 cards.
A. $E_{1}=$ Event of drawing a spade then $n\left(E_{1}\right)=13$
$E_{2}=$ Event of drawing an ace then $n\left(E_{2}\right)=4$
and $n\left(E_{1} \cap E_{2}\right)=1$

$$
\begin{aligned}
\therefore P\left(E_{1} \cup E_{2}\right) & =P\left(E_{1}\right)+P\left(E_{2}\right)-P\left(E_{1} \cap E_{2}\right) \\
& =\frac{13}{52}+\frac{4}{52}-\frac{1}{52}=\frac{17-1}{52}=\frac{16}{52}=\frac{4}{13}
\end{aligned}
$$

3. If one ticket is randomly selected from tickets numbered 1 to 30 , then find the probability that the number on the ticket is a multiple of 3 or 5 .
A: Let $A, B$ be events that the number on the ticket is a multiple of 3,5 respectively, when a ticket is selected from 1 to 30 .
$A=\{3,6,9,12,15,18,21,24,27,30\}$
$B=\{5,10,15,20,25,30\}$
$A \cap B=\{15,30\}$.
$n(A)=10, n(B)=6, n(A \cap B)=2$
Required probability by addition theorem $\mathrm{P}(\mathrm{A} \cup \mathrm{B})$
$P(A)+P(B)-P(A \cap B)=\frac{10}{30}+\frac{6}{30}-\frac{2}{30}=\frac{14}{30}=\frac{7}{15}$.
4. In a class of 60 boys and 20 girls half of the boys and half of the girls know cricket. Find the probability of the event that a person selected from the class is either 'a boy' or ' a girl who knows cricket'.
A: Number of boys $=60$ and Number of girls $=20$.
Let $A$ be the event that the selected person is a boy
Let $B$ be the event that the selected person is a girl who knows cricket.
Clearly $A, B$ are mutually exclusive events. i.e. $A \cap B=\phi$.
$P(A \cup B)=P(A)+P(B)=\frac{{ }^{60} C_{1}}{{ }^{80} C_{1}}+\frac{{ }^{10} C_{1}}{{ }^{80} C_{1}}=\frac{60}{80}+\frac{10}{80}=\frac{70}{80}=\frac{7}{8}$.
5. If $A, B, C$ are three events in a sample space $S$, then show that

$$
P(A \cup B \cup C)=P(A)+P(B)+P(C)-P(A \cap B)-P(B \cap C)-P(C \cap A)+P(A \cap B \cap C) .
$$

$A$ : Given that $A, B, C$ are three events in a sample space $S$.

$$
\begin{aligned}
P(A \cup B \cup C) & =P[A \cup(B \cup C)]=P(A)+P(B \cup C)-P[A \cap(B \cup C)] \\
& =P(A)+P(B)+P(C)-P(B \cap C)-\{P(A \cap B) \cup P(A \cap C)\} \\
& =P(A)+P(B)+P(C)-P(B \cap C)-\{P(A \cap B)+P(A \cap C)-P[(A \cap B) \cap(A \cap C)]\} \\
& =P(A)+P(B)+P(C)-P(B \cap C)-P(A \cap B)-P(A \cap C)+P(A \cap B \cap C) \\
& =P(A)+P(B)+P(C)-P(A \cap B)-P(B \cap C)-P(C \cap A)+P(A \cap B \cap C) .
\end{aligned}
$$

6. A speaks truth in $75 \%$ of the cases and $B$ in $80 \%$ of the cases. Find the percentage of the cases of which they likely to contradict each other in stating the same fact.
$A$ : Let $A$ and $B$ be the events that the persons $A, B$ respectively to speak truth about an incident.
Given that $P(A)=\frac{75}{100}=\frac{3}{4}$

$$
P(B)=\frac{80}{100}=\frac{4}{5}
$$

Clearly $A, B$ are independent events.
Now probability that their statements about an incident contradict each other $=P[(A \cap \bar{B}) \cap(\bar{A} \cap B)]$

$$
\begin{aligned}
P(A \cap \bar{B})+P(\bar{A} \cap B) & =P(A) P(\bar{B})+P(\bar{A}) P(B) \\
& =\frac{3}{4}\left(1-\frac{4}{5}\right)+\left(1-\frac{3}{4}\right)\left(\frac{4}{5}\right) \\
& =\frac{3}{4}\left(\frac{1}{5}\right)+\frac{1}{4}\left(\frac{4}{5}\right)=\frac{7}{20}
\end{aligned}
$$

Hence the percentage of the cases of which they likely to contradict each other $=\frac{7}{20} \times 100=35 \%$.
7. Two persons $A$ and $B$ are rolling a die on the condition that the person who gets 3 will win the game. If $A$ starts the game, then find the probabilities of $A$ and $B$ respectively to win the game.
A. Let $p$ be the probability of getting 3 on a die $=1 / 6$ $q$ be the probability of not getting 3

$$
q=1-p=1-\frac{1}{6} \quad \text { AND } q=\frac{5}{6}
$$

$A, B$ be the events that $A, B$ will win the game respectively.

## A starts the game

Then A will win in $1^{\text {st }}$ or $3^{\text {rd }}$ or $5^{\text {th }} \ldots$ chances.
The probabilities of $A$ will win the game is $P(A)=p+q q p+q q q q p+$ $\qquad$ $. \infty=p\left(1+q^{2}+q^{4}+\ldots.\right)$

$$
\begin{aligned}
& =p\left(\frac{1}{1-q^{2}}\right) \quad \because S_{\infty}=\frac{a}{1-r} \\
& =\frac{1}{6}\left[\frac{1}{1-\left(\frac{5}{6}\right)^{2}}\right]=\frac{1}{6}\left[\frac{1}{\frac{36-25}{36}}\right] \\
P(A) & =\frac{6}{11}
\end{aligned}
$$

Probability of B will win game is $P(B)=1-P(A)=1-\frac{6}{11}$

$$
\therefore P(B)=\frac{5}{11} \text {. }
$$

8. Find the probability that a non leap year contains (i) 53 Sundays (ii) 52 Sundays only.
A. Let ' $A$ ' be the event of a non-leap year contains 53 sundays.

365 days $=52$ weeks +1 day.
This 1 day may be Sun, Mon, Tue, Wed, Thu, Fri or Sat
$\mathrm{n}(\mathrm{S})=7$ and $\mathrm{n}(\mathrm{A})=1$
$\therefore P(A)=\frac{n(A)}{n(S)}=\frac{1}{7}$
ii) Let $B$ be the evnt of a non-leap yar having 52 sundays only.
$B=\{$ Mon, Tue, Wed, Thu, Fri, Sat $\}$
$n(B)=6$ and $n(S)=7$
$\therefore P(B)=\frac{n(B)}{n(S)}=\frac{6}{7}$
9. State and Prove multiplication theorem on probability.

A: Multiplication theorem on probability : Let $A, B$ be two events in a sample space $S$ such that $P(A) \neq 0, P(B) \neq 0$.
Then (i) $P(A \cap B)=P(A) P\left(\frac{B}{A}\right) \quad$ (ii) $P(A \cap B)=P(B) P\left(\frac{A}{B}\right)$
Let $n(A), n(B), n(A \cap B), n(S)$ be the number of sample points in $A, B, A \cap B$, $S$ respectively.
then $P(A)=\frac{n(A)}{n(S)}, P(B)=\frac{n(B)}{n(S)}, P(A \cap B)=\frac{n(A \cap B)}{n(S)} P\left(\frac{B}{A}\right)=\frac{n(A \cap B)}{n(A)}, P\left(\frac{A}{B}\right)=\frac{n(A \cap B)}{n(B)}$
(i) P
$P(A \cap B)=\frac{n(A \cap B)}{n(S)}$
multiply and divide by $n(A)$

$$
\begin{aligned}
& =\frac{n(A)}{n(S)} \cdot \frac{n(A \cap B)}{n(A)} \\
& =P(A) P\left(\frac{B}{A}\right)
\end{aligned}
$$

(ii) $\mathrm{P}(\mathrm{A} \cap \mathrm{B})=\frac{\mathrm{n}(\mathrm{A} \cap \mathrm{B})}{\mathrm{n}(\mathrm{S})}$
multiply and divide by $n(B)=\frac{n(B)}{n(S)} \cdot \frac{n(A \cap B)}{n(B)}=P(B) \cdot P\left(\frac{A}{B}\right)$.
10. If one card is drawn at random from a pack of cards, then show that the events of getting an ace and getting a heart are independent events.
A. Let $A$ be the event of getting an ace and $B$ be the event of getting a heart, when a card is drawn from a pack.
$P(A)=\frac{{ }^{4} \mathrm{C}_{1}}{{ }^{52} \mathrm{C}_{1}}=\frac{4}{52}$ and $\mathrm{P}(\mathrm{B})=\frac{{ }^{13} \mathrm{C}_{1}}{{ }^{52} \mathrm{C}_{1}}=\frac{13}{52}$
$P(A \cap B)=\frac{{ }^{1} C_{1}}{{ }^{52} C_{1}}=\frac{1}{52}$
$P(A) \cdot P(B)=\frac{4}{52} \times \frac{13}{52}=\frac{1}{52}$
$\therefore \mathrm{P}(\mathrm{A} \cap \mathrm{B})=\mathrm{P}(\mathrm{A}) \cdot \mathrm{P}(\mathrm{B})$
So, the events $A, B$ are independent events.
11. $A$ and $B$ seeking admission into IIT. If the probability for $A$ to be selected is 0.5 and both to be selected is 0.3 , then is it possible that the probability of $B$ to be selected is 0.9 ?
A: Let $P(A)$ and $P(B)$ be the probabilities of $A, B$ seeking admission into IIT resp.
$\mathrm{P}(\mathrm{A})=0.5, \mathrm{P}(\mathrm{A} \cap \mathrm{B})=0.3$
$P(A \cup B)=P(A)+P(B)-P(A \cap B)$
Then we have $P(A) \leq P(A \cup B) \leq 1$

$$
\begin{aligned}
P(A) \leq P(A)+P(B)-P(A \cap B) & \leq 1 \\
0.5 \leq 0.5+P(B)-0.3 & \leq 1 \\
0.5 \leq 0.2+P(B) & \leq 1 \\
0.5-0.2 \leq P(B) & \leq 1-0.2 \\
0.3 \leq P(B) & \leq 0.8
\end{aligned}
$$

The probability of $B$ to be selected is 0.9 is not possible.

