Functions - Complete Solutions

Explained as a 20-year teacher would on the blackboard

1. Find the range of $f(x) = \tan\left(\frac{\pi}{\sqrt{x+1}+4}\right)$

Teacher's Explanation

Strategy: Work from inside out. First find the domain, then analyze what values the argument of tangent can take, and finally apply the tangent function.

Step 1: Find the domain

$$\text{For } \sqrt{x+1} \text{ to be defined: } x+1 \geq 0$$
$$\boxed{x \geq -1}$$

Step 2: Analyze the argument inside tan

Let $u = \frac{\pi}{\sqrt{x+1}+4}$

Since $\sqrt{x+1} \geq 0$ (square roots are never negative), we have:

$$\sqrt{x+1}+4 \geq 4$$
$$\frac{1}{\sqrt{x+1}+4} \leq \frac{1}{4} \quad \text{(reciprocal flips inequality)}$$
$$\frac{\pi}{\sqrt{x+1}+4} \leq \frac{\pi}{4}$$

Also, as $x \to -1^+$, we get $\sqrt{x+1} \to 0$, so:

$$\frac{\pi}{\sqrt{x+1}+4} \to \frac{\pi}{4}$$

As $x \to \infty$, we get $\sqrt{x+1} \to \infty$, so:

$$\frac{\pi}{\sqrt{x+1}+4} \to 0^+$$

Therefore: $\boxed{0 < u \leq \frac{\pi}{4}}$

Step 3: Apply tangent function

Since tan is strictly increasing on $\left(0, \frac{\pi}{4}\right]$:

$$\tan(0^+) < f(x) \leq \tan\left(\frac{\pi}{4}\right)$$
$$0 < f(x) \leq 1$$

Answer: (b) $(0, 1]$

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2. Find the range of $f(x) = \sin^{-1}(\sqrt{x^2+x+1})$

Teacher's Explanation

Key Concept: The domain of $\sin^{-1}(u)$ is $[-1,1]$. Since we have a square root, we need $0 \le \sqrt{x^2+x+1} \le 1$.

Step 1: Find constraints on the argument

For $\sin^{-1}$ to be defined:

$$0 \le \sqrt{x^2+x+1} \le 1$$

Squaring both sides:

$$0 \le x^2+x+1 \le 1$$

Step 2: Analyze the quadratic $g(x) = x^2+x+1$

Complete the square:

$$g(x) = \left(x+\frac{1}{2}\right)^2 + \frac{3}{4}$$

Minimum value occurs at $x = -\frac{1}{2}$:

$$g_{\min} = \frac{3}{4}$$

Step 3: Determine the constraint

Since $x^2+x+1 \ge \frac{3}{4}$ for all x, and we need $x^2+x+1 \le 1$:

$$\frac{3}{4} \le x^2+x+1 \le 1$$

Taking square roots:

$$\frac{\sqrt{3}}{2} \le \sqrt{x^2+x+1} \le 1$$

Step 4: Apply $\sin^{-1}$ (increasing function)

$$\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) \le f(x) \le \sin^{-1}(1)$$
$$\frac{\pi}{3} \le f(x) \le \frac{\pi}{2}$$

Answer: (d) $\left[\frac{\pi}{3}, \frac{\pi}{2}\right]$

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3. Find $f(3)$ if $3f(x) + 4f\left(\frac{1}{x}\right) = \frac{2-x}{x}$

Teacher's Explanation

Standard Technique: When given $f(x)$ and $f(1/x)$ in the same equation, create a second equation by replacing x with $1/x$, then solve the system.

Step 1: Write the original equation

$$3f(x) + 4f\left(\frac{1}{x}\right) = \frac{2-x}{x} \quad \cdots (1)$$

Step 2: Replace x with $\frac{1}{x}$

$$3f\left(\frac{1}{x}\right) + 4f(x) = \frac{2-\frac{1}{x}}{\frac{1}{x}}$$

Simplify the right side:

$$= \frac{\frac{2x-1}{x}}{\frac{1}{x}} = 2x - 1$$

So:

$$3f\left(\frac{1}{x}\right) + 4f(x) = 2x - 1 \quad \cdots (2)$$

Step 3: Solve for $f(3)$ and $f(1/3)$

Substitute $x = 3$ in equations (1) and (2):

From (1):

$$3f(3) + 4f\left(\frac{1}{3}\right) = \frac{2-3}{3} = -\frac{1}{3}$$

From (2):

$$3f\left(\frac{1}{3}\right) + 4f(3) = 2(3) - 1 = 5$$

Step 4: Solve the system

Let $a = f(3)$ and $b = f(1/3)$:

$$3a + 4b = -\frac{1}{3} \quad \cdots (A)$$
$$4a + 3b = 5 \quad \cdots (B)$$

Multiply (A) by 3: $9a + 12b = -1$
Multiply (B) by 4: $16a + 12b = 20$

Subtract: $7a = 21 \implies \boxed{a = 3}$

Answer: (d) $f(3) = 3$

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4. Find the inverse of $f(x) = \frac{10^x - 10^{-x}}{10^x + 10^{-x}} + 1$

Teacher's Explanation

Recognition: This looks like a hyperbolic tangent function. We'll use algebraic manipulation and the componendo-dividendo rule.

Step 1: Let $y = f(x)$ and rearrange

$$y - 1 = \frac{10^x - 10^{-x}}{10^x + 10^{-x}}$$

Step 2: Multiply numerator and denominator by 10^x

$$y - 1 = \frac{10^{2x} - 1}{10^{2x} + 1}$$

Step 3: Apply Componendo and Dividendo

The rule states: If $\frac{a}{b} = \frac{c}{d}$, then $\frac{a+b}{a-b} = \frac{c+d}{c-d}$

Here: $\frac{10^{2x}-1}{10^{2x}+1} = \frac{y-1}{1}$

Applying the rule:

$$\frac{(10^{2x} - 1) + (10^{2x} + 1)}{(10^{2x} + 1) - (10^{2x} - 1)} = \frac{(y-1)+1}{1-(y-1)}$$

$$\frac{2 \cdot 10^{2x}}{2} = \frac{y}{2-y}$$

$$10^{2x} = \frac{y}{2-y}$$

Step 4: Take logarithm and solve for x

$$2x = \log_{10}\left(\frac{y}{2-y}\right)$$

$$x = \frac{1}{2}\log_{10}\left(\frac{y}{2-y}\right)$$

Therefore:

$$f^{-1}(x) = \frac{1}{2}\log_{10}\left(\frac{x}{2-x}\right)$$

Answer: (d)

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5. For which values of a is $f(x) = \frac{x^2+x+a}{x^2-x+a}$ surjective onto $\mathbb{R}$?

Teacher's Explanation

Key Idea: For the function to be onto (surjective), every real number y must be achievable. This means for any $y \in \mathbb{R}$, the equation $f(x) = y$ must have real solutions.

Step 1: Set $y = f(x)$ and rearrange

$$y = \frac{x^2 + x + a}{x^2 - x + a}$$

Cross-multiply:

$$y(x^2 - x + a) = x^2 + x + a$$

$$yx^2 - yx + ya = x^2 + x + a$$

$$x^2(y-1) - x(y+1) + a(y-1) = 0$$

Step 2: For real solutions, discriminant ≥ 0

For any value of y to be in the range, there must exist real x:

$$\Delta = [-(y+1)]^2 - 4(y-1)\cdot a(y-1) \geq 0$$

$$(y+1)^2 - 4a(y-1)^2 \geq 0$$

Step 3: This must hold for ALL $y \in \mathbb{R}$

Expand:

$$y^2 + 2y + 1 - 4a(y^2 - 2y + 1) \geq 0$$

$$y^2 + 2y + 1 - 4ay^2 + 8ay - 4a \geq 0$$

$$y^2(1-4a) + y(2+8a) + (1-4a) \geq 0$$

For this to be true for all y, the parabola in y must not dip below zero.

This happens when the discriminant (in y) is ≤ 0 and the coefficient of y^2 is positive:

Condition 1: $1 - 4a > 0 \implies a < \frac{1}{4}$

Condition 2: $(2+8a)^2 - 4(1-4a)(1-4a) \leq 0$

After simplification (which requires careful algebra), this yields: $a < 0$

Answer: (c) $a \in (-\infty, 0)$

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6. Find domain (a, b) and calculate $2b$ for $f(x) = \frac{1}{\sqrt{\log_{1/3}\left(\frac{x-1}{2-x}\right)}}$

Teacher's Explanation

Critical Point: Logarithms with base < 1 flip inequality signs when removing the log!

Step 1: Conditions for the function to be defined

For square root in denominator: $\log_{1/3}\left(\frac{x-1}{2-x}\right) > 0$

Since base $\frac{1}{3} < 1$, this means:

$$0 < \frac{x-1}{2-x} < \left(\frac{1}{3}\right)^0 = 1$$

Step 2: Solve the first inequality

$$\frac{x-1}{2-x} > 0$$

Critical points: $x = 1$ and $x = 2$

Using sign analysis:

- $x < 1$: negative/negative = positive
- $1 < x < 2$: positive/positive = positive ✓
- $x > 2$: positive/negative = negative

So: $x \in (1, 2)$

Step 3: Solve the second inequality

$$\frac{x-1}{2-x} < 1$$

$$\frac{x-1}{2-x} - 1 < 0$$

$$\frac{x-1-(2-x)}{2-x} < 0$$

$$\frac{2x-3}{2-x} < 0$$

Critical points: $x = \frac{3}{2}$ and $x = 2$

Using sign analysis:

- $x < \frac{3}{2}$: negative/positive = negative ✓
- $\frac{3}{2} < x < 2$: positive/positive = positive
- $x > 2$: positive/negative = negative ✓

So: $x \in \left(-\infty, \frac{3}{2}\right) \cup (2, \infty)$

Step 4: Find intersection

Domain $= (1, 2) \cap \left[\left(-\infty, \frac{3}{2}\right) \cup (2, \infty)\right]$

Domain $= \boxed{\left(1, \frac{3}{2}\right)}$

Therefore: $a = 1$, $b = \frac{3}{2}$

$$2b = 2 \times \frac{3}{2} = 3$$

Answer: (d) $2b = 3$

7. Determine if $f(x) = 5^{-|x|} + \text{sgn}(5^{-x})$ is one-one and/or onto

Teacher's Explanation

Key Facts:

- 5^{-x} is always positive (exponentials are always positive)
- $\text{sgn}(u) = 1$ if $u > 0$, 0 if $u = 0$, -1 if $u < 0$
- $|x|$ makes functions even: $f(-x) = f(x)$

Step 1: Simplify the function

Since $5^{-x} > 0$ for all x:

$$\text{sgn}(5^{-x}) = 1$$

Therefore:

$$f(x) = 5^{-|x|} + 1$$

Step 2: Check if one-one (injective)

Test: $f(-1) = 5^{-|-1|} + 1 = 5^{-1} + 1 = \frac{1}{5} + 1 = \frac{6}{5}$

Test: $f(1) = 5^{-|1|} + 1 = 5^{-1} + 1 = \frac{1}{5} + 1 = \frac{6}{5}$

Since $f(-1) = f(1)$ but $-1 \neq 1$:

$$\boxed{\text{NOT one-one}}$$

Step 3: Find the range (to check if onto)

Since $|x| \geq 0$:

$$5^{-|x|} \leq 5^0 = 1$$

As $|x| \to \infty$: $5^{-|x|} \to 0$

As $|x| = 0$: $5^{-|x|} = 1$

Therefore: $0 < 5^{-|x|} \leq 1$

Adding 1: $1 < f(x) \leq 2$

Range $= (1, 2]$

Since codomain is $\mathbb{R}$ and range $\neq \mathbb{R}$:

$$\boxed{\text{NOT onto}}$$

Answer: (d) Neither one-one nor onto

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8. If range of $f(x) = \frac{x^2+x+k}{x^2-x+k}$ is $\left[\frac{1}{3}, 3\right]$, find k

Teacher's Explanation

Strategy: The extreme values of the range occur when the discriminant equals zero (boundary condition for real x).

Step 1: Set $y = f(x)$ and form quadratic in x

$$y = \frac{x^2 + x + k}{x^2 - x + k}$$

$$y(x^2 - x + k) = x^2 + x + k$$

$$x^2(y-1) - x(y+1) + k(y-1) = 0$$

Step 2: For real x, discriminant ≥ 0

$$\Delta = (y+1)^2 - 4(y-1) \cdot k(y-1) \geq 0$$

$$(y+1)^2 - 4k(y-1)^2 \geq 0$$

Step 3: Boundary values satisfy $\Delta = 0$

At $y = 3$ (maximum):

$$(3+1)^2 - 4k(3-1)^2 = 0$$

$$16 - 4k(4) = 0$$

$$16 - 16k = 0$$

$$\boxed{k = 1}$$

Verification at $y = 1/3$:

$$\left(\frac{1}{3} + 1\right)^2 - 4(1)\left(\frac{1}{3} - 1\right)^2$$
$$= \left(\frac{4}{3}\right)^2 - 4\left(-\frac{2}{3}\right)^2$$
$$= \frac{16}{9} - 4 \cdot \frac{4}{9} = \frac{16}{9} - \frac{16}{9} = 0$$

✓

Answer: (c) $k = 1$

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9. If $f(x) = x^2 + bx + c$ and $f(1+k) = f(1-k)$, order: $f(-1), f(0), f(1)$

Teacher's Explanation

Symmetry Property: If $f(1+k) = f(1-k)$ for all k, the parabola is symmetric about $x = 1$ (this is the axis of symmetry).

Step 1: Identify the vertex

The condition $f(1+k) = f(1-k)$ means the function is symmetric about $x = 1$.

For parabola $f(x) = x^2 + bx + c$ with vertex at $x = -\frac{b}{2}$:

$$-\frac{b}{2} = 1 \implies b = -2$$

Step 2: Understand the parabola's behavior

Since the coefficient of x^2 is positive (equals 1), the parabola opens upward.

The function value increases as we move away from the vertex.

Step 3: Calculate distances from vertex at $x = 1$

- Distance from 1 to -1: $|1 - (-1)| = 2$
- Distance from 1 to 0: $|1 - 0| = 1$
- Distance from 1 to 1: $|1 - 1| = 0$ (at vertex, minimum)

Step 4: Order the values

Since the parabola opens upward and increases with distance from vertex:

$$\boxed{f(1) < f(0) < f(-1)}$$

Answer: (a) $f(1) < f(0) < f(-1)$

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10. Find domain of $f(x) = \log_{\sqrt{2}}(\sqrt{x^2+x} + \sqrt{x^2-x})$

Teacher's Explanation

Three Conditions:

(a) First square root defined: $x^2 + x \geq 0$

(b) Second square root defined: $x^2 - x \geq 0$

(c) Log argument positive: sum of roots > 0

Step 1: First square root

$$x^2 + x \geq 0$$
$$x(x+1) \geq 0$$

Sign analysis: $x \in (-\infty, -1] \cup [0, \infty)$

Step 2: Second square root

$$x^2 - x \geq 0$$
$$x(x-1) \geq 0$$

Sign analysis: $x \in (-\infty, 0] \cup [1, \infty)$

Step 3: Log argument must be positive

We need: $\sqrt{x^2+x} + \sqrt{x^2-x} > 0$

Since both terms are non-negative, the sum is zero only when both are zero.

Both are zero only at $x = 0$, so we must exclude $x = 0$.

Step 4: Find intersection

From Step 1: $(-\infty, -1] \cup [0, \infty)$

From Step 2: $(-\infty, 0] \cup [1, \infty)$

Intersection:

- $(-\infty, -1] \cap (-\infty, 0] = (-\infty, -1]$
- $[0, \infty) \cap [1, \infty) = [1, \infty)$

Combined: $(-\infty, -1] \cup [1, \infty)$

Excluding $x = 0$ (already excluded):

$$\boxed{\text{Domain} = (-\infty, -1] \cup [1, \infty)}$$

Answer: (b) $(-\infty, -1] \cup [1, \infty)$

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Functions - Complete Solutions (Part 2)

Problems 11-30

11. Find domain A, range B, and $A \cup B$ for $f(x) = \frac{1}{\sqrt{|x|-x^2}}$

Teacher's Explanation

Strategy: First find where the expression under the square root is positive (for domain), then analyze the function's output values (for range). The key is recognizing that $|x|$ behaves differently for positive and negative x.

Step 1: Find Domain A

For the square root to be defined in the denominator:

$$|x| - x^2 > 0$$

This means:

$$|x| > x^2$$

Since both sides are non-negative, we can analyze this as $|x| > |x|^2$

When does $t > t^2$ for $t \geq 0$?

$$t > t^2 \implies t - t^2 > 0 \implies t(1-t) > 0$$

This is true when: $0 < t < 1$

Therefore: $\boxed{0 < |x| < 1}$

This gives us: $x \in (-1, 0) \cup (0, 1)$

So: $\boxed{A = (-1, 0) \cup (0, 1)}$

Step 2: Find Range B

Let $g(x) = |x| - x^2$. We need to maximize this to find the minimum of $f(x)$.

For $x \in (-1, 1)$, we have $|x| = |x|$, so:

$$g(|x|) = |x| - |x|^2$$

This is a downward parabola in terms of $|x|$:

$$g'(|x|) = 1 - 2|x| = 0 \implies |x| = \frac{1}{2}$$

Maximum value: $g(1/2) = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$

As $|x| \to 0^+$ or $|x| \to 1^-$: $g(x) \to 0^+$

Therefore: $0 < g(x) \leq \frac{1}{4}$

Taking square roots: $0 < \sqrt{g(x)} \leq \frac{1}{2}$

Taking reciprocal (flips inequality): $f(x) \geq 2$

So: $\boxed{B = [2, \infty)}$

Step 3: Find $A \cup B$

$$A \cup B = [(-1, 0) \cup (0, 1)] \cup [2, \infty)$$

$$= \boxed{(-1, 0) \cup (0, 1) \cup [2, \infty)}$$

Answer: (c)

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12. Analyze $f(x) = \begin{cases} 2x + 3 & x \leq \frac{4}{3} \\ -3x^2 + 8x & x > \frac{4}{3} \end{cases}$

Teacher's Explanation

Key Concept: For piecewise functions, analyze each piece separately, then check what happens at the boundary point. A function is one-one if it never takes the same value twice.

Step 1: Analyze first piece ($x \leq 4/3$)

$$f(x) = 2x + 3$$

This is a line with slope 2 (positive), so it's strictly increasing.

At $x = 4/3$: $f(4/3) = 2(4/3) + 3 = \frac{8}{3} + 3 = \frac{17}{3}$

As $x \to -\infty$: $f(x) \to -\infty$

Range of first piece: $\boxed{(-\infty, 17/3]}$

Step 2: Analyze second piece ($x > 4/3$)

$$f(x) = -3x^2 + 8x$$

This is a downward parabola. Find the vertex:

$$x = -\frac{b}{2a} = -\frac{8}{2(-3)} = \frac{4}{3}$$

Since our domain is $x > 4/3$ (to the right of vertex), the function is strictly decreasing.

At $x = 4/3$: $f(4/3) = -3(16/9) + 8(4/3) = -\frac{16}{3} + \frac{32}{3} = \frac{16}{3}$

Wait, let me recalculate: At the boundary from the right:

$$\lim_{x \to (4/3)^+} f(x) = -3(4/3)^2 + 8(4/3) = -\frac{16}{3} + \frac{32}{3} = \frac{16}{3}$$

But from the left: $f(4/3) = \frac{17}{3}$

There's a jump discontinuity!

As $x \to \infty$: $f(x) \to -\infty$

Range of second piece: $\boxed{(-\infty, 16/3)}$

Step 3: Check if one-one

The function increases to 17/3, then drops to values approaching $-\infty$.

Any value less than 16/3 can be achieved in BOTH pieces!

For example, $y = 5$: - From first piece: $2x + 3 = 5 \implies x = 1 < 4/3$ ✓ - From second piece: $-3x^2 + 8x = 5$ has solutions $x > 4/3$ ✓

Therefore: $\boxed{\text{NOT one-one}}$

Step 4: Check if onto

Combined range: $(-\infty, 17/3]$

Since codomain is $\mathbb{R}$, values greater than 17/3 are not achieved.

Therefore: $\boxed{\text{NOT onto}}$

Answer: (b) Neither one-one nor onto

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13. Find $f_{32}(x)$ where $f(x) = \frac{2x-3}{3x-2}$

Teacher's Explanation

Key Insight: Instead of computing $f(f(f(...)))$ 32 times, look for a pattern! Compute $f(f(x))$ and see if the function is involutory (meaning applying it twice gives you back x).

Step 1: Compute $f(f(x))$

$$f(f(x)) = f\left(\frac{2x-3}{3x-2}\right)$$

$$= \frac{2\left(\frac{2x-3}{3x-2}\right) - 3}{3\left(\frac{2x-3}{3x-2}\right) - 2}$$

Step 2: Simplify numerator

Numerator: $2\left(\frac{2x-3}{3x-2}\right) - 3$

$$= \frac{2(2x-3) - 3(3x-2)}{3x-2}$$

$$= \frac{4x - 6 - 9x + 6}{3x-2}$$

$$= \frac{-5x}{3x-2}$$

Step 3: Simplify denominator

Denominator: $3\left(\frac{2x-3}{3x-2}\right)-2$

$$= \frac{3(2x-3)-2(3x-2)}{3x-2}$$

$$= \frac{6x-9-6x+4}{3x-2}$$

$$= \frac{-5}{3x-2}$$

Step 4: Complete the composition

$$f(f(x)) = \frac{\frac{-5x}{3x-2}}{\frac{-5}{3x-2}} = \frac{-5x}{3x-2}\cdot\frac{3x-2}{-5} = \frac{-5x}{-5} = x$$

Conclusion: $f(f(x)) = x$ (involutory function)

This means: - $f_2(x) = f(f(x)) = x$ - $f_4(x) = f_2(f_2(x)) = x$ - $f_{32}(x) = x$ (since 32 is even)

$$\boxed{f_{32}(x) = x}$$

Answer: (b) $f_{32}(x) = x$

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14. Find domain of $f(x) = \sqrt{\cos(\sin x)} + \cos^{-1}\left(\frac{1+x^2}{2x}\right)$

Teacher's Explanation

Critical Observation: The second term is the troublesome one! For $\cos^{-1}(u)$ to be defined, we need $-1 \le u \le 1$. Let's use the AM-GM inequality concept.

Step 1: First term analysis

$\sqrt{\cos(\sin x)}$ requires $\cos(\sin x) \ge 0$

Since $-1 \le \sin x \le 1$ and cos is positive on $[-1, 1]$, this is always defined.

Step 2: Second term - the constraint

For $\cos^{-1}\left(\frac{1+x^2}{2x}\right)$, we need:

$$-1 \le \frac{1+x^2}{2x} \le 1$$

Let's analyze: $\frac{1+x^2}{2x} = \frac{1}{2}\left(x+\frac{1}{x}\right)$

Step 3: Apply AM-GM inequality

For $x > 0$: By AM-GM, $x + \frac{1}{x} \ge 2\sqrt{x\cdot\frac{1}{x}} = 2$

So: $\frac{1}{2}\left(x+\frac{1}{x}\right) \ge 1$

For $x < 0$: Let $x = -t$ where $t > 0$

$$x + \frac{1}{x} = -t + \frac{1}{-t} = -t - \frac{1}{t} = -\left(t + \frac{1}{t}\right) \leq -2$$

So: $\frac{1}{2}\left(x + \frac{1}{x}\right) \leq -1$

Step 4: Find when equality holds

The expression equals 1 when:

$$\frac{1+x^2}{2x} = 1 \implies 1 + x^2 = 2x \implies x^2 - 2x + 1 = 0$$

$$(x-1)^2 = 0 \implies \boxed{x = 1}$$

The expression equals −1 when:

$$\frac{1+x^2}{2x} = -1 \implies 1 + x^2 = -2x \implies x^2 + 2x + 1 = 0$$

$$(x+1)^2 = 0 \implies \boxed{x = -1}$$

Conclusion: The only values where $\frac{1+x^2}{2x} \in [-1, 1]$ are $x = 1$ and $x = -1$.

$$\boxed{\text{Domain} = \{-1, 1\}}$$

Answer: (d) $\{-1, 1\}$

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15. Find domain of $f(x) = \sqrt[3]{\frac{\log(x^2-x-2)}{2x^2-7x+5}}$

Teacher's Explanation

Important Note: Cube roots (odd index) are defined for all real numbers, including negative ones! So we only worry about the logarithm and the denominator.

Step 1: Logarithm constraint

$$x^2 - x - 2 > 0$$

Factor: $(x-2)(x+1) > 0$

Sign analysis:

- $x < -1$: $(-)(-) = (+)$ ✓
- $-1 < x < 2$: $(-)(+) = (-)$
- $x > 2$: $(+)(+) = (+)$ ✓

From logarithm: $\boxed{x \in (-\infty, -1) \cup (2, \infty)}$

Step 2: Denominator cannot be zero

$$2x^2 - 7x + 5 \neq 0$$

Factor: $(2x - 5)(x - 1) \neq 0$

So: $x \neq \frac{5}{2}$ and $x \neq 1$

Step 3: Combine constraints

Start with: $(-\infty, -1) \cup (2, \infty)$

Remove $x = 1$: Already not in the set ✓

Remove $x = 5/2$: This IS in $(2, \infty)$!

Final domain: $\boxed{(-\infty, -1) \cup (2, 5/2) \cup (5/2, \infty)}$

Which can be written as: $(-\infty, -1) \cup (2, \infty) - \{5/2\}$

Answer: (a)

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16. If $f\left(3x + \frac{1}{2x}\right) = 9x^2 + \frac{1}{4x^2}$, solve $f\left(x + \frac{1}{x}\right) = 1$

Teacher's Explanation

Pattern Recognition: Notice the relationship between the LHS and RHS. The RHS looks like the square of the LHS minus some constant. Let's use substitution!

Step 1: Let $t = 3x + \frac{1}{2x}$ and square it

$$t^2 = \left(3x + \frac{1}{2x}\right)^2$$

$$= 9x^2 + 2 \cdot 3x \cdot \frac{1}{2x} + \frac{1}{4x^2}$$

$$= 9x^2 + 3 + \frac{1}{4x^2}$$

Step 2: Relate to the given function

From the given: $f(t) = 9x^2 + \frac{1}{4x^2}$

From Step 1: $t^2 = 9x^2 + \frac{1}{4x^2} + 3$

Therefore:

$$f(t) = t^2 - 3$$

Step 3: Solve the required equation

We need: $f\left(x + \frac{1}{x}\right) = 1$

Substitute:

$$\left(x + \frac{1}{x}\right)^2 - 3 = 1$$

$$\left(x + \frac{1}{x}\right)^2 = 4$$

$$x + \frac{1}{x} = \pm 2$$

Step 4: Solve each case

Case 1: $x + \frac{1}{x} = 2$

Multiply by x: $x^2 + 1 = 2x$

$x^2 - 2x + 1 = 0$

$(x-1)^2 = 0 \implies \boxed{x = 1}$

Case 2: $x + \frac{1}{x} = -2$

Multiply by x: $x^2 + 1 = -2x$

$x^2 + 2x + 1 = 0$

$(x+1)^2 = 0 \implies \boxed{x = -1}$

$$\boxed{\text{Solutions: } x = \pm 1}$$

Answer: (b) $x = \pm 1$

---

17. If domain of $f(x) = \sin^{-1}(x^2 - 1) + \log_3(3^x - 2)$ is not in $(-\infty, a) \cup (b, \infty)$, find $3^a + b^2$

Teacher's Explanation

Translation: The function IS defined on $[a, b]$. We need to find the intersection of domains from both terms.

Step 1: Domain from $\sin^{-1}(x^2 - 1)$

For $\sin^{-1}(u)$: $-1 \leq u \leq 1$

$$-1 \leq x^2 - 1 \leq 1$$

$$0 \leq x^2 \leq 2$$

$$-\sqrt{2} \leq x \leq \sqrt{2}$$

Domain 1: $\boxed{[-\sqrt{2}, \sqrt{2}]}$

Step 2: Domain from $\log_3(3^x - 2)$

Argument must be positive:

$$3^x - 2 > 0$$

$$3^x > 2$$

$$x > \log_3 2$$

Domain 2: $\boxed{(\log_3 2, \infty)}$

Step 3: Find intersection

$$[-\sqrt{2}, \sqrt{2}] \cap (\log_3 2, \infty)$$

Since $\log_3 2 \approx 0.631$ and $\sqrt{2} \approx 1.414$:

$$\boxed{[a, b] = (\log_3 2, \sqrt{2}]}$$

So: $a = \log_3 2$ and $b = \sqrt{2}$

Step 4: Calculate $3^a + b^2$

$$3^a = 3^{\log_3 2} = 2$$

$$b^2 = (\sqrt{2})^2 = 2$$

$$3^a + b^2 = 2 + 2 = \boxed{4}$$

Answer: (d) 4

---

18. Find $A \cap B$ where A is domain and B is range of $f(x) = \frac{1}{\sqrt{|x|-x}}$

Teacher's Explanation

Key Observation: When is $|x| - x > 0$? Think about when the absolute value differs from the number itself!

Step 1: Find Domain A

We need: $|x| - x > 0$, or $|x| > x$

Test cases:

- If $x > 0$: $|x| = x$, so $x > x$ is FALSE
- If $x = 0$: $0 > 0$ is FALSE
- If $x < 0$: $|x| = -x > 0$, and $x < 0$, so $-x > x$ is TRUE

Therefore: $\boxed{A = (-\infty, 0)}$

Step 2: Find Range B

For $x < 0$, let $x = -t$ where $t > 0$:

$$f(-t) = \frac{1}{\sqrt{|-t| - (-t)}} = \frac{1}{\sqrt{t+t}} = \frac{1}{\sqrt{2t}}$$

As $t \to 0^+$ (i.e., $x \to 0^-$): $f(x) \to \infty$

As $t \to \infty$ (i.e., $x \to -\infty$): $f(x) \to 0^+$

Therefore: $\boxed{B = (0, \infty)}$

Step 3: Find intersection

$$A \cap B = (-\infty, 0) \cap (0, \infty) = \boxed{\phi}$$

Answer: (a) ϕ (empty set)

---

19. Find domain of $f(x) = \sin^{-1}\left[\log_2\left(\frac{x^2}{2}\right)\right]$

Teacher's Explanation

Working backwards: Start from the outermost function and work inward. $\sin^{-1}$ requires its argument to be in $[-1, 1]$.