## PROBABILITY Q.NO 20

1. If $A$ and $B$ are independent events with $P(A)=0.6, P(B)=0.7$ then compute
(i) $P(A \cap B)$
(ii) $\mathbf{P}(A \cup B)$
(iii) $P\left(\frac{B}{A}\right)$
(iv) $\mathbf{P}(\overline{\mathbf{A}} \cap \overline{\mathbf{B}})$

A: Given that $A, B$ are independent events with $P(A)=0.6$, $P(B)=0.7$
i) $\mathrm{P}(\mathrm{A} \cap \mathrm{B})=\mathrm{P}(\mathrm{A}) \mathrm{P}(\mathrm{B})=(0.6)(0.7)=0.42$
ii) $P(A \cup B)=P(A)+P(B)-P(A \cap B)$

$$
=P(A)+P(B)-P(A) P(B)=0.6+0.7-(0.6)(0.7)=1.30-0.42=0.88
$$

iii) $P\left(\frac{B}{A}\right)=P(B)=0.7$
iv) $P(\bar{A} \cap \bar{B})=P(\bar{A}) P(\bar{B})$ $=(1-0.6)(1-0.7)=(0.4)(0.3)=0.12$.
2. If $A$ and $B$ are independent events with $P(A)=0.2, P(B)=0.5$ then find
(i) $\mathbf{P}\left(\frac{\mathbf{A}}{\mathbf{B}}\right)$
(ii) $\mathbf{P}\left(\frac{\mathbf{B}}{\mathbf{A}}\right)$
(iii) $\mathbf{P}(\mathbf{A} \cap \mathbf{B})$ (iv) $\mathbf{P}(\mathbf{A} \cup \mathbf{B})$
A. Given $A, B$ are independent events $P(A)=0.2$ and $P(B)=0.5$
(i) $P(A \cap B)=P(A) P(B)=(0.2)(0.5)=0.1$
(ii) $P\left(\frac{A}{B}\right)=\frac{P(A \cap B)}{P(B)}=\frac{0.1}{0.5}=\frac{1}{5}=0.2$
(iii) $P\left(\frac{B}{A}\right)=\frac{P(B \cap A)}{P(A)}=\frac{0.1}{0.2}=\frac{1}{2}=0.5$
(iv) $P(A \cup B)=P(A)+P(B)-P(A \cap B)=0.2+0.5-0.1=0.6$
3. For any two events, show that $P(\overline{\mathbf{A}} \cap \bar{B})=1+P(A \cap B)-P(A)-P(B)$

A: We know that $\overline{\mathrm{A}} \cap \overline{\mathrm{B}}=\overline{\mathrm{A} \cup \mathrm{B}}$

$$
\begin{aligned}
P(\bar{A} \cap \bar{B}) & =P(\overline{A \cup B}) \\
& =1-P(A \cup B) \\
& =1-\{P(A)+P(B)-P(A \cap B)\} \\
& =1+P(A \cap B)-P(A)-P(B)
\end{aligned}
$$

4. If $A, B$ are two events with $P(A \cup B)=0.65, P(A \cap B)=0.15$ then find $P(\bar{A})+P(\bar{B})$
$A$ : Given that $P(A \cup B)=0.65, P(A \cap B)=0.15$
By addition theorem,

$$
\begin{align*}
& P(A \cup B)=P(A)+P(B)-P(A \cap B) \\
& P(A \cup B)+P(A \cap B)=P(A)+P(B) \tag{1}
\end{align*}
$$

Now $P(\bar{A})+P(\bar{B})=1-P(A)+1-P(B)$

$$
\begin{aligned}
& =2-\{P(A)+P(B)\} \\
& =2-\{P(A \cup B)+P(A \cap B)\} \text { from }(1) \\
& =2-\{0.65+0.15\} \\
& =2-0.8 \\
& =1.2
\end{aligned}
$$

5. $A$ and $B$ are events with $P(A)=0.5, P(B)=0.4$ and $P(A \cap B)=0.3$. find the probability that (i) A does not occur (ii) neither A nor B occurs.

A: Given that $A, B$ are independent events with $P(A)=0.5, P(B)=0.4$ and $P(A \cap B)=0.3$
(i) The probability that A does not occur $P(\bar{A})=1-P(A)=1-0.5=0.5$

$$
\begin{aligned}
\mathrm{P}(\mathrm{~A} \cup \mathrm{~B}) & =\mathrm{P}(\mathrm{~A})+\mathrm{P}(\mathrm{~B})-\mathrm{P}(\mathrm{~A} \cap \mathrm{~B}) \\
& =0.5+0.4-0.3 \\
& =0.6
\end{aligned}
$$

(ii) The probability that neither $A$ nor $B$ occurs $=P(\bar{A} \cap \bar{B})$

$$
\begin{aligned}
P(\overline{A \cup B}) & =1-P(A \cup B) \\
& =1-0.6 \\
& =0.4
\end{aligned}
$$

6. A problem in calculus is given to two students $A$ and $B$ whose chances of solving it are $\frac{1}{3}$ and $\frac{1}{4}$. What is the probability that the problem will be solved if both of them try independently.
$A$ : Let $A$ and $B$ denote the events that the problem is solved by $A$ and $B$ respectively.
Here $P(A)=\frac{1}{3}, P(B)=\frac{1}{4}$
Required probability

$$
\begin{aligned}
P(A \cup B) & =P(A)+P(B)-P(A \cap B) \\
& =P(A)+P(B)-P(A) P(B) \\
& =\frac{1}{3}+\frac{1}{4}-\frac{1}{3} \cdot \frac{1}{4} \\
& =\frac{4+3-1}{12}=\frac{6}{12}=\frac{1}{2} .
\end{aligned}
$$

7. $A, B$ are twoindependent events such that the probability of both the events to occur is $1 / 6$ and the probability of both the events do not occur is $1 / 3$. Find $P(A)$.
$A$ : Given that $A, B$ are independent events
Let $P(A)=x$ and $P(B)=y$
Given that $P(A \cap B)=\frac{1}{6} \quad P(\bar{A} \cap \bar{B})=\frac{1}{3}$

$$
\begin{aligned}
P(A) P(B) & =\frac{1}{6} & P(\bar{A}) P(\bar{B})=\frac{1}{3} \\
x y & =\frac{1}{6} & {[1-P(A)][1-P(B)]=\frac{1}{3} }
\end{aligned}
$$

$$
\begin{aligned}
& y=1 / 6 x(1-x)(1-y)=\frac{1}{3} \\
& 1-x-y+x y=\frac{1}{3} \\
& 1-(x+y)+\frac{1}{6}= \frac{1}{3} \\
& x+y= \frac{7}{6}-\frac{1}{3} \\
& x+y=\frac{5}{6}
\end{aligned}
$$

Now $x+y=\frac{5}{6}$ and $x y=\frac{1}{6}$

$$
\begin{aligned}
& x+\frac{1}{6 x}=\frac{5}{6} \\
& \frac{6 x^{2}+1}{6 x}=\frac{5}{6} \\
& 6 x^{2}+1=5 x \\
& 6 x^{2}-5 x+1=0 \\
& 6 x^{2}-3 x-2 x+1=0 \\
& 3 x(2 x-1)-1(2 x-1)=0 \\
&(3 x-1)(2 x-1)=0 \\
& x=\frac{1}{3} \text { or } \frac{1}{2} \\
& \text { Hence } P(A)=\frac{1}{3} \text { or } \frac{1}{2}
\end{aligned}
$$

8. A bag $B_{1}$ contains 4 white and 2 black balls. Bag $B_{2}$ contains 3 white and 4 black balls. A bag is drawn at random and a ball is chosen at random from it. Then what is the probability that the ball is white?
$A$ : Let $A_{1}, A_{2}$ be the events of choosing bags $B_{1}, B_{2}$ respectively.
Here $A_{1}, A_{2}$ are equally likely events
Then $P\left(A_{1}\right)=\frac{1}{2}, P\left(A_{2}\right)=\frac{1}{2}$
Let $E$ be the event of drawing a white ball from the selected bag.
$P\left(\frac{E}{A_{1}}\right)=$ Probability of drawing a white ball from bag $B_{1}=\frac{4}{6}=\frac{2}{3}$
$P\left(\frac{E}{A_{2}}\right)=$ Probability of drawing a white ball from bag $B_{2}=\frac{3}{7}$
By total probability theorem, $P(E)=P\left(A_{1}\right) P\left(\frac{E}{A_{1}}\right)+P\left(A_{2}\right)\left(\frac{E}{A_{2}}\right)$

$$
\begin{aligned}
& =\frac{1}{2} \cdot \frac{2}{3}+\frac{1}{2} \cdot \frac{3}{7} \\
& =\frac{14+9}{42}=\frac{23}{42} .
\end{aligned}
$$

9. A number $x$ is drawn arbitrarily from the set of $\{1,2,3, \ldots \ldots 100\}$. Find the probability that $\left(\mathbf{x}+\frac{\mathbf{1 0 0}}{\mathbf{x}}\right)>29$.

A: The total points on the sample space are 100.
Let $A$ be the event that an $x$ is selected at random from the set $S=\{1,2,3, \ldots .100\}$ has the property $\left(x+\frac{100}{x}\right)>29$

$$
\begin{aligned}
\text { Now }\left(x+\frac{100}{x}\right) & >29 \\
x^{2}-29 x+100 & >0 \\
(x-4)(x-25) & >0 \\
x<4 \text { or } x & >25
\end{aligned}
$$

Since $x \in S$, it follows that $A=\{1,2,3,26,27, \ldots .100\}$
Thus the number of cases favourable to $A$ is 78
The required probability: $\mathrm{P}(\mathrm{A})=\frac{78}{100}=0.78$.

## VERY SHORT ANSWERS TYPE QUESTIONS

## COMPLEX NUMBERS Q.NO 1 AND Q.NO 2

1. Find the real and imaginary parts of the complex number $\frac{\mathbf{a}-\mathbf{i b}}{\mathbf{a}+\mathbf{i b}}$.
$A: \frac{a-i b}{a+i b}=\left(\frac{a-i b}{a+i b}\right)\left(\frac{a-i b}{a-i b}\right)$

$$
=\frac{\left(a^{2}-b^{2}\right)+(-2 a b) i}{a^{2}+b^{2}}
$$

Realpart $=\frac{a^{2}-b^{2}}{a^{2}+b^{2}}$, imaginary part $=\frac{-2 a b}{a^{2}+b^{2}}$
2. Find the square roots of $-5+12 i$.
$A$ : We know that $\sqrt{a+i b}= \pm\left[\sqrt{\frac{\sqrt{a^{2}+b^{2}}+a}{2}}+i \sqrt{\frac{\sqrt{a^{2}+b^{2}}-a}{2}}\right]$
Here $a=-5, b=12$
$\sqrt{-5+12 i}= \pm\left[\sqrt{\frac{\sqrt{25+144}+(-5)}{2}}+i \sqrt{\frac{\sqrt{25+144}-(-5)}{2}}\right]= \pm\left[\sqrt{\frac{13-5}{2}}+i \sqrt{\frac{13+5}{2}}\right]= \pm(2+3 i)$.
(ii) $7+24$ i.