Step 1: Domain of $\sin^{-1}$

$$-1 \le \log_2\left(\frac{x^2}{2}\right) \le 1$$

Step 2: Remove logarithm (base 2 ¿ 1)

Since base is 2 (greater than 1), we raise 2 to all parts:

$$2^{-1} \le \frac{x^2}{2} \le 2^1$$

$$\frac{1}{2} \le \frac{x^2}{2} \le 2$$

Step 3: Solve for x^2

Multiply all parts by 2:

$$1 \le x^2 \le 4$$

Step 4: Take square roots

$$\sqrt{1} \le |x| \le \sqrt{4}$$

$$1 \leq |x| \leq 2$$

This gives us two intervals:

$$\boxed{x \in [-2, -1] \cup [1, 2]}$$

Answer: (b) $[-2, -1] \cup [1, 2]$

---

20. Find range of $f(x) = \log_3(5 + 4x - x^2)$

Teacher's Explanation

Strategy: First find the range of the inner quadratic function, then apply the logarithm to that range.

Step 1: Analyze the quadratic $g(x) = 5 + 4x - x^2$

This is $g(x) = -x^2 + 4x + 5$ (downward parabola)

Vertex at: $x = -\frac{b}{2a} = -\frac{4}{2(-1)} = 2$

Maximum value: $g(2) = -(2)^2 + 4(2) + 5 = -4 + 8 + 5 = 9$

Step 2: Domain constraint

For logarithm, we need $g(x) > 0$:

$$5 + 4x - x^2 > 0$$

$$x^2 - 4x - 5 < 0$$

$$(x - 5)(x + 1) < 0$$

Domain: $x \in (-1, 5)$

Step 3: Range of the quadratic

On the domain $(-1, 5)$: - Maximum at $x = 2$: $g(2) = 9$ - As $x \to -1^+$ or $x \to 5^-$: $g(x) \to 0^+$

Range of $g(x)$: $\boxed{(0, 9]}$

Step 4: Apply logarithm (base 3 ¿ 1)

Since $\log_3$ is increasing:

$$\log_3(0^+) < f(x) \leq \log_3(9)$$

$$-\infty < f(x) \leq \log_3(3^2)$$

$$-\infty < f(x) \leq 2$$

Range: $\boxed{(-\infty, 2]}$

Answer: (c) $(-\infty, 2]$

---

21. If $f(x)f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right)$ and f is quadratic, find $\sqrt{f(2/3) + f(3/2)}$

Teacher's Explanation

Special Formula: For polynomials satisfying this equation, the solution is always $f(x) = 1 \pm x^n$ where n is the degree. For quadratic, $n = 2$.

Step 1: Apply the standard result

For a quadratic satisfying this functional equation:

$$f(x) = 1 + x^2$$

(We can verify: $f(x)f(1/x) = (1 + x^2)(1 + 1/x^2) = 1 + x^2 + 1/x^2 + 1$)

Step 2: Calculate $f(2/3)$

$$f(2/3) = 1 + \left(\frac{2}{3}\right)^2 = 1 + \frac{4}{9} = \frac{13}{9}$$

Step 3: Calculate $f(3/2)$

$$f(3/2) = 1 + \left(\frac{3}{2}\right)^2 = 1 + \frac{9}{4} = \frac{13}{4}$$

Step 4: Find the sum

$$f(2/3) + f(3/2) = \frac{13}{9} + \frac{13}{4}$$

$$= 13\left(\frac{1}{9} + \frac{1}{4}\right) = 13\left(\frac{4+9}{36}\right) = 13 \cdot \frac{13}{36} = \frac{169}{36}$$

Step 5: Take square root

$$\sqrt{f(2/3) + f(3/2)} = \sqrt{\frac{169}{36}} = \frac{13}{6}$$

Answer: (c) $\frac{13}{6}$

---

22. If $h(x) = f(x) + g(x)$ where f is even, g is odd, and $h(-2) = 0$, find $8p + 4q + 2r$

Given: $f(x) = ax^2 + bx + c$ and $g(x) = px^3 + qx^2 + rx$

Teacher's Explanation

Key Properties:

- Even function: $f(-x) = f(x)$ means only even powers (no $x, x^3, ...$)
- Odd function: $g(-x) = -g(x)$ means only odd powers (no $x^0, x^2, ...$)

Step 1: Apply even function property to $f(x)$

$$f(-x) = a(-x)^2 + b(-x) + c = ax^2 - bx + c$$

For $f(-x) = f(x)$:

$$ax^2 - bx + c = ax^2 + bx + c$$

This requires: $\boxed{b = 0}$

So: $f(x) = ax^2 + c$

Step 2: Apply odd function property to $g(x)$

$$g(-x) = p(-x)^3 + q(-x)^2 + r(-x) = -px^3 + qx^2 - rx$$

For $g(-x) = -g(x)$:

$$-px^3 + qx^2 - rx = -(px^3 + qx^2 + rx)$$
$$-px^3 + qx^2 - rx = -px^3 - qx^2 - rx$$

This requires: $\boxed{q = 0}$

So: $g(x) = px^3 + rx$

Step 3: Use the condition $h(-2) = 0$

$$h(-2) = f(-2) + g(-2) = 0$$

Since f is even: $f(-2) = f(2) = 4a + c$

Since g is odd: $g(-2) = -g(2) = -(8p + 2r)$

Therefore:

$$(4a + c) + (-(8p + 2r)) = 0$$
$$4a + c - 8p - 2r = 0$$
$$\boxed{4a + c = 8p + 2r}$$

Step 4: Evaluate the expression

$$8p + 4q + 2r = 8p + 4(0) + 2r = 8p + 2r = \boxed{4a + c}$$

Since $b = 0$:

$$= 4a + 2b + c$$

Answer: (c) $4a + 2b + c$

---

23. Find range of $f(x) = \log_{0.5}(x^4 - 2x^2 + 3)$

Teacher's Explanation

Warning: Base is 0.5 which is less than 1! This means the logarithm is a DECREASING function - maximum input gives minimum output.

Step 1: Analyze the inner function

Let $u = x^2$ (where $u \geq 0$)

$$g(u) = u^2 - 2u + 3$$

Complete the square:

$$g(u) = (u - 1)^2 + 2$$

Step 2: Find range of $g(u)$ **for** $u \geq 0$

Minimum at $u = 1$: $g(1) = 0 + 2 = 2$

As $u \to \infty$: $g(u) \to \infty$

Range of argument: $\boxed{[2, \infty)}$

Step 3: Apply logarithm with base 0.5

Since $0 < 0.5 < 1$, the function is DECREASING:

Maximum input (2) gives maximum output:

$$\log_{0.5}(2) = \log_{0.5}(0.5^{-1}) = -1$$

As input $\to \infty$: output $\to -\infty$

Range: $\boxed{(-\infty, -1]}$

Answer: (b) $(-\infty, -1]$

---

24. Find range of $f(x) = x^2 - 4x + 5$ **on domain** $[2, \infty)$

Teacher's Explanation

Key Point: When domain is restricted, find the vertex first. If the vertex is in the domain, that's where the minimum (for upward parabola) occurs.

Step 1: Find the vertex

$$x = -\frac{b}{2a} = -\frac{-4}{2(1)} = 2$$

The vertex is at $x = 2$, which is exactly at the left boundary of the domain!

Step 2: Determine behavior

Since the parabola opens upward ($a = 1 > 0$) and the domain starts at the vertex:

The function is increasing for all $x \geq 2$.

Step 3: Find minimum value

At $x = 2$:

$$f(2) = (2)^2 - 4(2) + 5 = 4 - 8 + 5 = 1$$

Step 4: Find behavior as $x \to \infty$

$$\lim_{x\to\infty} f(x) = \infty$$

Range: $\boxed{[1, \infty)}$

Answer: (b) $[1, \infty)$

---

25. Find $(f \circ f \circ f)(x) + (f \circ f \circ f)(-x)$ **where** $f(x) = -|x|$

Teacher's Explanation

Strategy: First see what happens when we compose f with itself. Look for a pattern!

Step 1: Compute $f(f(x))$

$$f(f(x)) = f(-|x|) = -|-|x||$$

Since $|-|x|| = ||x|| = |x|$:

$$f(f(x)) = -|x| = f(x)$$

Key observation: Applying f twice gives us f back!

Step 2: Compute $(f \circ f \circ f)(x)$

$$(f \circ f \circ f)(x) = f(f(f(x))) = f(f(x)) = f(x) = -|x|$$

Step 3: Compute $(f \circ f \circ f)(-x)$

$$(f \circ f \circ f)(-x) = f(-x) = -|-x| = -|x|$$

Step 4: Add them

$$(f \circ f \circ f)(x) + (f \circ f \circ f)(-x) = -|x| + (-|x|) = -2|x|$$

Note that $2f(x) = 2(-|x|) = -2|x|$

Therefore: $\boxed{(f \circ f \circ f)(x) + (f \circ f \circ f)(-x) = 2f(x)}$

Answer: (c) $2f(x)$

---

26. Find range of $f(x) = -\sqrt{-x^2 - 6x - 5}$

Teacher's Explanation

Careful: There's a negative sign outside the square root! This will flip our final range.

Step 1: Find domain

$$-x^2 - 6x - 5 \geq 0$$
$$x^2 + 6x + 5 \leq 0$$
$$(x + 1)(x + 5) \leq 0$$

Domain: $\boxed{[-5, -1]}$

Step 2: Maximize the expression inside the root

Let $g(x) = -x^2 - 6x - 5$

This is a downward parabola. Vertex at:

$$x = -\frac{-6}{2(-1)} = -3$$

Maximum: $g(-3) = -(-3)^2 - 6(-3) - 5 = -9 + 18 - 5 = 4$

Step 3: Find range of $\sqrt{g(x)}$

At endpoints: $g(-5) = g(-1) = 0$

At vertex: $g(-3) = 4$

So: $0 \leq g(x) \leq 4$

Therefore: $0 \leq \sqrt{g(x)} \leq 2$

Step 4: Apply the negative sign

$$f(x) = -\sqrt{g(x)}$$

Range: $\boxed{[-2, 0]}$

Answer: (b) $[-2, 0]$

---

27. Determine if $f(x) = 2x + \sin x$ **is one-one and/or onto**

Teacher's Explanation

Derivative Test: If $f'(x) > 0$ everywhere, the function is strictly increasing, hence one-one. For onto, check if the range is all of $\mathbb{R}$.

Step 1: Check injectivity (one-one)

Find derivative:

$$f'(x) = 2 + \cos x$$

Since $-1 \leq \cos x \leq 1$:

$$2 - 1 \leq f'(x) \leq 2 + 1$$
$$1 \leq f'(x) \leq 3$$

Since $f'(x) \geq 1 > 0$ for all x:

The function is strictly increasing.

Therefore: $\boxed{\text{ONE-ONE}}$

Step 2: Check surjectivity (onto)

Analyze limits:

As $x \to \infty$: $f(x) = 2x + \sin x \to \infty$ (since $2x$ dominates)

As $x \to -\infty$: $f(x) = 2x + \sin x \to -\infty$

Since f is continuous and strictly increasing from $-\infty$ to ∞:

Range $= \mathbb{R}$

Therefore: $\boxed{\text{ONTO}}$

Conclusion: The function is BIJECTIVE (both one-one and onto)

Answer: (a) One-one and onto

---

28. Analyze $f(x) = \begin{cases} 2x - 3 & x < -2 \\ x^2 - 1 & -2 \leq x \leq 2 \\ 3x + 2 & x > 2 \end{cases}$

Teacher's Explanation

Strategy: Check each piece separately. A function is NOT one-one if we can find two different inputs giving the same output.

Step 1: Check for one-one property

Look at the middle piece: $f(x) = x^2 - 1$ on $[-2, 2]$

This is a parabola (U-shaped), so it's not one-one on this interval.

Test: $f(-1) = (-1)^2 - 1 = 0$

Test: $f(1) = (1)^2 - 1 = 0$

Since $f(-1) = f(1) = 0$ but $-1 \neq 1$:

$\boxed{\text{NOT ONE-ONE}}$

Step 2: Find range of each piece

Piece 1: $f(x) = 2x - 3$ for $x < -2$

As $x \to -\infty$: $f(x) \to -\infty$

At $x = -2$: $f(-2^-) = 2(-2) - 3 = -7$

Range: $(-\infty, -7)$

Piece 2: $f(x) = x^2 - 1$ for $-2 \leq x \leq 2$

Minimum at $x = 0$: $f(0) = -1$

At boundaries: $f(\pm 2) = 4 - 1 = 3$

Range: $[-1, 3]$

Piece 3: $f(x) = 3x + 2$ for $x > 2$

At $x = 2$: $f(2^+) = 3(2) + 2 = 8$

As $x \to \infty$: $f(x) \to \infty$

Range: $(8, \infty)$

Step 3: Check surjectivity

Total range: $(-\infty, -7) \cup [-1, 3] \cup (8, \infty)$

Missing intervals: $[-7, -1)$ and $(3, 8]$

For example, $y = 5$ cannot be achieved.

$\boxed{\text{NOT ONTO}}$

Answer: (d) Neither one-one nor onto

---

29. Find domain of $f(x) = \sqrt{\frac{\log_{10}\left(\frac{x}{x-2}\right)}{\sqrt{[x]^2-5[x]+6}}}$

Teacher's Explanation

Multiple Constraints: We need: (1) denominator's square root positive, (2) log argument positive, (3) outer fraction positive. Work systematically!

Step 1: Denominator square root

$$[x]^2 - 5[x] + 6 > 0$$

$$([x] - 2)([x] - 3) > 0$$

Critical points: $[x] = 2$ and $[x] = 3$

This is positive when: $[x] < 2$ or $[x] > 3$

Since $[x]$ is an integer: - $[x] \leq 1$ means $x < 2$ - $[x] \geq 4$ means $x \geq 4$

From this: $\boxed{x \in (-\infty, 2) \cup [4, \infty)}$

Step 2: Log argument positive

$$\frac{x}{x-2} > 0$$

Critical points: $x = 0$ and $x = 2$

Sign analysis: - $x < 0$: $(-)/(-) = (+)$ ✓ - $0 < x < 2$: $(+)/(-) = (-)$ - $x > 2$: $(+)/(+) = (+)$ ✓

From this: $\boxed{x \in (-\infty, 0) \cup (2, \infty)}$

Step 3: Outer square root (numerator ≥ 0)

Since denominator is always positive (from Step 1), we need:

$$\log_{10}\left(\frac{x}{x-2}\right) \geq 0$$

$$\frac{x}{x-2} \geq 10^0 = 1$$

$$\frac{x}{x-2} - 1 \geq 0$$

$$\frac{x-(x-2)}{x-2} \geq 0$$

$$\frac{2}{x-2} \geq 0$$

This requires: $\boxed{x > 2}$

Step 4: Find intersection

We need all three conditions: - From Step 1: $(-\infty, 2) \cup [4, \infty)$ - From Step 2: $(-\infty, 0) \cup (2, \infty)$ - From Step 3: $(2, \infty)$

The intersection is dominated by Step 3: $x > 2$

Combined with Step 1: $(2, \infty) \cap [(-\infty, 2) \cup [4, \infty)] = [4, \infty)$

$\boxed{\text{Domain} = [4, \infty)}$

Answer: (d) $[4, \infty)$

---

30. Find domain and range of $f(x) = \frac{1}{x-[x]}$

Teacher's Explanation

Fractional Part: Recall that $x - [x] = \{x\}$ is the fractional part of x. It equals 0 for integers and is between 0 and 1 for non-integers.

Step 1: Recognize the fractional part

$$f(x) = \frac{1}{\{x\}}$$

where $\{x\} = x - [x]$

Step 2: Find domain

Denominator cannot be zero: $\{x\} \neq 0$

The fractional part is 0 when x is an integer.

Domain: $\boxed{\mathbb{R} - \mathbb{Z}}$ (all reals except integers)

Step 3: Find range

For non-integers: $0 < \{x\} < 1$

Taking reciprocal (flips inequality):

$$\frac{1}{1} < \frac{1}{\{x\}} < \frac{1}{0^+}$$

$$1 < f(x) < \infty$$

Range: $\boxed{(1, \infty)}$

Answer: (b) Domain: $\mathbb{R} - \mathbb{Z}$, Range: $(1, \infty)$

Continue to next page for solutions 31-50...

Functions - Complete Solutions (Part 3)

Problems 31-50

31. Find values NOT in range of $y = \frac{x^2-2x+1}{x^2+x-1}$

Teacher's Explanation

Range Finding Technique: Cross-multiply to get a quadratic in x. For y to be in the range, this quadratic must have real solutions, so discriminant ≥ 0.

Step 1: Cross-multiply and rearrange

$$y(x^2+x-1) = x^2-2x+1$$

$$yx^2+yx-y = x^2-2x+1$$

$$x^2(y-1)+x(y+2)+(-y-1) = 0$$

Step 2: Apply discriminant condition

For real solutions in x:

$$\Delta = (y+2)^2 - 4(y-1)(-y-1) \geq 0$$

$$= (y+2)^2 + 4(y-1)(y+1)$$

$$= y^2+4y+4+4(y^2-1)$$

$$= y^2+4y+4+4y^2-4$$

$$= 5y^2+4y \geq 0$$

$$= y(5y+4) \geq 0$$

Step 3: Solve the inequality

Critical points: $y = 0$ and $y = -\frac{4}{5}$

Sign analysis:

- $y < -4/5$: $(-)(-) = (+)$ ✓
- $-4/5 < y < 0$: $(-)(+) = (-)$
- $y > 0$: $(+)(+) = (+)$ ✓

Values in range: $\boxed{(-\infty, -4/5] \cup [0, \infty)}$

Values NOT in range: $\boxed{(-4/5, 0)}$

Answer: (a) $(-4/5, 0)$

---

32. Analyze $f : [0, 4] \to [0, 4]$ where $f(x) = \sqrt{16 - x^2}$

Teacher's Explanation

Geometric View: This is the upper semicircle of radius 4 centered at origin. When restricted to $[0, 4]$, it's a quarter circle in the first quadrant.

Step 1: Check injectivity (one-one)

Suppose $f(x_1) = f(x_2)$ where $x_1, x_2 \in [0, 4]$:

$$\sqrt{16 - x_1^2} = \sqrt{16 - x_2^2}$$

Squaring: $16 - x_1^2 = 16 - x_2^2$

$$x_1^2 = x_2^2$$

Since both $x_1, x_2 \geq 0$: $\boxed{x_1 = x_2}$

Therefore: $\boxed{\text{ONE-ONE}}$

Step 2: Check surjectivity (onto)

Find the range:

At $x = 0$: $f(0) = \sqrt{16 - 0} = 4$

At $x = 4$: $f(4) = \sqrt{16 - 16} = 0$

Since f is continuous and decreasing on $[0, 4]$:

Range $= [0, 4] =$ Codomain

Therefore: $\boxed{\text{ONTO}}$

Conclusion: The function is BIJECTIVE

Answer: (d) Bijection

---

33. Find domain of $f(x) = \frac{\sqrt{|x|-x}}{\sqrt{x-[x]}}$

Teacher's Explanation

Two Parts to Analyze:

- Numerator: when is $|x| - x \geq 0$?
- Denominator: when is $x - [x] > 0$ (strict, since it's in denominator)?

Step 1: Numerator condition

$$|x| - x \geq 0 \implies |x| \geq x$$

This is always true! (If $x \geq 0$: $x \geq x$ OK; if $x < 0$: $-x > x$ TRUE)

Step 2: Denominator condition

$$x - [x] > 0$$

This is the fractional part: $\{x\} > 0$

The fractional part is 0 when x is an integer, and positive otherwise.

Therefore: $\boxed{x \notin \mathbb{Z}}$

Domain: $\boxed{\mathbb{R} - \mathbb{Z}}$ (all reals except integers)

Answer: (c) $\mathbb{R} - \mathbb{Z}$

---

34. Find range of $f(x) = \begin{cases} 2x - 3 & x < -1 \\ 1 - x^2 & -1 \leq x \leq 1 \\ 3x^2 + 2 & x > 1 \end{cases}$

Teacher's Explanation

Piecewise Range: Find the range of each piece, being careful about whether boundary points are included.

Step 1: Range of first piece

$$f(x) = 2x - 3, \quad x < -1$$

This is a line with slope 2.

At $x = -1$: $f(-1^-) = 2(-1) - 3 = -5$ (not included)

As $x \to -\infty$: $f(x) \to -\infty$

Range 1: $\boxed{(-\infty, -5)}$

Step 2: Range of second piece

$$f(x) = 1 - x^2, \quad -1 \leq x \leq 1$$

This is a downward parabola with vertex at $x = 0$.

Maximum at $x = 0$: $f(0) = 1$

At boundaries: $f(\pm 1) = 1 - 1 = 0$

Range 2: $\boxed{[0, 1]}$

Step 3: Range of third piece

$$f(x) = 3x^2 + 2, \quad x > 1$$

This is an upward parabola.

At $x = 1$: $f(1^+) = 3(1) + 2 = 5$ (not included)

As $x \to \infty$: $f(x) \to \infty$

Range 3: $\boxed{(5, \infty)}$

Step 4: Combine ranges

Total range: $\boxed{(-\infty, -5) \cup [0, 1] \cup (5, \infty)}$

Answer: (b)

---

35. Find domain of $f(x) = {}^{16-x}C_{2x-1}$

Teacher's Explanation

Combination Requirements:

- $n \geq 0$ (total items non-negative)
- $r \geq 0$ (selection non-negative)
- $n \geq r$ (can't select more than available)
- Both n and r must be integers

Step 1: Condition $n \geq 0$

$$16 - x \geq 0 \implies x \leq 16$$

Step 2: Condition $r \geq 0$

$$2x - 1 \geq 0 \implies x \geq \frac{1}{2}$$

Step 3: Condition $n \geq r$

$$16 - x \geq 2x - 1$$

$$17 \geq 3x$$

$$x \leq \frac{17}{3} \approx 5.67$$

Step 4: Integer requirement

For combinations to be defined, both n and r must be integers.

From constraints: $\frac{1}{2} \leq x \leq \frac{17}{3}$

For $n = 16 - x$ to be integer: x must be integer

For $r = 2x - 1$ to be integer: $2x$ must be integer

Both conditions are satisfied when x is an integer.

Integers in $[\frac{1}{2}, \frac{17}{3}]$: $\boxed{\{1, 2, 3, 4, 5\}}$

Answer: (a) $\{1, 2, 3, 4, 5\}$

---

36. Analyze $f(A) = \det(A)$ where A is a 2×2 matrix

Teacher's Explanation

Key Concepts:

- Onto: Can we get any real number as a determinant?
- One-one: Do different matrices always have different determinants?

Step 1: Check if onto

For any $k \in \mathbb{R}$, can we find a matrix with $\det(A) = k$?

Yes! For example: $A = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$

$$\det(A) = k(1) - 0(0) = k$$

Therefore: $\boxed{\text{ONTO}}$

Step 2: Check if one-one

Can different matrices have the same determinant?

Example: $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ has $\det(A) = 1$

Also: $B = \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$ has $\det(B) = 1$

Since $A \neq B$ but $\det(A) = \det(B)$:

Therefore: $\boxed{\text{NOT ONE-ONE}}$

Answer: (b) Onto but not one-one

---

37. If $f(x + y) = f(x) + f(y)$ and $f(1) = 7$, find $\sum_{r=1}^{n} f(r)$

Teacher's Explanation

Cauchy's Functional Equation: This classic equation has the solution $f(x) = kx$ for some constant k.

Step 1: Find the function

The general solution is: $f(x) = kx$

Given $f(1) = 7$: $k(1) = 7 \implies k = 7$

Therefore: $\boxed{f(x) = 7x}$

Step 2: Compute the sum

$$\sum_{r=1}^{n} f(r) = \sum_{r=1}^{n} 7r = 7\sum_{r=1}^{n} r$$

Using the formula: $\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$

$$= 7 \cdot \frac{n(n+1)}{2} = \boxed{\frac{7n(n+1)}{2}}$$

Answer: (d) $\frac{7n(n+1)}{2}$

---

38. Find period of $f(x) = e^{\log(\sin x)} + \tan^3 x - \csc(3x - 5)$

Teacher's Explanation

Simplification First: $e^{\log(\sin x)} = \sin x$ (inverse operations cancel). Then find LCM of individual periods.

Step 1: Simplify

$$f(x) = \sin x + \tan^3 x - \csc(3x - 5)$$

Step 2: Period of each term

Term 1: $\sin x$ has period 2π

Term 2: $\tan^3 x$

Since $\tan x$ has period π, and raising to a power doesn't change the period:

Period $= \pi$

Term 3: $\csc(3x - 5)$

Basic period of csc is 2π. With coefficient 3:

Period $= \frac{2\pi}{3}$

Step 3: Find LCM

We need $\text{LCM}(2\pi, \pi, \frac{2\pi}{3})$

For fractions: $\text{LCM} = \frac{\text{LCM of numerators}}{\text{GCD of denominators}}$

Numerators: $2\pi, \pi, 2\pi \implies \text{LCM} = 2\pi$

Denominators: $1, 1, 3 \implies \text{GCD} = 1$

Period $= \boxed{\frac{2\pi}{1} = 2\pi}$

Answer: (c) 2π

---

39. Solve $f(x) = 8$ where $f(x) = \begin{cases} x^2 - 4x + 3 & x < 2 \\ x - 3 & x \geq 2 \end{cases}$

Teacher's Explanation

Piecewise Solving: Solve in each piece separately, then check if the solution satisfies the domain condition for that piece.

Step 1: Solve in first piece ($x < 2$)

$$x^2 - 4x + 3 = 8$$

$$x^2 - 4x - 5 = 0$$

Factor: $(x - 5)(x + 1) = 0$

Solutions: $x = 5$ or $x = -1$

Check domain: We need $x < 2$

- $x = 5$: NOT in domain ($5 \not< 2$) $\times$
- $x = -1$: In domain ($-1 < 2$) ✓

Valid solution from piece 1: $\boxed{x = -1}$

Step 2: Solve in second piece ($x \geq 2$)

$$x - 3 = 8$$

$$x = 11$$

Check domain: $11 \geq 2$ ✓

Valid solution from piece 2: $\boxed{x = 11}$

Total solutions: $\{-1, 11\} \implies 2$ solutions

Answer: (b) 2

---

40. Express $(g \circ f)(x) + (f \circ g)(x)$ in terms of f and g

Given: $f(x) = 3x - 2$ and $g(x) = x^2 + 2$

Teacher's Explanation

Strategy: Compute each composition separately, then add them and try to express in terms of the original functions.

Step 1: Compute $(g \circ f)(x)$

$$g(f(x)) = g(3x - 2) = (3x - 2)^2 + 2$$

$$= 9x^2 - 12x + 4 + 2 = 9x^2 - 12x + 6$$

Step 2: Compute $(f \circ g)(x)$

$$f(g(x)) = f(x^2 + 2) = 3(x^2 + 2) - 2$$

$$= 3x^2 + 6 - 2 = 3x^2 + 4$$

Step 3: Add them

$$(g \circ f)(x) + (f \circ g)(x) = (9x^2 - 12x + 6) + (3x^2 + 4)$$

$$= 12x^2 - 12x + 10$$

Step 4: Express in terms of f and g

Try option (b): $12g(x) - 4f(x) - 22$

$$= 12(x^2 + 2) - 4(3x - 2) - 22$$

$$= 12x^2 + 24 - 12x + 8 - 22$$

$$= 12x^2 - 12x + 10$$

✓

Answer: (b) $12g(x) - 4f(x) - 22$

---

41. If $f(4) = -4$ and $f(x) = \frac{ax^{10} + bx^8 + \ldots + ex^2 + 12x + 15}{x}$, find $f(-4)$

Teacher's Explanation

Key Observation: Dividing by x creates terms with odd powers plus constants. Odd functions satisfy $f(-x) = -f(x)$.

Step 1: Rewrite the function

$$f(x) = ax^9 + bx^7 + cx^5 + dx^3 + ex + 12 + \frac{15}{x}$$

Step 2: Identify odd function part

Let $g(x) = ax^9 + bx^7 + cx^5 + dx^3 + ex$

This is an odd function: $g(-x) = -g(x)$

Also, $\frac{15}{x}$ is odd: $\frac{15}{-x} = -\frac{15}{x}$

So: $f(x) = g(x) + 12 + \frac{15}{x}$

Step 3: Use given information

$$f(4) = g(4) + 12 + \frac{15}{4} = -4$$

$$g(4) = -4 - 12 - \frac{15}{4} = -16 - \frac{15}{4} = -\frac{79}{4}$$

Step 4: Find $f(-4)$

$$f(-4) = g(-4) + 12 + \frac{15}{-4}$$

Since g is odd: $g(-4) = -g(4) = -\left(-\frac{79}{4}\right) = \frac{79}{4}$

$$f(-4) = \frac{79}{4} + 12 - \frac{15}{4}$$

$$= \frac{79-15}{4} + 12 = \frac{64}{4} + 12 = 16 + 12 = \boxed{28}$$

Answer: (a) 28

---

42. If $f : A \to B$ is onto and $g : B \to C$ where $g(x) = \sqrt{3 + 4x - 4x^2}$, find range of f

Teacher's Explanation

Key Fact: If $f : A \to B$ is onto, then Range$(f) = B$. So we just need to find the domain of g, which equals B.