A: We know that $\sqrt{\mathbf{a}+\mathbf{i b}}= \pm\left[\sqrt{\frac{\sqrt{a^{2}+\mathrm{b}^{2}}+\mathrm{a}}{2}}+i \sqrt{\frac{\sqrt{\mathrm{a}^{2}+\mathrm{b}^{2}}-\mathrm{a}}{2}}\right][\because \mathbf{b}>\mathbf{0}]$

$$
\begin{aligned}
\therefore \sqrt{7+24 i} & = \pm\left[\sqrt{\frac{\sqrt{7^{2}+24^{2}}+7}{2}}+i \sqrt{\frac{\sqrt{7^{2}+24^{2}}-7}{2}}\right]= \pm\left[\sqrt{\frac{\sqrt{625}+7}{2}}+i \sqrt{\frac{\sqrt{625-7}}{2}}\right] \\
& = \pm\left[\sqrt{\frac{25+7}{2}}+i \sqrt{\frac{25-7}{2}}\right]= \pm\left[\sqrt{\frac{32}{2}}+i \sqrt{\frac{18}{2}}\right] \\
& = \pm[\sqrt{16}+i \sqrt{9}]= \pm(4+3 i)
\end{aligned}
$$

3. Find the complex conjugate of $(3+4 i)(2-3 i)$.

A: Let $z=(3+4 i)(2-3 i)$

$$
\begin{aligned}
& =6-9 i+8 i-12 i^{2} \\
& =18-i
\end{aligned}
$$

$$
\therefore \overline{\mathrm{z}}=18+\mathrm{i} .
$$

(ii) Show that $z_{1}=\frac{2+11 i}{25}, z_{2}=\frac{-2+i}{(1-2 i)^{2}}$ are conjugate to each other.
$A: z_{2}=\frac{-2+i}{(1-2 i)^{2}}=\frac{-2+i}{1-4-4 i}=\frac{-2+i}{-3-4 i}$

$$
\begin{aligned}
& =\frac{2-i}{3+4 i} \times \frac{3-4 i}{3-4 i}=\frac{6-8 i-3 i-4}{9+16} \quad\left(\because(a+i b)(a-i b)=a^{2}+b^{2}\right) \\
& =\frac{2-11 i}{25}
\end{aligned}
$$

$\therefore \mathrm{z}_{1}$ and $\mathrm{z}_{2}$ are conjugate to each other.
4. Find the additive inverse of $(\sqrt{3}, 5)$.

A: $(\sqrt{3}, 5)=\sqrt{3}+5 i$
Its additive inverse $=-(\sqrt{3}, 5)=-\sqrt{3}-5 i=(-\sqrt{3},-5)$.
5. Write the multiplicative inverse of ( 7,24 ).

A: $(7,24)=7+24 i$.
Multiplicative inverse of $7+24 i=\frac{1}{7+24 i}$.

$$
\frac{1}{7+24 i} \times \frac{7-24 i}{7-24 i}=\frac{7-24 i}{7^{2}-24^{2} i^{2}}=\frac{7-24 i}{49+576}=\frac{7-24 i}{625}=\frac{7}{625}-i \frac{24}{625}=\left(\frac{7}{625},-\frac{24}{625}\right) .
$$

(H/W) $\boldsymbol{\operatorname { s i n }} \theta, \boldsymbol{\operatorname { c o s }} \theta)$.
6. If $z=2-3 i$, show tht $z^{2}-4 z+13=0$.

A: Given that $z=2-3 i$

$$
z-2=-3 i
$$

Squaring on both sides,

$$
\begin{aligned}
(z-2)^{2} & =(-3 i)^{2} \\
z^{2}-4 z+4 & =-9 \\
z^{2}-4 z+13 & =0 .
\end{aligned}
$$

7. Find the least positive integer $n$, satisfying $\left(\frac{1+i}{1-i}\right)^{n}=1$.

A: Given that $\left(\frac{1+i}{1-i}\right)^{n}=1 \Rightarrow\left(\frac{1+i}{1-i} \times \frac{1+i}{1+i}\right)^{n}=1$

$$
\Rightarrow\left[\frac{(1+i)^{2}}{1^{2}-i^{2}}\right]^{n}=1 \Rightarrow\left(\frac{1+i^{2}+2 i}{1+1}\right)^{n}=1
$$

$$
\left(\frac{\not X-\lambda+2 i}{1+1}\right)^{n}=1 \Rightarrow\left[\frac{\not 2 i}{\not 2}\right]^{n}=1 \Rightarrow i^{n}=1
$$

$\mathrm{n}=\{4,8,12$, $\qquad$ $\infty\}$
$\therefore$ Required least positive integer is 4 .
8. If $z=(\cos \theta, \sin \theta)$ then find $z-\frac{\mathbf{1}}{\mathbf{z}}$.

A: Given that $z=(\cos \theta, \sin \theta)=\cos \theta+i \sin \theta$

$$
\begin{aligned}
& \text { then } \begin{aligned}
\frac{1}{z} & =\frac{1}{\cos \theta+i \sin \theta} \times \frac{\cos \theta-i \sin \theta}{\cos \theta-i \sin \theta} \\
& =\frac{\cos \theta-i \sin \theta}{(\cos \theta)^{2}-(i \sin \theta)^{2}}=\frac{\cos \theta-i \sin \theta}{\cos ^{2} \theta+\sin ^{2} \theta} \\
& =\cos \theta-i \sin \theta
\end{aligned} \\
& \begin{aligned}
\therefore z-\frac{1}{z} & =\cos \theta+i \sin \theta-(\cos \theta-i \sin \theta) \\
\Rightarrow z-\frac{1}{z} & =2 i \sin \theta
\end{aligned}
\end{aligned}
$$

9. If $(a+i b)^{2}=x+i y$, find $x^{2}+y^{2}$.

A: Given that $(a+i b)^{2}=x+i y$
Now, $|a+i b|^{2}=|x+i y|$

$$
\begin{array}{r}
\Rightarrow\left(\sqrt{a^{2}+b^{2}}\right)^{2}=\sqrt{x^{2}+y^{2}} \Rightarrow a^{2}+b^{2}=\sqrt{x^{2}+y^{2}} \\
\Rightarrow x^{2}+y^{2}=\left(a^{2}+b^{2}\right)^{2}
\end{array}
$$

10. Express $(1-i)^{3}(1+i)$ in the form $a+i b$.

Sol. Let $z=(1-i)^{2}(1-i)(1+i)$

$$
\begin{aligned}
& =\left(1+\mathrm{i}^{2}-2 \mathrm{i}\right)\left(1-\mathrm{i}^{2}\right) \\
& =(1-1-2 \mathrm{i})(1+1) \\
& =-4 \mathrm{i} \\
& =0+\mathrm{i}(-4)
\end{aligned}
$$

11. Simplify $i^{18}-3 . i^{7}+i^{2}\left(1+i^{4}\right)(-i)^{26}$.

Sol. $i^{18}-3 . i^{7}+i^{2}\left(1+i^{4}\right)(-i)^{26} .=\left(i^{2}\right)^{9}-3\left(i^{2}\right)^{3} \cdot 1+i^{2}\left(1+\left(i^{2}\right)^{2}\right)\left(i^{2}\right)^{13}$

$$
\begin{aligned}
& =(-1)^{9}-3(-1)^{3} \cdot 1+(-1)\left(1+(-1)^{2}\right)(-1)^{13} \\
& =-1+3 i-2=1+3 i .
\end{aligned}
$$

## DEMOIVRES THEOREM Q.NO 3

1. Find the value of $(1+i)^{16}$.
$A:(1+i)^{16}=\left[\sqrt{2}\left(\frac{1}{\sqrt{2}}+i \frac{1}{\sqrt{2}}\right)\right]^{16}$

$$
=(\sqrt{2})^{16}\left[\cos 45^{0}+i \sin 45^{0}\right]^{16}
$$

By applying De Moivre's theorem for an integral index.
$=2^{8}\left[\cos 16\left(45^{\circ}\right)-i \sin 16\left(45^{\circ}\right)\right]$
$=256\left[\cos 720^{\circ}+i \sin 720^{\circ}\right]=256[1-i .0]=256$
2. Find the value of $(1+i \sqrt{3})^{3}$.
$A:(1+i \sqrt{3})^{3}=\left[2\left(\frac{1}{2}+i \frac{\sqrt{3}}{2}\right)\right]^{3}=8\left(\frac{1}{2}+i \frac{\sqrt{3}}{2}\right)^{3}=8\left(\cos 60^{\circ}+i \sin 60^{\circ}\right)^{3}$
By applying De Moivre's theorem
$=8\left[\cos 3\left(60^{\circ}\right)+i \sin 3\left(60^{\circ}\right)\right]$
$=8\left(\cos 180^{\circ}+i \sin 180^{\circ}\right)=8[-1+i(0)]=-8$.
3. Find the value of $(1-i)^{8}$.
$A:(1-i)^{8}=\left[\sqrt{2}\left(\frac{1}{\sqrt{2}}-i \frac{1}{\sqrt{2}}\right)\right]^{8}=(\sqrt{2})^{8}\left(\cos 45^{0}-i \sin 45^{0}\right)^{8}$
By applying De Moivre's theorem

$$
\begin{aligned}
& =2^{4}\left[\cos 8\left(45^{\circ}\right)-i \sin 8\left(45^{\circ}\right)\right] \\
& =2^{4}\left[\cos 360^{\circ}-i \sin 360^{\circ}\right]=16[1-i(0)]=16
\end{aligned}
$$

4. Find the value of $\left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right)^{5}-\left(\frac{\sqrt{3}}{2}-\frac{i}{2}\right)^{5}$.

A: $\left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right)^{5}-\left(\frac{\sqrt{3}}{2}-\frac{i}{2}\right)^{5}=\left(\cos 30^{\circ}+i \sin 30^{\circ}\right)^{5}-\left(\cos 30^{\circ}-i \sin 30^{\circ}\right)^{5}$
By applying De Moivre's theorem
$=\cos 5\left(30^{\circ}\right)+i \sin 5\left(30^{\circ}\right)-\left[\cos 5\left(30^{\circ}\right)-i \sin 5\left(30^{\circ}\right)\right]$
$=\cos 450^{\circ}+i \sin 150^{\circ}-\left[\cos 450^{\circ}-i \sin 150^{\circ}\right]$
$=2 i \sin 150^{\circ}=2 i\left(\frac{\mathbf{1}}{\mathbf{2}}\right)=\mathbf{i}$.
5. If $A, B, C$ are the angles of a triangle such that $x=\operatorname{cis} A, y=\operatorname{cis} B, z=c i s C$, then find $x y z$.

A: Given that $x=\operatorname{cis} A, y=\operatorname{cis} B, z=\operatorname{cis} C$
Now $x y z=\operatorname{cis} A . c i s B . c i s C=\operatorname{cis}(A+B+C)$

$$
\begin{aligned}
& =\cos (A+B+C)+i \sin (A+B+C) \\
& =\cos 180^{\circ}+i \sin 180^{\circ}=-1+i(0)=-1 .
\end{aligned}
$$

6. If $x=\operatorname{cis} \theta$, then find the value of $\left(x^{6}+\frac{1}{x^{6}}\right)$.

A: Given that $x=\cos \theta+i \sin \theta$.

$$
x^{6}=(\cos \theta+i \sin \theta)^{6}=\cos 6 \theta+i \sin 6 \theta .
$$

Now, $\frac{1}{x^{6}}=\frac{1}{\cos 6 \theta+i \sin 6 \theta}=\cos 6 \theta-i \sin 6 \theta$.
Hence, $x^{6}+\frac{1}{x^{6}}==\cos 6 \theta+i \sin 6 \theta+\cos 6 \theta-i \sin 6 \theta=2 \cos 6 \theta$.
7. If $\alpha, \beta$ are the roots of the equation $x^{2}+x+1=0$, then prove that $\alpha^{4}+\beta^{4}+\alpha^{-1} \beta^{-1}=0$.
$A$ : Given equation is $x^{2}+x+1=0$

$$
\begin{aligned}
& x=\frac{-b \pm \sqrt{b^{2}-4 a c}}{2 a}=\frac{-1 \pm \sqrt{1-4}}{2}=\frac{-1 \pm \sqrt{3} i}{2}=\omega, \omega^{2} \\
& \alpha^{4}+\beta^{4}+\alpha^{-1} \beta^{-1}=\omega^{4}+\left(\omega^{2}\right)^{4}+\frac{1}{\omega} \cdot \frac{1}{\omega^{2}}=\omega+\omega^{2}+1=0
\end{aligned}
$$

8. Simplify $\frac{(\cos \alpha+i \sin \alpha)^{4}}{(\sin \beta+i \cos \beta)^{8}}$.
$A: \frac{(\cos \alpha+i \sin \alpha)^{4}}{\left(-i^{2} \sin \beta+i \cos \beta\right)^{8}}=\frac{(\cos \alpha+i \sin \alpha)^{4}}{[i(\cos \beta-i \sin \beta)]^{8}}$

$$
=\frac{(\cos \alpha+i \sin \alpha)^{4}}{i^{8}(\cos \beta-i \sin \beta)^{8}}=\frac{\cos 4 \alpha+i \sin 4 \alpha}{\cos 8 \beta-i \sin 8 \beta}
$$

$$
=(\cos 4 \alpha+i \sin 4 \alpha)(\cos 8 \beta+i \sin 8 \beta)
$$

$$
=\cos (4 \alpha+8 \beta)+i \sin (4 \alpha+8 \beta)=\operatorname{cis}(4 \alpha+8 \beta)
$$

9. If $1, \omega, \omega^{2}$ are the cube roots of unity, prove that $\left(1-\omega+\omega^{2}\right)^{6}+\left(1-\omega^{2}+\omega\right)^{6}=128$.