Step 1: Understand the setup

Since f is onto: Range of $f = B$

Since $g : B \to C$: Domain of $g = B$

Therefore: Range of f = Domain of g

Step 2: Find domain of g

$$3 + 4x - 4x^2 \geq 0$$

$$-4x^2 + 4x + 3 \geq 0$$

$$4x^2 - 4x - 3 \leq 0$$

Step 3: Factor the quadratic

Using the quadratic formula:

$$x = \frac{4 \pm \sqrt{16 + 48}}{8} = \frac{4 \pm 8}{8}$$

Roots: $x = \frac{12}{8} = \frac{3}{2}$ and $x = \frac{-4}{8} = -\frac{1}{2}$

Factor: $(2x-3)(2x+1) \leq 0$ or equivalently $4(x-\frac{3}{2})(x+\frac{1}{2}) \leq 0$

Step 4: Solve inequality

Values between roots: $\boxed{-\frac{1}{2} \leq x \leq \frac{3}{2}}$

Therefore: Range of $f = \boxed{\left[-\frac{1}{2}, \frac{3}{2}\right]}$

Answer: (c) $[-\frac{1}{2}, \frac{3}{2}]$

---

43. Find $A \cap B$ where A is domain and B is range of $f(x) = \sqrt{\frac{1-x^2}{1+x^2}}$

Teacher's Explanation

Strategy: Domain requires numerator ≥ 0 (denominator is always positive). For range, analyze the function's behavior.

Step 1: Find domain A

Denominator $1 + x^2 > 0$ always (OK)

Numerator: $1 - x^2 \geq 0$

$$x^2 \leq 1 \implies -1 \leq x \leq 1$$

Domain: $\boxed{A = [-1, 1]}$

Step 2: Find range B

At $x = 0$: $f(0) = \sqrt{\frac{1}{1}} = 1$ (maximum)

At $x = \pm 1$: $f(\pm 1) = \sqrt{\frac{0}{2}} = 0$ (minimum)

Since the function is continuous:

Range: $\boxed{B = [0, 1]}$

Step 3: Find intersection

$$A \cap B = [-1, 1] \cap [0, 1] = \boxed{[0, 1]}$$

Answer: (b) $[0, 1]$

---

44. Find domain of $f(x) = \log_2(2^x - 2) + \sqrt{1-x}$

Teacher's Explanation

Two Constraints:

- Logarithm: argument > 0
- Square root: argument ≥ 0

Then find intersection.

Step 1: Logarithm constraint

$$2^x - 2 > 0$$

$$2^x > 2 = 2^1$$

Since exponential is increasing: $\boxed{x > 1}$

Step 2: Square root constraint

$$1 - x \geq 0$$

$$\boxed{x \leq 1}$$

Step 3: Find intersection

We need: $x > 1$ AND $x \leq 1$

This is impossible!

Domain $= \boxed{\phi}$ (empty set)

Answer: (d) ϕ

---

45. If $f(u) = 1 - u$, $h(x) = \frac{1}{x}$, $g(x) = \frac{1}{1-x}$, and $f(F(h(x))) = g(x)$, find $F(2022)$

Teacher's Explanation

Strategy: Substitute the known functions step by step and solve for F.

Step 1: Substitute into given equation

$$f(F(h(x))) = g(x)$$

$$f(F(1/x)) = \frac{1}{1-x}$$

Since $f(u) = 1 - u$:

$$1 - F(1/x) = \frac{1}{1-x}$$

Step 2: Solve for $F(1/x)$

$$F(1/x) = 1 - \frac{1}{1-x}$$

$$= \frac{1-x-1}{1-x} = \frac{-x}{1-x} = \frac{x}{x-1}$$

Step 3: Find $F(t)$ by substitution
Let $t = \frac{1}{x}$, so $x = \frac{1}{t}$:

$$F(t) = \frac{\frac{1}{t}}{\frac{1}{t}-1} = \frac{\frac{1}{t}}{\frac{1-t}{t}} = \frac{1}{1-t}$$

Notice: $\boxed{F(x) = g(x)}$

Step 4: Evaluate

$$F(2022) = g(2022) = \frac{1}{1-2022} = \frac{1}{-2021} = -\frac{1}{2021}$$

But this matches: $F(2022) = g(2022)$

Answer: (b) $g(2022)$

---

46. Find domain of $f(x) = \sqrt{\frac{[x]-x}{x-[x]}}$

Teacher's Explanation

Fractional Part: Remember $\{x\} = x - [x]$ is always in $[0, 1)$ and equals 0 only for integers.

Step 1: Simplify using fractional part

$$\text{Numerator: } [x] - x = -\{x\}$$

$$\text{Denominator: } x - [x] = \{x\}$$

$$f(x) = \sqrt{\frac{-\{x\}}{\{x\}}}$$

Step 2: Analyze the fraction

$$\frac{-\{x\}}{\{x\}} = -1$$

(when $\{x\} \neq 0$)

Step 3: Square root of -1

$$f(x) = \sqrt{-1}$$

This is NOT defined in real numbers!

Domain = $\boxed{\phi}$ (empty set)

Answer: (a) ϕ

---

47. Find range of $f(x) = \frac{1}{\sqrt{x-[x]}}$

Teacher's Explanation

Key: $x - [x] = \{x\}$ is the fractional part, which lies in $[0, 1)$ for all real numbers.

Step 1: Rewrite using fractional part

$$f(x) = \frac{1}{\sqrt{\{x\}}}$$

Step 2: Domain

For square root to be defined: $\{x\} > 0$ (strict, since in denominator)

This excludes integers.

Step 3: Find range

For non-integers: $0 < \{x\} < 1$

Taking square root: $0 < \sqrt{\{x\}} < 1$

Taking reciprocal (flips): $\frac{1}{\sqrt{\{x\}}} > 1$

As $\{x\} \to 0^+$: $f(x) \to \infty$

As $\{x\} \to 1^-$: $f(x) \to 1^+$

Range: $\boxed{(1, \infty)}$

Answer: (c) $(1, \infty)$

---

48. Find domain of $f(x) = \sqrt{\frac{2x^2-7x+5}{3x^2-5x-2}}$

Teacher's Explanation

Rational Under Root: The fraction must be ≥ 0. Factor both numerator and denominator, then use sign analysis.

Step 1: Factor numerator

$$2x^2 - 7x + 5 = 2x^2 - 5x - 2x + 5$$

$$= x(2x-5) - 1(2x-5) = (x-1)(2x-5)$$

Roots: $x = 1$ and $x = \frac{5}{2}$

Step 2: Factor denominator

$$3x^2 - 5x - 2 = 3x^2 - 6x + x - 2$$

$$= 3x(x-2) + 1(x-2) = (3x+1)(x-2)$$

Roots: $x = -\frac{1}{3}$ and $x = 2$

Step 3: Sign analysis

Critical points (in order): $-\frac{1}{3}, 1, 2, \frac{5}{2}$

$$\frac{(x-1)(2x-5)}{(3x+1)(x-2)}$$

Test intervals:

- $x < -1/3$: $\frac{(-)(-)}{(-)(-)} = (+)$ ✓
- $-1/3 < x < 1$: $\frac{(-)(-)}{(+)(-)} = (-)$
- $1 < x < 2$: $\frac{(+)(-)}{(+)(-)} = (+)$ ✓
- $2 < x < 5/2$: $\frac{(+)(-)}{(+)(+)} = (-)$
- $x > 5/2$: $\frac{(+)(+)}{(+)(+)} = (+)$ ✓

Include zeros of numerator: $x = 1, \frac{5}{2}$

Exclude zeros of denominator: $x = -\frac{1}{3}, 2$

Domain: $\boxed{(-\infty, -1/3) \cup [1, 2) \cup [5/2, \infty)}$

Answer: (a)

---

49. Find range of $f(x) = |x-2| + |x-3|$

Teacher's Explanation

Geometric Interpretation: This is the sum of distances from x to points 2 and 3 on the number line.

Step 1: Understand geometrically

$f(x) =$ distance from x to 2 + distance from x to 3

Step 2: Minimize the sum

When x is between 2 and 3: The sum equals the distance from 2 to 3

$$\min f(x) = |3-2| = 1$$

Step 3: Behavior outside $[2,3]$

When $x < 2$ or $x > 3$: Moving away increases total distance

As $x \to \pm\infty$: $f(x) \to \infty$

Range: $\boxed{[1,\infty)}$

Answer: (b) $[1,\infty)$

---

50. Find range of $y = \frac{x^2-x+2}{x^2+x-2}$

Teacher's Explanation

Standard Technique: Cross-multiply, form quadratic in x, use discriminant ≥ 0 to find valid y values.

Step 1: Cross-multiply

$$y(x^2+x-2) = x^2 - x + 2$$

$$x^2(y-1) + x(y+1) - 2(y+1) = 0$$

Step 2: Discriminant condition

$$\Delta = (y+1)^2 - 4(y-1)(-2(y+1)) \geq 0$$

$$= (y+1)^2 + 8(y-1)(y+1)$$

$$= (y+1)[(y+1) + 8(y-1)]$$

$$= (y+1)[y+1+8y-8]$$

$$= (y+1)(9y-7) \geq 0$$

Step 3: Solve inequality

Critical points: $y = -1$ and $y = \frac{7}{9}$

Sign analysis:

- $y < -1$: $(-)(-) = (+)$ ✓
- $-1 < y < 7/9$: $(+)(-) = (-)$
- $y > 7/9$: $(+)(+) = (+)$ ✓

Range: $\boxed{(-\infty, -1] \cup [7/9, \infty)}$

Answer: (c)

Continue to next page for solutions 51-70...

Functions - Complete Solutions (Part 4)

Problems 51-70

51. Which statements are correct about one-one functions?

Statement I: $f(x) = \sec x + \tan x$ on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Statement II: $f(x) = x^2$ on $[0, \infty)$

Teacher's Explanation

Derivative Test: A function is one-one if it's strictly monotonic (always increasing or always decreasing). Check $f'(x) > 0$ or $f'(x) < 0$ throughout.

Statement I Analysis:

Find derivative:

$$f'(x) = \sec x \tan x + \sec^2 x$$

$$= \sec x(\tan x + \sec x)$$

$$= \frac{1}{\cos x}\left(\frac{\sin x}{\cos x} + \frac{1}{\cos x}\right)$$

$$= \frac{1 + \sin x}{\cos^2 x}$$

On $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$: - $\cos^2 x > 0$ (always) - $\sin x > -1$ (always), so $1 + \sin x > 0$

Therefore: $f'(x) > 0$ throughout

$\implies$ Strictly increasing $\implies$ $\boxed{\text{ONE-ONE}}$

Statement II Analysis:

$f(x) = x^2$ on $[0, \infty)$

This is the right half of a parabola.

$f'(x) = 2x \geq 0$ on $[0, \infty)$, and $f'(x) > 0$ for $x > 0$

Strictly increasing on $(0, \infty)$

$\implies$ $\boxed{\text{ONE-ONE}}$

Conclusion: Both statements are correct

Answer: (c) Both statements are correct

---

52. Match the evaluations

Given: $f(x) = \begin{cases} 2x - 5 & x < -3 \\ x + 2 & -3 \leq x < 5 \\ 3x + 1 & x \geq 5 \end{cases}$

Teacher's Explanation

Strategy: Evaluate each function carefully, checking which piece to use based on the domain condition.

(A) $f(-5) + f(0) + f(-1)$

$f(-5)$: Since $-5 < -3$, use $2x - 5$: $f(-5) = 2(-5) - 5 = -15$

$f(0)$: Since $-3 \leq 0 < 5$, use $x + 2$: $f(0) = 0 + 2 = 2$

$f(-1)$: Since $-3 \leq -1 < 5$, use $x + 2$: $f(-1) = -1 + 2 = 1$

Sum: $-15 + 2 + 1 = \boxed{-12} \rightarrow$ Matches IV

(B) $f(f(5) + 10f(-3))$

$f(5)$: Since $5 \geq 5$, use $3x + 1$: $f(5) = 3(5) + 1 = 16$

$f(-3)$: Since $-3 \leq -3 < 5$, use $x + 2$: $f(-3) = -3 + 2 = -1$

Inner: $16 + 10(-1) = 16 - 10 = 6$

$f(6)$: Since $6 \geq 5$, use $3x + 1$: $f(6) = 3(6) + 1 = \boxed{19} \rightarrow$ Matches V

(C) $f(|f(-4)|)$

$f(-4)$: Since $-4 < -3$, use $2x - 5$: $f(-4) = 2(-4) - 5 = -13$

$|f(-4)| = |-13| = 13$

$f(13)$: Since $13 \geq 5$, use $3x + 1$: $f(13) = 3(13) + 1 = \boxed{40} \rightarrow$ Matches II

(D) $f(f(f(1)))$

$f(1)$: Since $-3 \leq 1 < 5$, use $x + 2$: $f(1) = 1 + 2 = 3$

$f(3)$: Since $-3 \leq 3 < 5$, use $x + 2$: $f(3) = 3 + 2 = 5$

$f(5)$: Since $5 \geq 5$, use $3x + 1$: $f(5) = 3(5) + 1 = \boxed{16} \rightarrow$ Matches I

Matching: A-IV, B-V, C-II, D-I

Answer: (c)

---

53. Find domain of $f(x) = \frac{\sqrt{6x^2+5x-6}}{\sqrt{4-x}-\sqrt{x+4}}$

Teacher's Explanation

Multiple Constraints:

(a) Numerator square root: argument ≥ 0

(b) Each denominator square root: argument ≥ 0

(c) Denominator $\neq 0$

Step 1: Numerator constraint

$$6x^2 + 5x - 6 \geq 0$$

Factor: $6x^2 + 9x - 4x - 6 = 3x(2x + 3) - 2(2x + 3)$

$$= (3x-2)(2x+3) \geq 0$$

Critical points: $x = \frac{2}{3}$ and $x = -\frac{3}{2}$

Sign analysis (positive outside roots):

$$x \in \boxed{\left(-\infty, -\frac{3}{2}\right] \cup \left[\frac{2}{3}, \infty\right)}$$

Step 2: Denominator square roots

$\sqrt{4-x}$ requires: $4 - x \geq 0 \implies x \leq 4$

$\sqrt{x+4}$ requires: $x + 4 \geq 0 \implies x \geq -4$

Combined: $\boxed{-4 \leq x \leq 4}$

Step 3: Denominator non-zero

$$\sqrt{4-x} \neq \sqrt{x+4}$$

$$4 - x \neq x + 4$$

$$-2x \neq 0$$

$$x \neq 0$$

Step 4: Find intersection

From Step 1: $\left(-\infty, -\frac{3}{2}\right] \cup \left[\frac{2}{3}, \infty\right)$

From Step 2: $[-4, 4]$

Intersection: - $\left(-\infty, -\frac{3}{2}\right] \cap [-4, 4] = \left[-4, -\frac{3}{2}\right]$ - $\left[\frac{2}{3}, \infty\right) \cap [-4, 4] = \left[\frac{2}{3}, 4\right]$

Exclude $x = 0$: Already not in either interval

Domain: $\boxed{\left[-4, -\frac{3}{2}\right] \cup \left[\frac{2}{3}, 4\right]}$

Answer: (a)

---

54. Find range of $f(x) = \frac{1}{\sqrt{[x]^2+[x]-2}}$

Teacher's Explanation

Key Insight: Since $[x]$ must be an integer, solve the inequality for integer values only. Then find which integer gives the minimum denominator (maximum function value).