A: Given that $1, \omega, \omega^{2}$ are the cube roots of unity,

$$
\text { then } 1+\omega+\omega^{2}=0 \text { and } \omega^{3}=1
$$

(i) $\left(1-\omega+\omega^{2}\right)^{6}+\left(1-\omega+\omega^{2}\right)^{6}=(-\omega-\omega)^{6}+\left(-\omega^{2}-\omega^{2}\right)^{6}=(-2 \omega)^{6}+\left(-2 \omega^{2}\right)^{6}=(-2)^{6}\left[\omega^{6}+\omega^{12}\right]=64(1+1)=128$
(ii) $\left(1-\omega+\omega^{2}\right)^{7}+\left(1+\omega-\omega^{2}\right)^{7}=(-\omega-\omega)^{7}+\left(-\omega^{2}-\omega^{2}\right)^{7}=(-2 \omega)^{7}+\left(-2 \omega^{2}\right)^{7}=(-2)^{7}\left[\omega^{7}+\omega^{14}\right]$

$$
=(-128)\left(\omega+\omega^{2}\right)=(-128)(-1)=128
$$

## QUADRATIC EXPRESSIONS Q.NO 4

1. If $\alpha, \beta$ are the roots of the equation $a x^{2}+b x+c=0$. Find the value?

A: $\alpha, \beta$ are the roots of $a x^{2}+b x+c=0$
$\alpha+\beta=\frac{-b}{a}, \alpha \beta=\frac{c}{a}$
$\frac{1}{\alpha}+\frac{1}{\beta}=\frac{\beta+\alpha}{\alpha \beta}=\frac{-b / a}{c / a}=\frac{-b}{c}$.
(ii) $\frac{1}{\alpha^{2}}+\frac{1}{\beta^{2}}$.

A: $\alpha, \beta$ are the roots of $a x^{2}+b x+c=0$.

$$
\begin{aligned}
& \alpha+\beta=\frac{-b}{a}, \alpha \beta=\frac{c}{a} \\
& \frac{1}{\alpha^{2}}+\frac{1}{\beta^{2}}=\frac{\beta^{2}+\alpha^{2}}{(\alpha \beta)^{2}}=\frac{(\alpha+\beta)^{2}-2 \alpha \beta}{(\alpha \beta)^{2}}=\frac{(-b / a)^{2}-2 c / a}{(c / a)^{2}}=\frac{b^{2}-2 a c}{a^{2}} \cdot \frac{a^{2}}{c^{2}}=\frac{b^{2}-2 a c}{c^{2}} .
\end{aligned}
$$

(ii) $\alpha^{3}+\beta^{3}=(\alpha+\beta)^{3}-3 \alpha \beta(\alpha+\beta)$

$$
=\left(\frac{-b}{a}\right)^{3}-3\left(\frac{c}{a}\right)\left(\frac{-b}{a}\right)=-\frac{b^{3}}{a^{3}}+\frac{3 b c}{a^{2}}=\frac{3 a b c-b^{3}}{a^{3}} .
$$

(iv) $\alpha^{4} \beta^{7}+\alpha^{7} \beta^{4}=\alpha^{4} \beta^{4}\left(\alpha^{3}+\beta^{3}\right)$
$\alpha \beta=\frac{\mathrm{c}}{\mathrm{a}}$ and $\alpha^{3}+\beta^{3}=(\alpha+\beta)^{3}-3 \alpha \beta(\alpha+\beta)$

$$
\begin{aligned}
& =\left(\frac{-b}{a}\right)^{3}-3\left(\frac{c}{a}\right)\left(\frac{-b}{a}\right) \\
& =-\frac{b^{3}}{a^{3}}+\frac{3 b c}{a^{2}}=\frac{3 a b c-b^{3}}{a^{3}} .
\end{aligned}
$$

$\alpha^{4} \beta^{4}\left(\alpha^{3}+\beta^{3}\right)=\frac{c^{4}}{a^{4}}\left(\frac{3 a b c-b^{3}}{a^{3}}\right)=b c^{4}\left(\frac{3 a c-b^{2}}{a^{7}}\right)$
2. If $\alpha$ and $\beta$ are the roots of the equation $x^{2}+x+1=0$, find the value of $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}$.

A: $\alpha$ and $\beta$ are the roots of $x^{2}+x+1=0$.
$\alpha+\beta=-b / a=-1 ; \alpha \beta=c / a=1$
$\frac{\alpha}{\beta}+\frac{\beta}{\alpha}=\frac{\alpha^{2}+\beta^{2}}{\alpha \beta}=\frac{(\alpha+\beta)^{2}-2 \alpha \beta}{\alpha \beta}=\frac{(-1)^{2}-2(1)}{1}=-1$.
3. Obtain a quadratic equation whose roots are $7 \pm 2 \sqrt{5}$.
A. Given roots are $7+2 \sqrt{5}, 7-2 \sqrt{5}$.

Quadratic equation is $x^{2}-(\alpha+\beta) x+\alpha \beta=0$

$$
\begin{aligned}
\Rightarrow & x^{2}-(7+2 \sqrt{5}+7-2 \sqrt{5}) x+(7+2 \sqrt{5})(7-2 \sqrt{5})=0 \\
& x^{2}-14 x+29=0
\end{aligned}
$$

(ii) $\frac{p-q}{p+q}, \frac{-(p+q)}{p-q}$.

A: The quadratic equation whose roots are $\frac{p-q}{p+q}, \frac{-(p+q)}{p-q}$ is
$x^{2}-\left[\frac{p-q}{p+q}-\frac{(p+q)}{p-q}\right] x+\left(\frac{p-q}{p+q}\right)\left\{\frac{-(p+q)}{p-q}\right\}=0$
$x^{2}-\left\{\frac{(p-q)^{2}-(p+q)^{2}}{p^{2}-q^{2}}\right\} x-1=0$

$$
\left(p^{2}-q^{2}\right) x^{2}+4 p q x-\left(p^{2}-q^{2}\right)=0
$$

4. If $\alpha$ and $\beta$ are the roots of the equation $2 x^{2}+3 x+6=0$, find the quadratic equation whose roots are $\alpha^{3}$ and $\beta^{3}$.
A: $\alpha, \beta$ are the roots of $2 x^{2}+3 x+6=0$
$\alpha+\beta=\frac{-b}{a}=\frac{-3}{2} ; \alpha \beta=\frac{c}{a}=\frac{6}{2}=3$
$\alpha^{3}+\beta^{3}=(\alpha+\beta)^{3}-3 \alpha \beta(\alpha+\beta)=\frac{-27}{8}+\frac{27}{2}=\frac{-27+108}{8}=\frac{81}{8}$
$\alpha^{3} \beta^{3}=3^{3}=27$
Required quadratic equation is $x^{2}-\left(\alpha^{3}+\beta^{3}\right) x+\alpha^{3} \beta^{3}=0$

$$
x^{2}-\frac{81}{8} x+27=0
$$

$8 x^{2}-81 x+216=0$
5. Find the quadratic equation, the sum of whose roots is 7 and the sum of the squares of the roots is 25 .

A: Let $\alpha, \beta$ be the roots of the required quadratic equation.
Given that $\alpha+\beta=7, \alpha^{2}+\beta^{2}=25$.
$(\alpha+\beta)^{2}=7^{2}$
$\alpha^{2}+\beta^{2}+2 \alpha \beta=49$

$$
\begin{aligned}
25+2 \alpha \beta & =49 \\
2 \alpha \beta & =49-25=24 \\
\alpha \beta & =12
\end{aligned}
$$

$\therefore$ Required quadratic equation is $\mathrm{x}^{2}-(\alpha+\beta) \mathrm{x}+\alpha \beta=0$

$$
x^{2}-7 x+12=0
$$

5. If the equation $x^{2}-15-m(2 x-8)=0$ has equal roots, then find the values of $m$.

A: Given equation is $x^{2}-2 m x+(8 m-15)=0$

$$
\begin{gathered}
\text { since it has equal roots } b^{2}-4 a c=0 \\
(-2 m)^{2}-4(1)(8 m-15)=0 \\
4 m^{2}-4(8 m-15)=0 \\
m^{2}-8 m+15=0 \\
(m-3)(m-5)=0
\end{gathered}
$$

$\therefore \mathrm{m}=3$ or 5 .
6. Find the maximum value of $2 x-7-5 x^{2}$ for $x \in R$.

A: Comparing $2 x-7-5 x^{2}$ with $a x^{2}+b x+c$,
we get $a=-5, b=2, c=-7$.
Maximum value of $2 x-7-5 x^{2}=\frac{4 a c-b^{2}}{4 a}=\frac{4(-5)(-7)-2^{2}}{4(-5)}=\frac{140-4}{-20}=\frac{136}{-20}=-\frac{34}{5}$.
7. For what values of $x$, the expression $x^{2}-5 x-14$ is positive.

Sol: Given expression is $x^{2}-5 x-14$.
Consider the equation $x^{2}-5 x-14=0$

$$
\begin{aligned}
& x^{2}-7 x+2 x-14=0 \\
& x(x-7)+2(x-7)=0 \\
& (x+2)(x-7)=0
\end{aligned}
$$

$$
\alpha=-2, \beta=7
$$

Here coefficient of $x^{2}$ is 1 , which is positive.
So for $x \in R$ and $x<-2$ or $x>7$, then $x^{2}-5 x-14$ is positive.
(ii) $3 x^{2}+4 x+4$ is positive.

A: Given expression is $3 x^{2}+4 x+4$
Consider $3 x^{2}+4 x+4=0$
Roots are $x=\frac{-b \pm \sqrt{b^{2}-4 a c}}{2 a}=\frac{-4 \pm \sqrt{16-4(3)(4)}}{2(3)}=\frac{-4 \pm \sqrt{-32}}{6}=\frac{-4 \pm 4 \sqrt{2} i}{6}$.
8. For what values of $x$, the expression $15+4 x-3 x^{2}$ is negative.

A: Given expression is $15+4 x-3 x^{2}$.
Here $\mathrm{a}=-3<0$.
Consider $15+4 x-3 x^{2}=0$
$3 x^{2}-4 x-15=0$
$3 x^{2}-9 x+5 x-15=0$
$3 x(x-3)+5(x-3)=0$
$(3 x+5)(x-3)=0$
$\alpha=-5 / 3, \beta=3 \because \alpha<\beta$
Thus for $x \in R$ and $x<-5 / 3$ or $x>3$, then $15+4 x-3 x^{2}$ is negative.
9. If $x^{2}-6 x+5=0$ and $x^{2}-12 x+p=0$ have a common root, then find $p$.

A: $x^{2}-6 x+5=0$
$(x-1)(x-5)=0$
$x=1,5$
If $x=1,1-12+p=0 \Rightarrow p=11$
If $x=5,25-60+p=0 \Rightarrow p=35$
$\therefore \mathrm{p}=11$ or 35 .

## THEORY OF EQUATIONS Q.NO 5

1. Form a polynomial equation of lowest degree, whose roots are 1, -1, 3 .

A: Polynomial equation of lowest degree whose roots are $1,-1,3$ is $(x-1)(x+1)(x-3)=0$
$\Rightarrow\left(x^{2}-1\right)(x-3)=0$
$\Rightarrow x^{3}-3 x^{2}-x+3=0$.
2. Form the polynomial equation of the lowest degree with roots as $0,0,2,2,-2,-2$.
A. Given roots are $0,0,2,2,-2,-2$

Required $6^{\text {th }}$ degree equation is

$$
\begin{aligned}
& x^{2}\left(x^{2}-4\right)^{2}=0 \\
& \Rightarrow x^{2}\left(x^{4}-8 x^{2}+16\right)=0 \\
& \Rightarrow x^{6}-8 x^{4}+16 x^{2}=0
\end{aligned}
$$

3. Form a polynomial equation with rational coefficients and whose roots are $\mathbf{2} \pm \sqrt{\mathbf{3}}, \mathbf{1} \pm \mathbf{2 i}$.

A: Given roots are $2 \pm \sqrt{3}, 1 \pm 2 i$.
Required biquadratic equation is
$\left[x^{2}-(2+\sqrt{3}+2-\sqrt{3}) x+(2+\sqrt{3})(2-\sqrt{3})\right]\left[x^{2}-(1+2 i+1-2 i) x+(1+2 i)(1-2 i)\right]=0$
$\Rightarrow\left(\mathrm{x}^{2}-4 \mathrm{x}+1\right)\left(\mathrm{x}^{2}-2 \mathrm{x}+5\right)=0$
$\Rightarrow \mathrm{x}^{4}-2 \mathrm{x}^{3}+5 \mathrm{x}^{2}-4 \mathrm{x}^{3}+8 \mathrm{x}^{2}-20 \mathrm{x}+\mathrm{x}^{2}-2 \mathrm{x}+5=0$
$\Rightarrow x^{4}-6 x^{3}+14 x^{2}-22 x+5=0$.
4. If $-1,2, \alpha$ are the roots of $2 x^{3}+x^{2}-7 x-6=0$, then find $\alpha$.

A: Given that $-1,2, \alpha$ are the roots of $2 x^{3}+x^{2}-7 x-6=0$
$\Rightarrow s_{1}=\alpha+\beta+\gamma=-b / a$
$\Rightarrow-1+2+\alpha=-1 / 2 \Rightarrow \alpha=-\frac{1}{2}-1$
$\Rightarrow \alpha=-3 / 2$.
5. If the product of the roots of the equation $4 x^{3}+16 x^{2}-9 x-a=0$ is 9 , then find $a$.

A: Product of the roots of $4 x^{3}+16 x^{2}-9 x-a=0$ is 9 .
$\Rightarrow \mathrm{s}_{3}=\alpha \beta \gamma=-\mathrm{d} / \mathrm{a}=9$
$\Rightarrow-(-\mathrm{a}) / 4=9$
$\Rightarrow a=36$.
6. If $\alpha, \beta$ and 1 are the roots of $x^{3}-2 x^{2}-5 x+6=0$, then find $\alpha$ and $\beta$.

A: Given $\alpha, \beta, 1$ are the roots of $x^{3}-2 x^{2}-5 x+6=0$
$\Rightarrow \mathrm{s}_{1}=\alpha+\beta+1=2$
$\Rightarrow \alpha+\beta=1$.
Also $\mathrm{s}_{3}=\alpha \beta \gamma=\alpha \beta(1)=-\mathrm{d} / \mathrm{a}=-6 / 1=-6$
$\Rightarrow \alpha \beta=-6$
By observation $\alpha=3, \beta=-2$.
7. If $1,-2,3$ are the roots of $x^{3}-2 x^{2}+a x+6=0$, then find $a$.

A: Given that 1 is a root of $x^{3}-2 x^{2}+a x+6=0$
$\Rightarrow 1-2+a+6=0$
$\Rightarrow a=-5$.
8. Solve the equation $x^{3}-3 x^{2}-6 x+8=0$, given that the roots are in A.P.

A: Let the roots be $a-d, a, a+d$.
$s_{1}=a-d+a+a+d=3$
$3 a=3 \Rightarrow a=1$
$s_{3}=(a-d)(a)(a+d)=-8$
$(1-d)(1)(1+d)=-8$

$$
1-d^{2}=-8
$$

$$
d^{2}=9
$$

$\therefore \mathrm{d}= \pm 3$
If $d=3$, the roots are $1-3,1,1+3=-2,1,4$.
9. If $\alpha, \beta, \gamma$ are the roots of $4 x^{3}-6 x^{2}+7 x+3=0$, then find the value of $\alpha \beta+\beta \gamma+\gamma \alpha$.
A. Given that $\alpha, \beta, \gamma$ are the roots of $4 x^{3}-6 x^{2}+7 x+3=0$
$\therefore \alpha \beta+\beta \gamma+\gamma \alpha=\mathrm{s}_{2}=\frac{\mathrm{c}}{\mathrm{a}}=\frac{7}{4}$.
10. Find $s_{1}, s_{2}, s_{3}$ and $s_{4}$ for the equation $8 x^{4}-2 x^{3}-27 x^{2}+6 x+9=0$.

A: Given equation is $8 x^{4}-2 x^{3}-27 x^{2}+6 x+9=0$
Comparing this with $a x^{4}+b x^{3}+c x^{2}+d x+e=0$.
$\mathrm{s}_{1}=-\frac{\mathrm{b}}{\mathrm{a}}=\frac{2}{8}=\frac{1}{4}$
$s_{2}=\frac{c}{a}=\frac{-27}{8}$
$s_{3}=-\frac{d}{a}=\frac{-6}{8}=\frac{-3}{4}$
$\mathrm{s}_{4}=\frac{\mathrm{e}}{\mathrm{a}}=\frac{9}{8}$.
11. Solve the equation $x^{3}-3 x^{2}-16 x+48=0$, one root being 3 .

A: Given equation is $x^{3}-3 x^{2}-16 x+48=0$
$s_{1}=3+\beta+\gamma=3$

$$
\beta+\gamma=0
$$

$\mathrm{s}_{3}=3(\beta)(\gamma)=-48$

$$
\beta \gamma=-16
$$

The quadratic equation whose roots are $\beta, \gamma$ is $x^{2}-(\beta+\gamma) x+\beta \gamma=0$
$x^{2}-(0) x-16=0$
$x= \pm 4$.
$\therefore$ The other two roots are $4,-4$.
12.If $1,2,3,4$ are the roots of $x^{4}+a x^{3}+b x^{2}+c x+d=0$, then find the values of $a, b, c, d$.

A: Given that 1, 2, 3, 4 are the roots of
$x^{4}+a x^{3}+b x^{2}+c x+d=0$.
$a=-s_{1}=-(\alpha+\beta+\gamma+\delta)=-(1+2+3+4)=-10$
$b=s_{2}=\alpha \beta+\alpha \gamma+\alpha \delta+\beta \gamma+\beta \delta+\gamma \delta=1(2)+1(3)+1(4)+2(3)+2(4)+3(4)=35$
$\mathrm{c}=-\mathrm{s}_{3}=-(\alpha \beta \gamma+\alpha \beta \delta+\alpha \gamma \delta+\beta \gamma \delta)=-(1.2 .3+1.2 .4+1.3 .4+2.3 .4)=-50$
$\mathrm{d}=\mathrm{s}_{4}=\alpha \beta \gamma \delta=1.2 \cdot 3.4=24$
$\therefore \mathrm{a}=-10, \mathrm{~b}=35, \mathrm{c}=-50, \mathrm{~d}=24$
13. If $\alpha, \beta, \gamma$ are the roots of $x^{3}+p x^{2}+q x+r=0$, then find $\alpha^{2}+\beta^{2}+\gamma^{2}$.

A: Given that $\alpha, \beta, \gamma$ are the roots of $x^{3}+p x^{2}+q x+r=0$.
Now $\alpha^{2}+\beta^{2}+\gamma^{2}=(\alpha+\beta+\gamma)^{2}-2(\alpha \beta+\beta \gamma+\gamma \alpha)$

$$
=(-p)^{2}-2(q)=p^{2}-2 q
$$

14. If $\alpha, \beta, \gamma$ are the roots of $\mathbf{x}^{3}+\mathrm{px}^{2}+\mathrm{qx}+\mathrm{r}=0$, then find the value of $\alpha^{3}+\beta^{3}+\gamma^{3}$.

A: We know that $\alpha^{3}+\beta^{3}+\gamma^{3}-3 \alpha \beta \gamma=(\alpha+\beta+\gamma)\left[\alpha^{2}+\beta^{2}+\gamma^{2}-\alpha \beta-\beta \gamma-\gamma \alpha\right]$
$\alpha^{3}+\beta^{3}+\gamma^{3}=(\alpha+\beta+\gamma)\left[(\alpha+\beta+\gamma)^{2}-3(\alpha \beta+\beta \gamma+\gamma \alpha)\right]+3 \alpha \beta \gamma$

$$
\begin{aligned}
& =(-p)\left[(-p)^{2}-3(q)\right]+3(-r) \\
& =3 p q-p^{3}-3 r .
\end{aligned}
$$

15. If $\alpha, \beta, \gamma$ are the roots of $\mathbf{x}^{3}-2 x^{2}+3 x-4=0$, then find $\sum \alpha^{2} \beta^{2}$

A: Given that $\alpha, \beta, \gamma$ are the roots of $x^{3}-2 x^{2}+3 x-4=0$
$\Sigma \alpha^{2} \beta^{2}=\alpha^{2} \beta^{2}+\beta^{2} \gamma^{2}+\gamma^{2} \alpha^{2}$
$=(\alpha \beta+\beta \gamma+\gamma \alpha)^{2}-2(\alpha \beta \cdot \beta \gamma+\beta \gamma \cdot \gamma \alpha+\gamma \alpha \cdot \alpha \beta)$
$=(\alpha \beta+\beta \gamma+\gamma \alpha)^{2}-2 \alpha \beta \gamma(\beta+\gamma+\alpha)$
$=3^{2}-2(4)(2)$
= 9-16
$=-7$.
16. Find the quotient and remainder, when $2 x^{5}-3 x^{4}+5 x^{3}-3 x^{2}+7 x-9$ divided by $x^{2}-x-3$.