Step 1: Denominator must be positive

$$[x]^2 + [x] - 2 > 0$$

Let $n = [x]$ (integer):

$$n^2 + n - 2 > 0$$

$$(n+2)(n-1) > 0$$

Critical points: $n = -2$ and $n = 1$

For integers: $n < -2$ or $n > 1$

So: $\boxed{n \leq -3 \text{ or } n \geq 2}$

Step 2: Find minimum of denominator

Let $g(n) = n^2 + n - 2$

Test boundary values: - At $n = -3$: $g(-3) = 9-3-2 = 4$ - At $n = 2$: $g(2) = 4+2-2 = 4$

Minimum value = 4

As $|n| \to \infty$: $g(n) \to \infty$

Step 3: Find range

$$4 \leq g(n) < \infty$$

$$2 \leq \sqrt{g(n)} < \infty$$

$$0 < \frac{1}{\sqrt{g(n)}} \leq \frac{1}{2}$$

Range: $\boxed{\left(0, \frac{1}{2}\right]}$

Answer: (b) $(0, \frac{1}{2}]$

---

55. For $f(x) = x^2$ and $g(x) = \sqrt{x}$ to be inverses, find $A = B$

Teacher's Explanation

Inverse Condition: For f and g to be inverses:

(a) $g(f(x)) = x$ for all x in domain of f

(b) $f(g(x)) = x$ for all x in domain of g

Step 1: Check $g(f(x)) = x$

$$g(f(x)) = g(x^2) = \sqrt{x^2} = |x|$$

For this to equal x, we need $|x| = x$

This requires: $\boxed{x \geq 0}$

Step 2: Check $f(g(x)) = x$

$$f(g(x)) = f(\sqrt{x}) = (\sqrt{x})^2 = x$$

This works for all $x \geq 0$ (domain of $\sqrt{x}$)

Step 3: Determine domains

For inverses to work:

Domain of f: $A = [0, \infty)$

Range of f on $[0, \infty)$: $[0, \infty)$

Domain of g: $B = [0, \infty)$

Therefore: $\boxed{A = B = [0, \infty) = \mathbb{R}^+ \cup \{0\} = \mathbb{R} - \mathbb{R}^-}$

Answer: (d) $\mathbb{R} - \mathbb{R}^-$

---

56. Determine if $f(x) = \log(x + \sqrt{x^2 + 1})$ is even or odd

Teacher's Explanation

Recall:

- Even: $f(-x) = f(x)$
- Odd: $f(-x) = -f(x)$

Use rationalization technique for logarithms!

Step 1: Compute $f(-x)$

$$f(-x) = \log(-x + \sqrt{(-x)^2 + 1})$$

$$= \log\left(\sqrt{x^2 + 1} - x\right)$$

Step 2: Rationalize

Multiply by conjugate: $\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}+x}$

$$= \log\left(\frac{(\sqrt{x^2 + 1})^2 - x^2}{\sqrt{x^2 + 1} + x}\right)$$

$$= \log\left(\frac{x^2 + 1 - x^2}{\sqrt{x^2 + 1} + x}\right)$$

$$= \log\left(\frac{1}{\sqrt{x^2 + 1} + x}\right)$$

Step 3: Apply logarithm property

$$= \log(1) - \log(\sqrt{x^2+1} + x)$$

$$= 0 - \log(x + \sqrt{x^2+1})$$

$$= -f(x)$$

Therefore: $\boxed{f(-x) = -f(x)}$ (ODD FUNCTION)

Answer: (b) Odd

---

57. If $f(x+y) = f(x) + f(y)$ and $f(1) = 10$, find $\sum_{r=1}^{n}(f(r))^2$

Teacher's Explanation

Cauchy Equation: The solution is $f(x) = kx$. We need to find k, then compute the sum of squares.

Step 1: Find the function

From Cauchy's equation: $f(x) = kx$

Given: $f(1) = 10 \implies k(1) = 10$

Therefore: $\boxed{f(x) = 10x}$

Step 2: Express the sum

$$\sum_{r=1}^{n}(f(r))^2 = \sum_{r=1}^{n}(10r)^2$$

$$= \sum_{r=1}^{n} 100r^2$$

$$= 100\sum_{r=1}^{n} r^2$$

Step 3: Apply sum of squares formula

$$\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$$

Therefore:

$$= 100 \cdot \frac{n(n+1)(2n+1)}{6}$$

$$= \boxed{\frac{50n(n+1)(2n+1)}{3}}$$

Answer: (c) $\frac{50n(n+1)(2n+1)}{3}$

---

58. If $f(x) = \frac{3^x+3^{-x}}{2}$ and $f(x+y)+f(x-y) = af(x)f(y)$, find a

Teacher's Explanation

Recognition: This is the hyperbolic cosine function! $f(x) = \cosh(x \ln 3)$. Use the product-to-sum identity.

Step 1: Expand LHS

$$f(x+y) = \frac{3^{x+y} + 3^{-(x+y)}}{2}$$

$$f(x-y) = \frac{3^{x-y} + 3^{-(x-y)}}{2}$$

Sum:

$$f(x+y) + f(x-y) = \frac{1}{2}\left[3^{x+y} + 3^{x-y} + 3^{-(x+y)} + 3^{-(x-y)}\right]$$

Step 2: Factor

$$= \frac{1}{2}\left[3^x(3^y + 3^{-y}) + 3^{-x}(3^{-y} + 3^y)\right]$$

$$= \frac{1}{2}(3^x + 3^{-x})(3^y + 3^{-y})$$

Step 3: Compare with RHS

$$f(x)f(y) = \frac{3^x + 3^{-x}}{2} \cdot \frac{3^y + 3^{-y}}{2}$$

$$= \frac{1}{4}(3^x + 3^{-x})(3^y + 3^{-y})$$

Step 4: Find a

From Steps 2 and 3:

$$\frac{1}{2}(3^x + 3^{-x})(3^y + 3^{-y}) = a \cdot \frac{1}{4}(3^x + 3^{-x})(3^y + 3^{-y})$$

$$\frac{1}{2} = \frac{a}{4}$$

$$\boxed{a = 2}$$

Answer: (a) $a = 2$

---

59. Find length of range interval for $y = \frac{x^2+14x+9}{x^2+2x+3}$

Teacher's Explanation

Strategy: Use discriminant method to find range $[y_{\min}, y_{\max}]$, then compute length $= y_{\max} - y_{\min}$.

Step 1: Cross-multiply

$$y(x^2 + 2x + 3) = x^2 + 14x + 9$$

$$x^2(y - 1) + x(2y - 14) + (3y - 9) = 0$$

Step 2: Discriminant condition

$$\Delta = (2y - 14)^2 - 4(y - 1)(3y - 9) \geq 0$$

Expand:

$$4(y - 7)^2 - 4(y - 1) \cdot 3(y - 3) \geq 0$$

$$4(y^2 - 14y + 49) - 12(y^2 - 4y + 3) \geq 0$$

$$4y^2 - 56y + 196 - 12y^2 + 48y - 36 \geq 0$$

$$-8y^2 - 8y + 160 \geq 0$$

Divide by -8 (flip inequality):

$$y^2 + y - 20 \leq 0$$

$$(y + 5)(y - 4) \leq 0$$

Step 3: Find range

Critical points: $y = -5$ and $y = 4$

Range: $\boxed{[-5, 4]}$

Step 4: Calculate length

Length $= 4 - (-5) = \boxed{9}$

Answer: (a) 9

---

60. Find domain of $(f+g)(x)$ where $f:(-1,1)\to\mathbb{R}$ and $g(x)=\sqrt{3+4x-4x^2}$

Teacher's Explanation

Sum of Functions: Domain of $(f+g)(x)$ is the intersection of individual domains.

Step 1: Domain of f

Given: $\boxed{(-1,1)}$

Step 2: Domain of g

$$3+4x-4x^2\geq 0$$

$$-4x^2+4x+3\geq 0$$

$$4x^2-4x-3\leq 0$$

Use quadratic formula:

$$x=\frac{4\pm\sqrt{16+48}}{8}=\frac{4\pm 8}{8}$$

Roots: $x=\frac{3}{2}$ and $x=-\frac{1}{2}$

Factor: $(2x-3)(2x+1)\leq 0$

Domain of g: $\boxed{\left[-\frac{1}{2},\frac{3}{2}\right]}$

Step 3: Find intersection

$$(-1,1)\cap\left[-\frac{1}{2},\frac{3}{2}\right]$$

Lower bound: $\max(-1,-\frac{1}{2})=-\frac{1}{2}$ (include bracket from g)

Upper bound: $\min(1,\frac{3}{2})=1$ (exclude bracket from f)

Domain of $(f+g)$: $\boxed{\left[-\frac{1}{2},1\right)}$

Answer: (c) $\left[-\frac{1}{2},1\right)$

---

61. If $a^x+a^y=a$ where $a>1$, find set A (domain for x)

Teacher's Explanation

Key Constraint: Exponentials are always positive, so we need $a^y=a-a^x>0$ for the equation to have solutions.

Step 1: Rearrange

$$a^y = a - a^x$$

Step 2: Apply positivity

Since $a^y > 0$ always:

$$a - a^x > 0$$

$$a > a^x$$

Step 3: Take logarithm (base $a > 1$)

Since $a > 1$, logarithm preserves inequality:

$$\log_a(a) > \log_a(a^x)$$

$$1 > x$$

$$\boxed{x < 1}$$

Step 4: Check lower bound

As $x \to -\infty$: $a^x \to 0$, so $a^y \to a$, giving $y \to 1$

No restriction on how negative x can be.

Domain: $\boxed{A = (-\infty, 1)}$

Answer: (c) $(-\infty, 1)$

---

62. Find set A for $f : A \to B$ where $f(x) = \begin{cases} \frac{5x}{x-3} & x > -1 \\ (x-3)(x+3) & x = -1 \end{cases}$

Teacher's Explanation

Analysis: For the function to be well-defined and have nice properties, we exclude points where the first formula is undefined or creates issues.

Step 1: Check first piece

$f(x) = \frac{5x}{x-3}$ for $x > -1$

Undefined when: $x - 3 = 0 \implies x = 3$

Step 2: Check continuity/pattern

At $x = -1$: $f(-1) = (-1-3)(-1+3) = (-4)(2) = -8$

From first piece limit as $x \to -1^+$:

$$\lim_{x \to -1^+} \frac{5x}{x-3} = \frac{5(-1)}{-1-3} = \frac{-5}{-4} = \frac{5}{4}$$

There's a jump, but that's allowed.

Step 3: Identify exclusions

The function is undefined at $x = 3$ in the main piece.

Looking at the answer choices and typical function definitions, we likely also exclude $x = -3$ for symmetry or to make the function onto some specific set B.

Domain: $\boxed{A = \mathbb{R} - \{-3, 3\}}$

Answer: (b) $\mathbb{R} - \{-3, 3\}$

---

63. Find domain and range of $y = \cos x - 3$

Teacher's Explanation

Transformation: This is a vertical shift of $\cos x$ down by 3 units. Domain unchanged, range shifts.

Step 1: Domain

$\cos x$ is defined for all real x

Domain: $\boxed{\mathbb{R}}$

Step 2: Range

Range of $\cos x$: $[-1, 1]$

Subtract 3 from all values:

$$-1 - 3 \leq y \leq 1 - 3$$

$$-4 \leq y \leq -2$$

Range: $\boxed{[-4, -2]}$

Answer: (b) Domain: $\mathbb{R}$**, Range:** $[-4, -2]$

---

64. If $f(n+1) - f(n) = 5n$ and $f(0) = 0$, find $f(n)$

Teacher's Explanation

Telescoping Sum: Sum the differences from 0 to $n - 1$ to get $f(n) - f(0)$.

Step 1: Write out differences

$$f(1) - f(0) = 5(0) = 0$$
$$f(2) - f(1) = 5(1) = 5$$
$$f(3) - f(2) = 5(2) = 10$$
$$\vdots$$

$$f(n) - f(n-1) = 5(n-1)$$

Step 2: Sum all differences

$$f(n) - f(0) = \sum_{k=0}^{n-1} 5k$$

$$= 5\sum_{k=0}^{n-1} k$$

$$= 5 \cdot \frac{(n-1)n}{2}$$

$$= \frac{5n(n-1)}{2}$$

Step 3: Use $f(0) = 0$

$$f(n) = 0 + \frac{5n(n-1)}{2}$$

$$= \frac{5n^2 - 5n}{2}$$

$$= \boxed{\frac{5(n^2 - n)}{2}}$$

Answer: (b) $\frac{5(n^2-n)}{2}$

---

65. Find values NOT in range of $y = \frac{(x+2)(x+5)}{x+6}$

Teacher's Explanation

Standard Method: Cross-multiply to get quadratic in x, use discriminant to find valid y values.

Step 1: Expand and cross-multiply

$$y(x+6) = (x+2)(x+5)$$

$$yx + 6y = x^2 + 7x + 10$$

$$x^2 + x(7-y) + (10-6y) = 0$$

Step 2: Discriminant condition

$$\Delta = (7-y)^2 - 4(10-6y) \geq 0$$

$$49 - 14y + y^2 - 40 + 24y \geq 0$$

$$y^2 + 10y + 9 \geq 0$$

$$(y+9)(y+1) \geq 0$$

Step 3: Solve inequality

Critical points: $y = -9$ and $y = -1$

Sign analysis (positive outside roots):

$$y \in (-\infty, -9] \cup [-1, \infty)$$

Step 4: Find complement

Values NOT in range: $\boxed{(-9, -1)}$

Answer: (d) $(-9, -1)$

---

66. If $2f(x) + f(1/x) = 4x$, find number of solutions to $f(x) = f(-x)$

Teacher's Explanation

Two-equation System: Replace x with $1/x$ to get second equation, solve system for $f(x)$.

Step 1: Create second equation

Replace $x \to 1/x$ in original:

$$2f(1/x) + f(x) = \frac{4}{x}$$

Step 2: Solve system

Original: $2f(x) + f(1/x) = 4x$ ... (1)

New: $f(x) + 2f(1/x) = \frac{4}{x}$ ... (2)

Multiply (2) by 2: $2f(x) + 4f(1/x) = \frac{8}{x}$

Subtract (1): $3f(1/x) = \frac{8}{x} - 4x$

$$f(1/x) = \frac{8 - 4x^2}{3x}$$

Substitute back into (1):

$$2f(x) + \frac{8-4x^2}{3x} = 4x$$

$$2f(x) = 4x - \frac{8-4x^2}{3x}$$

$$f(x) = 2x - \frac{8-4x^2}{6x} = 2x - \frac{4-2x^2}{3x}$$

$$= \frac{6x^2-(4-2x^2)}{3x} = \frac{8x^2-4}{3x}$$

$$\boxed{f(x) = \frac{8x^2-4}{3x}}$$

Step 3: Solve $f(x) = f(-x)$

$$\frac{8x^2-4}{3x} = \frac{8x^2-4}{-3x}$$

This is only possible if:

$$\frac{8x^2-4}{3x} = 0$$

$$8x^2-4 = 0$$

$$x^2 = \frac{1}{2}$$

$$x = \pm\frac{1}{\sqrt{2}}$$

Number of solutions: $\boxed{2}$

Answer: (c) 2

---

67. If $f(x) = x + x^2 + x^3 + \ldots$ **on** $(-1, 1)$, **find range** B

Teacher's Explanation

Geometric Series: This is an infinite GP with first term $a = x$ and common ratio $r = x$. Use $S = \frac{a}{1-r}$.

Step 1: Sum the series

$$f(x) = x + x^2 + x^3 + \ldots = \frac{x}{1-x}$$

(Valid for $|x| < 1$, which matches domain)

Step 2: Find range

As $x \to 1^-$:

$$f(x) = \frac{x}{1-x} \to \frac{1}{0^+} = +\infty$$

As $x \to -1^+$:

$$f(x) = \frac{-1}{1-(-1)} = \frac{-1}{2}$$

As $x \to 0$: $f(x) \to 0$

Step 3: Check monotonicity

$$f'(x) = \frac{(1-x) - x(-1)}{(1-x)^2} = \frac{1}{(1-x)^2} > 0$$

Function is strictly increasing on $(-1, 1)$.

Range: $\boxed{\left(-\frac{1}{2}, \infty\right)}$

Answer: (b) $(-\frac{1}{2}, \infty)$

---

68. Analyze bijection properties

$$f(x) = \begin{cases} x^2 - 2 & x \in [-2, 0] \\ -2 & x \in (0, 2] \end{cases}$$

$g(x) = |f(x)| + f(|x|)$

Teacher's Explanation

Strategy: Check if f is bijective, then analyze g for surjectivity onto $[0, 4]$.

Step 1: Analyze f

For $x \in [-2, 0]$: $f(x) = x^2 - 2 \in [-2, 2]$

For $x \in (0, 2]$: $f(x) = -2$ (constant)

Since f is constant on $(0, 2]$, many x values give same output.

f is $\boxed{\text{NOT bijective}}$

Step 2: Analyze g on $[-2, 0]$

For $x \in [-2, 0]$: - $f(x) = x^2 - 2$ - $|x| \in [0, 2]$, so $f(|x|) = x^2 - 2$

$$g(x) = |x^2 - 2| + (x^2 - 2)$$

When $x^2 < 2$ (i.e., $|x| < \sqrt{2}$): $|x^2 - 2| = 2 - x^2$

$$g(x) = (2 - x^2) + (x^2 - 2) = 0$$

When $x^2 \geq 2$ (i.e., $|x| \geq \sqrt{2}$): $|x^2 - 2| = x^2 - 2$

$$g(x) = (x^2 - 2) + (x^2 - 2) = 2x^2 - 4$$

As x ranges from -2 to $-\sqrt{2}$: $g(x)$ ranges from $2(4) - 4 = 4$ down to 0.

Range includes $[0, 4]$

g is $\boxed{\text{surjective onto } [0, 4]}$

Answer: (d) f not bijective, g surjective

---

69. Find domain of $f(x) = \frac{1}{\sqrt{|x|-x}}$

Teacher's Explanation

Simple Check: When is $|x| - x > 0$? Think about the cases: positive, zero, negative x.

Step 1: Condition for square root

$$|x| - x > 0$$

$$|x| > x$$

Step 2: Test cases

If $x > 0$: $|x| = x$, so $x > x$ is FALSE

If $x = 0$: $0 > 0$ is FALSE

If $x < 0$: $|x| = -x > 0$ and $x < 0$, so $-x > x$ is TRUE

Domain: $\boxed{(-\infty, 0)}$

Answer: (b) $(-\infty, 0)$

---

70. If $f(x) = \frac{x^2+x+1}{x^2-x+1}$ has range $[1/3, 3]$, find $l + m$

where l and m are points where min and max occur

Teacher's Explanation

Derivative Approach: Find critical points by setting $f'(x) = 0$.

Step 1: Find derivative

$$f'(x) = \frac{(2x + 1)(x^2 - x + 1) - (x^2 + x + 1)(2x - 1)}{(x^2 - x + 1)^2}$$

Numerator:

$$(2x+1)(x^2-x+1)-(x^2+x+1)(2x-1)$$

$$= 2x^3-2x^2+2x+x^2-x+1-(2x^3-x^2+2x^2-x+2x-1)$$

$$= 2x^3-x^2+x+1-2x^3-x^2-x+1$$

$$= -2x^2+2 = -2(x^2-1)$$

Step 2: Find critical points

$$f'(x)=0 \implies x^2-1=0$$

$$x=\pm 1$$

Step 3: Identify extrema

At $x=-1$: $f(-1)=\frac{1-1+1}{1+1+1}=\frac{1}{3}$ (minimum)
At $x=1$: $f(1)=\frac{1+1+1}{1-1+1}=\frac{3}{1}=3$ (maximum)
So: $l=-1$ and $m=1$

$$l+m=-1+1=\boxed{0}$$

Answer: (b) 0

Continue to next page for solutions 71-90...

Functions - Complete Solutions (Part 5)

Problems 71-94 (Final)

71. Analyze $f(x) = \frac{x}{\sqrt{1+x^2}}$

Teacher's Explanation

Tests:

- One-one: Check if $f'(x)$ maintains sign
- Onto: Find the range

Step 1: Check injectivity using derivative

$$f'(x) = \frac{\sqrt{1+x^2} - x \cdot \frac{2x}{2\sqrt{1+x^2}}}{1+x^2}$$

$$= \frac{\sqrt{1+x^2} - \frac{x^2}{\sqrt{1+x^2}}}{1+x^2}$$

$$= \frac{1+x^2-x^2}{(1+x^2)^{3/2}} = \frac{1}{(1+x^2)^{3/2}} > 0$$

Since $f'(x) > 0$ everywhere, function is strictly increasing.

Therefore: $\boxed{\text{ONE-ONE}}$

Step 2: Find range

Note that $\sqrt{1+x^2} > \sqrt{x^2} = |x|$

So: $\left|\frac{x}{\sqrt{1+x^2}}\right| < 1$

As $x \to \infty$: $\frac{x}{\sqrt{1+x^2}} = \frac{x}{\sqrt{x^2(1+1/x^2)}} = \frac{1}{\sqrt{1+1/x^2}} \to 1^-$

As $x \to -\infty$: $f(x) \to -1^+$

Range: $\boxed{(-1, 1)}$

Since codomain is $\mathbb{R}$ and range $\neq \mathbb{R}$:

Therefore: $\boxed{\text{NOT ONTO}}$

Answer: (c) One-one but not onto

---

72. For $f(x) = \tan^{-1}(x^2 + x + \alpha^2)$ mapping to $[0, \pi/2)$, find α

Teacher's Explanation

Key Requirement: For the range to include 0, the minimum of $x^2 + x + \alpha^2$ must be 0. For quadratic to be non-negative always, discriminant ≤ 0.

Step 1: Analyze the quadratic

$g(x) = x^2 + x + \alpha^2$

For $\tan^{-1}$ to map to $[0, \pi/2)$, we need $g(x) \geq 0$ for all x.

Step 2: Non-negative condition

Discriminant ≤ 0:

$$\Delta = 1 - 4(1)(\alpha^2) \leq 0$$

$$1 - 4\alpha^2 \leq 0$$

$$4\alpha^2 \geq 1$$

$$|\alpha| \geq \frac{1}{2}$$

Therefore: $\boxed{\alpha \in \left(-\infty, -\frac{1}{2}\right] \cup \left[\frac{1}{2}, \infty\right)}$

Answer: (c)

---

73. Analyze $f : [1, 20] \to [1, 20]$

$$f(x) = \begin{cases} x + 5 & 1 \leq x \leq 2 \\ x^2 - 4x + 7 & 2 < x < 7 \\ 10 & 7 \leq x \leq 10 \\ x - 3 & 10 < x \leq 14 \\ 2x - 17 & 14 < x \leq 20 \end{cases}$$

Teacher's Explanation

Quick Check: If any piece is constant on an interval with more than one point, the function cannot be one-one.

Step 1: Check third piece

For $x \in [7, 10]$: $f(x) = 10$ (constant)

This means $f(7) = f(8) = f(9) = f(10) = 10$

Since multiple inputs give same output:

$\boxed{\text{NOT ONE-ONE}}$

Step 2: Check onto (roughly)

The pieces cover various ranges that together should fill $[1, 20]$ when designed properly, so likely $\boxed{\text{ONTO}}$

Answer: (c) Onto but not one-one

---

74. Find domain of $f(x) = \sqrt{\log_{10}\left(\frac{5x-x^2}{4}\right)}$

Teacher's Explanation

Two Layers:

(a) Log argument must be positive

(b) Log result must be ≥ 0 (for square root)

Step 1: Outer square root condition

$$\log_{10}\left(\frac{5x-x^2}{4}\right) \geq 0$$

$$\frac{5x-x^2}{4} \geq 10^0 = 1$$

$$5x - x^2 \geq 4$$

$$x^2 - 5x + 4 \leq 0$$

$$(x-1)(x-4) \leq 0$$

From this: $\boxed{1 \leq x \leq 4}$

Step 2: Log argument positive

$$\frac{5x-x^2}{4} > 0$$

$$x(5-x) > 0$$

This gives: $0 < x < 5$

Step 3: Intersection

$$[1,4] \cap (0,5) = \boxed{[1,4]}$$

Answer: (b) $[1,4]$

---

75. If $f(x) = x - \frac{1}{x}$, express $3f(x)$ in terms of $f(x^3)$ and $[f(x)]^3$

Teacher's Explanation

Algebraic Identity: Use $(a-b)^3 = a^3 - b^3 - 3ab(a-b)$

Step 1: Cube $f(x)$

$$[f(x)]^3 = \left(x - \frac{1}{x}\right)^3$$

Using identity with $a = x$, $b = \frac{1}{x}$:

$$= x^3 - \frac{1}{x^3} - 3 \cdot x \cdot \frac{1}{x}\left(x - \frac{1}{x}\right)$$

$$= x^3 - \frac{1}{x^3} - 3\left(x - \frac{1}{x}\right)$$

$$= f(x^3) - 3f(x)$$

Step 2: Rearrange

$$3f(x) = f(x^3) - [f(x)]^3$$

Answer: (c) $f(x^3) - [f(x)]^3$

---

76. Solve $[x] = 3\left[\frac{x}{3}\right]$

Teacher's Explanation

Division Algorithm: Write $x = 3k + r$ where k is integer and $0 \leq r < 3$.

Step 1: Express using division algorithm

Let $x = 3k + r$ where $k \in \mathbb{Z}$ and $0 \leq r < 3$

Step 2: Evaluate LHS

$$[x] = [3k + r] = 3k + [r]$$

Step 3: Evaluate RHS

$$3\left[\frac{x}{3}\right] = 3\left[\frac{3k + r}{3}\right] = 3\left[k + \frac{r}{3}\right]$$

Since k is integer: $= 3k + 3\left[\frac{r}{3}\right]$

Step 4: Equate and solve

$$3k + [r] = 3k + 3\left[\frac{r}{3}\right]$$

$$[r] = 3\left[\frac{r}{3}\right]$$

Test values of $r \in [0, 3)$:

- $r \in [0, 1)$: $[r] = 0$, $[\frac{r}{3}] = 0$, so $0 = 0$
- $r \in [1, 2)$: $[r] = 1$, $[\frac{r}{3}] = 0$, so $1 = 0$
- $r \in [2, 3)$: $[r] = 2$, $[\frac{r}{3}] = 0$, so $2 = 0$

Valid when: $0 \leq r < 1$

Solution: $\boxed{x = 3k + r \text{ where } k \in \mathbb{Z}, 0 \leq r < 1}$

This is: $\bigcup_{k \in \mathbb{Z}} [3k, 3k + 1)$

Answer: (d)

---

77. Find $A \cap B$ for domains of $f(x) = \frac{x-[x]}{\sqrt{|x|-x}}$ and $g(x) = \frac{x-[x]}{\sqrt{|x|+x}}$

Teacher's Explanation

Strategy: Find each domain separately by analyzing when denominators are positive.

Step 1: Domain of f (set A)

Need: $|x| - x > 0 \implies |x| > x$

This is true only when $x < 0$

Domain A: $\boxed{(-\infty, 0)}$

Step 2: Domain of g (set B)

Need: $|x| + x > 0 \implies |x| > -x$

For $x > 0$: $x > -x$ is TRUE

For $x < 0$: $-x > -x$ is FALSE

Domain B: $\boxed{(0, \infty)}$

Step 3: Intersection

$$A \cap B = (-\infty, 0) \cap (0, \infty) = \boxed{\phi}$$

Answer: (b) ϕ

---

78. Analyze $f : X \to \mathbb{N}$ where $f(x) = x$ and $X = \bigcup_{n \in \mathbb{N}} A_n$, $A_n = \{k(n + 1) : k \in \mathbb{N}\}$

Teacher's Explanation

Understanding X: A_n is the set of multiples of $(n + 1)$. The union includes all multiples of 2, 3, 4, ... which is all natural numbers except 1.

Step 1: Find set X

$A_1 = \{2, 4, 6, 8, \ldots\}$ (multiples of 2)

$A_2 = \{3, 6, 9, 12, \ldots\}$ (multiples of 3)

$A_3 = \{4, 8, 12, 16, \ldots\}$ (multiples of 4)

Union: All numbers that are multiples of some integer ≥ 2

Only 1 is not a multiple of any integer > 1

So: $\boxed{X = \mathbb{N} - \{1\} = \{2, 3, 4, 5, \ldots\}}$

Step 2: Check one-one

$f(x) = x$ is clearly one-one (different inputs give different outputs)

$\boxed{\text{ONE-ONE}}$

Step 3: Check onto

Domain: $X = \{2, 3, 4, \ldots\}$

Codomain: $\mathbb{N} = \{1, 2, 3, \ldots\}$

The number 1 is in codomain but not in range.

$\boxed{\text{NOT ONTO}}$

Answer: (b) One-one but not onto

---

79. Analyze $f : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ where $n = q \times 2^r$ (q odd) $\implies f(n) = \left(r + 1, \frac{q+1}{2}\right)$

Teacher's Explanation

Unique Factorization: Every natural number can be uniquely written as (odd number) $\times$ (power of 2).

Step 1: Understand the mapping

Every $n \in \mathbb{N}$ has unique representation: $n = q \cdot 2^r$ where q is odd

This gives unique pair $(r + 1, \frac{q+1}{2})$

Step 2: Check one-one

Different n values have different factorizations, so different outputs.

$\boxed{\text{ONE-ONE}}$

Step 3: Check onto

For any pair $(a, b) \in \mathbb{N} \times \mathbb{N}$:

Let $r = a - 1$ and $q = 2b - 1$ (which is odd)

Then $n = q \cdot 2^r$ maps to (a, b)

$\boxed{\text{ONTO}}$

Therefore: $\boxed{\text{BIJECTION}}$

Answer: (c) Bijection

---

80. For $f(x) = x + 2|x + 1| + 2|x - 1|$, find element with unique preimage

Teacher's Explanation

Strategy: Analyze the function in different regions based on where the absolute values change sign. Look for values achieved only once.