A: By synthetic division, dividing
$2 x^{5}-3 x^{4}+5 x^{3}-3 x^{2}+7 x-9$ is by $x^{2}-x-3$

|  | 2 | -3 | 5 | -3 | 7 | -9 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 0 | 2 | -1 | 10 | 4 | 0 |
| 3 | 0 | 0 | 6 | -3 | 30 | 12 |
|  | 2 | -1 | 10 | 4 | 41 | 3 |

Required quotient is $2 x^{3}-x^{2}+10 x+4$ and the remainder is $41 x+3$.
17. Find the polynomial equation of degree 4 whose roots are negatives of the roots of $x^{4}-6 x^{3}+7 x^{2}-2 x+1=0$.

A: Required transformed equation is $f(-x)=0$

$$
\begin{array}{r}
(-x)^{4}-6(-x)^{3}+7(-x)^{2}-2(-x)+1=0 \\
x^{4}+6 x^{3}+7 x^{2}+2 x+1=0
\end{array}
$$

18. Find the polynomial equation whose roots are the reciprocals of the roots of $x^{4}-3 x^{3}+7 x^{2}+5 x-2=0$.
A. Given equation is $f(x)=x^{4}-3 x^{3}+7 x^{2}+5 x-2=0$

Required transformed equation is $f\left(\frac{1}{x}\right)=0$
$\Rightarrow \frac{1}{x^{4}}-\frac{3}{x^{3}}+\frac{7}{x^{2}}+\frac{5}{x}-2=0$
$1-3 x+7 x^{2}+5 x^{3}-2 x^{4}=0$
$2 x^{4}-5 x^{3}-7 x^{2}+3 x-1=0$.
19. Find the algebraic equation whose roots are 3 times the roots of $x^{3}+2 x^{2}-4 x+1=0$.

A: Given equation is $f(x)=x^{3}+2 x^{2}-4 x+1=0$.
Required transformed equation is $f(x / 3)=0$
$\Rightarrow \frac{x^{3}}{27}+\frac{2 x^{2}}{9}-\frac{4 x}{3}+1=0$
$\Rightarrow x^{3}+6 x^{2}-36 x+27=0$.
20. If $\alpha, \beta, \gamma$ are the roots of the equation $x^{3}+2 x^{2}-4 x-3=0$, find the equation whose roots are $\frac{\alpha}{3}, \frac{\beta}{3}, \frac{\gamma}{3}$.

A: Given equation is $f(x)=x^{3}+2 x^{2}-4 x-3=0$
Required transformed equation is $f(3 x)=0$ $27 x^{3}+2\left(9 x^{2}\right)-4(3 x)-3=0 \quad \div 3$ $9 x^{3}+6 x^{2}-4 x-1=0$.
21. Find the equation whose roots are squares of the roots of $x^{3}+3 x^{2}-7 x+6=0$.

A: Given equation is $f(x)=x^{3}+3 x^{2}-7 x+6=0$.
Required transformed equation is $f(\sqrt{x})=0$

$$
\begin{aligned}
& (\sqrt{x})^{3}+3(\sqrt{x})^{2}-7 \sqrt{x}+6=0 \\
& x \sqrt{x}+3 x-7 \sqrt{x}+6=0 \\
& \sqrt{x}(x-7)=-(3 x+6)
\end{aligned}
$$

Squaring on both sides,

$$
\begin{aligned}
& x\left(x^{2}-14 x+49\right)=9 x^{2}+36 x+36 \\
& x^{3}-14 x^{2}+49 x-9 x^{2}-36 x-36=0 \\
& x^{3}-23 x^{2}+13 x-36=0
\end{aligned}
$$

## PERMUTATION O.NO 6

1. If ${ }^{n} P_{4}=1680$, then find $n$.

A: ${ }^{n} P_{4}=1680$

$$
\begin{aligned}
n(n-1)(n-2)(n-3) & =168 \times 10 \\
& =8 \times 21 \times 10 \\
& =8 \times 7 \times 3 \times 5 \times 2 \\
& =8 \times 7 \times 6 \times 5 \\
\therefore \quad n & =8 .
\end{aligned}
$$

2. If ${ }^{n} P_{7}=42 \cdot{ }^{n} P_{5}$, then find $n$.

A: Given that ${ }^{n} P_{7}=42$. ${ }^{n} P_{5}$

```
\(n(n-1)(n-2)(n-3)(n-4)(n-5)(n-6)=42 . n(n-1)(n-2)(n-3)(n-4)\)
\((\mathrm{n}-5)(\mathrm{n}-6)=42\)
\((\mathrm{n}-5)(\mathrm{n}-6)=(7)(6)\)
\(\mathrm{n}-5=7\)
    \(\therefore \mathrm{n}=12\).
```

3. If ${ }^{n+1} P_{5}:{ }^{n} P_{6}=2: 7$, find $n$

A: $\frac{{ }^{n+1} P_{5}{ }^{5}}{{ }^{n} P_{6}}=\frac{2}{7}$
7. ${ }^{n+1} P_{5}=2 .{ }^{n} P_{6}$
7. $(\mathrm{n}+1) \mathrm{n}(\mathrm{n}-1)(\mathrm{n}-2)(\mathrm{n}-3)=2 . \mathrm{n}(\mathrm{n}-1)(\mathrm{n}-2)(\mathrm{n}-3)(\mathrm{n}-4)(\mathrm{n}-5)$
$7(n+1)=2(n-4)(n-5)$
$7 n+7=2\left(n^{2}-9 n+20\right)$
$2 n^{2}-25 n+33=0$
$2 n^{2}-3 n-22 n+33=0$
$n(2 n-3)-11(2 n-3)=0$
$(n-11)(2 n-3)=0$
$\mathrm{n}=11$ or $\mathrm{n}=3 / 2$ is not possible
$\therefore \mathrm{n}=11$.
4. If ${ }^{18} P_{r-1}:{ }^{17} P_{r-1}=9: 7$, find $r$.

A: ${ }^{{ }^{18} P_{r-1}}=\frac{9}{7}$
7. ${ }^{18} P_{r-1}=9 .{ }^{17} P_{r-1}$
7. $\frac{18!}{[18-(r-1)]!}=9 \cdot \frac{17!}{[17-(r-1)]!}$
$\frac{7 .(18)(17!)}{(19-r)(18-r)!}=\frac{9(17!)}{(18-r)!}$
$14=19-r$
$\therefore r=5$.
5. If ${ }^{12} P_{r}=1320$, find $r$.

A: ${ }^{12} P_{r}=1320$
$=132 \times 10$
$=12 \times 11 \times 10$
$={ }^{12} \mathrm{P}_{3}$
$\therefore \mathrm{r}=3$.
6. If ${ }^{12} P_{5}+5 .{ }^{12} P_{4}={ }^{13} P_{r}$, find $r$.

A: Given that ${ }^{12} P_{5}^{4}+5 .{ }^{12} P_{4}={ }^{13} P$
Comparing this with ${ }^{n-1} P_{r}+r .{ }^{n-1} P_{r-1}={ }^{n} P_{r}$
Here $\mathrm{r}=5$.
7. Find the number of injections of set $A$ with 5 elements to a set $B$ with 7 elements.
A. Given that $n(A)=5, n(B)=7$.

Number of injections formed from set $A$ to set $B={ }^{n(B)} P_{n(A)}={ }^{7} P_{5}=7 \times 6 \times 5 \times 4 \times 3=2520$.
8. In a class there are 30 students. On the new year day, every student posts a greeting card to all his/her classmates. Find the total number of greeting cards posted by them.
A: Total number of greeting cards posted by the students $={ }^{30} \mathrm{P}_{2}=30 \times 29=870$.

## COMBINATION Q.NO 7

1. If ${ }^{n} C_{5}={ }^{n} C_{6}$, then find ${ }^{13} C_{n}$.

A: Given that ${ }^{n} C_{5}={ }^{n} C_{6}$
$r=s$ or $n=r+s$
Here $5 \neq 6, \mathrm{n}=5+6=11$
$\therefore{ }^{13} \mathrm{C}_{\mathrm{n}}={ }^{13} \mathrm{C}_{11}={ }^{13} \mathrm{C}_{2}=\frac{13.12}{2}=78$.
2. If ${ }^{12} C_{r+1}={ }^{12} C_{3 r-5}$, find $r$.

A: Now ${ }^{12} \mathrm{C}_{\mathrm{r}+1}={ }^{12} \mathrm{C}_{3 r-5}$

$$
\begin{array}{ll}
r=s \text { or } n=r+s . & \\
r+1=3 r-5 & 12=r+1+3 r-5 \\
2 r=6 & 4 r=16 \\
r=3 & r=4
\end{array}
$$

$$
\therefore r=3 \text { or } 4
$$

3. If $10 .{ }^{n} C_{2}=3 .{ }^{n+1} C_{3}$, find $n$.

A: Given that 10. ${ }^{n} C_{2}=3 .{ }^{n+1} C_{3}$

$$
\begin{aligned}
& 10 \frac{\mathrm{n}(\mathrm{n}-1)}{2}=\frac{3(\mathrm{n}+1) \mathrm{n}(\mathrm{n}-1)}{6} \\
& 10=\mathrm{n}+1 \quad \therefore \mathrm{n}=9 .
\end{aligned}
$$

4. If ${ }^{n} C_{4}=210$, find $n$.
$A:{ }^{n} C_{4}=\frac{210 \times 24}{24}=\frac{21 \times 10 \times 8 \times 3}{4!}=\frac{7 \times 3 \times 10 \times 8 \times 3}{4!}=\frac{10 \times 9 \times 8 \times 7}{4!}={ }^{10} \mathrm{C}_{4}$
$\therefore \mathrm{n}=10$.
5. If ${ }^{n} P_{r}=5040$ and ${ }^{n} C_{r}=210$, find $n$ and $r$.

A: We know that $r!=\frac{{ }^{n} P_{r}}{{ }^{n} C_{r}}=\frac{5040}{210}=24=4!$

$$
\therefore r=4 .
$$

Also ${ }^{n} \mathrm{P}_{4}=5040$

$$
=10 \times 504
$$

$$
=10 \times 9 \times 56
$$

$$
=10 \times 9 \times 8 \times 7
$$

$$
={ }^{10} \mathrm{P}_{4} \quad \therefore \mathrm{n}=10 .
$$

6. Find the value of ${ }^{10} \mathrm{C}_{5}+2 .{ }^{10} \mathrm{C}_{4}+{ }^{10} \mathrm{C}_{3}$

A: ${ }^{10} \mathrm{C}_{5}+2 .{ }^{10} \mathrm{C}_{4}+{ }^{10} \mathrm{C}_{3}=\left\{{ }^{10} \mathrm{C}_{5}+{ }^{10} \mathrm{C}_{4}\right\}+\left\{{ }^{10} \mathrm{C}_{4}+{ }^{10} \mathrm{C}_{3}\right\}$
$\because{ }^{n} C_{r}+{ }^{n} C_{r-1}={ }^{n+1} C_{r}={ }^{11} C_{5}+{ }^{11} C_{4}={ }^{12} C_{5}$
7. Find the number of ways of selecting 4 boys and 3 girls from a group of 8 boys and 5 girls.

A: The no. of ways of selecting 4 boys from 8 boys is ${ }^{8} \mathrm{C}_{4}$.
The no. of ways of selecting 3 girls from 5 girls is ${ }^{5} \mathrm{C}_{3}$.
$\therefore$ Total no. of selections $={ }^{8} \mathrm{C}_{4} \cdot{ }^{5} \mathrm{C}_{3}=\frac{8 \times 7 \times 6 \times 5}{4 \times 3 \times 2 \times 1} \cdot \frac{5 \times 4 \times 3}{3 \times 2 \times 1}=70 \times 10=700$.
8. Find the number of ways of selecting 3 vowels and 2 consonants from the letters of the word 'EQUATION'.

A: Given word 'EQUATION' contains 5 vowels and 3 consonants.
Number of ways of selecting 3 vowels and 2 consonants from 5 vowels and 3 consonants

$$
\begin{aligned}
& ={ }^{5} \mathrm{C}_{3} \times{ }^{3} \mathrm{C}_{2}=10 \times 3 \\
& =30
\end{aligned}
$$

9. Find the number of diagonals of a polygon with 12 sides.

A: Number of sides $=12$
No. of diagonals $=\frac{\mathrm{n}(\mathrm{n}-3)}{2}=\frac{12 \times 9}{2}=54$.
10. Find the number of positive divisiors of 1080.

A: $1080=10 \times 108$

$$
\begin{aligned}
& =10 \times 9 \times 12 \\
& =2 \times 5 \times 3^{2} \times 2^{2} \times 3 \\
& =2^{3} \times 3^{3} \times 5^{1}
\end{aligned}
$$

Number of positive divisors of $1080=(3+1)(3+1)(1+1)=32$.
11. If ${ }^{12} \mathrm{C}_{\mathrm{r}}=495$, find the possible values of r .
$\mathrm{A}:{ }^{12} \mathrm{C}_{\mathrm{r}}=\frac{495 \times 24}{24}=\frac{5 \times 99 \times 12 \times 2}{4!}=\frac{12 \times 11 \times 10 \times 9}{4!}={ }^{12} \mathrm{C}_{4}$ or ${ }^{12} \mathrm{C}_{8}$ $\therefore r=4$ or 8 .

## BINOMIAL THEOREM Q.NO 8

1. Write down and simplify $6^{\text {th }}$ term in $\left(\frac{2 x}{3}+\frac{3 y}{2}\right)^{9}$.

A: $6^{\text {th }}$ term $=T_{6}=T_{5+1}$.
$={ }^{9} C_{5}\left(\frac{2 x}{3}\right)^{9-5}\left(\frac{3 y}{2}\right)^{5}={ }^{9} C_{5}\left(\frac{2 x}{3}\right)^{4}\left(\frac{3 y}{2}\right)^{5}=126 .\left[\frac{3}{2}\right] x^{4} y^{5}=189 x^{4} y^{5}$.
2. Find the $3^{\text {rd }}$ term from the end in the expansion of $\left(x^{\frac{-2}{3}}-\frac{3}{x^{2}}\right)^{8}$.

A: $\ln \left(x^{\frac{-2}{3}}-\frac{3}{x^{2}}\right)^{8}, 3 r d$ term from the end $=T_{7}$

$$
\begin{aligned}
\mathrm{T}_{6+1} & ={ }^{8} \mathrm{C}_{6}\left(\mathrm{x}^{-2 / 3}\right)^{8-6}\left(-3 / \mathrm{x}^{2}\right)^{6} \\
& ={ }^{8} \mathrm{C}_{2} \times \mathrm{x}^{-4 / 3} \times 36 / \mathrm{x}^{12} \\
& ={ }^{8} \mathrm{C}_{2} \times 3^{6} / \mathrm{x}^{40 / 3} .
\end{aligned}
$$

3. Obtain the values of $x$ for which the binomial expansion of $(2+3 x)^{-2 / 3}$ is valid.

A: $(2+3 x)^{-2 / 3}=2^{-2 / 3}(1+3 x / 2)^{-2 / 3}$
The above expansion is valid if $|3 x / 2|<1$

$$
|x|<2 / 3
$$

$$
x \in\left(\frac{-2}{3}, \frac{2}{3}\right)
$$

4. Find the number of terms with non-zero coefficients in $(4 x+7 y)^{49}+(4 x-7 y)^{49}$.

A: The number of terms in the expansion $(x+y)^{n}+(x-y)^{n}$ when ' $n$ 'is odd is

$$
\frac{n+1}{2}=\frac{49+1}{2}=25 .
$$

5. Find the middle term in the expansion of $\left(\frac{3 x}{7}-2 y\right)^{10}$.

A: Here $n=10$, even
So, middle term $=T_{\frac{10}{2}+1}=T_{5+1}$.

$$
\begin{aligned}
\therefore T_{5+1} & ={ }^{10} C_{5} \cdot\left(\frac{3 x}{7}\right)^{10-5} \cdot(-2 y)^{5} \\
& =-{ }^{10} C_{5} \cdot\left(\frac{3 x}{7}\right)^{5} \cdot(2 y)^{5}==-{ }^{10} C_{5} \cdot\left(\frac{6}{7}\right)^{5} x^{5} y^{5} .
\end{aligned}
$$

6. Find the middle terms in the expansion of $\left(4 a+\frac{3}{2} b\right)^{11}$.

A: Given expansion is $\left(4 a+\frac{3}{2} b\right)^{11}$.
Here $n=11$, odd
So, middle terms are $\frac{T_{11+1}}{2}, \frac{T_{1+3}}{2}=T_{6}, T_{7}$.

$$
\begin{aligned}
& T_{6}=T_{5+1}={ }^{11} C_{5}(4 a)^{11-5}\left(\frac{3}{5} b\right)^{5} \\
& ={ }^{11} C_{5} 4^{6} \cdot a^{6} \cdot\left(\frac{3}{2}\right)^{5} \cdot b^{2}={ }^{11} C_{5} \cdot 4^{6} \cdot\left(\frac{3}{2}\right)^{5} a^{6} b^{6} \\
& \mathrm{~T}_{7}=\mathrm{T}_{6+1}={ }^{11} \mathrm{C}_{6}(4 \mathrm{a})^{11-6}\left(\frac{3}{5} \mathrm{~b}\right)^{6} \\
& ={ }^{11} C_{5} 4^{5} \cdot a^{5} \cdot\left(\frac{3}{2}\right)^{6} \cdot b^{6}={ }^{11} C_{6} \cdot 4^{5} \cdot\left(\frac{3}{2}\right)^{6} a^{5} b^{6} .
\end{aligned}
$$

7. If the coefficients of $(2 r+4)^{\text {th }}$ and $(r-2)^{\text {nd }}$ terms in the expansion of $(1+x)^{18}$ are equal, find $r$.

A: $\ln (1+x)^{18}, T_{2 r+4}=T_{(2 r+3)+1}={ }^{18} \mathrm{C}_{2 r+3}$

$$
\mathrm{T}_{\mathrm{r}-2}=\mathrm{T}_{(r-3)+1}={ }^{18} \mathrm{C}_{\mathrm{r}-3}
$$

But ${ }^{18} \mathrm{C}_{2 r+3}={ }^{18} \mathrm{C}_{\mathrm{r}-3}$

$$
\begin{array}{ll}
r=s & n=r+s \\
2 r+3=r-3 & 18=2 r+3+r-3 \\
r=-6 & 18=3 r \\
\text { is not possible } & r=6
\end{array}
$$

$$
\therefore \mathrm{r}=6 \text {. }
$$

8. If ${ }^{22} \mathrm{C}_{r}$ is the largest binomial coefficient in the expansion of $(1+\mathrm{x})^{22}$ find the value of ${ }^{13} \mathrm{C}_{r}$.

A: Largest binomial coefficient in $(1+x)^{22}={ }^{n} C_{n / 2}$ if $n$ is even $={ }^{22} C_{11}$
$r=11$.
Now ${ }^{13} \mathrm{C}_{\mathrm{r}}={ }^{13} \mathrm{C}_{11}={ }^{13} \mathrm{C}_{2}=\frac{13 \times 12}{2}=78$.
9. Find the term independent of $x$ in the expansion of $\left(\frac{3}{\sqrt[3]{x}}+5 \sqrt{x}\right)^{25}$.

A: General term $T_{r+1}={ }^{n} C_{r} x^{n-r} a^{r}$.

$$
\begin{aligned}
& ={ }^{25} C_{r}\left(\frac{3}{\sqrt[3]{x}}\right)^{25-r}(5 \sqrt{x})^{r}={ }^{25} C_{r} 3^{25-r}\left(\frac{1}{x^{\frac{25-r}{3}}}\right)(5 \sqrt{x})^{r} \\
& ={ }^{25} C_{r} 3^{25-r} 5^{r} x^{-\left(\frac{25-r}{3}\right)+\frac{r}{2}} \\
\text { take } \frac{-25+r}{3}+\frac{r}{2} & =0 \text { then }-50+2 r+3 r=0 \Rightarrow r=10 .
\end{aligned}
$$

$\therefore$ The term independent of x is $\mathrm{T}_{11}={ }^{25} \mathrm{C}_{10} 3{ }^{25-10} 5^{10}={ }^{25} \mathrm{C}_{10} 3^{15} 5^{10}$.
10. Find the coefficient of $x^{-6}$ in $(3 x-4 / x)^{10}$

A: $T_{r+1}={ }^{10} C_{r}(3 x)^{10-r}(-4 / x)^{r}$

$$
={ }^{10} C_{r} \cdot 3^{10-r} \cdot(-4) r \cdot x^{10-r-r}
$$

To get the coefficient of $x^{-6}, 10-2 r=-6$

$$
\begin{aligned}
2 r & =16 \\
r & =8
\end{aligned}
$$

Coefficient of $x^{-6}={ }^{10} \mathrm{C}_{8} \cdot 3^{2} \cdot(-4)^{8}={ }^{10} \mathrm{C}_{2} \cdot 3^{2} \cdot 4^{8}$.
11. Find the numerically greatest terms in the expansion of $(3+2 a)^{15}$ when $a=5 / 2$.
$A:(3+2 a)^{15}=3^{15}\left(1+\frac{2 a}{3}\right)^{15}$
$|x|=\left|\frac{2 \mathrm{a}}{3}\right|=\left|\frac{2}{3} \cdot \frac{5}{2}\right|=\frac{5}{3}$
Now $\frac{(n+1)|x|}{|x|+1}=\frac{(15+1) \cdot 5 / 3}{8 / 3}=10$
$\therefore\left|T_{10}\right|$ and $\left|T_{11}\right|$ are numerically greatest.
$\left|\mathrm{T}_{10}\right|={ }^{15} \mathrm{C}_{9} 3^{6}\left(2 \cdot \frac{5}{2}\right)^{9}={ }^{15} \mathrm{C}_{9} \cdot 3^{6} \cdot 5^{9}$
$\left|T_{11}\right|={ }^{55} C_{10} 3^{5}\left(2 \cdot \frac{5}{2}\right)^{10}={ }^{55} C_{0} \cdot 3^{5} \cdot 3^{10}$
and $\left|T_{10}\right|=\left|T_{11}\right|$.
12. Find the range of $x$ for which the binomial expansion $(3-4 x)^{3 / 4}$ is valid.