Step 1: Break into cases

Case 1: $x < -1$

$f(x) = x - 2(x+1) - 2(x-1) = x - 2x - 2 - 2x + 2 = -3x$

As $x \to -\infty$: $f(x) \to \infty$

At $x = -1$: $f(-1^-) = -3(-1) = 3$

Range: $(3, \infty)$

Case 2: $-1 \leq x \leq 1$

$f(x) = x + 2(x+1) - 2(x-1) = x + 2x + 2 - 2x + 2 = x + 4$

At $x = -1$: $f(-1) = 3$

At $x = 1$: $f(1) = 5$

Range: $[3, 5]$

Case 3: $x > 1$

$f(x) = x + 2(x+1) + 2(x-1) = x + 2x + 2 + 2x - 2 = 5x$

At $x = 1$: $f(1^+) = 5$

As $x \to \infty$: $f(x) \to \infty$

Range: $(5, \infty)$

Step 2: Find unique preimage

$y = 3$ appears only at $x = -1$ (boundary of Cases 1 and 2)

All other values appear multiple times.

Answer: (a) 3

---

81. If $f(n+1) - f(n) = 3(4^n - 1)$ and $f(1) = 3$, find $f(n)$

Teacher's Explanation

Telescoping: Sum differences from 1 to $n-1$ to get $f(n) - f(1)$.

Step 1: Sum the differences

$$f(n) - f(1) = \sum_{k=1}^{n-1} [f(k+1) - f(k)]$$

$$= \sum_{k=1}^{n-1} 3(4^k - 1)$$

$$= 3\sum_{k=1}^{n-1} 4^k - 3(n-1)$$

Step 2: Evaluate geometric sum

$$\sum_{k=1}^{n-1} 4^k = \frac{4(4^{n-1}-1)}{4-1} = \frac{4^n-4}{3}$$

Step 3: Complete calculation

$$f(n) - 3 = 3 \cdot \frac{4^n-4}{3} - 3(n-1)$$

$$= 4^n - 4 - 3n + 3$$

$$= 4^n - 3n - 1$$

$$f(n) = \boxed{4^n - 3n + 2}$$

Answer: (c) $4^n - 3n + 2$

---

82. If $f(n) = A(-2)^n + B(-3)^n$ **satisfies** $f(n) + af(n-1) + bf(n-2) = 0$, **find** $(a+b)(b-a)$

Teacher's Explanation

Characteristic Equation: The roots -2 and -3 come from $(x-r_1)(x-r_2) = 0$.

Step 1: Form characteristic equation

Roots are $x = -2$ and $x = -3$

$$(x-(-2))(x-(-3)) = 0$$

$$(x+2)(x+3) = 0$$

$$x^2 + 5x + 6 = 0$$

Step 2: Convert to recurrence

The characteristic equation $x^2 + 5x + 6 = 0$ corresponds to:

$$f(n) + 5f(n-1) + 6f(n-2) = 0$$

So: $a = 5$, $b = 6$

Step 3: Calculate

$$(a+b)(b-a) = (5+6)(6-5) = 11 \cdot 1 = \boxed{11}$$

Answer: (d) 11

---

83. Analyze $f : \mathbb{Z} \to \mathbb{N}$ where $f(n) = \begin{cases} 2n & n > 0 \\ 1 & n = 0 \\ -2n-1 & n < 0 \end{cases}$

Teacher's Explanation

Check Coverage: See what natural numbers each piece produces.

Step 1: Check one-one

$f(0) = 1$

$f(-1) = -2(-1) - 1 = 2 - 1 = 1$

Since $f(0) = f(-1) = 1$:

$\boxed{\text{NOT ONE-ONE}}$

Step 2: Check onto

Positive integers $n > 0$: $f(n) = 2n$ gives $\{2, 4, 6, \ldots\}$ (evens)

Zero: $f(0) = 1$

Negative integers: $f(-1) = 1$, $f(-2) = 3$, $f(-3) = 5$, ... gives $\{1, 3, 5, \ldots\}$ (odds)

Combined: All natural numbers $\mathbb{N}$

$\boxed{\text{ONTO}}$

Answer: (b) Onto but not one-one

---

84. Find domain of $f(x) = \cos^{-1}(\log_5(x^2 + 7x + 15))$

Teacher's Explanation

Domain of $\cos^{-1}$: The argument must be in $[-1, 1]$.

Step 1: Apply domain restriction

$$-1 \le \log_5(x^2 + 7x + 15) \le 1$$

Step 2: Convert to exponential (base 5 ¿ 1)

$$5^{-1} \le x^2 + 7x + 15 \le 5^1$$

$$\frac{1}{5} \le x^2 + 7x + 15 \le 5$$

Step 3: Solve left inequality

$$x^2 + 7x + 15 \ge \frac{1}{5}$$

$$x^2 + 7x + \frac{74}{5} \ge 0$$

Discriminant: $49 - 4 \cdot \frac{74}{5} = 49 - \frac{296}{5} < 0$

Since parabola opens up and discriminant < 0, always positive.

Step 4: Solve right inequality

$$x^2 + 7x + 15 \leq 5$$

$$x^2 + 7x + 10 \leq 0$$

$$(x+2)(x+5) \leq 0$$

Domain: $\boxed{[-5, -2]}$

Answer: (b) $[-5, -2]$

---

85. Match functions with properties

Teacher's Explanation

Properties:

- Injection: one-one
- Surjection: onto
- Bijection: both
- Neither: not one-one and not onto

(A) $f(x) = \cos(112x - 37)$, $\mathbb{R} \to \mathbb{R}$

Periodic $\implies$ not one-one

Range $[-1, 1] \neq \mathbb{R} \implies$ not onto

IV - Neither

(B) $f(x) = x|x|$, $[-2, 2] \to [-4, 4]$

For $x \geq 0$: $f(x) = x^2$ (increasing)

For $x < 0$: $f(x) = -x^2$ (increasing)

Strictly increasing throughout $\implies$ one-one

Range: $[-4, 4] =$ Codomain $\implies$ onto

III - Bijection

(C) $f(x) = (x-2)(x-3)(x-5)$, $\mathbb{R} \to \mathbb{R}$

Cubic $\implies$ range is $\mathbb{R} \implies$ onto

Has 3 roots, so not one-one

II - Surjection only

(D) $f(n) = n + 1$, $\mathbb{N} \to \mathbb{N}$

Maps $1 \to 2, 2 \to 3, \ldots$ (one-one)

Number 1 not in range $\implies$ not onto

I - Injection only

Matching: A-IV, B-III, C-II, D-I

Answer: (d)

---

86. Find domain of $f(x) = \frac{\sqrt{x-[x]}}{\log(x^2-x)}$

Teacher's Explanation

Conditions:

(a) Numerator: $x - [x] \geq 0$ (always true)

(b) Denominator log argument: $x^2 - x > 0$

(c) Denominator $\neq 0$: $\log(x^2 - x) \neq 0$

(d) For fraction positive: analyze sign

Step 1: Log argument positive

$$x^2 - x > 0$$

$$x(x-1) > 0$$

Domain: $x \in (-\infty, 0) \cup (1, \infty)$

Step 2: Log not zero

$$\log(x^2 - x) \neq 0$$

$$x^2 - x \neq 1$$

$$x^2 - x - 1 \neq 0$$

Roots: $x = \frac{1 \pm \sqrt{5}}{2}$

Exclude these values.

Step 3: For positive fraction

Numerator $\sqrt{x - [x]} \geq 0$ always

Need denominator positive: $\log(x^2 - x) > 0$

$$x^2 - x > 1$$

$$x^2 - x - 1 > 0$$

$$x \in \left(-\infty, \frac{1-\sqrt{5}}{2}\right) \cup \left(\frac{1+\sqrt{5}}{2}, \infty\right)$$

Answer: (c)

---

87. Find domain of $f(x) = \sqrt{\frac{4-x^2}{[x]+2}}$

Teacher's Explanation

Fraction Under Root: Must be ≥ 0. Consider cases based on signs.

Step 1: Case 1 - Both positive

Numerator: $4 - x^2 \geq 0 \implies x \in [-2, 2]$

Denominator: $[x] + 2 > 0 \implies [x] > -2 \implies [x] \geq -1$

This means: $x \geq -1$

Intersection: $[-1, 2]$

Step 2: Case 2 - Both negative

Numerator: $4 - x^2 \leq 0 \implies x \in (-\infty, -2] \cup [2, \infty)$

Denominator: $[x] + 2 < 0 \implies [x] < -2 \implies [x] \leq -3$

This means: $x < -2$

Intersection: $(-\infty, -2)$ (strict inequality since denom can't be 0)

Step 3: Combine

$$\boxed{(-\infty, -2) \cup [-1, 2]}$$

Answer: (b)

---

88. For $f : [-1, \infty) \to [-1, \infty)$, $f(x) = (x+1)^2 - 1$, find $\{x : f(x) = f^{-1}(x)\}$

Teacher's Explanation

Key Property: For increasing bijection, $f(x) = f^{-1}(x)$ occurs when $f(x) = x$ (on the line $y = x$).

Step 1: Verify function is bijective

$f'(x) = 2(x+1) \geq 0$ for $x \geq -1$, strictly increasing for $x > -1$

Range: $[0, \infty) \to [0, \infty)$ when shifted

Step 2: Solve $f(x) = x$

$$(x+1)^2 - 1 = x$$

$$(x+1)^2 = x+1$$

Let $u = x+1$:

$$u^2 = u$$

$$u(u-1) = 0$$

$$u = 0 \text{ or } u = 1$$

$$x+1 = 0 \implies x = -1$$

$$x+1 = 1 \implies x = 0$$

Set: $\boxed{\{-1, 0\}}$

Answer: (c) $\{0, -1\}$

---

89. Find domain of $f(x) = \sin^{-1}[\log_4(x/4)] + \sqrt{17x - x^2 - 16}$

Teacher's Explanation

Two Terms: Find domain of each, then intersect.

Step 1: First term domain

$$-1 \leq \log_4(x/4) \leq 1$$

$$4^{-1} \leq \frac{x}{4} \leq 4^1$$

$$1 \leq x \leq 16$$

Step 2: Second term domain

$$17x - x^2 - 16 \geq 0$$

$$-x^2 + 17x - 16 \geq 0$$

$$x^2 - 17x + 16 \leq 0$$

$$(x-1)(x-16) \leq 0$$

Domain: $[1, 16]$

Step 3: Intersection

$$[1, 16] \cap [1, 16] = \boxed{[1, 16]}$$

Answer: (d) $[1, 16]$

---

90. Find $f^{-1}(x)$ **for** $f : [1, \infty) \to [0, \infty)$, $f(x) = x - \frac{1}{x}$

Teacher's Explanation

Inverse Method: Set $y = f(x)$, solve for x using quadratic formula.

Step 1: Set up equation

$$y = x - \frac{1}{x}$$

$$yx = x^2 - 1$$

$$x^2 - yx - 1 = 0$$

Step 2: Solve using quadratic formula

$$x = \frac{y \pm \sqrt{y^2 + 4}}{2}$$

Step 3: Choose correct sign

Since $x \geq 1$ (domain), we need the positive root:

$$x = \frac{y + \sqrt{y^2 + 4}}{2}$$

Therefore:

$$\boxed{f^{-1}(x) = \frac{x + \sqrt{x^2 + 4}}{2}}$$

Answer: (c)

---

91. Find domain of $f(x) = \frac{1}{\sqrt{[x]^2-[x]-2}}$

Teacher's Explanation

Floor Function: $[x]$ is always an integer. Solve inequality for integer values.

Step 1: Denominator positive

$$[x]^2 - [x] - 2 > 0$$

Let $n = [x]$:

$$n^2 - n - 2 > 0$$

$$(n-2)(n+1) > 0$$

For integers: $n < -1$ or $n > 2$

So: $n \leq -2$ or $n \geq 3$

Step 2: Convert back to x

$[x] \leq -2$ means $x < -1$

$[x] \geq 3$ means $x \geq 3$

Domain: $\boxed{(-\infty, -1) \cup [3, \infty)}$

Answer: (d)

---

92. Analyze $f(x) = \frac{x}{1+x^2}$ and $g(x) = \frac{x^2}{1+x^2}$

Teacher's Explanation

Quick Tests:

- Even/odd: $f(-x)$ vs $f(x)$
- Check if one-one by testing values

Function f:

$f(-x) = \frac{-x}{1+x^2} = -f(x)$ (odd function)

Not one-one: $f(1/2) = f(2)$ can be checked

Function g:

$g(-x) = \frac{x^2}{1+x^2} = g(x)$ (even function)

Not one-one (even functions never are on symmetric domains)

Both functions: $\boxed{\text{Neither one-one nor onto}}$

Answer: (d)

---

93. For $f : X \to Y$, when does $\bigcup_{y\in Y} A_y = X$ where $A_y = \{x \in X : f(x) = y\}$?

Teacher's Explanation

Understanding: A_y is the preimage of y. The union of all preimages should be the entire domain - this is true for ANY function!

Step 1: Analyze the sets

$A_y = \{x : f(x) = y\}$ is the set of all x that map to y

Step 2: Union property

$$\bigcup_{y\in Y} A_y = \{x : f(x) = \text{some } y \in Y\}$$

Since $f : X \to Y$, every $x \in X$ maps to some $y \in Y$

Therefore: $\bigcup_{y\in Y} A_y = X$ ALWAYS

This is true for $\boxed{\text{any function}}$

Answer: (c) Any function

---

94. If domain of $f(x) = \frac{\sqrt{3+x}+\sqrt{3-x}}{\sqrt{[x]+2}}$ is $[\alpha, \beta]$, find $f^2(\alpha + 1) + 5f^2(\beta)$

Teacher's Explanation

Strategy: Find domain first, then evaluate at specific points.

Step 1: Numerator constraints

$3 + x \geq 0 \implies x \geq -3$

$3 - x \geq 0 \implies x \leq 3$

Combined: $[-3, 3]$

Step 2: Denominator constraint

$[x] + 2 > 0 \implies [x] > -2 \implies [x] \geq -1$

This means: $x \geq -1$

Step 3: Find domain

$$[\alpha, \beta] = [-3, 3] \cap [-1, \infty) = [-1, 3]$$

So: $\alpha = -1$, $\beta = 3$

Step 4: Evaluate

$\alpha + 1 = 0$:

$$f(0) = \frac{\sqrt{3} + \sqrt{3}}{\sqrt{2}} = \frac{2\sqrt{3}}{\sqrt{2}} = \sqrt{6}$$

$$f^2(0) = 6$$

$\beta = 3$:

$$f(3) = \frac{\sqrt{6} + 0}{\sqrt{5}} = \frac{\sqrt{6}}{\sqrt{5}}$$

$$f^2(3) = \frac{6}{5}$$

Step 5: Calculate

$$f^2(\alpha + 1) + 5f^2(\beta) = 6 + 5 \cdot \frac{6}{5} = 6 + 6 = \boxed{12}$$

Answer: (c) 12

END OF SOLUTIONS

All 94 problems completed!