A: $(3-4 x)^{3 / 4}=\left[3\left(1-\frac{4 x}{3}\right)\right]^{3 / 4}=3^{3 / 4}\left(1-\frac{4 x}{3}\right)^{3 / 4}$
The expansion is valid when $\left|\frac{-4 x}{3}\right|<1$
$\Rightarrow|x|<\frac{3}{4} \Rightarrow \frac{-3}{4}<x<\frac{3}{4}$ (or) $x \in\left(\frac{-3}{4}, \frac{3}{4}\right)$

## MEASURES OF DISPERSION Q.NO 9

1. Find the mean deviation from the mean of the following discrete data: 3, 6, 10, 4, 9, 10.

A: Mean of the data $3,6,10,4,9,10$ is

$$
\bar{x}=\frac{3+6+10+4+9+10}{6}=\frac{42}{6}=7
$$

$\therefore$ Mean deviation from the mean $=\frac{\sum_{i=1}^{6}\left|x_{i}-\bar{x}\right|}{n}=\frac{4+1+3+3+2+3}{6}=\frac{16}{6}=2.67$
2. Find the mean deviation about mean for the data $38,70,48,40,42,55,63,46,54,44$.
A. Given data is $38,70,48,40,42,55,63,46,54,44$.

Arithmetic mean $\overline{\mathrm{x}}=\frac{\Sigma \mathrm{x}_{\mathrm{i}}}{\mathrm{n}}=\frac{500}{10}=50$.
Mean deviation about mean M.D. $=\frac{\Sigma\left|\mathrm{x}_{\mathrm{i}}-\overline{\mathrm{x}}\right|}{\mathrm{n}}=\frac{12+20+2+10+8+5+13+4+4+6}{10}=\frac{84}{10}=8.4$.
3. Compute the mean deviation about the median of the data $6,7,10,12,13,4,12,16$.

A: Ascending order of the given data is $4,6,7,10,12,12,13,16$.
MEDIAN $=\frac{x 4+x 5}{2}=\frac{10+12}{2}=11$
$\therefore$ Mean deviation from the median $=\frac{\sum_{i=1}^{8}\left(\mathrm{x}_{\mathrm{i}}-\mathrm{M}\right)}{8}=\frac{7+5+4+1+1+1+2+5}{8}=\frac{26}{8}=3.25$.
4. Find the mean deviation about the median for the data: 4, 6, 9, 3, 10, 13, 2.

A: The ascending order of the data is $2,3,4,6,9,10,13$.
Median $M=x_{4}=6$.
$\therefore$ Mean deviation from the median $=\frac{\sum_{i=1}^{7}\left|x_{i}-M\right|}{7}=\frac{4+3+2+0+3+4+7}{7}=\frac{23}{7}=3.29$.
5. Find the mean deviation about median for the data $13,17,16,11,13,10,16,11,18,12,17$.
A. Given data in descending order is $10,11,11,12,13,13,16,16,17,17,18$

Median is $\frac{11+1^{\text {th }}}{2}$ observation
$M=13$.
Mean deviation about median, M.D. $=\frac{\Sigma\left|x_{i}-M\right|}{n}=\frac{3+2+2+1+0+0+3+3+4+4+5}{11}=\frac{27}{11}=2.45$.
6. Find the variance for the discrete data : $6,7,10,12,13,4,8,12$. DELETED
$A:$ Mean $=\frac{6+7+10+12+13+4+8+12}{8}=\frac{72}{8}=9$.
Variance $\sigma^{2}=\frac{\sum_{i=1}^{8}\left(x_{i}-\bar{x}\right)^{2}}{8}=\frac{9+4+1+9+16+25+1+9}{8}=\frac{74}{8}=9.25$.
7. Find the variance and standard deviation of the data $5,12,3,18,6,8,2,10$. DELETED
$A:$ Mean $\bar{x}=\frac{5+12+3+18+6+8+2+10}{8}$

$$
\bar{x}=\frac{64}{8}=8
$$

Variance $\sigma^{2}=\frac{\sum_{i=1}^{8}\left(x_{i}-\bar{x}\right)^{2}}{n}=\frac{9+16+25+100+4+36+4}{8}=\frac{194}{8}=24.25$. then Standard deviation $\sigma=\sqrt{24.25}$

## RANDOM VARIABLE \& DISTRIBUTIONS Q.NO 10

1. A random variable $X$ has the range $\{1,2,3 \ldots \ldots \ldots\}$. If $P(X=r)=\frac{C^{r}}{r!}$ for $r=1,2,3, \ldots \ldots$, then find $C$.

A: Given that $X$ is a random variable $\sum_{r=1}^{\infty} P(x=r)=1$
$\sum_{r=1}^{\infty} \frac{C^{r}}{r!}=1$
$\frac{C}{1!}+\frac{C^{2}}{2!}+\frac{C^{3}}{3!}+\ldots . \infty=1$
Adding 1 on bothsides, $1+\frac{C}{1!}+\frac{C^{2}}{2!}+\ldots . . \infty=1+1$
$\mathrm{e}^{\mathrm{c}}=2$
$\Rightarrow C=\log _{\mathrm{e}} 2$.
2. Find the constant $C$, so that $f(x)=C\left(\frac{2}{3}\right)^{x}, x=1,2,3 \ldots \ldots$ is the probability distribution function of a discrete random variable $X$.
A: Given that $X$ is a discrete random variable $\sum_{x=1}^{\infty} f(x)=1$
$\sum_{x=1}^{\infty} C\left(\frac{2}{3}\right)^{x}=1$
$C\left[\frac{2}{3}+\left(\frac{2}{3}\right)^{2}+\left(\frac{2}{3}\right)^{3}+\ldots . \infty\right]=1$
$C\left[\frac{\frac{2}{3}}{1-\frac{2}{3}}\right]=1$
$\because S_{\infty}=\frac{a}{1-r}$
$2 C=1$
$\therefore \mathrm{C}=\frac{1}{2}$.
3. The probability that a person choosen at random is left handed (in hand writing) is 0.1 . What is the probability that in a group of 10 people, there is one who is left handed.
A: Here $n=10, p=0.1=\frac{1}{10}, q=0.9=\frac{9}{10}$.
$\therefore$ The required probability that exactly one out of 10 is left handed is $P(X=1)={ }^{n} C_{r} p^{r} q^{n-r}$

$$
{ }^{10} C_{1} p^{1} q^{10-1}=10 \cdot\left(\frac{1}{10}\right)^{1} \cdot\left(\frac{9}{10}\right)^{9}=\left(\frac{9}{10}\right)^{9}
$$

4. The mean and variance of a binomial distribution are 4 and 3 respectively. Find $P(X \geq 1)$.

A: Let $n, p$ be the parameters of the binomial distribution.
$n p=4, n p q=3$
$q=\frac{n p q}{n p}=\frac{3}{4}$
$p=1-q=1-\frac{3}{4}=\frac{1}{4}$
Also $n p=4$
n. $1 / 4=4$
$\mathrm{n}=16$
Required probability $\mathrm{P}(\mathrm{X} \geq 1)$

$$
\begin{aligned}
& =\sum_{r=1}^{16} P(X=r) \\
& =\sum_{r=1}^{16}{ }^{16} C_{r}\left(\frac{3}{4}\right)^{16-r}\left(\frac{1}{4}\right)^{r} .
\end{aligned}
$$

5. The mean and variance of a binomial distribution are 4 and 3 respectively. Find $P(X \geq 1)$.

A: Let $n, p$ be the parameters of the binomial distribution.
$n p=4, n p q=3$
$q=\frac{n p q}{n p}=\frac{3}{4}$
$p=1-q=1-\frac{3}{4}=\frac{1}{4}$
Also $n p=4$

$$
\text { n. } 1 / 4=4
$$

$$
\mathrm{n}=16
$$

6. If the mean and variance of a binomial variate $X$ are 2.4 and 1.44 respectively, then find $p$ and $n$.

A: Let $\mathrm{n}, \mathrm{p}$ be the parameters of the binomial distribution.
Given that $n p=2.4, n p q=1.44$
$\mathrm{q}=\frac{\mathrm{npq}}{\mathrm{np}}=\frac{1.44}{2.4}=\frac{144}{240}=\frac{12}{20}=\frac{3}{5}$
$\mathrm{p}=1-\mathrm{q}=1-\frac{3}{5}=\frac{2}{5}$
Also $\mathrm{np}=2.4$
$\mathrm{n}\left(\frac{2}{5}\right)=2.4$
$n=6 \quad \because n=6, p=2 / 5$
7. For a binomial ditribution with mean 6 and variance 2, find the first two terms of the distribution..

A: Let $\mathrm{n}, \mathrm{p}$ be the paremeters of the binomial distribution.
$n p=6, n p q=2$.
$\mathrm{q}=\frac{\mathrm{npq}}{\mathrm{np}}=\frac{2}{6}=\frac{1}{3}$.
$p=1-q=1-1 / 3=2 / 3$.
Also $n p=6$
$n\left(\frac{2}{3}\right)=6$
$\mathrm{n}=9$
First two terms of the binomial distribution are $P(X=0), P(X=1)={ }^{9} C_{0}\left(\frac{1}{3}\right)^{9}\left(\frac{2}{3}\right)^{0},{ }^{9} C_{1}\left(\frac{1}{3}\right)^{8}\left(\frac{2}{3}\right)^{1}=\frac{1}{3^{9}}, \frac{2}{3^{7}}$.
8. $X$ follows Poisson distribution such that $P(X=1)=3 P(X=2)$. Find the variance of $X$.

A: Let $X$ be the parameter of the poisson distribution.
Given that $P(X=1)=3 P(X=2)$

$$
\begin{aligned}
\frac{e^{-\lambda} \lambda^{1}}{1!} & =3 \frac{e^{-\lambda} \lambda^{2}}{2!} \\
1 & =\frac{3 \lambda}{2} \\
\lambda & =\frac{2}{3}
\end{aligned}
$$

Variance of $X=\frac{2}{3}$.
9. A poisson variate $X$ satisfies $P(X=1)=P(X=2)$. Find $P(X=5)$.
$A$ : Let $\lambda$ be the parameter of the poisson distribution.
Given that $P(X=1)=P(X=2)$

$$
\begin{aligned}
\frac{\mathrm{e}^{-\lambda} \lambda^{1}}{1!} & =\frac{\mathrm{e}^{-\lambda} \lambda^{2}}{2!} \\
1 & =\frac{\lambda}{2} \\
\lambda & =2
\end{aligned}
$$

Now $P(X=5)=\frac{\mathrm{e}^{-\lambda} \lambda^{5}}{5!}=\frac{\mathrm{e}^{-2} \cdot 2^{5}}{5!}=\frac{32 \mathrm{e}^{-2}}{120}=\frac{4 \mathrm{e}^{-2}}{15}$.
10. On an average, rain falls on 12 days in every 30 days, find the probability that, rain will fall on just 3 days of a given week.
A: Let $X$ be the number of days rain falls in a week.
The probability that rain will fall on a day.
$\mathrm{p}=\frac{12}{30}=\frac{2}{5}$, Hence $\mathrm{q}=1-\frac{2}{5}=\frac{3}{5}$.
Now, $X$ follows the binomial distribution with parameters $n=7, p=\frac{2}{5}, q=\frac{3}{5}$
$\therefore$ The required probability is $\mathrm{P}(\mathrm{X}=3)={ }^{7} \mathrm{C}_{3}\left(\frac{2}{5}\right)^{3}\left(\frac{3}{5}\right)^{4}=35 \cdot \frac{2^{3} \cdot 3^{4}}{5^{7}}$

